

Distributive Lattices

1. Characterization and Representation Theorems

1.1 Characterization theorems

The two typical examples of nondistributive lattices are N_5 and M_3 , whose diagrams are given in [Figure 24](#). Our next result characterizes distributivity by the absence of these lattices as sublattices.

We introduce special names and notation for these lattices. A sublattice A of a lattice L is called a *pentagon*, respectively a *diamond*, if A is isomorphic to N_5 , respectively to M_3 . If we say that e_0, e_1, e_2, e_3, e_4 is a *pentagon* (respectively, a *diamond*), we also assume that $e_0 \mapsto o$, $e_1 \mapsto a$, $e_2 \mapsto b$, $e_3 \mapsto c$, $e_4 \mapsto i$ is an isomorphism of A with N_5 (respectively, with M_3).

The characterization theorem will be stated in two forms. Theorem 101 is a striking and useful characterization of distributive lattices; Theorem 102 is a more detailed version of Theorem 101 with some additional information.

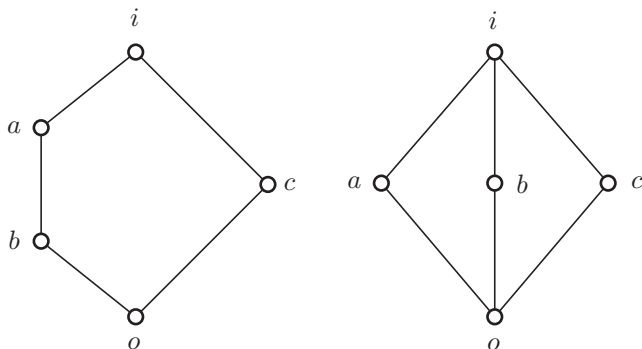
Theorem 101. *A lattice L is distributive iff L does not contain a *pentagon* or a *diamond*.*

Theorem 102.

- (i) *A lattice L is modular iff it does not contain a *pentagon*.*
- (ii) *A modular lattice L is distributive iff it does not contain a *diamond*.*

Proof.

(i) If L is modular, then every sublattice of L is also modular; N_5 is not modular, thus it cannot be isomorphic to a sublattice of L .

Figure 24. The lattices N_5 and M_3

Conversely, let L be nonmodular, let $a, b, c \in L$ with $a \geq b$ and let

$$(a \wedge c) \vee b \neq a \wedge (c \vee b).$$

The free lattice generated by a, b, c with $a \geq b$ is shown in Figure 6. Therefore, the sublattice of L generated by a, b, c must be a homomorphic image of the lattice of Figure 6. Observe that if two of the five elements

$$a \wedge c, (a \wedge c) \vee b, a \wedge (b \vee c), b \vee c, c$$

are identified under a homomorphism, then so are $(a \wedge c) \vee b$ and $a \wedge (b \vee c)$. Consequently, these five elements are distinct in L , and they form a pentagon.

(ii) Let L be modular, but nondistributive, and choose $x, y, z \in L$ such that

$$x \wedge (y \vee z) \neq (x \wedge y) \vee (x \wedge z).$$

The free modular lattice generated by x, y, z is shown in Figure 20. By inspecting the diagram we see that the elements u, x_1, y_1, z_1, v form a diamond. Thus in *any* modular lattice, they form a sublattice isomorphic to a quotient lattice of M_3 . But M_3 has only two quotient lattices: M_3 and the one-element lattice. In the former case, we have finished the proof. In the latter case, note that if u and v collapse, then so do $x \wedge (y \vee z)$ and $(x \wedge y) \vee (x \wedge z)$, contrary to our assumption. \square

Naturally, Theorems 101 and 102 could be proved without any reference to free lattices. A routine proof of (ii) runs as follows: Take x, y, z in a modular lattice L such that $x \wedge (y \vee z) \neq (x \wedge y) \vee (x \wedge z)$ and define the elements u, x_1, y_1, z_1, v as the corresponding terms of Figure 20. Then a direct computation shows that u, x_1, y_1, z_1, v form a diamond. There are some very natural objections to such a proof. How are the appropriate terms found? How is it

possible to guess the result? And there is only one answer: by working it out in the free lattice.

For some special classes of lattices, Theorems 101 and 102 have various stronger forms that claim the existence of very large or very small pentagons and diamonds. For instance, a bounded relatively complemented nonmodular lattice always contains a pentagon as a $\{0, 1\}$ -sublattice. The same is true of the diamond in certain complemented modular lattices; such results are implicit in J. von Neumann [552], [553]. If the lattice is finite, modular, and nondistributive, then it contains a *cover-preserving* diamond, that is, a diamond in which a, b, c cover o , and i covers a, b, c . (See E. Fried, G. Grätzer, and H. Lakser [201] for related results.) If L is finite and nonmodular, then the pentagon it contains can be required to satisfy $a \succ b$.

Corollary 103. *A lattice L is distributive iff every element has at most one relative complement in any interval.*

Proof. The “only if” part was proved in Section I.6.1. If L is nondistributive, then, by Theorem 102, it contains a pentagon or a diamond, and each has an element with two relative complements in some interval. \square

Corollary 104. *A lattice L is distributive iff, for any two ideals $I, J \in L$:*

$$I \vee J = \{i \vee j \mid i \in I, j \in J\}.$$

Proof. Let L be distributive. By Lemma 5(ii), if $t \in I \vee J$, then $t \leq i \vee j$ for some $i \in I$ and $j \in J$. Therefore,

$$t = t \wedge (i \vee j) = (t \wedge i) \vee (t \wedge j), \quad t \wedge i \in I, t \wedge j \in J.$$

Conversely, if L is nondistributive, then L contains elements a, b, c as in Figure 24. Let $I = \text{id}(b)$ and $J = \text{id}(c)$; observe that $a \in I \vee J$, since $a \leq b \vee c$. However, a has no representation as required in this corollary, because if $a = b_1 \vee c_1$ with $b_1 \in \text{id}(b)$ and $c_1 \in \text{id}(c)$, then $c_1 \leq a \wedge c = o$ would give that $a = b_1 \vee c_1 \leq b_1 \vee o = b_1 \leq b$, that is, $a \leq b$, a contradiction. \square

Another important property of ideals of a distributive lattice is the following statement.

Lemma 105. *Let I and J be ideals of a distributive lattice L . If $I \wedge J$ and $I \vee J$ are principal, then so are I and J .*

Proof. Let $I \wedge J = \text{id}(x)$ and $I \vee J = \text{id}(y)$. Then $y = i \vee j$ for some $i \in I$ and $j \in J$ by Corollary 104. Set $c = x \vee i$ and $b = x \vee j$; note that $c \in I$ and $b \in J$ (since $x \in I \wedge J = I \cap J$). We claim that $I = \text{id}(c)$ and $J = \text{id}(b)$. Indeed, if for instance, $J \neq \text{id}(b)$, then there is an $a > b$ with $a \in J$. It is easy to see that the elements x, a, b, c, y form a pentagon. \square

Theorem 106. *Let L be a distributive lattice and let $a \in L$. Then the map*

$$\varphi: x \mapsto (x \wedge a, x \vee a), \quad x \in L,$$

is an embedding of L into $\text{id}(a) \times \text{fil}(a)$; it is an isomorphism if a has a complement.

Proof. The map φ is one-to-one, since if $\varphi(x) = \varphi(y)$, then x and y are both relative complements of a in the same interval; thus $x = y$ by Corollary 103. Distributivity implies that φ is a homomorphism.

If a has a complement b and $(u, v) \in \text{id}(a) \times \text{fil}(a)$, then $\varphi(x) = (u, v)$ for $x = (u \vee b) \wedge v$; therefore, φ is an isomorphism. \square

1.2 Structure theorems, finite case

We start the detailed investigation of the structure of distributive lattices with the finite case. Our basic tool is the concept of down-sets introduced in Section I.1.6. Note that $\text{Down } P$ is a lattice in which join and meet are union and intersection, respectively, and thus $\text{Down } P$ is distributive.

In Section I.6.3, we introduced the order $\text{Ji } L$ of nonzero join-irreducible elements of a lattice L . Set

$$\text{spec}(a) = \{x \in \text{Ji } L \mid x \leq a\} = \text{id}(a) \cap \text{Ji } L = \downarrow a \cap \text{Ji } L,$$

the *spectrum* of a . (We give a variant of this definition in the proof of Theorem 119.)

The structure of finite distributive lattices is revealed by the following result:

Theorem 107. *Let L be a finite distributive lattice. Then the map*

$$\varphi: a \mapsto \text{spec}(a)$$

is an isomorphism between L and $\text{Down } \text{Ji } L$.

Proof. Since L is finite, every element is the join of nonzero join-irreducible elements; thus

$$a = \bigvee \text{spec}(a),$$

showing that the map φ is one-to-one. Obviously,

$$\text{spec}(a) \cap \text{spec}(b) = \text{spec}(a \wedge b),$$

and so $\varphi(a \wedge b) = \varphi(a) \cap \varphi(b)$. The formula $\varphi(a \vee b) = \varphi(a) \cup \varphi(b)$ is equivalent to

$$\text{spec}(a \vee b) = \text{spec}(a) \cup \text{spec}(b).$$

To verify this formula, note that $\text{spec}(a) \cup \text{spec}(b) \subseteq \text{spec}(a \vee b)$ is trivial. Now let $x \in \text{spec}(a \vee b)$. Then

$$x = x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b);$$

therefore, $x = x \wedge a$ or $x = x \wedge b$, since x is join-irreducible. Thus $x \in \text{spec}(a)$ or $x \in \text{spec}(b)$, that is, $x \in \text{spec}(a) \cup \text{spec}(b)$.

Finally, we have to show that if $A \in \text{Down Ji } L$, then $\varphi(a) = A$ for some $a \in L$. Set $a = \bigvee A$. Then $\text{spec}(a) \supseteq A$ is obvious. Let $x \in \text{spec}(a)$; then

$$x = x \wedge a = x \wedge \bigvee A = \bigvee (x \wedge y \mid y \in A).$$

Since x is join-irreducible, it follows that $x = x \wedge y$, for some $y \in A$, implying that $x \in A$, since A is a down-set. \square

Corollary 108. *The correspondence $L \mapsto \text{Ji } L$ makes the class of all finite distributive lattices with more than one element correspond to the class of all finite orders; isomorphic lattices correspond to isomorphic orders, and vice versa.*

Proof. This is obvious from $\text{Ji Down } P \cong P$ and $\text{Down Ji } L \cong L$. \square

A sublattice S of $\text{Pow } A$ is called a *ring of sets*. Since $\text{Down Ji } L$ is a ring of sets, we obtain:

Corollary 109. *A finite lattice is distributive iff it is isomorphic to a ring of sets.*

If Q is unordered, then $\text{Down } Q = \text{Pow } Q$; if B is finite and boolean, then $\text{Ji } B = \text{Atom}(B)$ and therefore, $\text{Ji } B$ is unordered. Thus we get:

Corollary 110. *A finite lattice is boolean iff it is isomorphic to the boolean lattice of all subsets of a finite set.*

For an element a of a lattice L , the representation

$$a = x_0 \vee \cdots \vee x_{n-1}$$

is *redundant* if

$$a = x_0 \vee \cdots \vee x_{i-1} \vee x_{i+1} \vee \cdots \vee x_{n-1},$$

for some $0 \leq i < n$; otherwise it is *irredundant*.

Corollary 111. *Every element of a finite distributive lattice has a unique irredundant representation as a join of join-irreducible elements.*

Proof. The existence of such a representation is obvious. If

$$a = x_0 \vee \cdots \vee x_{n-1}$$

is an irredundant representation, then

$$\text{spec}(a) = \bigcup (\text{spec}(x_i) \mid 0 \leq i < n).$$

Thus x occurs in such a representation iff x is a maximal element of $\text{spec}(a)$; hence the uniqueness. \square

Corollary 112. *Every maximal chain C of a finite distributive lattice L is of length $|\text{Ji } L|$.*

Proof. For $a \in \text{Ji } L$, let $m(a)$ be the smallest member of C majorizing a . Then

$$\varphi: a \mapsto m(a)$$

is a one-to-one map of $\text{Ji } L$ onto the nonzero elements of C .

To prove that φ is one-to-one, let $a \neq b \in \text{Ji } L$ and $m(a) = m(b)$. If $m(a) = m(b) = 0$, then $a = b = 0$, contradicting that $a \neq b$. So let $m(a) = m(b) > 0$. Then $m(a) \succ x$ for an element $x \in C$. Therefore, $x \vee a = m(a) = m(b) = x \vee b$; and so $a = a \wedge (x \vee b) = (a \wedge x) \vee (a \wedge b)$, implying that $a \leq x$ or $a \leq b$, because a is join-irreducible. But $a \leq x$ implies that $m(a) \leq x < m(a)$, a contradiction. Consequently, $a \leq b$; similarly, $b \leq a$; thus $a = b$.

To prove that φ is onto, let $y \succ z$ in C . Then $\text{spec}(y) \supset \text{spec}(z)$, by Theorem 107, and so $y = m(a)$ for every $a \in \text{spec}(y) - \text{spec}(z)$. \square

Corollary 112 and its dual yield

$$|\text{Ji } L| = |\text{Mi } L|.$$

This also holds in the modular case, see Section V.5.13.

For a finite distributive lattice L , what is the smallest k such that L is embeddable in a direct product of k chains? For $a \in L$, let n_a be the number of elements of L covering a . Then $k = \max\{n_a \mid a \in L\}$. This is an easy application of the result of R. P. Dilworth [157], discussed for $k \leq 2$ in Exercises 1.50–1.53 and in its full generality in Section 5.13. Note also that k is the same as the width of $\text{Ji } L$.

It seems hard to generalize the uniqueness of an irredundant join-representation of an element of a finite distributive lattice. The most useful generalization is in R. P. Dilworth [153] (utilized, for instance, in the theory of finite convex geometries). In my opinion, the best generalization is that of R. P. Dilworth and P. Crawley [161] to relatively atomic, distributive, algebraic lattices. See the survey article by R. P. Dilworth [160] and S. Kinugawa and J. Hashimoto [472]. Some results on, and references to, the modular and semimodular cases can be found in Chapter V.

1.3 \diamond Structure theorems, finite case, categorical variant

The following version of Corollary 108 gives a wealth of additional information on the correspondence between finite orders and finite distributive lattices.

\diamond **Theorem 113.** *Let P and Q be finite orders. Let*

$$L = \text{Down } P$$

$$K = \text{Down } Q$$

Then

- (i) *With every $\{0, 1\}$ -homomorphism $f: L \rightarrow K$ we can associate an isotone map $\text{Ji}(f): Q \rightarrow P$ defined by*

$$\text{Ji}(f)(y) = \inf \{ x \in P \mid y \in f(\downarrow x) \},$$

for $y \in Q$.

- (ii) *With every isotone map $\psi: Q \rightarrow P$ we can associate a $\{0, 1\}$ -homomorphism $\text{Down}(\psi): L \rightarrow K$ defined by*

$$\text{Down}(\psi)(a) = \psi^{-1}(a),$$

for $a \in L$.

- (iii) *The constructions of (i) and (ii) are inverse to one another, and so yield together a bijection between $\{0, 1\}$ -homomorphisms $L \rightarrow K$ and isotone maps $Q \rightarrow P$.*

- (iv) *f is one-to-one iff $\text{Ji}(f)$ is onto.*

- (v) *f is onto iff $\text{Ji}(f)$ is an order-embedding.*

This result tells us that there is a close relationship, like an isomorphism, between finite orders and finite distributive lattices. Category theory provides the language to formulate this mathematically. We give here an informal description of how this is done.

The finite orders form a *category* $\mathbf{Ord}_{\text{fin}}$; the *objects* are the finite orders, and for the finite orders P and Q , the category contains the set of *morphisms* $\text{Hom}(P, Q)$, the set of isotone maps from P to Q . If $\alpha \in \text{Hom}(P, Q)$ and $\beta \in \text{Hom}(Q, R)$, we can form the *composition* $\beta \circ \alpha$. We write $\beta\alpha$ for $\beta \circ \alpha$. Composition is associative.

Similarly, finite distributive lattices form a category \mathbf{D}_{fin} , where the morphisms are $\{0, 1\}$ -homomorphisms.

We have a *contravariant functor* $\text{Down}: \mathbf{Ord}_{\text{fin}} \rightarrow \mathbf{D}_{\text{fin}}$, that is, Down maps the objects of $\mathbf{Ord}_{\text{fin}}$ to objects of \mathbf{D}_{fin} and if it maps P to L and Q to K , then it maps $\text{Hom}(P, Q)$ to $\text{Hom}(K, L)$. (This reversal of direction is what is meant

by calling this functor “contravariant”. Functors that preserve the order of morphisms are called “covariant”.) Similarly, we have a contravariant functor $\text{Ji}: \mathbf{D}_{\text{fin}} \rightarrow \mathbf{Ord}_{\text{fin}}$, that is, Ji maps the objects of \mathbf{D}_{fin} to objects of $\mathbf{Ord}_{\text{fin}}$ and if it maps L to P and K to Q , then it maps $\text{Hom}(L, K)$ to $\text{Hom}(Q, P)$.

Clearly, the composition $\text{Down} \circ \text{Ji}$ is a *covariant functor* from the category \mathbf{D}_{fin} into itself; it is covariant, because if it maps L to L' and K to K' , then it maps $\text{Hom}(L, K)$ to $\text{Hom}(L', K')$.

Note that in Theorem 107, we get an isomorphism φ_L between L and $(\text{Down} \circ \text{Ji})(L)$.

Let $\text{Id}_{\mathbf{D}_{\text{fin}}}$ be the identity functor on \mathbf{D}_{fin} .

Theorem 114. *The family of isomorphisms $(\varphi_L \mid L \in \mathbf{D}_{\text{fin}})$ is a natural isomorphism between the functors $\text{Id}_{\mathbf{D}_{\text{fin}}}$ and $\text{Down} \circ \text{Ji}$, meaning that if $\varphi \in \text{Hom}(L, K)$, then the diagram*

$$\begin{array}{ccc} L & \xrightarrow{\varphi} & K \\ \cong \downarrow \varphi_L & & \cong \downarrow \varphi_K \\ (\text{Down} \circ \text{Ji})(L) & \xrightarrow{(\text{Down} \circ \text{Ji})(\varphi)} & (\text{Down} \circ \text{Ji})(K) \end{array}$$

is commutative.

And there is an analogous statement for a natural isomorphism between the functors $\text{Id}_{\mathbf{Ord}_{\text{fin}}}$ and $\text{Ji} \circ \text{Down}$.

1.4 Structure theorems, infinite case

The crucial Theorem 107 and its most important consequence, Corollary 109, depend on the existence of sufficiently many join-irreducible elements in a finite distributive lattice. In an infinite distributive lattice, there may be no join-irreducible element. Note that in a distributive lattice L , a nonzero element a is join-irreducible iff $L - \text{fil}(a)$ is a prime ideal. In the infinite case, the role of join-irreducible elements is taken by prime ideals. The crucial result is the existence of sufficiently many prime ideals (as illustrated in Figure 25).

For a distributive lattice L with more than one element, let $\text{Spec } L$ (the “spectrum” of L) denote the set of all prime ideals of L , regarded as an order under \subseteq . The importance of $\text{Spec } L$ should be clear from the following results. Topologies on $\text{Spec } L$ will be discussed in Section 5.

We start with the fundamental result of M. H. Stone [668]:

Theorem 115. *Let L be a distributive lattice, let I be an ideal, let D be a filter of L , and let $I \cap D = \emptyset$. Then there exists a prime ideal P of L such that $P \supseteq I$ and $P \cap D = \emptyset$.*

Proof. Some form of the Axiom of Choice is needed to prove this statement. The most convenient form for this proof is:

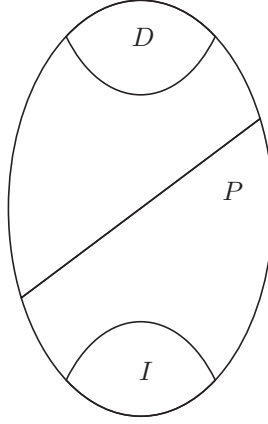


Figure 25. Illustrating Theorem 115

Zorn's Lemma. *Let A be a set and let \mathcal{X} be a nonempty subset of $\text{Pow } A$. Let us assume that \mathcal{X} has the following property: If \mathcal{C} is a chain in $(\mathcal{X}; \subseteq)$, then $\bigcup \mathcal{C} \in \mathcal{X}$. Then \mathcal{X} has a maximal member.*

We define

$$\mathcal{X} = \bigcap (P \in \text{Spec } L \mid P \supseteq I, P \cap D = \emptyset)$$

and verify that \mathcal{X} satisfies the hypothesis of Zorn's Lemma. The set \mathcal{X} is nonempty, since $I \in \mathcal{X}$. Let \mathcal{C} be a chain in \mathcal{X} and let $M = \bigcup \mathcal{C}$. If $a, b \in M$, then $a \in X$ and $b \in Y$ for some $X, Y \in \mathcal{C}$. Since \mathcal{C} is a chain, either $X \subseteq Y$ or $Y \subseteq X$ hold. If say, $X \subseteq Y$, then $a, b \in Y$, and so $a \vee b \in Y \subseteq M$, since Y is an ideal. Also, if $b \leq a \in M$, then $a \in X \in \mathcal{C}$; since X is an ideal, $b \in X \subseteq M$. Thus M is an ideal. It is obvious that $M \supseteq I$ and $M \cap D = \emptyset$, verifying that $M \in \mathcal{X}$. Therefore, by Zorn's Lemma, \mathcal{X} has a maximal element P .

We claim that P is a prime ideal. Indeed, if P is not prime, then there exist $a, b \in L$ such that $a, b \notin P$ but $a \wedge b \in P$. The maximality of P yields that $(P \vee \text{id}(a)) \cap D \neq \emptyset$ and $(P \vee \text{id}(b)) \cap D \neq \emptyset$. Thus there are $p, q \in P$ such that $p \vee a \in D$ and $q \vee b \in D$. Then $x = (p \vee a) \wedge (q \vee b) \in D$, since D is a filter. Expanding by distributivity,

$$x = (p \wedge q) \vee (p \wedge b) \vee (a \wedge q) \vee (a \wedge b) \in P;$$

thus $P \cap D \neq \emptyset$, a contradiction. □

Corollary 116. *Let L be a distributive lattice, let I be an ideal of L , and let $a \in L$ and $a \notin I$. Then there is a prime ideal P such that $P \supseteq I$ and $a \notin P$.*

Proof. Apply Theorem 115 to I and $D = \text{fil}(a)$. □

Corollary 117. *Let L be a distributive lattice, $a, b \in L$ and $a \neq b$. Then there is a prime ideal containing exactly one of a and b .*

Proof. Either $\text{id}(a) \cap \text{fil}(b) = \emptyset$ or $\text{fil}(a) \cap \text{id}(b) = \emptyset$, so we can apply Corollary 116. \square

Corollary 118. *Every ideal I of a distributive lattice is the intersection of all prime ideals containing it.*

Proof. Let

$$I_1 = \bigcap (P \in \text{Spec } L \mid P \supseteq I).$$

Clearly, every $a \in I$ belongs to I_1 . Conversely, if $a \notin I$, then by Corollary 117, there is a P in the family $\{P \in \text{Spec } L \mid P \supseteq I\}$ not containing a , so $a \notin I_1$. \square

As a final application, we get the celebrated result of G. Birkhoff [61] and M. H. Stone [668]:

Theorem 119. *A lattice is distributive iff it is isomorphic to a ring of sets.*

Proof. Let L be a distributive lattice. For $a \in L$, set

$$\text{spec}(a) = \{P \in \text{Spec } L \mid a \notin P\}.$$

the *spectrum* of a . Then the family of sets $\{\text{spec}(a) \mid a \in L\}$ is a ring of sets, and the map $a \mapsto \text{spec}(a)$ is an isomorphism. The details are similar to the proof of Theorem 107, except for the first step, which now uses Corollary 117. \square

1.5 Some applications

Corollary 120. *Let L be a distributive lattice with more than one element. An identity holds in L iff it holds in the two-element chain, C_2 .*

Proof. Let $p = q$ hold in L . Since $|L| > 1$, clearly $C_2 \leq L$, and so $p = q$ holds in C_2 . Conversely, let $p = q$ hold in C_2 . Note that $C_2 = \text{Pow } X$ with $|X| = 1$, and that $\text{Pow } A$ is isomorphic to the direct power $(\text{Pow } X)^{|A|}$. Therefore, $p = q$ holds in any $\text{Pow } A$. By Theorem 119, L is a sublattice of some $\text{Pow } A$; thus $p = q$ holds in L . \square

So now we have the result we claimed in Section I.5.5:

Theorem 121. *For any order P , a lattice completely freely generated by P , $\text{CFree}_{\mathbf{V}} P$, exists for any variety \mathbf{V} containing a two-element lattice.*

We can adapt Theorem 119 to boolean lattices, see M. H. Stone [668], using the concept of a *field of sets*: a ring of sets closed under set complementation.

Corollary 122. *A lattice is boolean iff it is isomorphic to a field of sets.*

Proof. Use the representation of Theorem 119. Obviously,

$$\text{spec}(a') = \text{Spec } L - \text{spec}(a),$$

and thus complements are also preserved. \square

Some interesting properties of L are reflected in $\text{Spec } L$. An important result of this type is the following theorem of L. Nachbin [537] (see also L. Rieger [611]):

Theorem 123. *Let L be a bounded distributive lattice with $0 \neq 1$. Then L is a boolean lattice iff $\text{Spec } L$ is unordered.*

Proof. Let L be boolean, $P, Q \in \text{Spec } L$, and $P \subset Q$. Choose $a \in Q - P$. Since $a \in Q$, clearly $a' \notin Q$, and thus $a' \notin P$. Therefore, $a, a' \notin P$, but $a \wedge a' = 0 \in P$, a contradiction, showing that $\text{Spec } L$ is unordered. This proof, in fact, verifies that in a boolean algebra every prime ideal is maximal.

Now let $\text{Spec } L$ be unordered and $a \in L$, and let us assume that a has no complement. Set

$$D = \{x \mid a \vee x = 1\}.$$

By distributivity, D is a filter. Take

$$D_1 = D \vee \text{fil}(a) = \{x \mid x \geq d \wedge a, \text{ for some } d \in D\}.$$

The filter D_1 does not contain 0, since $0 = d \wedge a$ and $a \vee d = 1$ would mean that d is a complement of a . Thus there exists a prime ideal P disjoint from D_1 .

Note that $1 \notin \text{id}(a) \vee P$, otherwise $1 = a \vee p$, for some $p \in P$, contradicting that $P \cap D = \emptyset$. Thus some prime ideal Q contains $\text{id}(a) \vee P$; and so $P \subset Q$, which is impossible since $\text{Spec } L$ is unordered. \square

According to Corollary 118, every ideal is an intersection of prime ideals. When is this representation unique? This question was answered in J. Hashimoto [375].

Theorem 124. *Let L be a bounded distributive lattice with $0 \neq 1$. Every ideal has a unique representation as an intersection of prime ideals iff L is a finite boolean lattice.*

Proof. If L is a finite boolean lattice, then P is a prime ideal iff $P = \text{id}(a)$, where a is a dual atom; the uniqueness follows from Corollary 111 (or it is obvious by direct computation).

Now let every ideal of L have a unique representation as a meet of prime ideals. We claim that $\text{Id } L$ is boolean. Let $I \in \text{Id } L$; define

$$J = \bigcap (P \in \text{Spec } L \mid P \not\supseteq I).$$

Then

$$I \wedge J = \bigcap (P \mid P \in \text{Spec } L) = \text{id}(0).$$

If $L \neq I \vee J$, then there is a prime ideal $P_0 \supseteq I \vee J$, and consequently J has two representations:

$$\bigcap (P \mid P \not\supseteq I) = P_0 \cap \bigcap (P \mid P \not\supseteq I).$$

Thus $L = I \vee J$ and J is a complement of I in $\text{Id } L$.

So $I \vee J = L = \text{id}(1)$ and $I \wedge J = \text{id}(0)$, both principal. Thus by Lemma 105, every ideal of L is principal. We conclude that $L \cong \text{Id } L$, and so L is boolean. By Exercise I.6.24, L satisfies the Ascending Chain Condition; thus every element of L other than the unit is majorized by a dual atom. Since the complement of a dual atom is an atom, by taking complements, we find that every nonzero element of L majorizes an atom.

If $p_0, p_1, \dots, p_n, \dots \in \text{Atom}(L)$, then the ascending chain

$$p_0, p_0 \vee p_1, \dots, p_0 \vee p_1 \vee \dots \vee p_n, \dots$$

does not terminate, contradicting that L satisfies the Ascending Chain Condition. Thus $\text{Atom}(L)$ is finite, $\text{Atom}(L) = \{p_0, \dots, p_{n-1}\}$. Define the element $a = p_0 \vee \dots \vee p_{n-1}$. If $a' \neq 0$, then a' has to majorize an atom, which is impossible. Therefore, $a' = 0$, $a = 1$, and $L \cong \text{Pow } X$ with $|X| = n$. \square

1.6 Automorphism groups

Let L be a lattice and let $\text{Aut } L$ be the automorphism group of L (see Section I.3.1). In this section, we prove the characterization theorem of automorphism groups, in fact, as in G. Birkhoff [68], we prove here more. (This proof is from G. Grätzer, E. T. Schmidt, and D. Wang [351].)

Theorem 125. *Every group G can be represented as the automorphism group of a distributive lattice D . If G is finite, D can be chosen to be finite.*

Proof. Let $G = \{g_\gamma \mid \gamma < \alpha\}$ with $g_0 = 1$, the unit element of the group; we assume that $|G| > 1$. We view ordinals as well-ordered chains. In particular, $\gamma \cong \delta$ iff $\gamma = \delta$ for any ordinals γ and δ .

For every $x, y \in G$ with $y \neq 1$ (equivalently, with $x \neq yx$), we construct the order $P(x, y)$ of Figure 26, defined on the set $\{x, yx\} \cup \{(x, y, a_\gamma) \mid \gamma < \beta\}$, where $y = g_\beta$. Note that $G \cap P(x, y) = \{x, yx\}$, where yx is the product of y and x in G . We order this set by

$$\begin{aligned} x < (x, y, 1) < (x, y, 2) < \dots < (x, y, \gamma) < \dots, & \text{ for } \gamma < \beta, \\ yx < (x, y, 0) < (x, y, 1). \end{aligned}$$

The two minimal elements of $P(x, y)$ are x and yx , both in G .

Let $P = \bigcup (P(x, y) \mid x, y \in G, y \neq 1)$ be ordered by $u < v$ in P iff $u < v$ in some $P(x, y)$. It is sufficient to prove that $\text{Aut } P \cong G$. Indeed, let L be the distributive lattice completely freely generated by P ; this lattice L exists by Theorem 92. Then $\text{Aut } P \cong \text{Aut } L$; moreover, if G is finite, then both P and L are finite.

To prove that $\text{Aut } P \cong G$, let σ be an automorphism of P . The set of minimal elements of P is G ; it follows that the map σ permutes G . Let $a = \sigma(1)$ and let $b \in G$. We want to show that $\sigma(b) = ba$.

If $b = 1$, this holds by the definition of a . So let us assume that $b \neq 1$. Let $b = g_\beta$ with $\beta < \alpha$. Then the order $P(1, b)$, with minimal elements 1 and b , is defined (since $1 \neq b$). Also, $\sigma(b) \neq a$ and so $\sigma(b) = ua$ for some $u \in G$ with $u \neq 1$. Therefore, $P(a, u)$ with minimal elements $a = \sigma(1)$ and $ua = \sigma(b)$, is defined.

Thus σ takes the minimal elements of $P(1, b)$ into the minimal elements of $P(a, u)$, hence it must take all of $P(1, b)$ to $P(a, u)$, so $P(1, b) \cong P(a, u)$. Thus the top chain of $P(a, u)$ is the same as the top chain of $P(1, b)$, that is, β , and so $u = b$, proving that $\sigma(b) = ba$.

For every $u \in G$, define the permutation σ_u of G by $\sigma_u(v) = vu$. Then we have just proved that every automorphism of P restricted to G is of this form; the converse is trivial. This completes the proof of the theorem. \square

For an alternative short proof, producing a surprisingly nice distributive lattice, see G. Grätzer, H. Lakser, and E. T. Schmidt [315], Exercises 1.55–1.57.

Small lattices with given automorphism groups are considered in R. Frucht [207] and [208]. R. N. McKenzie and J. Sichler have some related results for lattices of finite length. Two sample results: every group is the automorphism

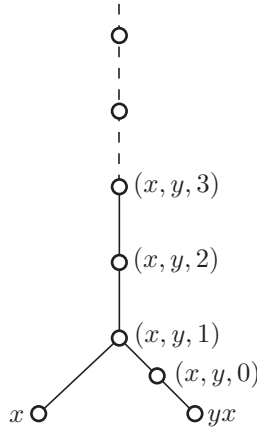


Figure 26. The order $P(x, y)$

group of a lattice of finite length; for every lattice L , there exists a bounded lattice K such that $\text{End } L \cong \text{End}_{\{0,1\}} K$ and if L is finite or finite length, then so is K , where $\text{End } L$ is the endomorphism monoid (that is, semigroup with identity) of L and $\text{End}_{\{0,1\}} K$ is the monoid of those endomorphism of K that fix 0 and 1. See also J. Sichler [645] and Section VII.3.4.

1.7 \diamond Distributive lattices and general algebra

R. Dedekind found the distributive identity by investigating ideals of number fields. Rings with a distributive lattice of ideals have been investigated by E. Noether [554], L. Fuchs [210] (who named such rings *arithmetical rings*—*MathSciNet* lists 61 papers on arithmetical rings alone), I. S. Cohen [93], and C. U. Jensen [426]. Varieties of rings with distributive ideal lattices were considered in G. Michler and R. Wille [530] and in H. Werner and R. Wille [718]. E. A. Behrens [52] and [53] considered rings in which one-sided ideals form a distributive lattice. Rings with a distributive lattice of subrings were classified in P. A. Freidman [192]. In this context, G. M. Bergman's work on the distributive-divisor-lattice of free algebras should be mentioned, see Chapter 4 of P. M. Cohn [94] and P. M. Cohn [95].

For an overview of distributive modules and rings, see A. A. Tuganbaev [682].

H. L. Silcock [648] proved that every finite distributive lattice is isomorphic to the lattice of normal subgroups of a group G . P. P. Pálffy [575] improved this result: G may be taken to be finite solvable. P. Růžička, J. Tůma, and F. Wehrung [623] proved that every distributive algebraic lattice with at most \aleph_1 compact elements is isomorphic to the normal subgroup lattice of some locally finite group and to the submodule lattice of some right module (over a non-commutative ring). Furthermore, they proved that the \aleph_1 bound is optimal: for example, the congruence lattice of the free lattice on \aleph_2 generators is not isomorphic to the congruence lattice of any congruence-permutable algebra.

The subgroup lattice of a group G is distributive iff G is locally cyclic, see O. Ore [559] and [560].

The distributivity of congruence lattices of lattices has a number of important consequences, for instance, Jónsson's Lemma (Theorem 475). B. Jónsson [444] discovered that many of these results hold for arbitrary universal algebras with distributive congruence lattices. His result has found applications that go far beyond lattice theory—it has been applied to lattice-ordered algebras, closure algebras, nonassociative lattices, cylindric algebras, monadic algebras, lattices with pseudocomplementation, primal algebras, and multi-valued logics. (Jónsson's Lemma is referenced in 55 papers according to *MathSciNet*.)

The foregoing examples show the central role played by distributive lattices in applications of the lattice concept.

Exercises

- 1.1. Consider the three lattices whose diagrams are shown in [Figure 7](#). Which are distributive? Show that the nondistributive ones contain a pentagon.
- 1.2. Work out a direct proof of Theorem 102(i).
- 1.3. Work out a direct proof of Theorem 102(ii).
- 1.4. Let K be a five-element distributive lattice. Is there an identity $p = q$ such that $p = q$ holds in a lattice L iff L has no sublattice isomorphic to K ?
- 1.5. Does the property stated in Lemma 105 characterize distributive lattices?
- 1.6. Let L be a distributive lattice with zero and unit. Prove that the direct decompositions $L_0 \times L_1$ of L are in one-to-one correspondence with the complemented elements of L .
- 1.7. Prove that the complemented elements of a distributive lattice form a sublattice.
- 1.8. Let L be a distributive lattice with zero and unit. Let

$$L \cong L_0 \times L_1 \cong K_0 \times K_1.$$

Show that there is a direct decomposition

$$L \cong A_0 \times A_1 \times A_2 \times A_3$$

such that

$$A_0 \times A_1 \cong L_0,$$

$$A_2 \times A_3 \cong L_1,$$

$$A_0 \times A_2 \cong K_0,$$

$$A_1 \times A_3 \cong K_1.$$

- 1.9. Let $L = \mathbf{B}_3$. Describe the orders $\text{Ji } L$ and $\text{Down Ji } L$.
- 1.10. Let $L = \text{Free}_{\mathbf{D}}(3)$, see [Figure 19](#). Describe $\text{Ji } L$ and $\text{Down Ji } L$. Compare $|\text{Free}_{\mathbf{D}}(3)|$ with $|\text{Ji Free}_{\mathbf{D}}(3)|$.
- 1.11. Verify Theorem 107 for the distributive lattices of Exercises 1.9 and 1.10.
- 1.12. Does Theorem 107 hold for countable chains?
- 1.13. Consider the modular lattice $L = \text{Free}_{\mathbf{M}}(3)$. How many diamonds are in L ?
- 1.14. Extend Theorem 107 to distributive lattices satisfying the Descending Chain Condition (see Exercise I.1.16).
- 1.15. Extend Corollary 108 to distributive lattices satisfying the Descending Chain Condition.

- 1.16. Can Exercises 1.14 and 1.15 be further sharpened?
- 1.17. Let L be a distributive lattice with zero and unit. Let C_0 and C_1 be finite chains in L . Show that there exist chains $D_0 \supseteq C_0$ and $D_1 \supseteq C_1$ such that $|D_0| = |D_1|$.
- 1.18. Derive from Exercise 1.17 the result that all maximal chains of a finite distributive lattice have the same length.
- 1.19. Find examples showing that Exercise 1.17 is not valid if “finite” is omitted.
- 1.20. For a finite distributive lattice L and $a \in L$, let n_a be the number of elements of L covering a . Prove that

$$\max\{n_a \mid a \in L\} = \text{width}(\text{Ji } L).$$

- 1.21. For a finite distributive lattice L , what is the smallest k such that L is embeddable in a direct product of k chains? (Hint: the number in Exercise 1.20.)
- 1.22. Prove the theorem “ L is modular iff $\text{Id } L$ is modular” by showing that “ L contains a pentagon iff $\text{Id } L$ contains a pentagon”.
- *1.23. Is the second statement of Exercise 1.22 true for the diamond rather than for the pentagon?
- 1.24. Let L be a distributive lattice, $a, b, c \in L$, and $a \leq b$. Is it true that $[a, b]$ is boolean iff $[a \wedge c, b \wedge c]$ and $[a \vee c, b \vee c]$ are boolean?
- 1.25. For an order P , let $\text{Down}_{\text{fin}} P$ denote the lattice of all subsets of P of the form $\downarrow H$, where $H \subseteq P$ is finite. Does Theorem 107 hold for $\text{Down}_{\text{fin}} P$?
- 1.26. Show that the Ascending Chain Condition is equivalent to the Descending Chain Condition for boolean lattices.
- 1.27. Show that Exercise 1.26 fails to hold for *generalized boolean lattices* (that is, relatively complemented distributive lattices with zero).
- 1.28. Let L be a lattice, let P be a prime ideal of L , and let $a, b, c \in L$. Prove that if $a \vee (b \wedge c) \in P$, then $(a \vee b) \wedge (a \vee c) \in P$.
- 1.29. Using Exercise 1.28, show that the lattice L is distributive iff, for all $x, y \in L$ with $x < y$, there exists a prime ideal P satisfying $x \in P$ and $y \notin P$.
- 1.30. Verify the statement of Exercise 1.29 using Theorem 101.
- 1.31. Let L be a distributive lattice. Then L is relatively complemented iff $\text{Spec } L$ is unordered.
- 1.32. Prove Theorem 115 by well-ordering the lattice, $L = \{a_\gamma \mid \gamma < \alpha\}$, and deciding one by one for each a_γ whether $a_\gamma \in P$ or $a_\gamma \notin P$ (M.H. Stone [668]).
- 1.33. Let L be a distributive lattice with unit. Show that every prime ideal P is contained in a *maximal prime ideal* Q (a prime ideal R is maximal if $R \subseteq S \in \text{Spec } L$ implies that $R = S$).
- 1.34. Let L be a distributive lattice with zero. Verify that every prime ideal P contains a *minimal prime ideal* Q (a prime ideal Q is minimal if

$Q \supseteq S \in \text{Spec } L$ implies that $Q = S$).

- 1.35. Find a distributive lattice L with no minimal and no maximal prime ideals.
- 1.36. Investigate the connections among the Ascending Chain Condition (and Descending Chain Condition) for a distributive lattice L , for the ideal lattice $\text{Id } L$, and for the order $\text{Spec } L$.
- 1.37. Let L be a distributive lattice with zero and let $I \in \text{Id } L$. Show that

$$\{x \mid \text{id}(x) \wedge I = \text{id}(0)\}$$

is the pseudocomplement of the ideal I in $\text{Id } L$. Conclude that $\text{Id } L$ is pseudocomplemented.

- 1.38. Let L be a distributive lattice with zero and let $I \in \text{Id } L$. Prove that $I = I^{**}$, for every $I \in \text{Id } L$, iff L is a generalized boolean lattice satisfying the Descending Chain Condition.
- 1.39. The congruence relations α and β *permute* if $\alpha \circ \beta = \beta \circ \alpha$. Show that the congruences of a relatively complemented lattice permute.
- 1.40. Prove the converse of Exercise 1.39 for distributive lattices.
- 1.41. Generalize Theorem 124 to distributive lattices without 0 and 1.
- *1.42. Let L be a distributive lattice, let $a \in L$, let $S \leq L$, and let $a \notin S$. Show that there exists a prime ideal P and a prime filter Q such that $a \notin P \cup Q \supseteq S$, provided that a is not the 0 or 1 of L (J. Hashimoto [375]).
- 1.43. Let L be a relatively complemented distributive lattice. A sublattice K of L is *proper* if $K \neq L$. Show that every proper sublattice of L can be extended to a maximal proper sublattice of L (K. Takeuchi [671]; see also J. Hashimoto [375] and G. Grätzer and E. T. Schmidt [333]).
- 1.44. Show that the statement of Exercise 1.43 is not valid in general if L is not relatively complemented (K. Takeuchi [671]; see also M. E. Adams [2]).
- 1.45. Generalize Corollary 111 to infinite distributive lattices, claiming the unique irredundant representation of certain ideals as a meet of prime ideals.
- 1.46. If P is a prime ideal of L , then $\text{id}(P)$ is a principal prime ideal of $\text{Id } L$. Is the converse true?
- 1.47. Show that Corollary 117 characterizes distributivity.
- 1.48. Let C be a chain in an order P . If $C \subseteq D$ implies that $C = D$, for every chain D in P , then C is called *maximal*. Using Zorn's Lemma, show that every chain is contained in a maximal chain.
- 1.49. Prove that a finite distributive lattice is planar iff no element is covered by three elements.
- 1.50. Show that a finite distributive lattice is planar iff it is dismantlable (see Exercise I.6.39).

- 1.51. Show that we can obtain every planar distributive lattice D in the following way. We start with a direct product of two finite chains, $L_0 = C_1 \times C_2$. We obtain L_1 by removing a doubly irreducible element from the *boundary* of L_0 . For $i > 1$, we obtain L_i by removing a doubly irreducible element from the *boundary* of L_{i-1} . In finitely many steps, we obtain D .
- 1.52. Show that D is a *cover-preserving sublattice* of $C_1 \times C_2$ in Exercise 1.51, that is, if $a \prec b$ in D , then $a \prec b$ in $C_1 \times C_2$.
- 1.53. Let S be a sublattice of the finite lattice L . Then S can be represented in the form

$$L - \bigcup ([a_i, b_i] \mid i \in I),$$

where a_i is join-irreducible and b_i is meet-irreducible for all $i \in I$.

- 1.54. Prove the converse of Exercise 1.53 for distributive lattices. (Exercises 1.53 and 1.54 are from I. Rival [612]; see also I. Rival [613].)
- 1.55. Derive from Exercises VII.3.1–VII.3.16, that for every finite group G , there exists a finite graph $(V; E)$ (that is, V is a nonempty set and $E \subseteq V^2$, as in Section I.1.5) such that G is isomorphic to the automorphism group of $(V; E)$.
- 1.56. Let G and V be as in Exercise 1.55. Let F be the free distributive lattice generated by V with zero and unit. Define in F :

$$o = \bigvee (x \wedge y \mid \{x, y\} \in E).$$

Define the finite distributive lattice

$$D = [o, 1].$$

Prove that $\text{Aut } D \cong G$ (G. Grätzer, H. Lakser, and E. T. Schmidt [315]).

- 1.57. Extend the construction of Exercise 1.56 to arbitrary groups, re-proving Birkhoff's result (Theorem 125).

2. Terms and Freeness

2.1 Terms for distributive lattices

We can introduce an equivalence relation $\equiv_{\mathbf{D}}$ for lattice terms: for $p, q \in \text{Term}(n)$, let $p \equiv_{\mathbf{D}} q$ iff p and q define the same functions in the class \mathbf{D} of distributive lattices. More formally, if p and q are n -ary terms (see Section I.4.1), then $p \equiv_{\mathbf{D}} q$ if, for every *distributive lattice* L and $a_1, \dots, a_n \in L$, the equality $p(a_1, \dots, a_n) = q(a_1, \dots, a_n)$ holds (see Definitions 52 and 53).

For an n -ary lattice term p , let p/\mathbf{D} denote the set of all n -ary lattice terms q satisfying $p \equiv_{\mathbf{D}} q$ and let $\text{Term}_{\mathbf{D}}(n)$ denote the set of all these blocks, that is,

$$\text{Term}_{\mathbf{D}}(n) = \{p/\mathbf{D} \mid p \in \text{Term}(n)\}.$$

Observe that, for any $p, p_1, q, q_1 \in \text{Term}(n)$, if $p \equiv_{\mathbf{D}} p_1$ and $q \equiv_{\mathbf{D}} q_1$, then $p \vee q \equiv_{\mathbf{D}} p_1 \vee q_1$ and $p \wedge q \equiv_{\mathbf{D}} p_1 \wedge q_1$. Thus

$$\begin{aligned} p/\mathbf{D} \vee q/\mathbf{D} &= (p \vee q)/\mathbf{D}, \\ p/\mathbf{D} \wedge q/\mathbf{D} &= (p \wedge q)/\mathbf{D} \end{aligned}$$

define the operations \vee and \wedge on $\text{Term}_{\mathbf{D}}(n)$. It is easily seen that $\text{Term}_{\mathbf{D}}(n)$ is a distributive lattice and $p/\mathbf{D} \leq q/\mathbf{D}$ iff the inequality $p \leq q$ holds in the class \mathbf{D} .

To describe the structure of $\text{Term}_{\mathbf{D}}(n)$, for $n > 0$, let $Q(n)$ denote the dual of the order of all proper nonempty subsets of $\{0, 1, \dots, n-1\}$.

Theorem 126. *Let $n > 0$. Then*

- (i) $\text{Term}_{\mathbf{D}}(n)$ is a free distributive lattice on n generators.
- (ii) $\text{Term}_{\mathbf{D}}(n)$ is isomorphic with $\text{Down } Q(n)$.
- (iii) $2^n - 2 \leq |\text{Term}_{\mathbf{D}}(n)| \leq 2^{2^n - 2}$.
- (iv) A finitely generated distributive lattice is finite.

Proof.

(i) Let L be a distributive lattice, $a_0, \dots, a_{n-1} \in L$. Then the map $x_i \mapsto a_i$ can be extended to the homomorphism

$$p/\mathbf{D} \mapsto p(a_0, \dots, a_{n-1}),$$

proving (i).

(ii) A lattice term p is called a *meet-term* if it is of the form $x_{i_0} \wedge \dots \wedge x_{i_{k-1}}$. (Recall from Section I.4 that we omit the outside parentheses and also the internal parentheses in iterated meets and iterated joins.)

For $\emptyset \neq J \subseteq \{0, \dots, n-1\}$, set

$$p_J = \bigwedge (x_i \mid i \in J).$$

We claim that, for any nonempty $J, K \subseteq \{0, \dots, n-1\}$, the inequality

$$p_J/\mathbf{D} \leq p_K/\mathbf{D}$$

holds iff the containment $J \supseteq K$ holds. The “if” part is obvious. Now assume that $J \not\supseteq K$; then there exists an $i \in K$ such that $i \notin J$. Consider the two-element chain \mathbf{C}_2 and substitute $x_i = 0$ and $x_j = 1$ for all $j \neq i$. Obviously, $p_J = 1$ and $p_K = 0$; thus the inequality $p_J \leq p_K$ fails in \mathbf{C}_2 , and therefore, in \mathbf{D} .

We claim that every lattice term is equivalent under $\equiv_{\mathbf{D}}$ to one of the form $\bigvee p_J$ for some family of nonempty sets $J \subseteq \{0, \dots, n-1\}$.

Indeed, every x_i is of this form (for a single J , which is a singleton), so it suffices to show that the set of terms equivalent to terms of this form is closed under \vee and \wedge . Closure under \vee is clear. To see closure under \wedge , we note that by distributivity,

$$\bigvee p_{J_i} \wedge \bigvee p_{K_j} \equiv_{\mathbf{D}} \bigvee (p_{J_i} \wedge p_{K_j})$$

and

$$p_{J_i} \wedge p_{K_j} \equiv_{\mathbf{D}} p_{J_i \cup K_j}.$$

Next we claim that p/\mathbf{D} is join-irreducible in $\text{Term}_{\mathbf{D}}(n)$ iff it is a p_J/\mathbf{D} . Since every $p/\mathbf{D} \in \text{Term}_{\mathbf{D}}(n)$ is a join of terms p_J/\mathbf{D} , it suffices to prove that each p_J/\mathbf{D} is join-irreducible. Let

$$p_J \equiv_{\mathbf{D}} \bigvee (p_{J_k} \mid k \in K),$$

where each J_k satisfies that $\emptyset \subseteq J_k \subset \{0, \dots, n-1\}$. Then $J \subseteq J_k$ follows from $p_J/\mathbf{D} \geq p_{J_k}/\mathbf{D}$. If $p_J/\mathbf{D} > p_{J_k}/\mathbf{D}$, holds for some $k \in K$, then $J \subset J_k$ holds.

In \mathbf{C}_2 , put $x_i = 1$, for all $i \in J$, and $x_i = 0$, otherwise. Then $p_J = 1$, and $\bigvee (p_{J_i} \mid i \in K) = 0$, which is a contradiction.

A reference to Theorem 107 completes the proof of (ii).

(iii) This proof is obvious from (ii).

(iv) This proof is obvious from (iii). □

Figure 19 is a diagram of $\text{Term}_{\mathbf{D}}(3)$.

The problem of determining $|\text{Free}_{\mathbf{D}}(n)|$ goes back to R. Dedekind [149]. For a modern survey of the field, see A. D. Korshunov [480]; the article has 356 references.

Free distributive lattices (on a finite or infinite generating set) have many interesting properties. All chains are finite or countable (the proof of this is similar to that of Theorem 550). If a and H are such that $x \wedge y = a$, for all $x, y \in H$ with $x \neq y$, call H *a-disjoint*. In a free distributive lattice, all *a-disjoint* sets are finite, see R. Balbes [44].

2.2 Boolean terms

Boolean terms are defined exactly like lattice terms except that all five operations $\vee, \wedge, ', 0, 1$ are used in the formation of the terms. A formal definition is the same as Definition 52 with two clauses added: If p is a boolean term, so is p' ; 0 and 1 are boolean terms. An n -ary boolean term p defines a function in n variables on any boolean algebra B ; we define $p(a_0, \dots, a_{n-1})$ imitating Definition 53.

For the boolean terms p and q , set $p \equiv_{\mathbf{B}} q$ if, for every boolean algebra B and $a_0, \dots, a_{n-1} \in B$, the equality $p(a_0, \dots, a_{n-1}) = q(a_0, \dots, a_{n-1})$ holds. Let p/\mathbf{B} denote the block containing p . Observe that $p \equiv_{\mathbf{B}} q$ is equivalent to the identity $p = q$ holding in the class \mathbf{B} of all boolean algebras.

Let $\text{Term}_{\mathbf{B}}(n)$ denote the set of all p/\mathbf{B} , where p is an n -ary boolean term. It is easily seen that we can define the boolean operations on $\text{Term}_{\mathbf{B}}(n)$:

$$\begin{aligned} p/\mathbf{B} \vee q/\mathbf{B} &= (p \vee q)/\mathbf{B}, \\ p/\mathbf{B} \wedge q/\mathbf{B} &= (p \wedge q)/\mathbf{B}, \\ (p/\mathbf{B})' &= p'/\mathbf{B}, \\ 0 &= 0/\mathbf{B}, \\ 1 &= 1/\mathbf{B}; \end{aligned}$$

thus $\text{Term}_{\mathbf{B}}(n)$ is a boolean algebra.

Theorem 127.

- (i) $\text{Term}_{\mathbf{B}}(n)$ is a free boolean algebra on n generators.
- (ii) $\text{Term}_{\mathbf{B}}(n)$ is isomorphic to $(\mathbf{B}_1)^{2^n}$.
- (iii) $|\text{Term}_{\mathbf{B}}(n)| = 2^{2^n}$.
- (iv) A finitely generated boolean algebra is finite.

Proof. The proof of (i) is routine (same proof as in Theorem 126).

A boolean term in x_0, \dots, x_{n-1} is called *atomic* if it is of the form

$$x_0^{i_0} \wedge \dots \wedge x_{n-1}^{i_{n-1}},$$

where $i_j = 0$ or 1 , x^0 denotes x , and x^1 denotes x' . There is an atomic term p_J , for every $J \subseteq \{0, \dots, n-1\}$, for which $i_j = 0$ iff $j \in J$. The crucial statement is:

$$p_{J_0}/\mathbf{B} \leq p_{J_1}/\mathbf{B} \quad \text{iff} \quad J_0 = J_1.$$

Indeed, let $J_0 \neq J_1$. We make the following substitutions in \mathbf{B}_1 :

$$x_i = 1 \text{ if } i \in J_0 \text{ and } x_i = 0 \text{ if } i \notin J_0.$$

This makes $p_{J_0} = 1$ and $p_{J_1} = 0$, contradicting that $p_{J_0}/\mathbf{B} \leq p_{J_1}/\mathbf{B}$.

Let $B(n)$ be the set of all boolean terms that are equivalent to one of the form $\bigvee (p_{J_i} \mid i \in K)$. Then $B(n)$ is closed under \vee and \wedge , since

$$\bigvee p_{J_i} \wedge \bigvee p_{I_k} \equiv_{\mathbf{B}} \bigvee (p_{J_i} \wedge p_{I_k})$$

and $p_{J_i} \wedge p_{I_k} \equiv_{\mathbf{B}} p_{J_i}$, if $J_i = I_k$, and $p_{J_i} \wedge p_{I_k} \equiv_{\mathbf{B}} 0$, otherwise.

Now we prove by induction on n that $x_i, x'_i \in B(n)$ for all $i < n$. If $n = 1$, then x_0 and x'_0 are atomic terms, so $x_0, x'_0 \in B(1)$. By induction,

$$x_0 \equiv_{\mathbf{B}} \bigvee (p_{J_i} \mid i \in K),$$

where the p_{J_i} are atomic $(n-1)$ -ary terms; then

$$\begin{aligned} x_0 &\equiv_{\mathbf{B}} x_0 \wedge (x_{n-1} \vee x'_{n-1}) \equiv_{\mathbf{B}} (x_0 \wedge x_{n-1}) \vee (x_0 \wedge x'_{n-1}) \\ &\equiv_{\mathbf{B}} \bigvee (p_{J_i} \wedge x_{n-1} \mid i \in K) \vee \bigvee (p_{J_i} \wedge x'_{n-1} \mid i \in K), \end{aligned}$$

and similarly for x'_0 . Thus $x_0, x'_0 \in B(n)$, and, by symmetry, $x_i, x'_i \in B(n)$ for all $i < n$. Since

$$(p_J)' \equiv_{\mathbf{B}} \bigvee (x'_i \mid i \in J) \vee \bigvee (x_i \mid i \notin J),$$

we conclude that $p'_J \in B(n)$; therefore, $B(n)$ is closed under $'$. Thus $B(n)$ is closed under $\vee, \wedge, '$ and clearly, under $0, 1$. Since $B(n)$ includes x_i , for all $i < n$, it is the set of all n -ary boolean terms.

Consequently, every p/\mathbf{B} is a join of atomic terms, the p/\mathbf{B} for p atomic terms are unordered and 2^n in number, implying (ii) and (iii). Finally, (iv) follows trivially from (iii). \square

I. Reznikoff [608] and A. Horn [400] prove that all chains of a free boolean algebra are finite or countable.

Infinitary boolean terms are considered in H. Gaifman [215] and A. W. Hales [369]; they prove that free complete boolean algebras on infinitely many generators do not exist.

2.3 Free constructs

We can use Theorems 126 and 127 to characterize free distributive lattices and free boolean algebras, respectively.

Theorem 128. *Let L be a distributive lattice generated by I . The lattice L is distributive freely generated by I iff the validity in L of a relation of the form*

$$\bigwedge I_0 \leq \bigvee I_1$$

implies that $I_0 \cap I_1 \neq \emptyset$ for finite nonempty subsets I_0 and I_1 of I .

Proof. The “only if” part can be easily verified by using substitutions in \mathbf{C}_2 . For the converse, let F be the distributive lattice freely generated by I , and let φ be the homomorphism of F into (in fact, onto) L satisfying $\varphi(i) = i$ for all $i \in I$. It suffices to prove that for the lattice terms p and q , the inequality $\varphi(p) \leq \varphi(q)$ implies that $p/\mathbf{D} \leq q/\mathbf{D}$. (We think of the elements of F as blocks of terms in I .)

Let

$$\begin{aligned} p &\equiv_{\mathbf{D}} \bigvee (\bigwedge I_j \mid j \in J), \\ q &\equiv_{\mathbf{D}} \bigwedge (\bigvee K_t \mid t \in T). \end{aligned}$$

Then $\varphi(p) \leq \varphi(q)$ takes the form

$$\bigvee (\bigwedge I_j \mid j \in J) \leq \bigwedge (\bigvee K_t \mid t \in T)$$

in L , which is equivalent to

$$\bigwedge I_j \leq \bigvee K_t$$

for all $j \in J$ and $t \in T$. By assumption, this implies that $I_j \cap K_t \neq \emptyset$ for all $j \in J$ and $t \in T$; thus

$$\bigwedge J_j \leq \bigvee K_t$$

as polynomials for all $j \in J$ and $t \in T$. This implies that $p/\mathbf{D} \leq q/\mathbf{D}$. \square

Theorem 129. *Let B be a boolean algebra generated by I . Then B is freely generated by I iff, whenever I_0, I_1, J_0, J_1 are finite subsets of I with $I_0 \cup I_1 = J_0 \cup J_1$ and $I_0 \cap I_1 = \emptyset$, then*

$$\bigwedge I_0 \wedge \bigwedge I_1 \leq \bigwedge J_0 \wedge \bigwedge J_1$$

implies that $I_0 = J_0$ and $I_1 = J_1$.

Proof. Again, the “only if” part is by substitution into B_1 . On the other hand, clearly B is freely generated by I iff, for every finite subset K of I , the subalgebra $\text{sub}(K)$ is freely generated by K . By Theorem 127, the latter holds iff the substitution map

$$\text{Free}_B(|K|) \rightarrow \text{sub}(K)$$

is one-to-one, equivalently, iff $\text{sub}(K)$ has $2^{2^{|K|}}$ elements, which, in turn by Corollary 111, is equivalent to $\text{sub}(K)$ having $2^{|K|}$ atoms. Using the proof of Theorem 127 and the present hypothesis for $I_0 \cup I_1 = K$, we can see that the elements of the form

$$\bigwedge I_0 \wedge \bigwedge I_1,$$

where $I_0 \cup I_1 = K$ and $I_0 \cap I_1 = \emptyset$, are distinct atoms in $\text{sub}(K)$, thus completing the proof. \square

2.4 Boolean homomorphisms

Now we turn our attention to an important application of terms: finding homomorphisms of boolean algebras.

Theorem 130. *Let the boolean algebra B be generated by the subalgebra D_1 and the element a . Let D_2 be a boolean algebra and let φ be a homomorphism of D_1 into D_2 . The extensions of φ to homomorphisms of B into D_2 are in one-to-one correspondence with the elements p of D_2 satisfying the following conditions:*

- (i) *If $x \in D_1$ and $x \leq a$, then $\varphi(x) \leq p$.*
- (ii) *If $x \in D_1$ and $x \geq a$, then $\varphi(x) \geq p$.*

To prepare for the proof of this theorem we verify a simple lemma, in which $+$ denotes the *symmetric difference*; that is,

$$x + y = (x' \wedge y) \vee (x \wedge y').$$

Lemma 131. *Let the boolean algebra B be generated by the subalgebra D_1 and the element a . Then every element x of B can be represented in the form*

$$x = (a \wedge x_0) \vee (a' \wedge x_1), \quad x_0, x_1 \in D_1.$$

This representation is not unique. Rather,

$$(a \wedge x_0) \vee (a' \wedge x_1) = (a \wedge y_0) \vee (a' \wedge y_1), \quad x_0, x_1, y_0, y_1 \in D_1,$$

iff

$$a \leq (x_0 + y_0)' \quad \text{and} \quad x_1 + y_1 \leq a.$$

Proof. Let D_0 denote the set of all elements of B having such a representation. If $x \in D_1$, then $x = (a \wedge x) \vee (a' \wedge x)$; thus $D_1 \subseteq D_0$. Also $a = (a \wedge 1) \vee (a' \wedge 0)$, and so $a \in D_0$. Therefore, to show that $D_0 = B$, it suffices to verify that D_0 is a subalgebra, which is left as an exercise. Now note that for all $p, q \in B$, the equality $p = q$ holds iff $p \wedge a = q \wedge a$ and $p \wedge a' = q \wedge a'$; thus

$$(a \wedge x_0) \vee (a' \wedge x_1) = (a \wedge y_0) \vee (a' \wedge y_1)$$

iff

$$a \wedge x_0 = a \wedge y_0 \quad \text{and} \quad a' \wedge x_1 = a' \wedge y_1.$$

However, $a \wedge x_0 = a \wedge y_0$ is equivalent to $(a \wedge x_0) + (a \wedge y_0) = 0$; that is, to $a \wedge (x_0 + y_0) = 0$ (see Exercise 2.13), which is the same as $a \leq (x_0 + y_0)'$. Similarly, $a' \wedge x_1 = a' \wedge y_1$ iff $x_1 + y_1 \leq a$. \square

Proof of Theorem 130. Let p be an element as specified and define the map $\psi: B \rightarrow D_2$ as follows:

$$(a \wedge x_0) \vee (a' \wedge x_1) \mapsto (p \wedge \varphi(x_0)) \vee (p' \wedge \varphi(x_1)).$$

By Lemma 131, the set of values at which ψ is defined is all of B . The map ψ is well defined, because if

$$(a \wedge x_0) \vee (a' \wedge x_1) = (a \wedge y_0) \vee (a' \wedge y_1),$$

then

$$x_1 + y_1 \leq a \leq (x_0 + y_0)';$$

thus

$$\varphi(x_1 + y_1) \leq p \leq (\varphi(x_0 + y_0))',$$

and therefore

$$\varphi(x_1) + \varphi(y_1) \leq p \leq (\varphi(x_0) + \varphi(y_0))',$$

implying that

$$(p \wedge \varphi(x_0)) \vee (p' \wedge \varphi(x_1)) = (p \wedge \varphi(y_0)) \vee (p' \wedge \varphi(y_1)).$$

It is routine to check that ψ is a homomorphism. Conversely, if ψ is an extension of φ to B , then ψ is uniquely determined by $p = \psi(a)$, and p satisfies (i) and (ii). \square

Corollary 132. *Let us assume the conditions of Theorem 130. In addition, let D_2 be complete. Set*

$$\begin{aligned} x_0 &= \bigvee (\varphi(x) \mid x \in D_1, x \leq a), \\ x_1 &= \bigwedge (\varphi(x) \mid x \in D_1, x \geq a). \end{aligned}$$

Then the extensions of φ to B are in one-to-one correspondence with the elements of the interval $[x_0, x_1]$. In particular, there is always at least one such extension.

A more general form of Theorem 130 can be found in R. Sikorski [647]; for the universal algebraic background, see Theorem 12.2 in G. Grätzer [254].

2.5 \diamond Polynomial completeness of lattices by Kalle Kaarli

Let D be a bounded distributive lattice. Let us say that an n -ary function f on L is *congruence compatible* if for any congruence θ of L and

$$a_i \equiv b_i \pmod{\theta}, \text{ for } i = 1, \dots, n,$$

the congruence

$$f(a_1, \dots, a_n) \equiv f(b_1, \dots, b_n) \pmod{\theta}$$

holds.

In Section I.4.1, we introduced polynomials. Clearly, polynomials are congruence compatible. The converse often fails. If D is boolean, the unary function $f(x) = x'$ is congruence compatible but is not a polynomial, in fact, it is not even isotone.

Let us call the lattice D *affine complete* if every congruence compatible function on D is a polynomial.

\diamond **Theorem 133.** *A bounded distributive lattice D is affine complete iff it does not have any nontrivial boolean interval.*

This result of G. Grätzer [251], see also J.D. Farley [177], started an interesting chapter in lattice theory and universal algebra, covered in great depth in the book K. Kaarli and A.F. Pixley [455]. To provide some of the highlights, we start with some definitions.

An n -ary function on a lattice L is a *local polynomial* if its restriction to any finite subset H of L^n equals a polynomial restricted to H . We denote by $\mathcal{P}(L)$ and $\mathcal{LP}(L)$ the set of all polynomial and local polynomial functions on L , respectively. Obviously, $\mathcal{P}(L) \subseteq \mathcal{LP}(L)$ for every lattice L .

Let us assume that we assign to every lattice L , a set $\mathcal{F}(L)$ of finitary functions on L containing $\mathcal{LP}(L)$. The lattice L is *\mathcal{F} -polynomially complete* if $\mathcal{F}(L) = \mathcal{P}(L)$, that is, every function in $\mathcal{F}(L)$ is a polynomial; similarly, the lattice L is *locally \mathcal{F} -polynomially complete* if $\mathcal{F}(L) = \mathcal{LP}(L)$, that is, every function in $\mathcal{F}(L)$ is a local polynomial.

In our example results, $\mathcal{F} \in \{\mathcal{O}, \mathcal{I}, \mathcal{C}, \mathcal{I} \cap \mathcal{C}\}$ where:

$\mathcal{O}(L)$	is the set of all functions on L ;
$\mathcal{C}(L)$	is the set of all congruence compatible functions on L ;
$\mathcal{I}(L)$	is the set of all isotone functions on L .

The polynomial completeness properties corresponding to these sets of functions are named as follows:

\mathcal{O}	(local) polynomial completeness;
\mathcal{C}	(local) affine completeness;
\mathcal{I}	(local) order polynomial completeness;
$\mathcal{I} \cap \mathcal{C}$	(local) order affine completeness.

Obviously no nontrivial lattice can be (locally) polynomially complete because all (local) polynomial functions on lattices are isotone. As Theorem 133 states, the class of affine complete lattices does contain nontrivial lattices. This result was generalized in different directions by D. Dorninger, G. Eigenthaler, and M. Ploščica. The first two of them proved in [166] that $\mathcal{LP}(L) = \mathcal{C}(L) \cap \mathcal{I}(L)$ for any distributive lattice L . Since we can construct from nontrivial boolean intervals congruence compatible functions that are not isotone (see G. Grätzer [251]), D. Dorninger and G. Eigenthaler [166] obtained the following result.

◆ **Theorem 134.** *A distributive lattice is locally affine complete iff it has no nontrivial boolean intervals.*

An ideal I of a lattice L is *almost principal* if its intersection with any principal ideal of L is principal. If L has a unit, then every almost principal ideal of L is principal. An *almost principal filter* of L is defined dually. It is easy to observe that, from almost principal but not principal ideals and filters, we can construct locally polynomial functions that are not polynomials. The converse also holds by M. Ploščica [585].

◇ **Theorem 135.** *A distributive lattice is affine complete iff it has no nontrivial boolean interval and all of its almost principal ideals and almost principal filters are principal.*

The necessity part of this stronger form of Theorem 133 holds for arbitrary lattices, thus every affine complete lattice must be infinite. Indeed, such a lattice cannot have prime intervals; in particular, it has no atoms or dual atoms. An obvious example of an affine complete distributive lattice is the chain \mathbb{R} . It is not known whether there exist nondistributive affine complete lattices.

We know considerably more about order polynomially complete lattices, see M. Kindermann [471].

◇ **Theorem 136.** *A finite lattice is order polynomially complete iff it has no nontrivial tolerances.*

In the modular case, R. Wille [737] provides the following nice description.

◇ **Theorem 137.** *A finite, simple, modular lattice is order polynomially complete iff it is complemented.*

Thus all finite irreducible projective geometries viewed as lattices are order polynomially complete. It also follows from Theorem 136 that every finite order polynomially complete lattice is simple. The question whether there exist infinite order polynomially complete lattices remained open until M. Goldstern and S. Shelah [236, 237] answered it in the negative.

It was proved by K. Kaarli and A. Pixley [455] that the “local” versions of Theorems 136 and 137 remain valid for lattices of finite height.

Next we consider (locally) order affine complete lattices. In view of Theorems 2.5 and 135, and the observations preceding them, we have the following result.

◇ **Theorem 138.** *Every distributive lattice is locally order affine complete. Every bounded distributive lattice is order affine complete.*

The theory of non-distributive (locally) order affine complete lattices is based on the following two observations:

- (1) An isotone function on a lattice L is a local polynomial iff it preserves all tolerances of L .
- (2) A sublattice L of a direct product $L_1 \times \cdots \times L_n$ is locally order affine complete iff so are all 2-fold coordinate projections L_{ij} of L .

Observation (1) first appeared for finite lattices in M. Kindermann [471]; it was used for proving Theorem 136. R. Wille’s characterization of finite order affine complete lattices in [738] is based on the same observation; it says,

in essence, that a finite lattice is order affine complete iff all of its tolerances are obtainable in a certain way from congruences. A more general version of this result is presented in K. Kaarli and A. F. Pixley [455, Theorem 5.3.28].

Observation (2) for finite lattices appeared in R. Wille [738]; the general form is due to K. Kaarli and V. Kuchmei [454]. It allows one to reduce the study of locally order affine complete lattices of finite height to the case of subdirect products of two subdirectly irreducible lattices. (For this concept, see Section 6.5.) By K. Kaarli and V. Kuchmei [454], relatively few subdirect products of two simple lattices are locally order affine complete.

◇ **Theorem 139.** *Let L be a subdirect product of simple lattices L_1 and L_2 of finite height. The lattice L is locally order affine complete iff L_1 and L_2 have no nontrivial tolerances and one of the following cases occurs:*

- (1) $L = L_1 \times L_2$;
- (2) L is a maximal sublattice of $L_1 \times L_2$;
- (3) L is the intersection of two maximal sublattices of $L_1 \times L_2$, one containing $(0, 1)$ and the other $(1, 0)$.

This result is especially useful for modular lattices because subdirectly irreducible modular lattices of finite height are simple.

In conclusion, we consider another version of local order affine completeness that is defined using partial functions and has several good properties. We call a lattice L *strictly locally order affine complete* if any isotone congruence compatible function $f: X \rightarrow L$, where X is a finite meet (or join) subsemilattice of some power L^n , is the restriction of some polynomial of L . The following results were obtained by K. Kaarli and K. Täht [456].

◇ **Theorem 140.**

- (1) *A lattice is strictly locally order affine complete iff all of its tolerances are congruences.*
- (2) *Every relatively complemented lattice is strictly locally order affine complete.*
- (3) *Every strictly locally order affine complete lattice is congruence permutable.*
- (4) *A modular lattice of finite height is strictly locally order affine complete iff it is relatively complemented.*

Exercises

- 2.1. Regard $\text{Term}(n)$ as an algebra with the binary operations \vee and \wedge . Define the concept of a congruence relation α on $\text{Term}(n)$ and the corresponding quotient algebra $\text{Term}(n)/\alpha$.

- Prove that $\equiv_{\mathbf{D}}$ is a congruence relation on $\text{Term}(n)$ and the corresponding quotient algebra is isomorphic to $\text{Term}_{\mathbf{D}}(n)$.
- 2.2. Work out Exercise 2.1 for Boolean terms.
 - 2.3. Get lower and upper bounds for $|\text{Term}_{\mathbf{B}}(n)|$ that are sharper than those given by Theorem 126(iii).
 - 2.4. Work out the details of the last steps in the proof of Theorem 126.
 - 2.5. Let p_J be an atomic boolean term. Show that under the substitution $x_i = 1$, for all $i \in J$, and $x_i = 0$, for all $i \notin J$, we get $p_J = 1$ and $p_{J_0} = 0$ for all $J_0 \neq J$.
 - 2.6. Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$. Prove that there is an n -ary boolean term p that defines the function f on \mathbf{B}_1 , as in Definition 53.
 - 2.7. Let B be a boolean algebra. Prove that there is a one-to-one correspondence between n -ary boolean terms over B (up to equivalence) and maps $\{0, 1\}^n \rightarrow \{0, 1\}$. In other words, all $\{0, 1\}$ substitutions take 0 and 1 as values and determine the term p .
 - 2.8. A *polynomial* over a boolean algebra B is built up inductively from the variables and the elements of B using \vee , \wedge , and $'$. A polynomial on B defines a function on B , called a *boolean polynomial*. Show that the n -ary polynomials are in one-to-one correspondence with maps $\{0, 1\}^n \rightarrow B$.
 - 2.9. Let p be an n -ary term over the boolean algebra B and α a congruence relation on B . Show that $a_i \equiv b_i \pmod{\alpha}$, for all $i < n$, implies that $p(a_0, \dots, a_{n-1}) \equiv p(b_0, \dots, b_{n-1}) \pmod{\alpha}$. (p has the Substitution Property.)
 - 2.10. Show that the property described in Exercise 2.9 characterizes boolean polynomials (G. Grätzer [248]).
 - *2.11. Use the property described in Exercise 2.9 to define *boolean polynomials* over a distributive lattice. Show that, for bounded distributive lattices, Exercise 2.8 holds without any change (G. Grätzer [251]).
 - 2.12. Show that a free boolean algebra on countably many generators has no atoms.
 - 2.13. Show that $a \wedge (b + c) = (a \wedge b) + (a \wedge c)$ holds in any boolean algebra.
 - 2.14. Let B be the boolean algebra freely generated by I . Let L be the sublattice generated by I . Prove that L is the free distributive lattice freely generated by I .
 - 2.15. Let L and L_1 be distributive lattices, let $L = \text{sub}(A)$, and let φ be a map of A into L_1 . Show that there is a homomorphism of L into L_1 extending φ iff, for every pair of finite nonempty subsets A_1 and A_2 of A ,

$$\bigwedge A_1 \leq \bigvee A_2 \quad \text{implies that} \quad \bigwedge \varphi(A_1) \leq \bigvee \varphi(A_2).$$

(Compare this with Exercise I.5.45.)

- 2.16. State and prove Exercise 2.15 for boolean algebras.

- 2.17. Interpret Lemma 131 using Exercise 2.16.
- 2.18. Extend the last statement of Corollary 132 to the case in which D_1 is generated by B and $a_0, \dots, a_{n-1} \in D_1$ for some $n > 1$.
- 2.19. Let p and q be lattice terms. Since p and q can also be regarded as boolean terms, $p \equiv q$ was defined in two ways: as $p \equiv_{\mathbf{D}} q$ and as $p \equiv_{\mathbf{B}} q$. Show that the two definitions are equivalent for lattice terms.
- 2.20. Define $\equiv_{\mathbf{K}}$ for lattice terms with respect to a class \mathbf{K} of lattices closed under isomorphisms. Show that $\text{Term}_{\mathbf{K}}(n) \in \mathbf{K}$ iff the free lattice over \mathbf{K} with n generators exists, in which case $\text{Term}_{\mathbf{K}}(n)$ is a free lattice with n generators.

3. Congruence Relations

3.1 Principal congruences

In distributive lattices, the following description of the principal congruence $\text{con}(a, b)$ (the notation introduced in Section I.3.6) is important (G. Grätzer and E. T. Schmidt [334]):

Theorem 141. *Let L be a distributive lattice, $a, b, x, y \in L$, and let $a \leq b$. Then*

$$x \equiv y \pmod{\text{con}(a, b)} \quad \text{iff} \quad x \vee b = y \vee b \text{ and } x \wedge a = y \wedge a.$$

Remark. This result is illustrated in [Figure 27](#).

Proof. Let β denote the binary relation under which $x \equiv y \pmod{\beta}$ iff $x \vee b = y \vee b$ and $x \wedge a = y \wedge a$. The binary relation β is obviously an equivalence relation. If $x \equiv y \pmod{\beta}$ and $z \in L$, then

$$(x \vee z) \wedge a = (x \wedge a) \vee (z \wedge a) = (y \wedge a) \vee (z \wedge a) = (y \vee z) \wedge a,$$

and

$$(x \vee z) \vee b = z \vee (x \vee b) = z \vee (y \vee b) = (y \vee z) \vee b;$$

thus $x \vee z \equiv y \vee z \pmod{\beta}$. Similarly, $x \wedge z \equiv y \wedge z \pmod{\beta}$. We conclude that β is a congruence relation. The congruence $a \equiv b \pmod{\beta}$ is obvious. Finally, let α be any congruence relation such that $a \equiv b \pmod{\alpha}$ and let $x \equiv y \pmod{\beta}$. Then

$$\begin{aligned} x \vee a &= y \vee a, \\ x \wedge b &= y \wedge b, \\ x \vee a &\equiv x \vee b \pmod{\alpha}, \\ x \wedge b &\equiv x \wedge a \pmod{\alpha}. \end{aligned}$$

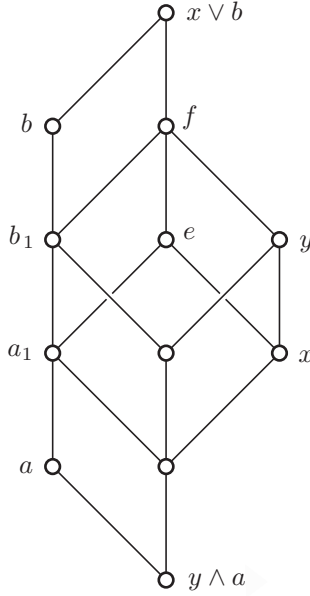


Figure 27. $x \equiv y \pmod{\text{con}(a, b)}$ in a distributive lattice for $a \leq b$ and $x \leq y$

Computing modulo α , we obtain

$$\begin{aligned} x &= x \vee (x \wedge a) = x \vee (y \wedge a) = (x \vee y) \wedge (x \vee a) \equiv (x \vee y) \wedge (x \vee b) \\ &= (x \vee y) \wedge (y \vee b) = y \vee (x \wedge b) \equiv y \vee (x \wedge a) = y \vee (y \wedge a) = y, \end{aligned}$$

that is, $x \equiv y \pmod{\alpha}$, proving that $\beta \leq \alpha$. \square

Explanation. Since $a \equiv b$ implies that $(a \vee p) \wedge q \equiv (b \vee p) \wedge q$, we must have $x \equiv y \pmod{\text{con}(a, b)}$ if

$$\begin{aligned} x \vee y &= (b \vee p) \wedge q, \\ x \wedge y &= (a \vee p) \wedge q. \end{aligned}$$

It is easy to check (see Exercise 3.1) that the x and y satisfying the conditions of Theorem 141 are exactly the same as those for which such p and q exist. Thus Theorem 141 can be interpreted as follows: We get all pairs $x \leq y$ with $x \equiv y \pmod{\text{con}(a, b)}$, by applying the Substitution Property “twice” to a subinterval of $[a, b]$. No further application of the Substitution Property is required nor is transitivity needed.

From the point of view of perspectivity and projectivity of intervals, see Section I.3.5, we get that in a distributive lattice L with elements $a \leq b$ and

$x \leq y$, the congruence $x \equiv y \pmod{\text{con}(a, b)}$ holds iff the interval $[a, b]$ has a subinterval $[a_1, b_1]$ and there is an interval $[e, f]$ such that

$$[a_1, b_1] \stackrel{\text{up}}{\sim} [e, f] \stackrel{\text{dn}}{\sim} [x, y];$$

in particular, projectivity is equivalent to a two-step projectivity.

The description of $\text{con}(a, b)$ in Theorem 141 is equivalent to the following two conditions:

$$\begin{aligned} x \vee y &\leq b \vee (x \wedge y), \\ (a \vee (x \wedge y)) \wedge (x \vee y) &= x \wedge y. \end{aligned}$$

Some applications of Theorem 141 follow.

Corollary 142. *Let I be an ideal of the distributive lattice L . Then the following two conditions are equivalent:*

- (i) $x \equiv y \pmod{\text{con}(I)}$;
- (ii) $x \vee y = (x \wedge y) \vee i$ for some $i \in I$.

Therefore, I is a block modulo $\text{con}(I)$.

Remark. This situation is illustrated in Figure 28, in which the dotted line indicates congruence modulo $\text{con}(I)$.

Some properties of $\text{con}(I)$ can be generalized to certain ideals of a general lattice, see Chapter III.

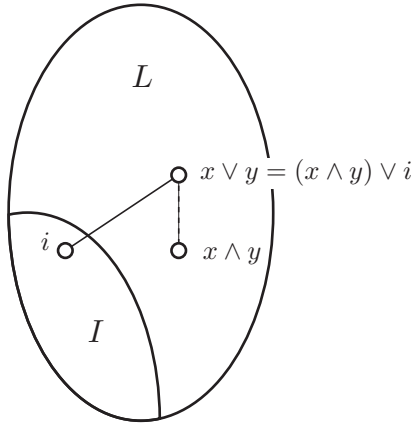


Figure 28. x and y are congruent modulo $\text{con}(I)$ in a distributive lattice

Proof. If $x \vee y = (x \wedge y) \vee i$, then

$$x \equiv y \pmod{\text{con}(x \wedge y \wedge i, i)}$$

with $x \wedge y \wedge i, i \in I$, and so $x \equiv y \pmod{\text{con}(I)}$. Conversely,

$$\text{con}(I) = \bigvee (\text{con}(u, v) \mid u, v \in I)$$

by Lemma 14. However,

$$\text{con}(u, v) \vee \text{con}(u_1, v_1) \leq \text{con}(u \wedge v \wedge u_1 \wedge v_1, u \vee v \vee u_1 \vee v_1);$$

therefore,

$$\text{con}(I) = \bigcup (\text{con}(u, v) \mid u, v \in I).$$

If $x \equiv y \pmod{\text{con}(u, v)}$, for $u, v \in I$ with $u \leq v$, then $x \vee v = y \vee v$, and so $(x \wedge y) \vee (v \wedge (x \vee y)) = x \vee y$; thus Corollary 142(ii) is satisfied with $i = v \wedge (x \vee y) \in I$. Finally, if $a \in I$ and $a \equiv b \pmod{\text{con}(I)}$, then $a \vee b = (a \wedge b) \vee i$ for some $i \in I$; so $a \vee b \in I$ and $b \in I$, showing that I is a full block. \square

Corollary 143. *Let L be a distributive lattice, $x, y, a, b \in L$, and let*

$$x \leq y \leq a \leq b$$

or

$$a \leq b \leq x \leq y.$$

Then $x \equiv y \pmod{\text{con}(a, b)}$ implies that $x = y$.

3.2 Prime ideals

A very important congruence relation has already been used in the proof of Lemma 8(ii): Given a prime ideal P of the lattice L , we can construct a congruence relation that has exactly two blocks, P and $L - P$. This statement can be generalized as follows: Let \mathcal{A} be a set of prime ideals of a lattice L and let us call two elements x and y congruent modulo \mathcal{A} if either $x, y \in P$ or $x, y \in L - P$ for every $P \in \mathcal{A}$; this describes a congruence relation on L . For instance, if $\mathcal{A} = \{P, Q, R\}$ with $Q \subset P$ and $R \subset P$, then we get five blocks as shown in Figure 29; the quotient lattice is shown in Figure 21.

This principle will be used often. An interesting application to the Congruence Extension Property (see Section I.3.8) is the following statement:

Theorem 144. *A distributive lattice L has the Congruence Extension Property. Therefore, the class **D** of distributive lattices has the CEP.*

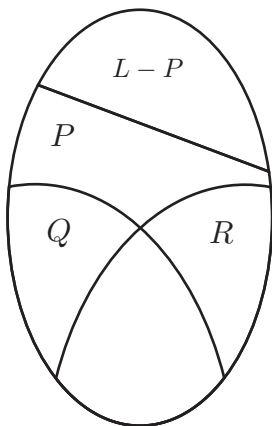


Figure 29. An important congruence

Proof. Let α be a congruence of K and let $\alpha: x \mapsto x/\alpha$ be the natural homomorphism of K onto K/α ; then $\alpha^{-1}(P)$ is a prime ideal of K for every prime ideal P of K/α . Therefore, $\text{id}(\alpha^{-1}(P))$ is an ideal of L , $\text{fil}(K - \alpha^{-1}(P))$ is a filter of L , and they are disjoint. Thus by Theorem 115, we can choose a prime ideal P_1 of L such that $P_1 \supseteq \alpha^{-1}(P)$ and $P_1 \cap (K - \alpha^{-1}(P)) = \emptyset$.

For every prime ideal P of K/α , we choose such a prime ideal P_1 of L . Let \mathcal{A} denote the collection of all such prime ideals. Let β be the congruence relation associated with \mathcal{A} , as previously described.

Now for $x, y \in K$, the congruence $x \equiv y \pmod{\alpha}$ is equivalent to the condition $\alpha(x) = \alpha(y)$, and so, for every $P_1 \in \mathcal{A}$, either $x, y \in P_1$ or $x, y \notin P_1$; thus $x \equiv y \pmod{\beta}$. Conversely, if $x \equiv y \pmod{\beta}$, then, for every $P_1 \in \mathcal{A}$, either $x, y \in P_1$ or $x, y \notin P_1$, and so either $\alpha(x), \alpha(y) \in P$ or $\alpha(x), \alpha(y) \notin P$. Since every pair of distinct elements of K/α is separated by a prime ideal (Corollary 117), we conclude that $\alpha(x) = \alpha(y)$ and thus $x \equiv y \pmod{\alpha}$. \square

An alternative proof would proceed as in Lemma 17. Form in L the congruence

$$\beta = \text{con}(\alpha) = \bigvee (\text{con}(x, y) \mid x, y \in \alpha).$$

Now if $\beta|_K = \alpha$ fails, then some $\text{con}_L(x, y)|_K = \text{con}_K(x, y)$ would fail, an easy contradiction with Theorem 141.

3.3 Boolean lattices

It is well known that in rings, ideals are in a one-to-one correspondence with congruence relations. In one class of lattices the situation is exactly the same.

Theorem 145. *Let L be a boolean lattice. Then $\alpha \mapsto 0/\alpha$ is a one-to-one correspondence between congruence relations and ideals of L .*

Proof. By Corollary 142, the map is onto; therefore, we have only to prove that it is one-to-one, that is, that $I = 0/\alpha$ determines α . This fact, however, is obvious, since $a \equiv b \pmod{\alpha}$ iff $a \wedge b \equiv a \vee b \pmod{\alpha}$, which, in turn, is equivalent to $c \equiv 0 \pmod{\alpha}$, where c is the relative complement of $a \wedge b$ in $[0, a \vee b]$ (see Figure 30). Thus $a \equiv b \pmod{\alpha}$ iff $c \in 0/\alpha$. \square

This proof does not make full use of the hypothesis that L is a complemented distributive lattice. In fact, all we need to make the proof work is that L has a zero and is relatively complemented. Such a distributive lattice is called a *generalized boolean lattice*. The following result of J. Hashimoto [375] demonstrates the importance of the class of generalized boolean lattices. For the proof we present, see G. Grätzer and E. T. Schmidt [334].

Theorem 146. *Let L be a lattice. There is a one-to-one correspondence between ideals and congruence relations of L under which the ideal I corresponding to a congruence relation α is a whole block under α iff L is a generalized boolean lattice.*

Proof. The “if” part is in the proof of Theorem 145. We proceed with the “only if” part. The ideal corresponding to $\mathbf{0}$ has to be $\{0\}$, and thus L has a 0 . If L contains a diamond, $\{o, a, b, c, i\}$, then $\text{id}(a)$ cannot be a block, because $a \equiv o$ implies that

$$\begin{aligned} i &= a \vee c \equiv o \vee c = c, \\ b &= b \wedge i \equiv b \wedge c = o. \end{aligned}$$

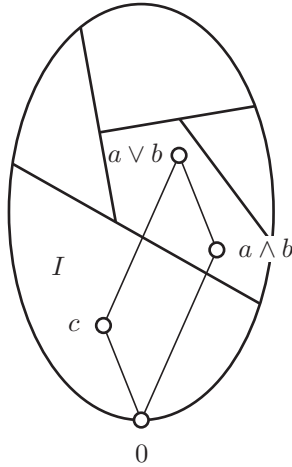


Figure 30. Illustrating the proof of Theorem 145

But $o \in \text{id}(a)$, and thus any block containing $\text{id}(a)$ contains $b \notin \text{id}(a)$. Similarly, if L contains a pentagon, $\{o, a, b, c, i\}$, and a block contains $\text{id}(b)$, then $b \equiv o$; thus

$$i = b \vee c \equiv o \vee c = c,$$

and so

$$a = a \wedge i \equiv a \wedge c = o.$$

Therefore, this block has to contain a , and $a \notin \text{id}(b)$. Thus by Theorem 101, L is distributive. Let $a < b$ and $I = 0/\text{con}(a, b)$. By Corollary 142, $\text{con}(I)$ is also a congruence relation of L having I as a whole block; consequently, we obtain that $\text{con}(I) = \text{con}(a, b)$, and so $a \equiv b \pmod{\text{con}(I)}$. Thus again by Corollary 142, $b = a \vee i$ and $i \equiv 0 \pmod{\text{con}(a, b)}$ for some $i \in I$. The latter is equivalent to $i \vee b = 0 \vee b$ and $i \wedge a = 0 \wedge a$. We conclude that $a \vee i = b$ and $a \wedge i = 0$, and so i is a relative complement of a in $[0, b]$. \square

It is no coincidence that, in the class of generalized boolean lattices, congruences and ideals behave as they do in rings. Indeed, generalized boolean lattices are rings in disguise as demonstrated in M. H. Stone [668]:

Theorem 147.

- (i) Let $\mathfrak{B} = (B; \vee, \wedge)$ be a generalized boolean lattice. Define the binary operations \cdot and $+$ on B by setting

$$x \cdot y = x \wedge y$$

and by defining $x + y$ as a relative complement of $x \wedge y$ in $[0, x \vee y]$ (see Figure 31). Then $\mathfrak{B}^{\text{ring}} = (B; +, \cdot)$ is a boolean ring—that is, an (associative) ring satisfying $x^2 = x$, for all $x \in B$ (and, consequently, satisfying $xy = yx$ and $x + x = 0$ for all $x, y \in B$).

- (ii) Let $\mathfrak{B} = (B; +, \cdot)$ be a boolean ring. Define the binary operations \vee and \wedge in B by

$$\begin{aligned} x \vee y &= x + y + x \cdot y, \\ x \wedge y &= x \cdot y. \end{aligned}$$

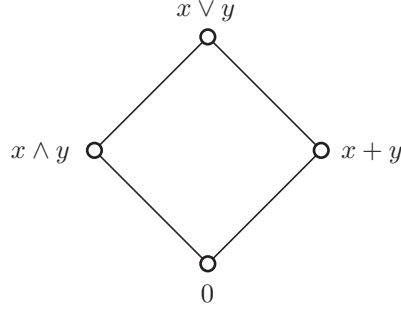
Then $\mathfrak{B}^{\text{lat}} = (B; \vee, \wedge)$ is a generalized boolean lattice.

- (iii) Let \mathfrak{B} be a generalized boolean lattice. Then $(\mathfrak{B}^{\text{ring}})^{\text{lat}} = \mathfrak{B}$.

- (iv) Let \mathfrak{B} be a boolean ring. Then $(\mathfrak{B}^{\text{lat}})^{\text{ring}} = \mathfrak{B}$.

The proof of this theorem is purely computational. Some steps will be given in the Exercises.

The method given in Theorem 147 is not the only one used to introduce ring operations in a generalized boolean lattice. G. Grätzer and E. T. Schmidt

Figure 31. Defining $x + y$

[334] prove that ring operations $+$ and \cdot can be introduced on a distributive lattice L such that $+$ and \cdot satisfy the Substitution Property iff L is relatively complemented. Furthermore, $+$ and \cdot are uniquely determined by the zero of the ring, which can be an arbitrary element of L .

The correspondence between boolean rings and generalized boolean lattices preserves many algebraic properties.

Theorem 148. *Let \mathfrak{B}_0 and \mathfrak{B}_1 be generalized boolean lattices.*

- (i) *Let $I \subseteq B_0$. Then I is an ideal of \mathfrak{B}_0 iff I is an ideal of $\mathfrak{B}_0^{\text{ring}}$.*
- (ii) *Let $\varphi: B_0 \rightarrow B_1$. Then φ is a $\{0\}$ -homomorphism of \mathfrak{B}_0 into \mathfrak{B}_1 iff φ is a homomorphism of $\mathfrak{B}_0^{\text{ring}}$ into $\mathfrak{B}_1^{\text{ring}}$.*
- (iii) *\mathfrak{B}_0 is a $\{0\}$ -sublattice of \mathfrak{B}_1 iff $\mathfrak{B}_0^{\text{ring}}$ is a subring of $\mathfrak{B}_1^{\text{ring}}$.*

The proof is again left to the reader.

3.4 Congruence lattices

N. Funayama and T. Nakayama [214] proves that congruence relations on an arbitrary lattice have an interesting connection with distributive lattices:

Theorem 149. *Let L be an arbitrary lattice. Then $\text{Con } L$, the lattice of all congruence relations of L , is distributive.*

Proof. Let $\alpha, \beta, \gamma \in \text{Con } L$. Since

$$\alpha \wedge (\beta \vee \gamma) \geq (\alpha \wedge \beta) \vee (\alpha \wedge \gamma),$$

it suffices to prove that

$$a \equiv b \pmod{\alpha \wedge (\beta \vee \gamma)} \text{ implies that } a \equiv b \pmod{(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}.$$

So let $a \equiv b \pmod{\alpha \wedge (\beta \vee \gamma)}$; that is, $a \equiv b \pmod{\alpha}$ and $a \equiv b \pmod{\beta \vee \gamma}$. By Theorem 12, there exists a sequence

$$a \wedge b = z_0 \leq \cdots \leq z_n = a \vee b$$

such that

$$z_i \equiv z_{i+1} \pmod{\beta} \quad \text{or} \quad z_i \equiv z_{i+1} \pmod{\gamma}$$

for every $0 \leq i < n$. Since $a \equiv b \pmod{\alpha}$, the congruence $a \wedge b \equiv a \vee b \pmod{\alpha}$ also holds, and so $z_i \equiv z_{i+1} \pmod{\alpha}$ for every $0 \leq i < n$. Thus

$$z_i \equiv z_{i+1} \pmod{\alpha \wedge \beta} \quad \text{or} \quad z_i \equiv z_{i+1} \pmod{\alpha \wedge \gamma},$$

for every $0 \leq i < n$, implying that

$$a \equiv b \pmod{(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}. \quad \square$$

Now we connect the foregoing with algebraic lattices, see Definition 41.

Lemma 150. *Every principal congruence relation is compact.*

Proof. Let L be a lattice, let $a, b \in L$. Let $\Lambda \subseteq \text{Con } L$ and

$$\text{con}(a, b) \leq \bigvee \Lambda.$$

Then $a \equiv b \pmod{\bigvee \Lambda}$, and thus (just as in Theorem 37) there exists a sequence

$$a = x_0, x_1, \dots, x_n = b$$

with

$$x_i \equiv x_{i+1} \pmod{\alpha_i},$$

for some $\alpha_i \in \Lambda$, and for all i with $0 \leq i < n$. Therefore, $a \equiv b \pmod{\bigvee \Lambda_0}$, where

$$\Lambda_0 = \{\alpha_0, \dots, \alpha_{n-1}\},$$

and so $\text{con}(a, b) \leq \bigvee \Lambda_0$, where Λ_0 is a finite subset of Λ . \square

Theorem 151. *Let L be an arbitrary lattice. Then $\text{Con } L$ is an algebraic lattice.*

Proof. For every $\alpha \in \text{Con } L$,

$$\alpha = \bigvee (\text{con}(a, b) \mid a \equiv b \pmod{\alpha}).$$

Consequently, this theorem follows from Lemma 150. \square

Combining Theorems 149 and 151 we get:

Corollary 152. *Let L be an arbitrary lattice. Then $\text{Con } L$ is a distributive algebraic lattice.*

The converse of Corollary 152 for the finite case is proved in Section IV.4.1.

Exercises

- 3.1. Let L be a distributive lattice and let $u, v, a, b \in L$. Prove that if $a \leq b$ and $x \leq v$, then

$$u \vee b = v \vee b \quad \text{and} \quad u \wedge a = v \wedge a$$

is equivalent to

$$(a \vee p) \wedge q = u \quad \text{and} \quad (b \vee p) \wedge q = v$$

for some p, q in L .

- 3.2. Use Theorem 141 to prove that the class **D** of distributive lattices has the CEP (Theorem 144).
 3.3. Verify Corollary 142 directly.
 3.4. Let K be a sublattice of the distributive lattice L and let P be a prime ideal of K . Prove that there exists a prime ideal Q of L with $Q \cap K = P$.
 3.5. Prove that if Corollary 143 holds for a lattice L , then L is distributive.
 3.6. Show that if Theorem 144 holds for a lattice L , then L is distributive.
 3.7. Let L be a *sectionally complemented lattice* (see Section I.6.1). Prove that $\alpha \mapsto 0/\alpha$ is a one-to-one correspondence between congruences and *certain* ideals of L .
 *3.8. Show that the “certain ideals” that appear in Exercise 3.7 form a sublattice of $\text{Id } L$. (See Section III.3.)
 3.9. Prove that every (principal) ideal of L is of the form $0/\alpha$ for a suitable congruence α of L iff L is distributive.
 3.10. Let L be a distributive lattice and let I be an ideal of L . Define a binary relation $\beta(I)$ on L :

$$x \equiv y \pmod{\beta(I)} \quad \text{iff} \quad \text{there is no } a \in I \text{ with } a \leq x \vee y, \ x \wedge y \wedge a \in I, \ a \notin I.$$

Prove that $\beta(I)$ is the largest congruence relation of L under which the ideal I is a block.

- 3.11. Let L be a distributive lattice with zero. Prove that there is a one-to-one correspondence between ideals and congruence relations (in the sense of Theorem 146) iff $\text{con}(I) = \beta(I)$ for all $I \in \text{Id } L$.
 3.12. Prove Theorem 146 using Exercises 3.10 and 3.11 (G. Ya. Areškin [31]).
 *3.13. Let L be a lattice and let a be an element of L . Show that every convex sublattice of L containing a is a block under exactly one congruence relation iff L is distributive and all the intervals $[b, a]$ ($b \in L$ with $b \leq a$) and $[a, c]$ ($c \in L$ with $a \leq c$) are complemented (G. Grätzer and E. T. Schmidt [334]).

- 3.14. Derive Theorem 146 (and also, a variant of Theorem 146) by taking $a = 0$ (and arbitrary $a \in L$) in Exercise 3.13.
- 3.15. Let L be a relatively complemented lattice, let $I, J \in \text{Id } L$, and let $I \subseteq J$. Prove that if I is an intersection of prime ideals, then so is J (J. Hashimoto [375]).
- 3.16. Use Exercises 3.14 and 3.15 to get the following theorem: Let L be a relatively complemented lattice. Then L is distributive iff, for some element a of L , the ideal $\text{id}(a)$ is an intersection of prime ideals and the filter $\text{fil}(a)$ is an intersection of prime filters (J. Hashimoto [375]).
- 3.17. Prove that the verification of Theorem 147(i) can be reduced to the boolean lattice case and that in this case

$$x + y = (x \wedge y') \vee (x' \wedge y).$$

- 3.18. Let B be a boolean lattice. Verify that

$$x + y = (x \vee y) \wedge (x' \vee y').$$

- 3.19. Let B be a boolean lattice. Verify that

$$(x + y) + z = (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z)$$

and conclude that $+$ is associative.

- 3.20. Prove that $x(y + z) = xy + xz$ in a boolean lattice.
- 3.21. Prove Theorem 147(i).
- 3.22. Prove Theorem 147(ii).
- 3.23. Let \mathfrak{B} be a generalized boolean lattice. For any $x, y \in B$, observe that the meet $x \wedge y$ is the same in \mathfrak{B} as in $(\mathfrak{B}^{\text{ring}})^{\text{lat}}$ (namely, $x \cdot y$); conclude that $\mathfrak{B} = (\mathfrak{B}^{\text{ring}})^{\text{lat}}$.
- 3.24. Verify Theorem 147(iv).
- 3.25. Verify Theorem 148.
- 3.26. Show that, using the concept of a distributive semilattice (see Section 5.1), Corollary 152 can be reformulated as follows: Let L be an arbitrary lattice. Then there exists a distributive join-semilattice F with zero such that $\text{Con } L$ is isomorphic to $\text{Id } F$.
- 3.27. Characterize the lattice of all ideals of a lattice using the concept of an algebraic lattice.
- 3.28. Characterize the lattice of all ideals of a boolean lattice as a special type of algebraic lattices.
- *3.29. Show that a chain C is the congruence lattice of a lattice iff C is algebraic. (This exercise and the next are from G. Grätzer and E. T. Schmidt [332].)
- 3.30. Prove that a boolean lattice B is the congruence lattice of a lattice iff B is algebraic.
- 3.31. Let L be a distributive lattice. Show that $a \mapsto \text{con}(\text{id}(a))$ embeds L into $\text{Con } L$.

- 3.32. Let L be a bounded distributive lattice. For $a, b \in L$ with $a \leq b$, show that the congruence $\text{con}(a, b)$ has a complement in $\text{Con } L$, namely, the congruence $\text{con}(0, a) \vee \text{con}(b, 1)$.
- 3.33. Generalize Exercise 3.32 to arbitrary distributive lattices.
- 3.34. Let L be a bounded distributive lattice. Show that the compact elements of $\text{Con } L$ form a boolean lattice (J. Hashimoto [375], G. Grätzer and E. T. Schmidt [335]).
- 3.35. Let B be a boolean algebra freely generated by X and let $L \leq B$ be the sublattice generated by X . Is L freely generated by X in \mathbf{D} (see Exercise 2.14)?

4. Boolean Algebras R-generated by Distributive Lattices

4.1 Embedding results

The following result is the fundamental embedding theorem for distributive lattices.

Theorem 153. *Every distributive lattice can be embedded in a boolean lattice.*

Proof. By Theorem 119, every distributive lattice L is isomorphic to a ring of subsets of some set X . Obviously, L can be embedded into $\text{Pow } X$. \square

Definition 154. Let L be a $\{0\}$ -sublattice of the generalized boolean lattice B . Then B is *R-generated* by L if L generates B as a ring.

Note that if L has a unit element, then the same element is the unit element of B ; equivalently, if $\bigvee L$ exists, then $\bigvee B$ exists and $\bigvee L = \bigvee B$.

Our goal is to show the uniqueness of the generalized boolean lattice R-generated by L . The first result is essentially due to H. M. MacNeille [518]:

Lemma 155. *Let B be R-generated by L . Then every $a \in B$ can be expressed in the form*

$$a_0 + a_1 + \cdots + a_{n-1}, \quad a_0 \leq a_1 \leq \cdots \leq a_{n-1}, \quad a_0, a_1, \dots, a_{n-1} \in L.$$

Remark. Let B be the boolean lattice shown in Figure 32 with the sublattice $L = \{0, a_0, a_1, a_2\}$. Then L R-generates B .

Proof. Let B_1 denote the set of all elements that can be represented in the form $a_0 + \cdots + a_{n-1}$, where $a_0, \dots, a_{n-1} \in L$. Then $L \subseteq B_1$, and B_1 is closed under $+$ and $-$ (since $x - y = x + y$). Furthermore,

$$(a_0 + \cdots + a_{n-1})(b_0 + \cdots + b_{m-1}) = \sum a_i b_j,$$

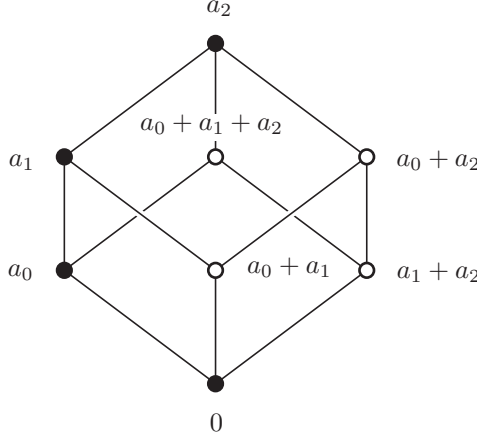


Figure 32. Illustrating “R-generated”

and each term $a_i b_j = a_i \wedge b_j \in L$, so B_1 is closed under multiplication. We conclude that $B_1 = B$.

Note that L is a sublattice of B ; therefore, for the elements $a, b \in L$, the join $a \vee b$ in L is the same as the join in B . Thus $a \vee b = a + b + ab$, and so

$$a + b = ab + (a \vee b) = (a \wedge b) + (a \vee b).$$

Take $a_0 + \cdots + a_{n-1} \in B$. We prove by induction on n that the summands can be made to form an increasing sequence. For $n = 1$, this is obvious. Let us assume that $a_1 \leq \cdots \leq a_{n-1}$. Then

$$\begin{aligned} & a_0 + a_1 + \cdots + a_{n-1} \\ &= (a_0 \wedge a_1) + (a_0 \vee a_1) + a_2 + \cdots + a_{n-1} \\ &= (a_0 \wedge a_1) + ((a_0 \vee a_1) \wedge a_2) + (a_0 \vee a_2) + a_3 + \cdots + a_{n-1} \\ &= (a_0 \wedge a_1) + ((a_0 \vee a_1) \wedge a_2) + ((a_0 \vee a_2) \wedge a_3) + (a_0 \vee a_3) + \cdots + a_{n-1} \\ &\dots \\ &= (a_0 \wedge a_1) + ((a_0 \vee a_1) \wedge a_2) + \cdots + ((a_0 \vee a_{n-2}) \wedge a_{n-1}) + (a_0 \vee a_{n-1}), \end{aligned}$$

and

$$a_0 \wedge a_1 \leq (a_0 \vee a_1) \wedge a_2 \leq \cdots \leq (a_0 \vee a_{n-2}) \wedge a_{n-1} \leq a_0 \vee a_{n-1}. \quad \square$$

Lemma 156. *Let L be a distributive lattice with zero. Then there exists a generalized boolean lattice B freely R-generated by L , that is, a generalized boolean lattice B with the following properties:*

- (i) B is R-generated by L .

- (ii) If B_1 is R -generated by L , then there is a homomorphism φ of B onto B_1 that is the identity map on L .

Proof. The existence of B can be proved by copying the proof of Theorem 69 (or Theorem 89), *mutatis mutandis*. \square

An interesting property of generalized boolean lattices R -generated by distributive lattices is proved in J. Hashimoto [375].

Lemma 157. *Let B be a generalized boolean lattice R -generated by the distributive lattice L with zero. Then B is a congruence-preserving extension of L .*

Proof. Let α be a congruence of L . The existence of an extension of α to B was proved in Theorem 144. By Theorems 145 and 148(i), the following statement implies the uniqueness of the extension:

If I and J are (ring) ideals of B with $I \subset J$, then there are elements $a, b \in L$ with $a \neq b$, such that $a \equiv b \pmod{J}$ and $a \not\equiv b \pmod{I}$.

Indeed, let $x \in J - I$. By Lemma 155, x can be represented in the form

$$x = x_0 + \cdots + x_{n-1}, \quad x_0 \leq \cdots \leq x_{n-1}, \quad x_0, \dots, x_{n-1} \in L.$$

If n is odd, then $x_0 = x \cdot x_0 \leq x \in J$, and thus $x_0 \in J$; also,

$$x_0 + x_1 + x_2 = x \cdot x_2 \in J,$$

therefore

$$x_1 + x_2 = x_0 + (x_0 + x_1 + x_2) \in J.$$

Similarly,

$$x_3 + x_4, x_5 + x_6, \dots \in J.$$

Since

$$x_0 + (x_1 + x_2) + (x_3 + x_4) + \cdots \in J - I,$$

we conclude that either $x_0 \in J - I$, or $x_{2i-1} + x_{2i} \in J - I$ for some $2i < n$. If n is even, then we obtain $x_0 + x_1, x_2 + x_3, \dots \in J$ (by multiplying x by x_1, x_3, \dots), and we conclude that $x_{2i-1} + x_{2i} \in J - I$ for some $2i < n$.

Now if $x_{2i-1} + x_{2i} \in J - I$, then $x_{2i-1} \equiv x_{2i} \pmod{J}$, but $x_{2i-1} \not\equiv x_{2i} \pmod{I}$ with $x_{2i-1}, x_{2i} \in L$. Finally, if $x_0 \in J - I$, then $x_0 \equiv 0 \pmod{J}$ and $x_0 \not\equiv 0 \pmod{I}$. \square

Theorem 158. *If D_1 and D_2 are generalized boolean lattices R -generated by a distributive lattice L with zero, then D_1 and D_2 are isomorphic.*

Proof. Let B be a free generalized boolean lattice R -generated by L (as defined in Lemma 156). Let φ be a homomorphism of B onto D_1 such that φ is the identity on L , see Lemma 156(ii). We want to show that φ is an isomorphism. Indeed, if φ is not an isomorphism, then the ideal kernel I of φ is not 0. Thus by Lemma 157, $a \equiv b \pmod{I}$ for some $a, b \in L$ with $a \neq b$. This means that $\varphi(a) = \varphi(b)$, contrary to our assumptions. Similarly, there is an isomorphism ψ between B and D_2 . Obviously, $\psi\varphi^{-1}$ is an isomorphism between D_1 and D_2 . \square

Remark. For a distributive lattice L with zero, we shall denote by $\text{BR } L$ a generalized boolean lattice R -generated by L .

Corollary 159. *Let L_0 and L_1 be distributive lattices with zero and let φ be a $\{0\}$ -homomorphism of L_0 onto L_1 . Then φ can be extended to a homomorphism of $\text{BR } L_0$ onto $\text{BR } L_1$.*

Proof. Let α be the congruence kernel of φ , and let $\bar{\alpha}$ be the extension of α to $\text{BR } L_0$ (by Lemma 157). Then $(\text{BR } L_0)/\bar{\alpha}$ is a generalized boolean lattice R -generated by $L_0/\alpha \cong L_1$. Thus

$$(\text{BR } L_0)/\bar{\alpha} \cong \text{BR } L_1$$

by Theorem 158. Now it is trivial to prove this corollary. \square

Corollary 160. *Let L_0 be a $\{0\}$ -sublattice of the distributive lattice L_1 with zero. Let B denote the subalgebra of $\text{BR } L_1$ R -generated by L_0 . Then $\text{BR } L_0 \cong B$.*

Proof. The proof is trivial. \square

Let L_0 and L_1 be given as in Corollary 160. It is natural to ask: Under what conditions does L_0 R -generate $\text{BR } L_1$? Let \bar{L}_0 denote the generalized boolean sublattice of $\text{BR } L_1$ R -generated by L_0 . We can answer our query by determining $L_1 \cap \bar{L}_0$.

Lemma 161. *Let L_0 and L_1 be given as in Corollary 160. Then $L_1 \cap \bar{L}_0$ is the smallest sublattice of L_1 containing L_0 that is closed under taking relative complements in L_1 . Therefore, L_0 R -generates $\text{BR } L_1$ iff the smallest sublattice of L_1 containing L_0 and closed under relative complementation in L_1 is L_1 itself.*

Proof. It is obvious that $L_0 \subseteq L_1 \cap \bar{L}_0$. If $a, b, c \in L_1 \cap \bar{L}_0$, $d \in L_1$, and d is a relative complement of b in $[a, c]$, then $d = a + b + c \in L_1 \cap \bar{L}_0$, since (see Figure 33) d is a relative complement of $a + b$ in the interval $[0, c]$. Thus $d \in L_1 \cap \bar{L}_0$. Now let us assume that L is a sublattice of L_1 containing L_0

and closed under relative complementation in L_1 . If $x \in L_1 \cap \bar{L}_0$, then by Lemma 155, we can represent x as

$$x = a_0 + \cdots + a_{n-1}, \quad a_0, \dots, a_{n-1} \in L_0, \quad a_0 \leq \cdots \leq a_{n-1}.$$

We prove that $x \in L$ by induction on n .

If $n = 1$, then $x = a_0 \in L_0 \subseteq L$.

If $n = 2$, then x is a relative complement of a_0 in $[0, a_1]$ with $0, a_0, a_1 \in L_0$, thus $x \in L$.

If $n = 3$, then (see Figure 32) $x = a_0 + a_1 + a_2$ is a relative complement of a_1 in $[a_0, a_2]$, and so $x \in L$.

Now let $n > 3$, and let $y \in L$ be proved for all $y = b_0 + \cdots + b_{k-1}$ for the elements $b_0, \dots, b_{k-1} \in L_0$ with $b_0 \leq \cdots \leq b_{k-1}$ and $k < n$. Note that $x \in L_1$ and $a_{n-3} \in L_0$ imply that

$$xa_{n-3} = a_0 + \cdots + a_{n-3} + a_{n-3} + a_{n-3} = a_0 + \cdots + a_{n-3} \in L_1$$

and

$$\begin{aligned} x \vee a_{n-3} &= x + a_{n-3} + xa_{n-3} \\ &= a_0 + \cdots + a_{n-1} + a_{n-3} + a_0 + \cdots + a_{n-3} \\ &= a_{n-3} + a_{n-2} + a_{n-1} \in L_1. \end{aligned}$$

By the induction hypothesis,

$$a_0 + \cdots + a_{n-3} \in L \text{ and } a_{n-3} + a_{n-2} + a_{n-1} \in L;$$

therefore, x is a relative complement in L_1 of an element (namely, of a_{n-3}) of L in an interval in L , namely, in

$$[a_0 + \cdots + a_{n-3}, a_{n-3} + a_{n-2} + a_{n-1}],$$

and so, by assumption, $x \in L$. Thus $L_1 \cap \bar{L}_0 \subseteq L$. \square

Some of the results presented above were first published in G. Grätzer [257].

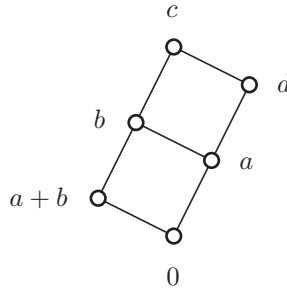


Figure 33. Illustrating the position of d

4.2 The complete case

In Theorem 153, we embedded L into $\text{Pow } X$, which is a complete boolean lattice. The question arises whether we can require this embedding to be complete, that is, to preserve arbitrary joins and meets, if they exist in L .

It is easy to see that not every complete distributive lattice has a complete embedding into a complete boolean lattice (J. von Neumann [552], [553]).

Lemma 162. *Let B be a complete boolean lattice. Then B satisfies the Join Infinite Distributive Identity*

$$(JID) \quad x \wedge \bigvee Y = \bigvee (x \wedge y \mid y \in Y),$$

for $x \in B$ and $Y \subseteq B$, and its dual, the Meet Infinite Distributive Identity, (MID).

Of course, (JID) is not an identity in the sense of Section I.4.2, only an infinitary analogue. The proof of Theorem 149 easily yields that (JID) holds for $\text{Con } L$, the lattice of all congruence relations of the lattice L .

Proof. Obviously, $\bigvee (x \wedge y \mid y \in Y) \leq x \wedge \bigvee Y$. Now let u be any upper bound of $\{x \wedge y \mid y \in Y\}$, that is, $x \wedge y \leq u$ for all $y \in Y$. Then

$$y = y \wedge (x \vee x') = (y \wedge x) \vee (y \wedge x') \leq u \vee x',$$

and so $\bigvee Y \leq u \vee x'$. Thus

$$x \wedge \bigvee Y \leq x \wedge (u \vee x') = (x \wedge u) \vee (x \wedge x') = x \wedge u \leq u,$$

showing that $x \wedge \bigvee Y$ is the least upper bound for $\{x \wedge y \mid y \in Y\}$. By duality, condition (MID) follows. \square

Corollary 163. *Any complete distributive lattice that has a complete embedding into a complete boolean lattice satisfies both (JID) and (MID).*

Easy examples show that (JID) and (MID) need not hold in a complete distributive lattice.

Our task now is to show the converse of Corollary 163 (N. Funayama [213]). The construction of V. Glivenko [233] depends on a property of $\text{BR } L$, on Theorem 100, and on the concept of the skeleton introduced in Section I.6.2.

Lemma 164. *Let L be a distributive lattice with zero. Then $\text{Id } L$ is a pseudo-complemented lattice in which*

$$I^* = \{x \mid x \wedge i = 0, \text{ for all } i \in I\}.$$

Let

$$\text{Skel}(\text{Id } L) = \{I^* \mid I \in \text{Id } L\}.$$

If L is a boolean lattice, then $\text{Skel}(\text{Id } L)$ is a complete boolean lattice and the map $a \mapsto \text{id}(a)$ embeds L into $\text{Skel}(\text{Id } L)$; this embedding preserves all existing meets and joins.

Proof. The first statement is trivial. Now let L be boolean. It follows from Theorem 100 that $\text{Skel}(\text{Id } L)$ is a boolean lattice. Furthermore, it is easily seen that for any $X \subseteq \text{Id } L$, the sup and inf of X in $\text{Skel}(\text{Id } L)$ are $(\bigvee X)^{**}$ and $\bigwedge X$, respectively, where \bigvee and \bigwedge are the join and meet of X in $\text{Id } L$, respectively. For $x, a \in L$, observe that $x \wedge a' = 0$ iff $x \leq a$, and so

$$\text{id}(a) = \text{id}(a')^* \in \text{Skel}(\text{Id } L).$$

Since

$$\bigwedge (\text{id}(x) \mid x \in X) = \text{id}(\inf X),$$

whenever $\inf X$ exists in L , the map $a \mapsto \text{id}(a)$ of L into $\text{Skel}(\text{Id } L)$ preserves all existing meets in L . Now let $a = \sup X$ in L and set

$$I = \text{id}(X) \quad (= \bigvee (\text{id}(x) \mid x \in X)).$$

To show that $x \mapsto \text{id}(x)$ is join-preserving, we have to verify that $I^{**} = \text{id}(a)$, or equivalently, that $I^* = \text{id}(a')$.

Indeed, if $b \in I^*$, then $b \wedge x = 0$, for all $x \in I$, and thus $x \leq b'$. Therefore,

$$a = \sup X \leq b',$$

proving $a' \geq b$, that is, $b \in \text{id}(a')$. Conversely, let $b \in \text{id}(a')$. Then $b' \geq a$; therefore,

$$b' \geq a = \sup X \geq x,$$

for all $x \in X$, and so $b \wedge x = 0$ for all $x \in X$. This shows that $b \in I^*$, proving that $I^* = \text{id}(a')$. \square

Lemma 165. *Let L be a complete lattice satisfying (JID) and (MID). Then the identity map is a complete embedding of L into $\text{BR } L$.*

Proof. Let us write $a \in \text{BR } L$ in the form

$$a = a_0 + \cdots + a_{n-1}, \quad a_0 \leq \cdots \leq a_{n-1}, \quad a_0, \dots, a_{n-1} \in L.$$

If n is even, let us replace a_0 by $0 + a_0$; thus we can assume that n is odd.

We claim that, for any $x \in L$ and $a \in \text{BR } L$, the inequality $x \leq a$ holds iff $x \wedge a_0 = x \wedge a_1$ and $x \leq a_2 + \cdots + a_{n-1}$.

Indeed, let $x \leq a$. Then

$$xa_1 = xa_1(a_0 + \cdots + a_{n-1}) = x(a_0 + a_1 + a_1 + \cdots + a_1) = xa_0;$$

therefore, $x \wedge a_0 = x \wedge a_1$. Thus

$$x(a_2 + \cdots + a_{n-1}) = (xa_0 + xa_1) + x(a_2 + \cdots + a_{n-1}) = xa = x,$$

and so $x \leq a_2 + \cdots + a_{n-1}$. Conversely, if $x \wedge a_0 = x \wedge a_1$ and $x \leq a_2 + \cdots + a_{n-1}$, then

$$xa = xa_0 + xa_1 + x(a_2 + \cdots + a_{n-1}) = x,$$

proving that $x \leq a$.

Now a simple induction proves that $x \leq a$ holds iff

$$\begin{aligned} x \wedge a_0 &= x \wedge a_1, \\ x \wedge a_2 &= x \wedge a_3, \\ &\dots \\ x \wedge a_{n-3} &= x \wedge a_{n-2}, \\ x &\leq a_{n-1}. \end{aligned}$$

Let $X \subseteq L$, let $y = \sup X$ in L , and let $a \in \text{BR } L$. If $x \leq a$, for all $x \in X$, then the formulas last displayed hold for all x , hence by (JID), for the element y , proving that $y \leq a$. Thus $y = \sup X$ in $\text{BR } L$. The dual argument, using (MID), completes the proof. \square

So finally we obtained the following result of N. Funayama [213].

Theorem 166. *A complete lattice L has a complete embedding into a complete boolean lattice iff L satisfies (JID) and (MID).*

Proof. Combine Lemma 162, Corollary 163, and Lemmas 164, 165. \square

4.3 Boolean lattices generated by chains

The representation for $a \in \text{BR } L$ given in Lemma 155 is not unique in general; the only exception is when L is a chain. Since this case is of special interest, we shall investigate it in detail.

Repeating the definition, a boolean lattice B is *R-generated by a chain C* with zero if $B = \text{BR } C$. This concept is due to A. Mostowski and A. Tarski [535] and can be extended to distributive lattices as follows.

A distributive lattice L with zero is *R-generated by a chain C* ($\subseteq L$) with zero if C R-generates $\text{BR } L$.

Lemma 167. *Let L be a distributive lattice with zero and let C be a chain in L with $0 \in C$. Then C R-generates L iff L is the smallest sublattice of itself containing C and closed under formation of relative complements.*

Proof. Apply Lemma 161 to C . \square

An explicit representation of $\text{BR } C$ is given as follows: for a chain C with zero, let $B[C]$ be the set of all subsets of C of the form

$$\text{id}(a_0) + \text{id}(a_1) + \cdots + \text{id}(a_{n-1}), \quad 0 < a_0 < a_1 < \cdots < a_{n-1}, \quad a_0, \dots, a_{n-1} \in C,$$

where $+$ is the symmetric difference in $\text{Pow } X$. We consider $B[C]$ as an order with the ordering \subseteq . We identify $a \in C$ with $\text{id}(a) \cap (C - \{0\})$. Thus $C \subseteq B[C]$.

Lemma 168. *$B[C]$ is the generalized boolean lattice R-generated by C .*

Proof. The proof is obvious, by construction and by Theorem 158. \square

Note that every nonempty element a of $B[C]$ can be represented in the form

$$a = (b_0, a_0] \cup (b_1, a_1] \cup \cdots \cup (b_{n-1}, a_{n-1}], \\ 0 \leq b_0 < a_0 < b_1 < a_1 < \cdots < b_{n-1} < a_{n-1},$$

where the union is disjoint union and $(x, y]$ is a *half-open interval*:

$$(x, y] = \{t \mid x < t \leq y\}$$

for the elements $x, y \in C$. Note that $(x, y]$ can be written $\text{id}(x) + \text{id}(y)$, which, under our identification of an element $x \in C$ with the ideal $\text{id}(x) \in B[C]$, becomes $x + y$. Thus

$$a = a_0 + b_1 + a_1 + \cdots + b_{n-1} + a_{n-1},$$

and so we conclude:

Corollary 169. *In $\text{BR } C$, every nonzero element a has a unique representation in the form*

$$a = a_0 + a_1 + \cdots + a_{n-1}, \quad 0 < a_0 < a_1 < \cdots < a_{n-1}, \quad a_0, a_1, \dots, a_{n-1} \in C.$$

The following results show that many distributive lattices can be R-generated by chains.

Lemma 170. *Every finite boolean lattice B can be R-generated by a chain; in fact, $B = \text{BR } C$ for every maximal chain C of B .*

Proof. Let B_1 be the subalgebra of B R-generated by C . Using the notation of Corollary 112, the length of C equals $|\text{Ji } B|$; also, the length of C equals $|\text{Ji } B_1|$; thus $|\text{Ji } B| = |\text{Ji } B_1| = n$. We conclude that both B and B_1 have 2^n elements. Since $B_1 \subseteq B$, we conclude that $B = B_1$. \square

Corollary 171. *Every finite distributive lattice L can be R-generated by a chain, in fact, by any maximal chain of L .*

Proof. Let C be a maximal chain in L and let $B = \text{BR } L$. Then $|\text{Ji } L| = |\text{Ji } B|$. By Corollary 112, C is maximal in B . Thus $B = \text{BR } C \supseteq L$. \square

Theorem 172. *Let L be a countable distributive lattice with zero. Then L can be R -generated by a chain.*

Proof. Let

$$L = \{a_0 = 0, a_1, a_2, \dots, a_n, \dots\},$$

and let L_n be the sublattice of L generated by a_0, \dots, a_n . Let A_0 be a maximal chain of L_0 , and, inductively, let A_n be a maximal chain of L_n containing A_{n-1} . Set

$$A = \bigcup (A_i \mid i < \omega).$$

Obviously, $0 \in A$. We claim that A R -generates L . Take $a \in \text{BR } L$;

$$a = x_0 + \dots + x_{m-1}, \quad x_0, \dots, x_{m-1} \in L.$$

Clearly,

$$L = \bigcup (L_i \mid i < \omega);$$

thus $x_0, \dots, x_{m-1} \in L_n$, for some n , and so $a \in \text{BR } L_n$. Since L_n is finite, we get $\text{BR } L_n = \text{BR } A_n$; therefore, $a \in \text{BR } A_n \subseteq \text{BR } A$, which proves that $L \subseteq \text{BR } A$. \square

Corollary 173. *The correspondence $C \mapsto \text{BR } C$ maps the class of countable chains with zero onto the class of countable generalized boolean lattices. Under this correspondence, $\{0\}$ -subchains and $\{0\}$ -homomorphic images correspond to $\{0\}$ -subalgebras and $\{0\}$ -homomorphic images.*

Note, however, that $C \cong C'$ is not implied by $\text{BR } C \cong \text{BR } C'$ (see Exercise 4.25).

W. Hanf [372] proves that there is no algorithmic way to find a generating chain in all countable boolean algebras. However, by R. S. Pierce [580], there always are generating chains of a rather special order type.

Much is known about countable chains. Utilizing the previous results, such information can be used to prove results on countable generalized boolean lattices.

Lemma 174. *Every countable chain C can be embedded in the chain \mathbb{Q} of rational numbers. Every countable chain not containing any prime interval is isomorphic to one of the intervals $(0, 1)$, $[0, 1)$, $(0, 1]$, and $[0, 1]$ of \mathbb{Q} .*

Proof. Let $C = \{x_0, x_1, \dots, x_{n-1}, \dots\}$. We define the map φ inductively as follows: Pick an arbitrary $r_0 \in \mathbb{Q}$ and set $\varphi(x_0) = r_0$. If $\varphi(x_0), \dots, \varphi(x_{n-1})$ have already been defined, we define $\varphi(x_n)$ as follows: Let

$$L_n = \bigcup (\text{id}(\varphi(x_i)) \mid x_i < x_n, i < n),$$

$$U_n = \bigcup (\text{fil}(\varphi(x_i)) \mid x_i > x_n, i < n);$$

observe that $L_n = \emptyset$ or $U_n = \emptyset$ is possible. Note that if $L_n \neq \emptyset$, then it has a greatest element l_n , and if $U_n \neq \emptyset$, then it has a smallest element u_n . If both are nonempty, then $l_n < u_n$. In any case, we can choose an $r_n \in \mathbb{Q}$ satisfying $r_n \notin L_n \cup U_n$. We set $\varphi(x_n) = r_n$. Obviously, φ is an embedding.

To prove the second assertion, we may adjoin a zero and/or a unit if these are absent; it is then easy to see that the desired result is equivalent to the statement that any two *bounded* countable chains C and D with no prime intervals satisfy $C \cong D$.

To prove this, let $C = \{c_0, c_1, \dots\}$ and $D = \{d_0, d_1, \dots\}$. We define two maps: $\varphi: C \rightarrow D$ and $\psi: D \rightarrow C$.

Let us assume that $c_0 = 0$, $c_1 = 1$ and $d_0 = 0$, $d_1 = 1$. For each $n < \omega$, we shall define inductively two finite chains $C^{(n)} \subseteq C$ and $D^{(n)} \subseteq D$, and an isomorphism $\varphi_n: C^{(n)} \rightarrow D^{(n)}$ with inverse $\psi_n: D^{(n)} \rightarrow C^{(n)}$.

Set $C^{(0)} = \{c_0, c_1\} = \{0, 1\}$ and $D^{(0)} = \{d_0, d_1\} = \{0, 1\}$; set $\varphi_0(i) = i$ and $\psi_0(i) = i$ for $i = 0, 1$.

Given $C^{(n)}$, $D^{(n)}$, φ_n , ψ_n , and n even, let k be the smallest integer with $c_k \notin C^{(n)}$. Define

$$u_k = \bigwedge (\text{fil}(c_k) \cap C^{(n)}),$$

$$l_k = \bigvee (\text{id}(c_k) \cap C^{(n)}).$$

Then $l_n < c_k < u_k$, and so $\varphi_n(l_k) < \varphi_n(u_k)$. Since D contains no prime intervals, we can choose a $d \in D$ satisfying the inequalities

$$\varphi_n(l_k) < d < \varphi_n(u_k).$$

Since ψ_n is isotone, $d \notin D^{(n)}$. Define

$$C^{(n+1)} = C^{(n)} \cup \{c_k\},$$

$$D^{(n+1)} = D^{(n)} \cup \{d\}.$$

Let φ_{n+1} restricted to $C^{(n)}$ be φ_n , and let $\varphi_{n+1}(c_k) = d$. Let ψ_{n+1} restricted to $D^{(n)}$ be ψ_n and let $\psi_{n+1}(d) = c_k$. If n is odd, then we proceed in a similar way, but we interchange the role of C and D , $C^{(n)}$ and $D^{(n)}$, φ_n and ψ_n , respectively.

Finally, put $\varphi = \bigcup (\varphi_n \mid n < \omega)$. Clearly,

$$C = \bigcup (C^{(n)} \mid n < \omega),$$

$$D = \bigcup (D^{(n)} \mid n < \omega),$$

and φ is the required isomorphism. □

Corollary 175. *Up to isomorphism, there is exactly one countable boolean lattice with no atoms and exactly one countable generalized boolean lattice with no atoms and no unit element, $\text{BR}[0, 1]_{\mathbb{Q}}$.*

Proof. Take the rational intervals $[0, 1]$ and $[0, 1)$. The generalized boolean lattices in question are $\text{BR}[0, 1]$ and $\text{BR}[0, 1)$. This follows from the observation that $[a, b]$ is a prime interval in C iff $a + b$ is an atom in $\text{BR } C$. The results follow from Lemmas 168 and 174 and Theorem 172. \square

Theorem 176. *Let B be a countable boolean algebra. Then B has either \aleph_0 or 2^{\aleph_0} prime ideals.*

Remark. This is obvious if we assume the Continuum Hypothesis. Interestingly, we can give a proof without it.

Proof. For a boolean algebra B and an ideal I of B , we shall write B/I for $B/\text{con}(I)$. If J is an ideal of B with $J \supseteq I$, then

$$J/I = \{x/\text{con}(I) \mid x \in J\}$$

is an ideal of B/I . (J/I is the usual notation in ring theory.)

Let B be a boolean algebra. We define the ideals I_γ by transfinite induction. Let $I_0 = \text{id}(0)$, and let I_1 be the ideal generated by the set $\text{Atom}(B)$. Given I_γ , let I be the ideal of B/I_γ generated by $\text{Atom}(B/I_\gamma)$. Let

$$\varphi: x \mapsto x + I_\gamma$$

be the homomorphism of B onto B/I_γ ; we set

$$I_{\gamma+1} = \varphi^{-1}(I).$$

Finally, if γ is a limit ordinal, set

$$I_\gamma = \bigcup (I_\delta \mid \delta < \gamma).$$

The *rank* of B is defined to be the smallest ordinal α such that $I_\alpha = I_{\alpha+1}$.

No element of B can have an image which is an atom in more than one of the quotient lattices B/I_γ , hence, the cardinality of α is at most $|B|$.

Claim 177. *Let B be countable. If $I_\alpha \neq B$, then $|\text{Spec } B| = 2^{\aleph_0}$.*

Indeed, if $I_\alpha \neq B$, then $\text{Atom}(B/I_\alpha) = \emptyset$, hence by the proof of Corollary 175, the isomorphism $B/I_\alpha \cong \text{BR } C$ holds, where C is the rational interval $[0, 1]$. By Lemma 157 and Exercise 4.32,

$$|B/I_\alpha| = |\text{Spec } C| = |\text{Id } C| = 2^{\aleph_0}.$$

Claim 178. *Let B be countable. If $I_\alpha = B$, then $|\text{Spec } B| = \aleph_0$.*

Indeed, for an ordinal $\gamma < \alpha$, let $\text{Spec}_\gamma B$ be the set of prime ideals P of B for which $I_\gamma \subseteq P$ and $I_{\gamma+1} \not\subseteq P$. It is easy to see that every prime ideal of B lies in one of the sets $\text{Spec}_\gamma B$. Since α is finite or countable, it suffices to show that $|\text{Spec}_\gamma B| = \aleph_0$. If $P \in \text{Spec}_\gamma B$, then, by Corollary 116 and Theorem 123, the equality $P \vee I_{\gamma+1} = B$ holds. It follows that

$$P \cap I_{\gamma+1} \neq Q \cap I_{\gamma+1}$$

for $P, Q \in \text{Spec}_\gamma B$ with $P \neq Q$. Thus

$$P \mapsto (P \cap [I_{\gamma+1}]_R) / I_\gamma$$

is a one-to-one correspondence of $\text{Spec}_\gamma B$ into (in fact, onto) $\text{Spec}([I_{\gamma+1}] / I_\gamma)$; but $[I_{\gamma+1}]_R / I_\gamma$ is just the generalized boolean lattice of all finite subsets of a countable set. Therefore, $|\text{Spec}_\gamma B| = \aleph_0$. \square

In order to avoid giving the impression that most boolean algebras can be R-generated by chains, we state:

Lemma 179. *Let B be a complete boolean algebra R-generated by a chain C with zero. Then B is finite.*

Proof. Let $B = \text{BR}C$ and let the chain C be infinite. Then by Exercise I.6.20, C cannot satisfy both the Ascending and the Descending Chain Conditions. Assume that the former fails. (If the latter fails, replace C by the chain of complements of its elements.) Thus C contains a subchain

$$0 < x_0 < x_1 < \cdots < x_n < \cdots$$

Then we define

$$a_n = x_0 + x_1 + \cdots + x_{2n} + x_{2n+1}$$

for all $n < \omega$. We claim that $\bigvee (a_n \mid n < \omega)$ does not exist. Indeed, let a majorize $\{a_n \mid n < \omega\}$. By the remarks immediately following Lemma 168, we can represent each a_n by a set

$$(x_0, x_1] \cup (x_2, x_3] \cup \cdots \cup (x_{2n}, x_{2n+1}],$$

and we can represent a in the form

$$a = (b_0, a_0] \cup (b_1, a_1] \cup \cdots \cup (b_{m-1}, a_{m-1}],$$

where $m < \omega$ and

$$0 \leq b_0 < a_0 < b_1 < a_1 < \cdots < b_{m-1} < a_{m-1}$$

with $a_i, b_i \in C$ for all $i < m$. Since a contains each a_n , there must exist an n and a $j < m$ such that both $(x_{2n}, x_{2n+1}]$ and $(x_{2n+2}, x_{2n+3}]$ are contained in $(b_{j-1}, b_j]$ or in $(0, b_0]$. Therefore, the interval $(x_{2n+1}, x_{2n+2}]$ can be deleted from a , and it will still contain all the a_n , that is, $a + x_{2n+1} + x_{2n+2}$ majorizes both $\{a_n \mid n < \omega\}$ and $a + x_{2n+1} + x_{2n+2} < a$. We conclude that $\{a_n \mid n < \omega\}$ does not have a least upper bound. \square

Next we consider which chains with zero can be R-generating chains of a given distributive lattice.

Lemma 180. *Let L be a distributive lattice with zero and let C be a chain in L with $0 \in C$. If L is R-generated by C , then C is maximal in L .*

Proof. If C is not maximal in L , then we can find an element a in L not in C , such that $C \cup \{a\}$ is a chain. Write

$$a = a_0 + a_1 + \cdots + a_{n-1},$$

with $0 < a_0 < a_1 < \cdots < a_{n-1}$ and $a_i \in C$ for all $i < n$. Since $a \notin C$, it follows that $n > 1$. Now

$$a \wedge a_0 = a_0 + a_0 + \cdots + a_0,$$

which is a_0 if n is odd and 0 if n is even. But $a_0 \neq a$ and $0 \neq a$, therefore, since a and a_0 are comparable, $a \wedge a_0 = a_0$ and n is odd. Then

$$a \wedge a_1 = a_0 + a_1 + \cdots + a_1 = a_0,$$

contradicting the comparability of a and a_1 . □

The converse of Lemma 180 is false by Lemma 179. To settle the matter, we need a new concept.

Definition 181. Let L be a distributive lattice with zero and let C be a chain in L with $0 \in C$. The chain C is called *strongly maximal in L* if, for every homomorphism φ of L onto a distributive lattice L_1 , the chain $\varphi(C)$ is maximal in L_1 .

Now the following theorem resolves our problem.

Theorem 182. *Let L be a distributive lattice with zero and let C be a chain in L with $0 \in C$. Then C R-generates L iff C is strongly maximal in L .*

Proof. If C R-generates L , then $\varphi(C)$ R-generates $\varphi(L)$ for every onto homomorphism φ . By Lemma 180, $\varphi(C)$ is maximal in $\varphi(L)$, so C is strongly maximal in L .

Next assume that C is strongly maximal in L but does not R-generate L . Without any loss of generality, we can assume that L and C have a greatest element. (Otherwise, add one. Then $C \cup \{1\}$ is strongly maximal in $L \cup \{1\}$ but does not R-generate $L \cup \{1\}$.) Let $B_1 = \text{BR } L$ and let $B_0 = \text{BR } C$. By hypothesis, $B_0 \neq B_1$, so there exists an $a \in B_1 - B_0$.

We claim that there exist prime ideals $P_1 \neq P_2$ of B_1 with

$$B_0 \cap P_1 = B_0 \cap P_2.$$

With $I = \text{id}(\text{id}(a) \cap B_0)$ and $D = \text{fil}(a)$ (formed in B_1), the equality $I \cap D = \emptyset$ holds, so by Theorem 115, there is a prime ideal P_1 such that $I \subseteq P_1$ and $P_1 \cap D = \emptyset$. Then let $I_1 = \text{id}(a)$ and $D_1 = \text{fil}(B_0 - P_1)$. Since $\text{id}(a) \cap B_0 \subseteq P_1$, it follows that $I_1 \cap D_1 = \emptyset$. Let P_2 be a prime ideal with $I_1 \subseteq P_2$ and $P_2 \cap D_1 = \emptyset$. Then $a \in P_2 - P_1$, so $P_1 \neq P_2$. Because $P_2 \cap (B_0 - P_1) = \emptyset$, it follows that $P_2 \cap B_0 \subseteq P_1 \cap B_0$. Since prime ideals of a boolean lattice are unordered (Theorem 123), it follows that $P_1 \cap B_0 = P_2 \cap B_0$, proving our claim.

Now we can map B_1 onto B_2 by a homomorphism ψ :

$$\psi(x) = \begin{cases} (0, 0), & \text{for } x \in P_1 \cap P_2; \\ (0, 1), & \text{for } x \in P_2 - P_1; \\ (1, 0), & \text{for } x \in P_1 - P_2; \\ (1, 1), & \text{for } x \notin P_1 \cup P_2. \end{cases}$$

Since $\psi(C) \subseteq \psi(B_0) = \{(0, 0), (1, 1)\}$ is not maximal, we conclude that C is not strongly maximal in L . \square

Thus a distributive lattice L is R-generated by a chain iff it has a strongly maximal chain. Theorem 172 shows that such chains exist if L is countable, while Lemma 179 shows that they do not always exist.

Corollary 183. *Let C and D be strongly maximal chains of the distributive lattice L with zero. Then $|C| = |D|$ and $|\text{Id } C| = |\text{Id } D|$.*

Proof. If L is finite, these conclusions follow from Corollary 112. If $|L|$ is infinite, then C and D generate $\text{BR } L$ as a generalized boolean lattice, and so $|C| = |D| = |L|$. By Lemma 157,

$$|\text{Spec } C| = |\text{Spec}(\text{BR } L)| = |\text{Spec } D|;$$

also $\text{Spec } C = \text{Id } C$ and $\text{Spec } D = \text{Id } D$, hence the second statement. \square

Corollary 183 is the strongest known extension of Corollary 112 to the infinite case. The second statement of Corollary 183 is from G. Grätzer and E. T. Schmidt [330].

A. Mostowski and A. Tarski [535] were the first to investigate boolean algebras generated by chains. Theorem 172 for boolean lattices and Theorem 176 were communicated to the author by J. R. Büchi. These results have been known for some time in topology (via the Stone topological representation theorem, see Section 5.2). Some of the other results appeared first in G. Grätzer [257].

Exercises

- 4.1. In a boolean lattice B , prove that

$$b_0 + \cdots + b_{n-1} = \sum_{1 \leq m \leq n} \bigvee (b_{i_0} \wedge \cdots \wedge b_{i_{m-1}} \mid i_0 < \cdots < i_{m-1})$$

for $b_0, \dots, b_{n-1} \in B$. Observe that the terms being summed on the right side of the formula form a chain (G. M. Bergman).

- 4.2. Use Exercise 4.1 to prove Lemma 155.
 4.3. Give a detailed proof of Lemma 156.
 4.4. Try to describe the most general situation to which the idea of the proof of Theorem 69 (Theorem 89) could be applied.
 4.5. Can you redefine “the boolean lattice B generated by a distributive lattice L ”, so that Lemma 157 remains valid?
 4.6. Find necessary and sufficient conditions on a distributive lattice L in order that L have a boolean congruence-preserving extension B .
 4.7. Work out Corollaries 159 and 160 for the boolean lattice R -generated by a distributive lattice L .
 4.8. Let B be a generalized boolean lattice and let L be a sublattice of B . Let $x \in L$ be written in B as the sum of a chain of elements of L : $x = x_0 + \cdots + x_{n-1}$ with $x_0 \leq \cdots \leq x_{n-1}$ and $x_0, \dots, x_{n-1} \in L$. Then

$$x_0 + \cdots + x_{m-1} \in L$$

holds for each $m \leq n$ with $m \equiv n \pmod{2}$.

- 4.9. Use Exercise 4.8 to prove Lemma 161.
 4.10. Let L be a finite lattice. Under what conditions on L is the map

$$x \mapsto \{u \in \text{Mi } L \mid x \not\leq u\}$$

a meet-embedding into the boolean lattice $\text{Pow}(\text{Ji } L)$.

- 4.11. How do you modify Exercise 4.10 to get a join-embedding? (M. Wild [729] characterizes finite lattices that have a cover-preserving embedding into a boolean lattice; but the embedding usually is neither join- nor meet-embedding.)
 4.12. Let L be the lattice of closed subsets of the real unit interval $[0, 1]$. Does (JID) or (MID) hold in L ?
 4.13. Show that in any complete distributive lattice, (JID) holds whenever x is a complemented element.
 4.14. Can you generalize Exercise 4.13 to “dual semi complements”, that is, to elements \bar{x} with $x \vee \bar{x} = 1$?
 *4.15. The *Complete Infinite Distributive Identity* is (for $I, J \neq \emptyset$):

$$\begin{aligned} \text{(CID)} \quad & \bigwedge \left(\bigvee (a_{i,j} \mid j \in J) \mid i \in I \right) \\ & = \bigvee \left(\bigwedge (a_{i,\varphi(i)} \mid i \in I) \mid \varphi: I \rightarrow J \right). \end{aligned}$$

Show that (CID) holds in a complete boolean lattice B iff it is atomic (A. Tarski [673]). (Hint: apply (CID) to $\bigwedge (a \vee a' \mid a \in B) = 1$.)

- 4.16. Prove that (CID) is selfdual for Boolean lattices.
- 4.17. Let B be a boolean lattice and let I be an ideal of B . Show that I is normal iff $I = I^{**}$ (for these concepts, see Exercise I.3.71–I.3.74 and Lemma 164).
- 4.18. Prove that the boolean lattice $\text{Skel}(\text{Id } L)$ of Lemma 164 is the MacNeille completion of the boolean lattice L .
- *4.19. Show that the MacNeille completion of a distributive lattice need not even be modular.
- 4.20. Let L be a distributive algebraic lattice. Show that L satisfies (JID). (Thus $\text{Con } K$ satisfies (JID) for every lattice K .)
- 4.21. Let L be a distributive lattice, $a_i, b_i \in L$, for all $i < \omega$, and

$$[a_0, b_0] \supset [a_1, b_1] \supset \cdots$$

Define

$$\alpha = \bigvee (\text{con}(a_0, a_i) \vee \text{con}(b_0, b_i) \mid i < \omega).$$

Show that

$$\alpha \vee \bigwedge (\text{con}(a_i, b_i) \mid i < \omega) \neq \bigwedge (\alpha \vee \text{con}(a_i, b_i) \mid i < \omega).$$

- 4.22. Let L be a distributive lattice. Use Exercise 4.21 to show that (MID) holds in $\text{Con } L$ iff every interval in L is finite (G. Grätzer and E. T. Schmidt [332]).
- *4.23. Prove the converse of Lemma 170: If every maximal chain R-generates the boolean lattice B , then B is finite.
- 4.24. Why is it not possible to use transfinite induction to extend Theorem 172 to the uncountable case?
- 4.25. Let C be a bounded chain and let $a \in C - \{0, 1\}$. Define

$$C' = [a, 1] \dot{+} [0, a].$$

Then C' is a chain, and $\text{BR } C \cong \text{BR } C'$, but, in general, $C \cong C'$ does not hold.

- 4.26. Describe a countable family of pairwise nonisomorphic countable boolean algebras.
- 4.27. Prove that a distributive lattice L_1 is R-generated by a sublattice L_0 iff distinct prime ideals of L_1 restrict to distinct prime ideals of L_0 .
- 4.28. Relate Exercise 4.27 to Theorem 182.
- 4.29. Give an example of a bounded distributive lattice L with a maximal chain C such that C is not maximal in $\text{BR } L$ (G. W. Day [142]).
- 4.30. Let L_0 be the $[0, 1]$ rational interval and let L_1 be the $[0, 1]$ real interval. Let

$$C = \{(x, x) \mid 0 \leq x \leq 1, x \text{ rational}\}.$$

Then C is a maximal chain in $L_0 \times L_1$. Show that C is not strongly maximal (G. Grätzer and E. T. Schmidt [330]).

- 4.31. In $L_0 \times L_1$ of Exercise 4.30, find a maximal chain of cardinality \aleph_0 ; find another of cardinality 2^{\aleph_0} . Show that $L_0 \times L_1$ has strongly maximal chains. What are their cardinalities?
- 4.32. Let L be a distributive lattice with zero and let $B = \text{BR } L$. Show that

$$P \mapsto P \cap L, \quad \text{for } P \in \text{Spec } B$$

is a one-to-one correspondence between the prime ideals of L and B .

- 4.33. Let A be a countably infinite set, and let $B = \text{Pow } A$. Prove that B has maximal chains of cardinality \aleph_0 and 2^{\aleph_0} .
- *4.34. Using the Generalized Continuum Hypothesis, generalize Exercise 4.33 to arbitrary infinite sets.
- 4.35. Construct an example in which the sequence of ideals I_γ of Theorem 176 does not terminate in finitely many steps.
- 4.36. Let C be the $[0, 1]$ interval of the rational numbers. Show that $\text{BR } C$ is $\text{Free}_{\mathbf{B}}(\aleph_0)$.
- 4.37. Let L be a distributive lattice with zero and let B be a generalized boolean lattice. Then every $\{0\}$ -homomorphism $\varphi: L \rightarrow B$ can be extended to a unique $\{0\}$ -homomorphism $\bar{\varphi}: \text{BR } L \rightarrow B$.
- 4.38. Let L_0 and L_1 be distributive lattices with zero. Then every $\{0\}$ -homomorphism $\varphi: L_0 \rightarrow L_1$ can be extended to a unique $\{0\}$ -homomorphism

$$\text{BR}(\varphi): \text{BR } L_0 \rightarrow \text{BR } L_1.$$

- 4.39. The assignment $L \mapsto \text{BR } L$, $\varphi \mapsto \text{BR}(\varphi)$ described in Exercise 4.38 is a functor from the category of distributive lattices with zero with $\{0\}$ -homomorphisms to the category of generalized boolean lattices with ring homomorphisms. Show that this functor preserves direct limits (F. Wehrung).

5. Topological Representation

The order $\text{Spec } L$ of prime ideals does give a great deal of information about the distributive lattice L , but obviously it does not characterize L . For instance, for a countably infinite boolean algebra L , the order $\text{Spec } L$ is an unordered set of cardinality \aleph_0 or 2^{\aleph_0} , whereas there are surely more than two such boolean algebras up to isomorphism.

Therefore, it is necessary to endow $\text{Spec } L$ with more structure if we want it to characterize L . M. H. Stone [669] endowed $\text{Spec } L$ with a topology; see also L. Rieger [611]. In most of this section, we shall discuss this approach in a slightly more general but, in our opinion, more natural framework. Then we follow H. A. Priestley and also endow $\text{Spec } L$ with an ordering: \subseteq . (See Section VI.2.8 for a related topic.)

These two approaches use topology to better understand distributive lattices. We conclude this section with a brief section on frames; how distributive lattices can be used to better understand topology.

5.1 Distributive join-semilattices

Let us call a join-semilattice L *distributive* if

$$a \leq b_0 \vee b_1, \quad \text{for } a, b_0, b_1 \in L,$$

implies that

$$a = a_0 \vee a_1 \quad \text{for some } a_0, a_1 \in L \text{ with } a_0 \leq b_0 \text{ and } a_1 \leq b_1;$$

see Figure 34. Note that a_0 and a_1 need not be unique.

Some elementary properties of a distributive join-semilattice are as follows (see the basic concepts following Definition 41):

Lemma 184.

- (i) *If $(L; \vee, \wedge)$ is a lattice, then the join-semilattice $(L; \vee)$ is distributive iff the lattice $(L; \vee, \wedge)$ is distributive.*
- (ii) *If a join-semilattice L is distributive, then for every $a, b \in L$, there is an element $d \in L$ with $d \leq a$ and $d \leq b$. Consequently, $\text{Id } L$ is a lattice.*
- (iii) *A join-semilattice L is distributive iff $\text{Id } L$, as a lattice, is distributive.*

Proof.

(i) If $(L; \vee, \wedge)$ is distributive, and $a \leq b_0 \vee b_1$, then with $a_0 = a \wedge b_0$ and $a_1 = a \wedge b_1$, we obtain that $a = a_0 \vee a_1$. Conversely, if $(L; \vee)$ is distributive, and the lattice L contains a diamond or a pentagon $\{o, a, b, c, i\}$, then $a \leq b \vee c$,

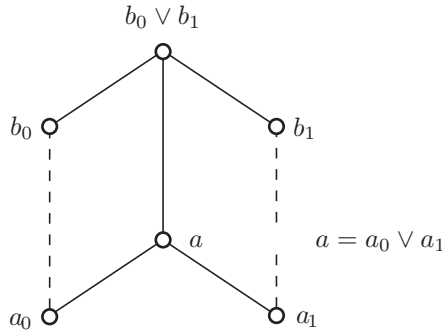


Figure 34. The distributivity of a semilattice

but a cannot be represented as $a = a_0 \vee a_1$ with $a_0 \leq b$ and $a_1 \leq c$, a contradiction.

(ii) $a \leq a \vee b$, thus $a = a_0 \vee b_0$, where $a_0 \leq a$ and $b_0 \leq b$. Since, in addition, the inequality $b_0 \leq a$ holds, it follows that b_0 is a lower bound for a and b .

(iii) First we observe that, for $I, J \in \text{Id } L$,

$$I \vee J = \{i \vee j \mid i \in I, j \in J\}$$

follows from the assumption that the join-semilattice L is distributive. Therefore, the distributivity of $\text{Id } L$ can be easily proved. Conversely, if $\text{Id } L$ is distributive and $a \leq b_0 \vee b_1$, then

$$\text{id}(a) = \text{id}(a) \wedge (\text{id}(b_0) \vee \text{id}(b_1)) = (\text{id}(a) \wedge \text{id}(b_0)) \vee (\text{id}(a) \wedge \text{id}(b_1)),$$

and so $a = a_0 \vee a_1$ with $a_0 \in \text{id}(b_0)$ and $a_1 \in \text{id}(b_1)$, which is distributivity for the join-semilattice L . \square

A nonempty subset D of a join-semilattice L is called a *filter* if the following two conditions hold:

- (i) $a, b \in D$ implies that there exists a lower bound $d \in D$ of a and b ;
- (ii) $a \in D$, $x \in L$, and $x \geq a$ imply that $x \in D$.

An ideal I of L is *prime* if $I \neq L$ and $L - I$ is a filter. Again, let $\text{Spec } L$ denote the set of all prime ideals of L .

Lemma 185. *Let I be an ideal and let D be a filter of a distributive join-semilattice L . If $I \cap D = \emptyset$, then there exists a prime ideal P of L with $P \supseteq I$ and $P \cap D = \emptyset$.*

Proof. The proof is a routine modification of the proof of Theorem 115. \square

In the rest of this section, unless stated otherwise, let L stand for a distributive join-semilattice with zero.

5.2 Stone spaces

We shall now develop a representation of L as a join-semilattice of subsets of the set $\text{Spec } L$. Note that larger elements of L lie in fewer prime ideals; hence under our representation, each $a \in L$ will be mapped to the set of prime ideals *not* containing it:

$$\text{spec}(a) = \{P \in \text{Spec } L \mid a \notin P\}.$$

We shall introduce a topology on $\text{Spec } L$ in which all sets $\text{spec}(a)$ are open.

We also denote by $\text{Spec } L$ the *topological space* defined on $\text{Spec } L$ by postulating that the sets of the form $\text{spec}(a)$ be a *subbase* for the open sets; we shall call $\text{Spec } L$ the *Stone space* of L . (Exercises 5.1–5.23 review the basic topological concepts used in this section.)

Lemma 186. *For an ideal I of L , define*

$$\text{spec}(I) = \{P \in \text{Spec } L \mid P \not\supseteq I\}.$$

Then $\text{spec}(I)$ is open in $\text{Spec } L$. Conversely, every open set U of $\text{Spec } L$ can be uniquely represented as $\text{spec}(I)$ for some ideal I of L .

Proof. We simply observe that

$$\begin{aligned} \text{spec}(I) \cap \text{spec}(J) &= \text{spec}(I \wedge J), \\ \text{spec}\left(\bigvee (I_j \mid j \in K)\right) &= \bigcup (\text{spec}(I_j) \mid j \in K), \end{aligned}$$

and $\text{spec}(\text{id}(a)) = \text{spec}(a)$, from which it follows that the $\text{spec}(I)$ form the smallest collection of sets closed under finite intersection and arbitrary union containing all the sets $\text{spec}(a)$ for $a \in L$. Observe that $a \in I$ iff $\text{spec}(a) \subseteq \text{spec}(I)$. Thus $\text{spec}(I) = \text{spec}(J)$ holds iff $a \in I$ is equivalent to $a \in J$, which in turn is equivalent to $I = J$. \square

Lemma 187. *The subsets of $\text{Spec } L$ of the form $\text{spec}(a)$ can be characterized as compact open sets.*

Proof. Indeed, if a family of open sets $\{\text{spec}(I_k) \mid k \in K\}$ is a cover for $\text{spec}(a)$, that is,

$$\text{spec}(a) \subseteq \bigcup (\text{spec}(I_k) \mid k \in K) = \text{spec}\left(\bigvee (I_k \mid k \in K)\right),$$

then $a \in \bigvee (I_k \mid k \in K)$. This implies that $a \in \bigvee (I_k \mid k \in K_0)$, for some finite $K_0 \subseteq K$, proving that $\text{spec}(a) \subseteq \bigcup (\text{spec}(I_k) \mid k \in K_0)$. Thus $\text{spec}(a)$ is compact. Conversely, if I is not principal, then

$$\text{spec}(I) \subseteq \bigcup (\text{spec}(a) \mid a \in I),$$

but being nonprincipal, I will not be finitely generated, hence

$$\text{spec}(I) \not\subseteq \bigcup (\text{spec}(a) \mid a \in I_0)$$

for any finite $I_0 \subseteq I$. \square

From Lemma 187, we immediately conclude:

Theorem 188. *The Stone space $\text{Spec } L$ determines L up to isomorphism.*

5.3 The characterization of Stone spaces

Stone spaces are characterized in Theorem 191. To prepare for the proof of Theorem 191, we first prove Lemma 189.

Let P be a prime ideal of L . Then P is represented as an element of $\text{Spec } L$ and also by $\text{spec}(P)$, an open set not containing that element. The connection between P and $\text{spec}(P)$ is given in Lemma 189 and is illustrated by Figure 35.

Lemma 189. *For every prime ideal P of L ,*

$$\overline{\{P\}} = \text{Spec } L - \text{spec}(P),$$

where $\overline{\{P\}}$ is the topological closure of the set $\{P\}$.

Proof. By the definition of closure,

$$\begin{aligned} \overline{\{P\}} &= \{Q \in \text{Spec } L \mid Q \in \text{spec}(a) \text{ implies that } P \in \text{spec}(a)\} \\ &= \{Q \in \text{Spec } L \mid Q \supseteq P\} = \text{Spec } L - \{Q \mid Q \not\supseteq P\} \\ &= \text{Spec } L - \text{spec}(P). \end{aligned} \quad \square$$

Corollary 190. *If $P \neq Q$, then $\overline{\{P\}} \neq \overline{\{Q\}}$.*

Proof. Combine Lemmas 186 and 189. \square

So $\text{Spec } L$ is a T_0 -space.

Lemma 189 also shows that if P is a prime ideal, then $\text{Spec } L - \text{spec}(P)$ must be the closure of a singleton. In other words:

(GC) Let U be a proper open set. Let us assume that for any pair of compact open sets U_0 and U_1 satisfying $U_0 \cap U_1 \subseteq U$, it follows that $U_0 \subseteq U$ or $U_1 \subseteq U$. Then $\text{Spec } L - U = \overline{\{P\}}$ for some element P .

Now we can state the characterization theorem.

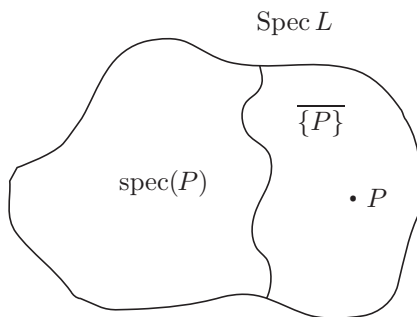


Figure 35. The connection between P and $\text{spec}(P)$

Theorem 191. *Let \mathcal{S} be a topological space. Then there exists a distributive join-semilattice L such that up to homeomorphism, $\text{Spec } L = \mathcal{S}$ iff the following two conditions hold.*

- (Stone1) \mathcal{S} is a T_0 -space in which the compact open sets form a base for the open sets.
- (Stone2) Let F be a closed set in \mathcal{S} , let $\{U_k \mid k \in K\}$ be a dually directed family (that is, $K \neq \emptyset$ and, for every $k, l \in K$, there exists a $t \in K$ such that $U_t \subseteq U_k \cap U_l$) of compact open sets of \mathcal{S} , and let $U_k \cap F \neq \emptyset$ for all $k \in K$; then $\bigcap (U_k \mid k \in K) \cap F \neq \emptyset$.

Remark. The meaning of condition (Stone1) is clear. Condition (Stone2) is a complicated way of ensuring that condition (GC) holds and that Lemma 185 holds for the join-semilattice of compact open sets of $\text{Spec } L$.

Proof. To show that condition (Stone1) holds for $\text{Spec } L$, we have to verify that the $\text{spec}(a)$, for $a \in L$, form a base (not only a subbase) for the open sets of $\text{Spec } L$. In other words, for $a, b \in L$ and $P \in \text{spec}(a) \cap \text{spec}(b)$, we have to find an element $c \in L$ with $P \in \text{spec}(c)$ and $\text{spec}(c) \subseteq \text{spec}(a) \cap \text{spec}(b)$. By assumption, $a \notin P$ and $b \notin P$. Since P is prime, there exists an element $c \in L$ with $c \notin P$ and with $c \leq a$, $c \leq b$. Then

$$P \in \text{spec}(c), \text{spec}(c) \subseteq \text{spec}(a), \text{spec}(c) \subseteq \text{spec}(b),$$

as required.

To verify condition (Stone2) for $\text{Spec } L$, let $F = \text{Spec } L - \text{spec}(I)$ and $U_k = \text{spec}(a_k)$. Thus

$$\begin{aligned} F &= \{P \mid P \supseteq I\}, \\ U_k &= \{P \mid a_k \notin P\}. \end{aligned}$$

The assumption that the $\{U_k \mid k \in K\}$ is a dually directed family implies that

$$D = \{x \mid x \geq a_k \text{ for some } k \in K\}$$

is a filter; since $U_k \cap F \neq \emptyset$, we have $\text{spec}(a_k) \not\subseteq \text{spec}(I)$; that is, $a_k \notin I$, showing that $D \cap I = \emptyset$. Therefore, by Lemma 185, there exists a prime ideal P with $P \supseteq I$ and $P \cap D = \emptyset$. Then $a_k \notin P$, and so $P \in \text{spec}(a_k)$ for all $k \in K$. Also $P \supseteq I$, thus $P \notin \text{spec}(I)$, and so $P \in F$, proving that

$$P \in F \cap \bigcap (U_k \mid k \in K),$$

verifying (Stone2).

Conversely, let \mathcal{S} be a topological space satisfying conditions (Stone1) and (Stone2). Let L be the set of compact open sets of \mathcal{S} . Obviously, $\emptyset \in L$;

moreover, if $A, B \in L$, then $A \cup B \in L$, and thus L is a join-semilattice with zero. Let

$$A \subseteq B_0 \cup B_1, \text{ with } A, B_0, B_1 \in L.$$

Then $A \cap B_i$ is open, and therefore

$$A \cap B_i = \bigcup (A_j^i \mid j \in J_i), \quad i = 0, 1,$$

where the A_j^i are compact open sets. Since

$$A = (A \cap B_0) \cup (A \cap B_1) \subseteq \bigcup (A_j^i \mid j \in J_0 \cup J_1, i = 0, 1),$$

by the compactness of A , we get

$$A \subseteq \bigcup (A_j^i \mid j \in J_0^* \text{ or } j \in J_1^*),$$

where J_i^* is a finite subset of J_i for $i = 0, 1$. Set

$$A_i = \bigcup (A_j^i \mid j \in J_i^*), \quad \text{for } i = 0, 1.$$

Then $A_0, A_1 \in L$, $A = A_0 \cup A_1$, and $A_0 \subseteq B_0$, $A_1 \subseteq B_1$, showing that L is distributive.

It follows from (Stone1) that the open sets of \mathcal{S} are uniquely associated with ideals of L : for an ideal I of L , let

$$U(I) = \bigcup (a \mid a \in I)$$

(keep in mind that an $a \in L$ is a subset of \mathcal{S} , as illustrated in [Figure 36](#)). Note that $a \in I$ iff $a \subseteq U(I)$ for any $a \in L$.

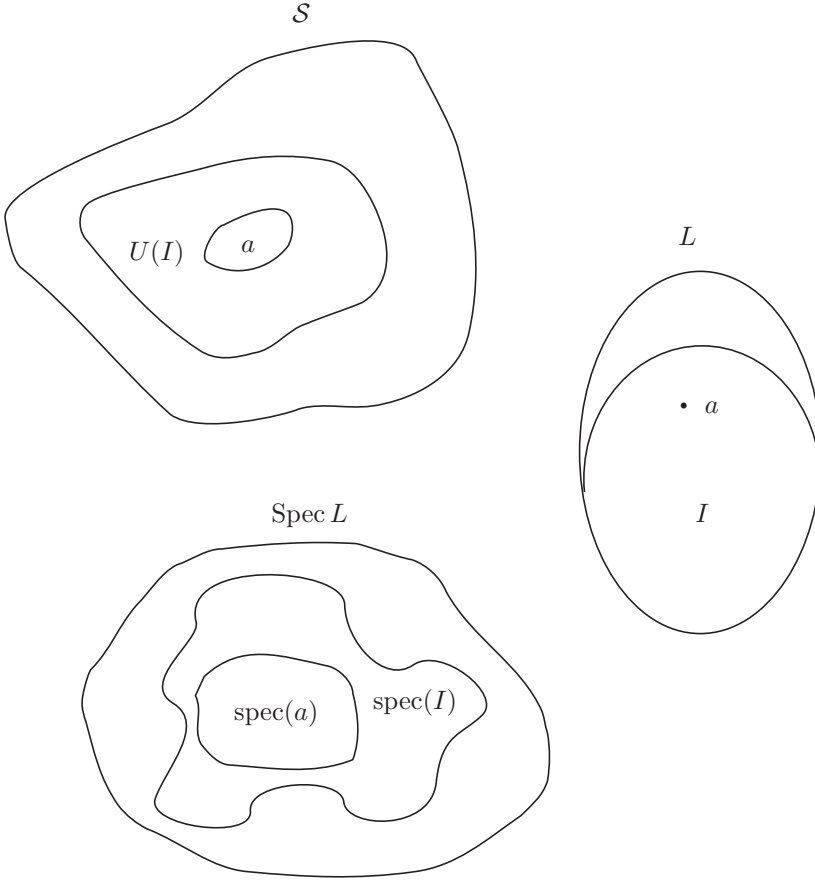
Now let P be a prime ideal of L , let $F = \mathcal{S} - U(P)$, and let $\{U_k \mid k \in K\}$ be the set of all compact open sets of \mathcal{S} that have nonempty intersections with F . Thus the U_k are exactly those elements of L that are not in P . Therefore, by the definition of a prime ideal, given $k, l \in K$, there exists $t \in K$ with $U_t \subseteq U_k$ and $U_t \subseteq U_l$, proving that F and $\{U_k \mid k \in K\}$ satisfy the hypothesis of (Stone2). By (Stone2), we conclude that there exists an element

$$p \in F \cap \bigcap (U_k \mid k \in K).$$

If $q \in F$, then $U \cap F \neq \emptyset$ for every compact open set U with $q \in U$; thus $p \in U$, proving that $\overline{\{p\}} = F$. Note that \mathcal{S} is a T_0 -space; therefore, p is unique. We shall write $p = \varphi(P)$.

Conversely, if $p \in \mathcal{S}$, let

$$I = \{a \in L \mid a \subseteq \mathcal{S} - \overline{\{p\}}\}.$$

Figure 36. a , $\text{spec}(a)$; I , $\text{spec}(I)$; L , \mathcal{S} , and $\text{Spec } L$

Then I is an ideal of L , and $\mathcal{S} - \overline{\{p\}} = U(I)$. We claim that I is prime. Indeed, if $U, V \in L$, $U \notin I$, $V \notin I$, then $U \cap \overline{\{p\}} \neq \emptyset$, $V \cap \overline{\{p\}} \neq \emptyset$, and therefore, $p \in U$ and $p \in V$. Thus $p \in U \cap V$ and so $U \cap V \not\subseteq U(I)$. By (Stone1), there exists a $W \in L$ with $W \subseteq U \cap V$ and $W \not\subseteq U(I)$. Therefore, $W \notin I$, and so I is prime.

Summing up, the map $\varphi: P \mapsto p$ is a bijection between $\text{Spec } L$ and \mathcal{S} . To show that φ is a homeomorphism, it suffices to show that U is open in $\text{Spec } L$ iff $\varphi(U)$ is open in \mathcal{S} .

A typical open set in $\text{Spec } L$ is of the form $\text{spec}(I)$, for $I \in \text{Id } L$, and an open set of \mathcal{S} is of the form $U(I)$, therefore, we need only prove that

$$\begin{aligned}\varphi(\operatorname{spec}(I)) &= U(I), \\ \varphi^{-1}(U(I)) &= \operatorname{spec}(I);\end{aligned}$$

in other words, $P \in \operatorname{spec}(I)$ iff $(\varphi(P) =) p \in U(I)$. Indeed, $P \in \operatorname{spec}(I)$ means that $P \not\leq I$, which is equivalent to $U(P) \not\leq U(I)$; this, in turn, is the same as

$$U(I) \cap (\mathcal{S} - U(P)) \neq \emptyset.$$

Since $\mathcal{S} - U(P) = \overline{\{p\}}$ with $p = \varphi(P)$, the last condition means that

$$U(I) \cap \overline{\{p\}} \neq \emptyset,$$

which holds iff $p \in U(I)$. Indeed, if $p \notin U(I)$, then $U(I) \subseteq U(P)$, and so $U(I) \cap \overline{\{p\}} = \emptyset$. \square

For distributive lattices and for boolean lattices, we now get the celebrated results of M. H. Stone [668], [669].

Corollary 192. *The Stone spaces of distributive lattices can be characterized by conditions (Stone1), (Stone2), and*

(Stone3) *The intersection of two compact open sets is compact.*

Proof. Theorem 191 shows that if a topological space is the spectrum of a distributive lattice L , conditions (Stone1) and (Stone2) must hold. Theorem 188 then shows that the lattice L must be isomorphic to the distributive join-semilattice of compact open subsets of \mathcal{S} . Thus we need to know when this join-semilattice is a lattice; that is, when any two compact open sets A, B have a greatest lower bound among the compact open sets.

Now by condition (Stone1), $A \cap B$ is a union of compact open subsets U_i , hence any greatest lower bound of A and B among compact open subsets must contain all these U_i , hence equal $A \cap B$. So such a greatest lower bound will exist iff $A \cap B$ is itself a compact open subset, which is guaranteed by condition (Stone3). \square

Corollary 193. *The Stone spaces of boolean lattices (called boolean spaces) can be characterized as the compact Hausdorff spaces in which the closed open (clopen) sets form a base for the open sets. (In other words, they are totally disconnected compact Hausdorff spaces.)*

Proof. Let $\mathcal{S} = \operatorname{Spec} B$, where B is a boolean lattice. Then $\mathcal{S} = \operatorname{spec}(1)$, and thus \mathcal{S} is compact. Let $P, Q \in \mathcal{S}$ and $P \neq Q$; by symmetry, we can take an element $a \in P - Q$. Then $Q \in \operatorname{spec}(a)$ and $P \in \operatorname{spec}(a')$; therefore, every pair of elements of \mathcal{S} can be separated by clopen sets, verifying that \mathcal{S} is Hausdorff. This also shows that \mathcal{S} is totally disconnected.

Conversely, let \mathcal{S} be compact, Hausdorff, and totally disconnected. Then condition (Stone1) is obvious. Condition (Stone2) follows from the observation that F and the U_i , for all $i \in I$, are now closed sets having the finite intersection property; therefore, by compactness, they have an element in common. Thus applying Theorem 191, \mathcal{S} has the form $\text{Spec } L$ for some distributive join-semilattice L ; and by Lemma 187, L is the semilattice of compact open subsets of \mathcal{S} . Now in a compact Hausdorff space, the compact open subsets are the clopen subsets, and form a boolean lattice; so \mathcal{S} is indeed homeomorphic to the Stone space of a boolean lattice. \square

5.4 Applications

As an interesting application, we prove:

Theorem 194. *Let B be an infinite boolean lattice. Then $|\text{Spec } B| \geq |B|$.*

Proof. Let \mathcal{S} be a totally disconnected compact Hausdorff space. For $a, b \in \mathcal{S}$ with $a \neq b$, fix a pair of clopen sets $U_{a,b}$ and $U_{b,a}$ such that $a \in U_{a,b}$, $b \in U_{b,a}$, and $U_{a,b} \cap U_{b,a} = \emptyset$. Now let U be clopen and $a \in U$. Then

$$\mathcal{S} - U \subseteq \bigcup (U_{b,a} \mid b \in \mathcal{S} - U),$$

and so, by the compactness of $\mathcal{S} - U$,

$$\mathcal{S} - U \subseteq \bigcup (U_{b,a} \mid b \in X),$$

for some finite $X \subseteq \mathcal{S} - U$. Then $V_\alpha = \bigcap (V_a \mid b \in X)$ is open and $a \in V_\alpha \subseteq U$. Thus $U = \bigcup (V_a \mid a \in U)$, so by the compactness of U , for some finite $A \subseteq U$, we obtain that $U = \bigcup (V_a \mid a \in A)$.

Thus every clopen set is a finite union of finite intersections of $U_{a,b}$, and so there are no more clopen sets than there are finite sequences of elements of \mathcal{S} ; this cardinality is $|\mathcal{S}|$, provided that $|\mathcal{S}|$ is infinite. \square

It might be illuminating to compare this to an algebraic proof, see Exercise 5.37.

Theorem 191 and its corollaries provide topological representations for distributive join-semilattices, distributive lattices, and boolean lattices, respectively. It is also possible to give a topological representation for homomorphisms. We do it here only for $\{0, 1\}$ -homomorphisms of bounded distributive lattices.

Lemma 195. *Let L_0 and L_1 be bounded distributive lattices and let φ be a $\{0, 1\}$ -homomorphism of L_0 into L_1 . Then*

$$\text{Spec}(\varphi): P \mapsto \varphi^{-1}(P)$$

maps $\text{Spec } L_1$ into $\text{Spec } L_0$; the map $\text{Spec}(\varphi)$ is a continuous function with the property that if U is compact open in $\text{Spec } L_0$, then $\text{Spec}(\varphi)^{-1}(U)$ is compact in $\text{Spec } L_1$. Conversely, if $\psi: \text{Spec } L_1 \rightarrow \text{Spec } L_0$ has these properties, then $\psi = \text{Spec}(\varphi)$ for exactly one $\varphi: L_0 \rightarrow L_1$.

Proof. If $U = \text{spec}(a)$, for some $a \in L_0$, then

$$\begin{aligned} \text{Spec}(\varphi)^{-1}(U) &= \{ P \in \text{Spec } L_1 \mid \varphi^{-1}(P) \in \text{spec}(a) \} \\ &= \{ P \in \text{Spec } L_1 \mid a \notin \varphi^{-1}(P) \} \\ &= \{ P \in \text{Spec } L_1 \mid \varphi(a) \notin P \} \\ &= \text{spec}(\varphi(a)), \end{aligned}$$

and so $\text{Spec}(\varphi)$ is continuous, and has the desired property.

Conversely, if such a map ψ is given and $U = \text{spec}(a)$, for some $a \in L_0$, then $\psi^{-1}(U)$ is compact open, and so $\psi^{-1}(U) = \text{spec}(b)$ for a unique $b \in L_1$. The map $\varphi: a \mapsto b$ is a $\{0, 1\}$ -homomorphism, and $\psi = \text{Spec}(\varphi)$. \square

The following interpretation of conditions (Stone1), (Stone2), and (Stone3) will be useful. Let \mathcal{S} be a topological space. The *booleanization* of \mathcal{S} is a topological space \mathcal{S}^B on \mathcal{S} that has the compact open sets of \mathcal{S} and their complements as a subbase for open sets. (For a similar construction on the prime spectrum of a commutative ring, see M. Hochster [396].)

Lemma 196. *A compact topological space \mathcal{S} satisfies conditions (Stone1), (Stone2), and (Stone3) iff \mathcal{S}^B is a boolean space.*

Proof. Let \mathcal{S} satisfy (Stone1), (Stone2), and (Stone3). Then \mathcal{S}^B is obviously Hausdorff and totally disconnected.

To verify the compactness of \mathcal{S}^B , let \mathcal{F}_0 be a collection of compact open sets of \mathcal{S} , and let \mathcal{F}_1 be a collection of complements of compact open sets of \mathcal{S} such that in $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ no finite intersection is void. Because of (Stone3), we can assume that \mathcal{F}_0 is closed under finite intersection. Since members of \mathcal{F}_1 are closed in \mathcal{S} and \mathcal{S} is compact, the set

$$\bigcap (X \mid X \in \mathcal{F}_1) = F$$

is nonempty. Also, $U \cap X$ is closed in U , for every $U \in \mathcal{F}_0$ and $X \in \mathcal{F}_1$, and thus by compactness of U ,

$$U \cap F = \bigcap (U \cap X \mid X \in \mathcal{F}_1) \neq \emptyset.$$

Applying (Stone2) to F and \mathcal{F}_0 , we conclude that $\bigcap \mathcal{F} \neq \emptyset$, which, by Alexander's Theorem (see Exercise 5.15), proves compactness.

Conversely, if \mathcal{S}^B is boolean, then the compact open sets of \mathcal{S}^B form a boolean lattice L . Moreover, every compact open subset of \mathcal{S} is closed in \mathcal{S}^B , hence so is the intersection of any two such subsets, hence such an intersection, as a closed subset of the compact space \mathcal{S}^B , will be compact in the topology of \mathcal{S}^B . Hence it must also be compact in the weaker topology of \mathcal{S} , showing that \mathcal{S} satisfies (Stone3). Hence the compact open sets of \mathcal{S} form a sublattice L_1 of L . Thus L_1 is a distributive lattice, and by Theorem 191, the homeomorphism $\mathcal{S} \cong \text{Spec } L_1$ holds, and \mathcal{S} also satisfies conditions (Stone1) and (Stone2). \square

5.5 Free distributive products

Let L_i , for $i \in I$, be pairwise disjoint distributive lattices. Then

$$Q = \bigcup (L_i \mid i \in I)$$

is a partial lattice. A free lattice generated by Q over the class \mathbf{D} of all distributive lattices is called a *free distributive product* of the L_i for $i \in I$. To prove the existence of free distributive products, it suffices by Theorem 89 to show that there exists a distributive lattice L containing Q as a partial sublattice. This is easily done: Let L be the direct product of the $L_i \cup \{0\}$, for $i \in I$, where 0 is a new zero element of L_i . Identify $x \in L_i$ with $f \in L$ defined by $f(i) = x$ and $f(j) = 0$ for all $j \neq i$. Then Q becomes a partial sublattice of L .

An equivalent definition is:

Definition 197. Let \mathbf{K} be a class of lattices and let L_i , for $i \in I$, be lattices in \mathbf{K} . A lattice L in \mathbf{K} is called a *free \mathbf{K} -product* of the L_i , for $i \in I$, if every L_i has an embedding ε_i into L such that:

- (i) L is generated by $\bigcup (\varepsilon_i(L_i) \mid i \in I)$.
- (ii) If K is any lattice in \mathbf{K} and φ_i is a homomorphism of L_i into K , for all $i \in I$, then there exists a homomorphism φ of L into K satisfying $\varphi_i = \varphi \varepsilon_i$ for all $i \in I$ (see Figure 37).

For distributive lattices, this is equivalent to the first definition. In most cases, we will assume that each $L_i \leq L$ and that ε_i is the inclusion map; then (ii) will simply state that the φ_i have a common extension. Note that in all cases we shall consider, (i) can be replaced by the requirement that the φ in (ii) be unique.

If, in Definition 197, \mathbf{K} is a class of bounded lattices and all homomorphisms are assumed to be $\{0, 1\}$ -homomorphisms, we get the concept of a *free \mathbf{K}*

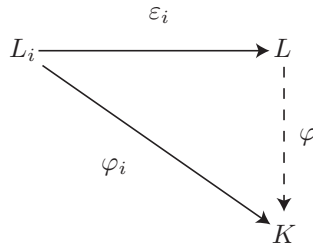


Figure 37. Illustrating condition (ii) in Definition 197

$\{0, 1\}$ -product. In particular, if $\mathbf{K} = \mathbf{L}$, we get the concept of a *free* $\{0, 1\}$ -product, see Section VII.1.12, and if $\mathbf{K} = \mathbf{D}$, we obtain the concept of a *free* $\{0, 1\}$ -distributive product.

Our final result is the existence and description of a free $\{0, 1\}$ -distributive product of a family of bounded distributive lattices, see A. Nerode [548].

A *Stone space* is a topological space satisfying the conditions (Stone1), (Stone2), and (Stone3).

Theorem 198. *Let L_i , for $i \in I$, be distributive lattices with zero and unit. Let $\mathcal{S} = \prod (\text{Spec } L_i \mid i \in I)$ (see Exercise 5.16). Then \mathcal{S} is a Stone space, and thus $\mathcal{S} \cong \text{Spec } L$ for some distributive lattice L . Such a lattice L is a free $\{0, 1\}$ -distributive product of the L_i for $i \in I$.*

The proof of Theorem 198 will be preceded by two lemmas.

Lemma 199. *Let \mathcal{S}_i , for $i \in I$, be compact Stone spaces. Then*

$$\prod (\mathcal{S}_i^{\mathbf{B}} \mid i \in I) = (\prod (\mathcal{S}_i \mid i \in I))^{\mathbf{B}}.$$

Proof. For $U \subseteq \mathcal{S}_j$, let

$$E(U) = \{f \in \prod \mathcal{S}_i \mid f(j) \in U\}$$

(see Exercise 5.16). The compact open sets form a base for open sets in \mathcal{S}_j ; therefore,

$$\{E(U) \mid U \text{ compact open in some } \mathcal{S}_j\}$$

is a subbase for open sets in $\prod (\mathcal{S}_i \mid i \in I)$. Note that all the sets $E(U)$ in the above family are compact open in $\prod \mathcal{S}_i$; therefore, $V \subseteq \prod \mathcal{S}_i$ is compact open iff it is a finite union of finite intersections of some of the $E(U)$. Consequently, declaring also the complements of compact open sets to be open (when forming $(\prod \mathcal{S}_i)^{\mathbf{B}}$) is equivalent to making the complements of the sets $E(U)$ open. But the complement of $E(U)$ is $E(\mathcal{S}_j - U)$, and $\mathcal{S}_j - U$ is an open set of $\mathcal{S}_j^{\mathbf{B}}$. Thus $\prod \mathcal{S}_i^{\mathbf{B}}$ and $(\prod \mathcal{S}_i)^{\mathbf{B}}$ have the same topology. \square

Lemma 200. *A product of compact Stone spaces is again a compact Stone space.*

Proof. Let \mathcal{S}_i , for $i \in I$, be Stone spaces. Then $\mathcal{S} = \prod \mathcal{S}_i$ is T_0 and compact (use Exercises 5.17 and 5.22). Since $\mathcal{S}_j^{\mathbf{B}}$ is boolean (see Lemma 196), so is $\prod \mathcal{S}_i^{\mathbf{B}}$ (by Exercises 5.21–5.23). By Lemma 199, the homeomorphism $\mathcal{S}^{\mathbf{B}} = \prod \mathcal{S}_i^{\mathbf{B}}$ holds, and thus $\mathcal{S}^{\mathbf{B}}$ is boolean. Therefore, \mathcal{S} is a Stone space by Lemma 196. \square

Proof of Theorem 198. Let e_i be the i th projection ($e_i: \text{Spec } L \rightarrow \text{Spec } L_i$ is given by $e_i(f) = f(i)$). By Lemma 195, there is a unique $\{0, 1\}$ -homomorphism $\varepsilon_i: L_i \rightarrow L$ satisfying $\text{Spec}(\varepsilon_i) = e_i$. It is easy to visualize ε_i ; think of the

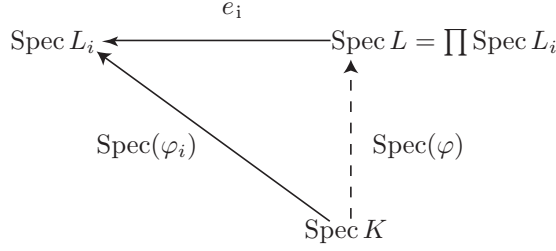


Figure 38. Proving Theorem 198

elements of L_i as compact open sets of \mathcal{S}_i ; then $\varepsilon_i(U) = E(U) = e_i^{-1}(U)$. It is obvious from this that the map ε_i is an embedding.

Now let K be a bounded distributive lattice and let $\varphi_i : L_i \rightarrow K$ be $\{0, 1\}$ -homomorphisms as in Figure 37. By applying Spec , we obtain Figure 38, where the dashed arrow is a continuous map; we have yet to show that it satisfies the conditions of the last sentence of Lemma 195, and so arises from a lattice homomorphism φ .

Thus the method of defining $\text{Spec}(\varphi)$ is clear. For $x \in \text{Spec } K$, the element $\text{Spec}(\varphi)(x)$ is a member of $\prod \text{Spec } L_i$, and

$$\text{Spec}(\varphi)(x)(i) = \text{Spec}(\varphi_i)(x)$$

for $i \in I$.

To show that this correspondence is indeed of the form $\text{Spec}(\varphi)$, for some homomorphism $\varphi : L \rightarrow K$, we have to verify the following:

(a) the map we labeled $\text{Spec}(\varphi)$ is continuous (this statement follows from Exercise 5.19),

(b) if V is compact open in $\text{Spec } L$, then $\text{Spec}(\varphi)^{-1}(V)$ is compact open in $\text{Spec } K$.

Let us first verify (b) for $V = E(U)$, where U is compact open in some $\text{Spec } L_i$. In this case

$$\begin{aligned}
 \text{Spec}(\varphi)^{-1}(V) &= \text{Spec}(\varphi)^{-1}(E(U)) = \text{Spec}(\varphi)^{-1}(e_i^{-1}(U)) \\
 &= (e_i \text{Spec}(\varphi))^{-1}(U) = \text{Spec}(\varphi_i)^{-1}(U),
 \end{aligned}$$

and therefore, $\text{Spec}(\varphi)^{-1}(V)$ is compact open since $\text{Spec}(\varphi_i)$ satisfies the condition of Lemma 195.

Next consider the case where V is a finite intersection of sets $E(U)$. From the facts that inverse images respect intersections, and that $\text{Spec } K$ satisfies condition (Stone3) by Corollary 192, we see that $\text{Spec}(\varphi)^{-1}(V)$ will again be compact. Finally, let V be an arbitrary compact open subset of $\text{Spec } L$. Then because it is open, it is a union of such finite intersections, hence by compactness, it is a union of finitely many of them; and since inverse images

respect unions, and finite unions of compact sets are compact, we again conclude that $\text{Spec}(\varphi)^{-1}(V)$ is compact, as required. \square

5.6 \diamond Priestley spaces by Hilary A. Priestley

Priestley duality for distributive lattices is Stone duality in different clothes. But in terms of outward appearance the difference is significant. In outline, Priestley's formulation makes order overt and has a Hausdorff topology in place of a T_0 -topology. This better reveals how the finite and boolean cases fit into the overall picture: for the former we need only order (the topology is discrete and can be suppressed) and for the latter we need only topology (the order is discrete and can be suppressed). At the time of Stone's pioneering work [669], non-Hausdorff topologies were rather an alien concept and Stone's representation for distributive lattices was relatively little exploited. The T_0 -spaces it uses came into vogue only much later, through the development of domain theory (see Section I.3.16).

We focus on the class \mathbf{D} of bounded distributive lattices with $\{0, 1\}$ -preserving homomorphisms, leaving aside the adaptations to encompass lattices lacking one or both bounds. So consider a member L of \mathbf{D} and its spectrum $\text{Spec } L$ of prime ideals. To obtain Priestley's representation, we order $\text{Spec } L$ by inclusion and take the topology \mathcal{T} having as a subbase for the open sets the sets of the forms

$$\text{spec}(a) \quad \text{and} \quad (\text{Spec } L - \text{spec}(b)) \quad (a, b \in L).$$

We now form the ordered space $X_L = \langle \text{Spec } L; \leq, \mathcal{T} \rangle$, where \leq is the inclusion ordering on prime ideals. Then X_L is a *Priestley space*: \mathcal{T} is compact and the space X_L is *totally order-disconnected* in the sense that given $x \not\leq y$ in X_L there is a \mathcal{T} -clopen down-set U with $x \in U$ and $y \notin U$. Here the latter property is immediate since we can just take $U = \text{spec}(a)$ with $a \in x - y$; compactness is proved via Alexander's Subbase Lemma. Priestley's representation theorem for \mathbf{D} then asserts that each L in \mathbf{D} is isomorphic to the lattice of all clopen down-sets of its Priestley dual space X_L . Furthermore, every Priestley space is isomorphic, topologically and order-theoretically, to the dual space of its lattice of clopen down-sets. We arrive at a dual equivalence between \mathbf{D} and the category \mathbf{P} of Priestley spaces (in which the morphisms are the continuous order-preserving maps). With this equivalence to hand many other results tumble out. Some of the most useful are collected together in Theorem 201. An account, with proofs, of Priestley duality in simple dress can be seen in the textbook by B. A. Davey and H. A. Priestley [131].

Theorem 201. *Let L be a bounded distributive lattice and*

$$X_L = \langle \text{Spec } L; \leq, \mathcal{T} \rangle,$$

as defined above, be its Priestley dual space. Then, up to isomorphism,

- (i) the order dual L^δ of L is the lattice of clopen up-sets;
- (ii) the minimal boolean extension of L is the lattice of all clopen sets;
- (iii) the ideal lattice $\text{Id } L$ is the lattice of open down-sets, with principal and prime ideals corresponding to open down-sets which are, respectively, closed and of the form $X_L - \uparrow x$ ($x \in X_L$);
- (iv) the filter lattice $\text{Fil } L$ is the order dual of the lattice of closed down-sets;
- (v) the congruence lattice $\text{Con } L$ is the lattice of open sets, that is, \mathcal{J} .

To clarify the relationship between Stone duality and Priestley duality we indicate how to pass to and fro between Priestley spaces and spectral spaces, using a bijection under which the Priestley dual space of L in \mathbf{D} corresponds to its Stone space $\text{Spec } L$. By a *spectral space* we mean a compact space satisfying conditions (Stone1), (Stone2) and (Stone3) from Section II.5.3 (see Corollary 192). Given a Priestley space $\langle X; \leq, \mathcal{J} \rangle$, the space $\langle X; \tau \rangle$ is a spectral space, where τ is the topology consisting of the \mathcal{J} -open down-sets.

In the other direction, let $\langle X; \tau \rangle$ be a spectral space. Let \leq_τ be the associated specialization order: $x \leq_\tau y$ iff $x \in \text{cl}_\tau\{y\}$. Consider the *dual topology* τ^* . This has as a subbase for its closed sets the τ -compact sets which are saturated with respect to \geq_τ , that is, which are intersections of τ -open sets. Then $\langle X; \tau^* \rangle$ is also a spectral space. Let \mathcal{J} be the *patch topology* formed by taking the join of τ and τ^* . Then, finally, $\langle X; \geq_\tau, \mathcal{J} \rangle$ is a Priestley space whose family of open down-sets is exactly τ .

An interesting example of the above correspondence comes from domain theory. An algebraic lattice, or more generally a Scott domain, is a spectral space in its Scott topology and the associated Priestley space topology is the Lawson topology; see Section I.3.16 and [225]. More formally, the category of Stone spaces (the morphisms being the continuous maps under which inverse images of compact open subsets are compact) is isomorphic (and not merely equivalent) to the category \mathbf{P} of Priestley spaces; see W. H. Cornish [98].

It is a moot point in duality theory whether it is preferable to order $\text{Spec } L$ by \subseteq or its opposite, and whether to use down-sets or up-sets. Indeed, the Priestley representation for L can equally well be set up so as to identify L with the clopen up-sets of a Priestley space, rather than the clopen down-sets. The research literature concerning Priestley duality and its applications is divided roughly equally between these alternatives—a source of minor irritation.

The down-sets version of the duality fits well with Birkhoff's representation for the finite case; see Section II.1.2. The up-sets version naturally arises if one bases the representation of a lattice L not on the prime ideals but on the prime filters or equivalently on the $\{0, 1\}$ -homomorphisms into the lattice $2 = \{0, 1\}$ with $0 < 1$. This last approach was the one initially adopted by H. A. Priestley [591] and it is functorially by far the smoothest. An account is given by D. M. Clark and B. A. Davey in [91, Chapter 1], where, adapted

to the non-bounded case, it is used as a prototype example for the theory of natural dualities (see also Section VI.2.8 by B. A. Davey and M. Haviar).

The passage from T_0 -spaces to ordered spaces with a compact Hausdorff topology is critical here. Natural duality theory (in its vanilla form) applies to quasivarieties $\mathbf{ISP}(M)$, where M is a finite algebra. The dual structures belong to a category $\mathbf{IS}_c\mathbf{P}^+(\widetilde{M})$ of structured boolean spaces built from an *alter ego* \widetilde{M} , which is a relational structure on the underlying set of M carrying the discrete topology. Here $\mathbf{IS}_c\mathbf{P}^+(\widetilde{M})$ is the class of isomorphic copies of closed substructures of nonzero powers of \widetilde{M} .

Stone duality for boolean algebras is an example of a natural duality, but because no relational structure on the alter ego is needed the way to generalize this duality was not clearly apparent. And in [669] Stone had, in a different way, also concealed the role of relational structures by working with topology alone.

It is well known that $\mathbf{D} = \mathbf{ISP}(2)$. It can also be shown (see for example [91]) that $\mathbf{P} = \mathbf{IS}_c\mathbf{P}^+(\widetilde{2})$, where $\widetilde{2} = (\{0, 1\}; \leq, \mathcal{T})$, with \leq the underlying ordering and \mathcal{T} the discrete topology. We can now present Priestley duality for \mathbf{D} in full regalia.

Theorem 202. *There are natural hom-functors $D: \mathbf{D} \rightarrow \mathbf{P}$ and $E: \mathbf{P} \rightarrow \mathbf{D}$:*

$$\begin{aligned} \text{on objects:} \quad & D: L \mapsto \mathbf{D}(L, 2) \leq \widetilde{2}^L \quad \text{and} \quad E: X \mapsto \mathbf{P}(X, \widetilde{2}) \leq 2^X; \\ \text{on morphisms:} \quad & D: f \mapsto - \circ f \quad \text{and} \quad E: \phi \mapsto - \circ \phi. \end{aligned}$$

Then D and E set up a dual equivalence between \mathbf{D} and \mathbf{P} with the unit and co-unit the evaluation maps $e_L: L \rightarrow ED(L)$ and $\varepsilon_X: X \rightarrow DE(X)$, where $e_L(a)(x) = x(a)$ (for all $a \in L$, $x \in X$) and $\varepsilon_X(x)(\alpha) = \alpha(x)$ (for all $\alpha \in \mathbf{P}(X, \widetilde{2})$ and $x \in X$).

In addition, the free lattice in \mathbf{D} on κ generators is (isomorphic to) $E(\widetilde{2}^\kappa)$ and, more generally, coproducts in \mathbf{D} correspond to direct (concrete) products in \mathbf{P} .

We conclude this section with a few remarks on the application of Priestley duality to lattice-based algebras. Given a variety of algebras having \mathbf{D} -reducts, one may seek to enrich the Priestley dual spaces with operations or relations capturing the non-lattice operations and to find a first-order description of the resulting dual structures. No comprehensive account exists of the myriad of dualities developed for \mathbf{D} -based algebras. A full bibliography up to 1985 was compiled by M. E. Adams and W. Dziobiak [6]. Among later work we draw attention to W. H. Cornish's systematic treatment of dualities for classes of algebras whose non-lattice operations are dual endomorphisms; his monograph [99] encompasses *inter alia* De Morgan algebras, Kleene algebras, Stone algebras and more generally Ockham algebras. We also note R. Goldblatt's paper [235] which investigates n -ary operations which are coordinatewise \vee - or \wedge -preserving. This paper includes too a Priestley-type duality for Heyting

algebras, also obtained independently by M. E. Adams [3]. A Priestley space $\langle X; \leq, \mathcal{T} \rangle$ is a *Heyting space*, that is, the dual of a Heyting algebra, iff $\uparrow U$ is \mathcal{T} -open whenever U is \mathcal{T} -open; for a, b clopen down-sets, $a \rightarrow b$ is given by $X - \uparrow(a - b)$. Here the topological condition is exactly what is needed to ensure that the formula for the relative pseudocomplement valid in the (topology-free) finite case also works in general.

Building on Boole's original ideas on classical propositional calculus, lattice-based algebras are extensively used in logic as models for propositional logics. Join and meet model disjunction and conjunction and additional operations are used to model a non-classical negation or implication, or, traditionally on boolean algebras, a modal operator. In particular Heyting algebras model IPC (intuitionistic propositional calculus) and unary operations preserving join or meet represent modalities.

Fifty years ago S. Kripke famously introduced relational semantics for modal logic and for IPC. Kripke's ideas were hugely influential in modal logic, leading to the development of powerful semantic techniques in a subject which had hitherto been studied syntactically; see the textbook by P. Blackburn, M. de Rijke, and Y. Venema [76] and the earlier monograph by A. Chagrov and M. Zakharyashev [83].

For both modal logic and IPC, Kripke semantics used relational frames of 'possible worlds', which were in each case, sets carrying an 'accessibility relation' R . The underlying sets of the frames are the prime filters (ultrafilters in the boolean setting) of the lattice reducts of the algebras they serve to represent. But mathematically the role of R is quite different in the two cases. For Heyting algebras this relation is an order, as in Heyting spaces; we note that these algebras are special in that the Heyting implication is determined by the underlying lattice ordering. For modal logic R is a binary relation used to capture via the frames the modal operator; no ordering is needed since the underlying lattices are boolean.

Our remarks hint at a bigger picture, of which Goldblatt gave a first glimpse in his 1989 paper. Blackburn, de Rijke and Venema [76, pp. 41 and 328] comment on the parallel but separate developments of Kripke semantics on the one hand and Jónsson and Tarski's theory of canonical extensions of boolean algebras with operators on the other. The latter theory has now been vastly extended by B. Jónsson, M. Gehrke, and many others, so that it now encompasses very many classes of lattice-based algebras. It supplies, in a systematized way, purely relational models. Loosely, adding topology gives Priestley-type topologies for these classes. These connections are developed in a monograph in preparation *Lattices in Logic* by M. Gehrke and H. A. Priestley.

5.7 \diamond Frames by Aleš Pultr

A *frame* is a complete lattice L satisfying (JID), introduced in Section 4.2. A *frame homomorphism* $h: L \rightarrow M$ preserves *all* joins and *all finite* meets.

The most important example is given by the lattices $\text{Open } X$ of open sets of a topological space X , and the maps

$$\text{Open}(f): \text{Open } Y \rightarrow \text{Open } X$$

we get from continuous maps $f: X \rightarrow Y$ by the formula

$$\text{Open}(f)(U) = f^{-1}(U).$$

Viewing spaces as systems of “places” or “spots” with their interrelations—rather than as a structured set of points—was one of the strongest motivations for developing this theory. Instead of frames we often speak of *locales*—which terminology inverts the direction of homomorphisms to bring them into agreement with the continuous maps they represent.

How much information is lost? How well are spaces and continuous maps represented as frames? The answer is pleasing:

\diamond **Lemma 203.** *Let Y be a Hausdorff space and let X be an arbitrary topological space. Then the homomorphisms $h: \text{Open } Y \rightarrow \text{Open } X$ are precisely the maps $\text{Open}(f)$, where $f: X \rightarrow Y$ is a continuous map. The map f is uniquely determined by h .*

This theorem also holds for *sober spaces*, which are more general than the Hausdorff spaces. (In the lattice $\text{Open } X$, each filter $U(x)$ of all open neighborhoods of a point x is, trivially, completely prime; the space X is sober if each completely prime filter in $\text{Open } X$ is of the form $U(x)$.)

This allows us to reconstruct a space Y from the lattice $\text{Open } Y$.

Not every frame is isomorphic to an $\text{Open } X$. When studying frames, we deal with a larger class of generalized (“point-free”) spaces. Is this good or bad? It has proved to be useful; nevertheless, it is always good to know whether a frame is *spatial*, that is, isomorphic to an $\text{Open } X$ (K.H. Hofmann and J.D. Lawson [397]).

\diamond **Theorem 204 (Hofmann-Lawson duality).** *The formulas*

$$X, f \mapsto \text{Open } X, \text{Open}(f)$$

provide a one-one correspondence between the class of all locally compact sober spaces and their continuous maps, and the locally compact frames and their frame homomorphisms.

It should be noted that locally compact frames coincide with distributive continuous lattices in the sense of Scott (G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott [225]).

Topological concepts and phenomena (like regularity, complete regularity, normality, or compactness, paracompactness or local compactness) are, as a rule, easily translated into the “point-free” language.

Sometimes the extended classes have better properties than the original topological concepts. For instance: in classical topology, Tychonoff’s Theorem (products of compact spaces are compact) is equivalent to the Axiom of Choice. Here is the point-free counterpart:

◆ **Theorem 205.** *Products of compact locales are compact.*

This is fully constructive (“choice-free”), see P. T. Johnstone [428]. Furthermore, the counterpart of the Čech-Stone compactification can be described by a simple formula (B. Banaschewski and C. J. Mulvey [48]); prime filters are not involved.

This result combined with the Hofmann-Lawson duality gives Tychonoff’s theorem. The duality is, of course, heavily choice dependent; so the choice aspect of the product of spaces is not in the compactness but rather whether it has enough points—another fact revealed by point-free reasoning.

In the point-free context, we can also work with the richer structures. Thus for instance, a *uniformity* on a frame L can be viewed as a system of covers (a *cover* of L is a subset $A \subseteq L$ such that $\bigvee A = 1$, the top) with specific natural properties. One has a concept of *completeness*, parallel with the classical one, and of *completion*; like the compactification, this completion is constructive.

Here is an interesting fact that holds in the point-free context but not in the classical one (J. R. Isbell [419]):

◆ **Theorem 206.** *A frame is paracompact iff it admits a complete uniformity.*

While in the classical context, paracompact spaces often misbehave in constructions (even a product of a paracompact space with a metric space is not necessarily paracompact), for locales we have the following nice result.

◆ **Theorem 207.** Paracompact locales are reflective in the category of all locales.

Hence, in particular, paracompactness is preserved by all products (and similar constructions). This is one of the instances where we see that it is useful to have more “spaces” than before; the situation is strongly reminiscent of the extension of reals to complex numbers, allowing solutions of problems unsolvable in the real case.

For the basic ideas, and for the early history of the area, see the excellent surveys P. T. Johnstone [429] and [431].

For more about frames and for further references, see P. T. Johnstone [430], A. Pultr [600], and S. Vickers [691].

Exercises

The first 22 exercises review the basics of topology that is utilized in this section.

- 5.1. A *topological space* is a set A and a collection \mathcal{T} of subsets of A , satisfying the properties:

- (i) $A \in \mathcal{T}$;
- (ii) \mathcal{T} is closed under finite intersections;
- (iii) \mathcal{T} is closed under unions (empty, nonempty, finite, infinite).

A member of \mathcal{T} is called an *open set*. Call a set *closed* if its complement is open. Characterize those subsets of $\text{Pow } A$ that are the families of all closed subsets under topologies on A .

- 5.2. A family of nonempty sets \mathcal{B} in \mathcal{T} is a *base* for open sets iff every open set is a union of members of \mathcal{B} . Show that for a set A , a collection \mathcal{B} of subsets of A is a base of open sets of some topological space defined on A iff $\bigcup \mathcal{B} = A$, and for $X, Y \in \mathcal{B}$ and $p \in X \cap Y$, there exists a $Z \in \mathcal{B}$ with $p \in Z$, such that $Z \subseteq X$ and $Z \subseteq Y$.
- 5.3. A family of nonempty sets $\mathcal{C} \subseteq \text{Pow } A$ is a *subbase* for open sets if the finite intersections of members of \mathcal{C} form a base for open sets. Show that $\mathcal{C} \subseteq \text{Pow } A$ is a subbase of some topology defined on A iff $\bigcup \mathcal{C} = A$.
- 5.4. Let A be a topological space and let $X \subseteq A$. Then there exists a smallest closed set \overline{X} containing X , called the *closure* of X . Show that $\overline{\emptyset} = \emptyset$ and that, for all $X, Y \subseteq A$,
- (a) $X \subseteq Y$ implies that $\overline{X} \subseteq \overline{Y}$,
 - (b) $X \subseteq \overline{X}$,
 - (c) $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$,
 - (d) $\overline{\overline{X}} = \overline{X}$.
- 5.5. In Section I.3.12, we introduced closure operators (Definition 30). Relate Exercise 5.4 to closure operators.
- 5.6. Prove that the conditions of Exercise 5.4 characterize an operation $\overline{} : \text{Pow } A \rightarrow \text{Pow } A$ that is the topological closure with respect to a topology on A .
- 5.7. Show that $a \in \overline{X}$ iff every open set (in a given subbase) containing a has a nonempty intersection with X .
- 5.8. A space A is a T_0 -space if $\overline{\{x\}} = \overline{\{y\}}$ implies that $x = y$ for $x, y \in A$. Show that A is a T_0 -space iff, for every $x, y \in A$ with $x \neq y$, there exists an open set (in a given base) containing exactly one of x and y .
- 5.9. A space A is a T_1 -space if $\overline{\{x\}} = \{x\}$ for all $x \in A$. A T_1 -space is a T_0 -space. Show that A is a T_1 -space iff, for $x, y \in A$ with $x \neq y$, there exists an open set (in a given subbase) containing x but not y .

- 5.10. Let A and B be topological spaces and $f: A \rightarrow B$. Then f is called *continuous* if $f^{-1}(U)$ is open in A for every open set U of B . The map f is a *homeomorphism* if f is a bijection and if both f and f^{-1} are continuous. Show that continuity can be checked by considering only those $f^{-1}(U)$, where U belongs to a given subbase.
- 5.11. Show that $f: A \rightarrow B$ is continuous iff $f(\overline{X}) \subseteq \overline{f(X)}$ for all $X \subseteq A$.
- 5.12. A subset X of a topological space A is *compact* if whenever

$$X \subseteq \bigcup (U_i \mid i \in I),$$

where the U_i , for $i \in I$, are open sets, implies that

$$X \subseteq \bigcup (U_i \mid i \in I')$$

for some finite $I' \subseteq I$. The space A is *compact* if $X = A$ is compact. Show that A is compact, iff, for every family \mathcal{F} of closed sets, if $\bigcap \mathcal{F}_1 \neq \emptyset$, for all finite $\mathcal{F}_1 \subseteq \mathcal{F}$, then $\bigcap \mathcal{F} \neq \emptyset$.

- 5.13. Let A be a compact topological space and let X be a closed set in A . Show that X is compact.
- 5.14. Prove that a space A is compact iff, in the lattice of closed sets of A , every maximal filter is principal.
- *5.15. Show that a space A is compact iff it has a subbase \mathcal{C} of closed sets (that is, $\{A - X \mid X \in \mathcal{C}\}$ is a subbase for open sets) with the property: If $\bigcap \mathcal{D} = \emptyset$ for some $\mathcal{D} \subseteq \mathcal{C}$, then $\bigcap \mathcal{D}_1 = \emptyset$ for some finite $\mathcal{D}_1 \subseteq \mathcal{D}$ (J. W. Alexander [28]).
- 5.16. Let A_i , for $i \in I$, be topological spaces and set $A = \prod (A_i \mid i \in I)$. For $U \subseteq A_i$, set $E(U) = \{f \in A \mid f(i) \in U\}$. The *product topology* on A is the topology determined by taking all the sets $E(U)$ as a subbase for open sets, where U ranges over all open sets of A_i for all $i \in I$. Show that the projection map $e_i: f \mapsto f(i)$ is a continuous map of A onto A_i . (As a rule, a product of topological spaces will be understood to have the product topology.)
- 5.17. Show that if A_i , for $i \in I$, are T_0 -spaces (T_1 -spaces), so is

$$A = \prod (A_i \mid i \in I).$$

- 5.18. A map $f: A \rightarrow B$ is *open* if $f(U)$ is open in B for every open $U \subseteq A$. Show that the projection maps (see Exercise 5.16) are open.
- 5.19. Prove that a function $f: B \rightarrow \prod A_i$ is continuous iff $e_i f: B \rightarrow A_i$ is continuous for every $i \in I$.
- 5.20. A space A is a *Hausdorff space* (T_2 -space) if, for all $x, y \in A$ with $x \neq y$, there exist open sets U and V such that $x \in U$, $y \in V$, $U \cap V = \emptyset$. Show that:

(a) A is Hausdorff iff $\Delta = \{(x, x) \mid x \in A\}$ is closed in $A \times A$.

(b) A compact subset of a T_2 -space is closed.

5.21. Prove that a product of Hausdorff spaces is a Hausdorff space.

5.22. Show that

Theorem 208 (Tychonoff's Theorem). *A product of compact spaces is compact.*

(Hint: use Exercise 5.15.)

5.23. A space A is *totally disconnected* if, for all $x, y \in A$ with $x \neq y$, there exists a clopen set U with $x \in U$ and $y \notin U$. Show that the product of any family of totally disconnected sets is totally disconnected.

* * *

5.24. Let I and J be ideals of a join-semilattice. Verify that

$$I \vee J = \{t \mid t \leq i \vee j, i \in I, j \in J\}.$$

5.25. Let L be a join-semilattice. Show that $\text{Id } L$ is a lattice iff any two elements of L have a common lower bound.

5.26. Give a detailed proof of Lemma 185.

5.27. Prove that every join-semilattice can be embedded in a boolean lattice (considered as a join-semilattice).

5.28. Show that a finite distributive join-semilattice is a distributive lattice.

5.29. Let L be a join-semilattice and let α be a *join-congruence*, that is, an equivalence relation on L having the Substitution Property for join. Then L/α is also a join-semilattice. Show that the distributivity of L does not imply the distributivity of L/α .

5.30. Let F be a free join-semilattice on a set S ; let F_0 be F with a new zero added. Show that F_0 is a distributive join-semilattice.

5.31. Let φ be a join-homomorphism of the join-semilattice F_0 onto the join-semilattice F_1 . For distributive join-semilattices F_0 and F_1 , is the proper homomorphism concept the one requiring that if P is a prime ideal of F_1 , then $\varphi^{-1}(P)$ is a prime ideal of F_0 ?

5.32. Show that there is no “free distributive join-semilattice” with the homomorphism concept of Exercise 5.31.

5.33. Does Theorem 123 generalize to bounded distributive join-semilattices?

5.34. Characterize the Stone spaces of finite boolean lattices and of finite chains.

5.35. Let \mathcal{S}_0 and \mathcal{S}_1 be disjoint topological spaces; let $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$ and call $U \subseteq \mathcal{S}$ open if $U \cap \mathcal{S}_0$ and $U \cap \mathcal{S}_1$ are open. Show that if \mathcal{S}_0 and \mathcal{S}_1 are Stone spaces, then so is \mathcal{S} .

5.36. If in Exercise 5.35, $\mathcal{S}_i = \text{Spec } L_i$, for $i = 0, 1$, then

$$\mathcal{S} = \text{Spec}(L_0 \times L_1).$$

- 5.37. As an alternative proof of Theorem 194, pick an element

$$a(P, Q) \in P - Q$$

for all $P, Q \in \text{Spec } B$ with $P \neq Q$. Show that the elements $a(P, Q)$ R-generate all of B .

- 5.38. Give necessary and sufficient conditions for a map $\varphi: L_0 \rightarrow L_1$ to be one-to-one, respectively, onto, in terms of the induced map

$$\text{Spec}(\varphi): \text{Spec } L_1 \rightarrow \text{Spec } L_0.$$

- 5.39. Determine the connection between the Stone space of a lattice and the Stone space of a sublattice.
- 5.40. Call the Stone space of a generalized boolean lattice a *generalized boolean space*; characterize such spaces. (Compactness of \mathcal{S} should be replaced by *local compactness*: For every $p \in \mathcal{S}$, there exists an open set U with $p \in U$ and a set V with $U \subseteq V$ such that V is compact.)
- 5.41. Show that the product of (generalized) boolean spaces is (generalized) boolean.
- 5.42. Call the join-semilattice L *modular* if, for all elements $a, b, c \in L$ satisfying $a \leq b$ and $b \leq a \vee c$, there exists an element $c_1 \in L$ with $c_1 \leq c$ and $b = a \vee c_1$. Show that a distributive join-semilattice is modular.
- 5.43. Show that Lemma 184 remains valid if all occurrences of the word “distributive” are replaced by the word “modular”.
- 5.44. Show that the set of all finitely generated normal subgroups of a group (and also the finitely generated ideals of a ring) form a modular join-semilattice.
- 5.45. The lattice of congruence relations of a join-semilattice L is distributive iff any pair of elements of L with a lower bound is comparable (D. Papert [577], R. A. Dean and R. H. Oehmke [148]).

* * *

- 5.46. Define the concepts of subalgebra, term, identity, and variety for algebras of a given type τ . Show that if \mathbf{K} is a variety, \mathfrak{A} is an algebra in \mathbf{K} , and \mathfrak{B} is a subalgebra of \mathfrak{A} , then \mathfrak{B} is in \mathbf{K} .
- 5.47. Define the concepts of homomorphism, homomorphic image, and direct product for algebras of a given type. Show that a variety is closed under the formation of homomorphic images and direct products.
- 5.48. Let $\mathfrak{A} = (A; F)$ be an algebra, let $H \subseteq A$, and let $H \neq \emptyset$. Show that there exists a smallest subset $\text{sub}(H)$ of A with $\text{sub}(H) \supseteq H$ such that $(\text{sub}(H); F)$ is a subalgebra of \mathfrak{A} . (This subalgebra is said to be *generated by* H .)

- 5.49. Show that $|\text{sub}(H)| \leq |H| + |F| + \aleph_0$.
- 5.50. Modify Definition 197 for algebras. Show that the φ in (ii) is unique.
- 5.51. Let \mathfrak{B} and \mathfrak{C} be free \mathbf{K} -products of \mathfrak{A}_i , for $i \in I$, with embeddings ε_i and χ_i , for $i \in I$, respectively. Show that there exists an isomorphism $\alpha: B \rightarrow C$ such that $\alpha\varepsilon_i = \chi_i$ for all $i \in I$.
- 5.52. Let \mathbf{K} be a variety of algebras and let $\mathfrak{A}_i \in \mathbf{K}$ for all $i \in I$. Choose a set S satisfying

$$|S| \geq \sum(|A_i| \mid i \in I) + |F| + \aleph_0.$$

Let Q be the set of all pairs $(\mathfrak{B}, (\varphi_i \mid i \in I))$ such that $B \subseteq S$, the map φ_i is a homomorphism of \mathfrak{A}_i into \mathfrak{B} , and

$$B = \text{sub}(\bigcup(\varphi(A_i) \mid i \in I)).$$

Form

$$\mathfrak{A} = \prod(\mathfrak{B} \mid (\mathfrak{B}, (\varphi_i \mid i \in I)) \in Q)$$

(direct product), and, for all $a \in A_i$, define $f_a \in A$ by

$$f_a((\mathfrak{B}, (\varphi_i \mid i \in I))) = \varphi_i(a).$$

Finally, let \mathfrak{N} be the subalgebra generated by the f_a for all $a \in A_i$ and $i \in I$. Show that $\mathfrak{N} \in \mathbf{K}$, the map $a \mapsto f_a$ is a homomorphism ε_i of \mathfrak{A}_i into \mathfrak{N} , for every $i \in I$, and that \mathfrak{N} is generated by $\bigcup(\varepsilon_i(A_i) \mid i \in I)$.

- 5.53. Show that ε_i is one-to-one iff, for all $i \in I$ and for all $a, b \in A_i$ with $a \neq b$, there exists an algebra $\mathfrak{C} \in \mathbf{K}$ and homomorphisms $\psi_j: \mathfrak{A}_j \rightarrow \mathfrak{C}$, for all $j \in I$, such that $\psi_i(a) \neq \psi_i(b)$.
- 5.54. Combine the previous exercises to prove the following result.

Theorem 209 (Existence Theorem for Free Products). *Let \mathbf{K} be a variety of algebras, let \mathfrak{A}_i , be algebras in \mathbf{K} , for $i \in I$. A free \mathbf{K} -product of the algebras \mathfrak{A}_i , for $i \in I$, exists iff, for all $i \in I$ and for all $a, b \in A_i$ with $a \neq b$, there exists an algebra $\mathfrak{C} \in \mathbf{K}$, and there exist homomorphisms $\psi_j: \mathfrak{A}_j \rightarrow \mathfrak{C}$, for all $j \in I$, such that $\psi_i(a) \neq \psi_i(b)$.*

- 5.55. Show that in proving the existence of free distributive products and free $\{0, 1\}$ -distributive products, we can always choose $\mathfrak{C} = \mathbf{C}_2$, the two-element chain, in applying Exercise 5.54.
- 5.56. Show that the free boolean algebra on \mathfrak{m} generators is a free $\{0, 1\}$ -distributive product of \mathfrak{m} copies of the free boolean algebra on one generator.
- 5.57. Prove that the free boolean algebra on \mathfrak{m} generators can be represented by the clopen subsets of $\{0, 1\}^{\mathfrak{m}}$, where $\{0, 1\}$ is the two-element discrete topological space.

- 5.58. Find a topological representation for a free distributive lattice on m generators (G. Ya. Areskin [32]).
- 5.59. For an order P , let $\text{Down}_{\text{fin}} P$ denote the set of all subsets of P of the form $\downarrow H$ for a finite set $H \subseteq P$ and order this set by inclusion. Show that $\text{Down}_{\text{fin}} P$ is a join-semilattice.
- 5.60. Find examples of orders P for which $\text{Down}_{\text{fin}} P$ is not a distributive lattice. Is there a “smallest” such example?
- 5.61. Show that if we define $\text{Ji } L$ in the obvious way for a join-semilattice L , then for $L = \text{Down}_{\text{fin}} P$, the isomorphism $\text{Ji } L \cong P$ holds.
- 5.62. Deduce that for any join-semilattice L of the form $\text{Down}_{\text{fin}} P$, the analog of Corollary 108 holds.
- 5.63. Let L be a distributive algebraic lattice and let F be the set of compact elements of L . Show that the join-semilattice F is distributive.
- 5.64. What is the converse of Exercise 5.63?

6. Distributive Lattices with Pseudocomplementation

6.1 Definitions and examples

In this section, we shall deal exclusively with pseudocomplemented distributive lattices. There are two distinct concepts: a lattice, $(L; \vee, \wedge)$, in which every element has a pseudocomplement; and an algebra $(L; \vee, \wedge, *, 0, 1)$, where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and where, for every $a \in L$, the element a^* is the pseudocomplement of a . We shall call the former a *pseudocomplemented lattice* and the latter a *lattice with pseudocomplementation* (as an operation)—the same kind of distinction we make between boolean lattices and boolean algebras.

As defined in the Exercises of Section 5, a pseudocomplemented lattice is an algebra of type $(2, 2)$, whereas a lattice with pseudocomplementation is an algebra of type $(2, 2, 1, 0, 0)$. To see the difference in viewpoint, consider the lattice of Figure 39. As a distributive lattice, it has twenty-five sublattices and eight congruences; as a lattice with pseudocomplementation, it has three subalgebras and five congruences.

Thus for a lattice with pseudocomplementation L , a *subalgebra* L_1 is a $\{0, 1\}$ -sublattice of L closed under $*$ (that is, $a \in L_1$ implies that $a^* \in L_1$). A *homomorphism* φ is a $\{0, 1\}$ -homomorphism that also satisfies

$$(\varphi(x))^* = \varphi(x^*).$$

Similarly, a *congruence relation* α shall have the Substitution Property also for $*$, that is, $a \equiv b \pmod{\alpha}$ implies that $a^* \equiv b^* \pmod{\alpha}$.

A wide class of examples is provided by

Theorem 210. *Any complete lattice that satisfies the Join Infinite Distributive Identity (JID) is a pseudocomplemented distributive lattice.*

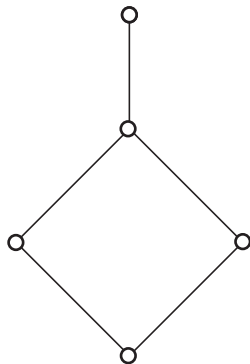


Figure 39. A small example

Proof. Let L be such a lattice. For $a \in L$, set

$$a^* = \bigvee (x \in L \mid a \wedge x = 0).$$

Then, by (JID),

$$a \wedge a^* = a \wedge \bigvee (x \mid a \wedge x = 0) = \bigvee (a \wedge x \mid a \wedge x = 0) = \bigvee 0 = 0.$$

Furthermore, if $a \wedge x = 0$, then $x \leq a^*$ by the definition of a^* ; thus a^* is indeed the pseudocomplement of a . \square

Corollary 211. *Every distributive algebraic lattice is pseudocomplemented.*

Proof. Let L be a distributive algebraic lattice. By Theorem 42 and Lemma 184, represent L as $\text{Id } S$, where S is a distributive join-semilattice with zero. Let I and I_j , for $j \in J$, be ideals of S . Then

$$\bigvee (I \wedge I_j \mid j \in J) \subseteq I \wedge \bigvee (I_j \mid j \in J)$$

is obvious. To prove the reverse inclusion, let

$$a \in I \wedge \bigvee (I_j \mid j \in J),$$

that is, $a \in I$ and $a \in \bigvee (I_j \mid j \in J)$. The latter implies that

$$a \leq t_1 \vee \cdots \vee t_n, \text{ where } t_1 \in I_{j_1}, \dots, t_n \in I_{j_n}, j_1, \dots, j_n \in J.$$

Thus $a \in I_{j_1} \vee \cdots \vee I_{j_n}$ and so, using the distributivity of $\text{Id } L$, we obtain that

$$a \in I \wedge (I_{j_1} \vee \cdots \vee I_{j_n}) = (I \wedge I_{j_1}) \vee \cdots \vee (I \wedge I_{j_n}) \subseteq \bigvee (I \wedge I_j \mid j \in J),$$

completing the proof of (JID). The statement now follows from Theorem 210. \square

Note that we have remarked this much already in Exercise 4.20.

Thus the lattice of all congruence relations of an arbitrary lattice and the lattice of all ideals of a distributive lattice (or semilattice) with zero are examples of pseudocomplemented distributive lattices. Note that

$$I^* = \{x \in K \mid x \wedge i = 0 \text{ for all } i \in I\}$$

for any ideal I of a distributive lattice K . Also, any finite distributive lattice is pseudocomplemented. Therefore, our investigations include all finite distributive lattices.

6.2 Stone algebras

The class of Stone algebras (named in G. Grätzer and E. T. Schmidt [331]) was the first class of distributive lattices with pseudocomplementation, other than the class of boolean algebras, to be examined in detail. A distributive lattice with pseudocomplementation L is called a *Stone algebra* if it satisfies the *Stone identity*:

$$a^* \vee a^{**} = 1.$$

The corresponding pseudocomplemented lattice is called a *Stone lattice*.

For a Stone algebra L , the skeleton $\text{Skel } L$ is a subalgebra of L :

Lemma 212. *For a distributive lattice with pseudocomplementation L , the following conditions are equivalent:*

- (i) L is a Stone algebra.
- (ii) $(a \wedge b)^* = a^* \vee b^*$ for all $a, b \in L$.
- (iii) $a, b \in \text{Skel } L$ implies that $a \vee b \in \text{Skel } L$.
- (iv) $\text{Skel } L$ is a subalgebra of L .

Proof. The proofs that (ii) implies (iii), that (iii) implies (iv), and that (iv) implies (i) are trivial. To prove that (i) implies (ii), let L be a Stone algebra. We show that $a^* \vee b^*$ is the pseudocomplement of $a \wedge b$, verifying (ii). First,

$$(a \wedge b) \wedge (a^* \vee b^*) = (a \wedge b \wedge a^*) \vee (a \wedge b \wedge b^*) = 0 \vee 0 = 0.$$

Second, if $(a \wedge b) \wedge x = 0$, then $(b \wedge x) \wedge a = 0$, and so $b \wedge x \leq a^*$. Meeting both sides by a^{**} yields

$$b \wedge x \wedge a^{**} \leq a^* \wedge a^{**} = 0;$$

that is, $x \wedge a^{**} \wedge b = 0$, implying that $x \wedge a^{**} \leq b^*$. Then $a^* \vee a^{**} = 1$, by the Stone identity, and thus

$$x = x \wedge 1 = x \wedge (a^* \vee a^{**}) = (x \wedge a^*) \vee (x \wedge a^{**}) \leq a^* \vee b^*. \quad \square$$

This is already enough to yield the structure theorem for finite Stone algebras (G. Grätzer and E. T. Schmidt [331]):

Corollary 213. *A finite distributive lattice L is a Stone lattice iff it is the direct product of finite distributive dense lattices, that is, finite distributive lattices with only one atom.*

Proof. By Lemma 212, a Stone lattice L has a complemented element a different from 0 and 1 iff $\text{Skel } L \neq \{0, 1\}$; thus the decomposition of Theorem 106 can be repeated until each factor L_i satisfies $\text{Skel } L_i = \{0, 1\}$. In a direct product, the pseudocomplementation $*$ is formed componentwise; therefore, all the lattices L_i are Stone lattices.

For a finite distributive lattice K with $\text{Skel } K = \{0, 1\}$, the condition that the lattice K has exactly one atom is equivalent to K being a Stone lattice. \square

6.3 Triple construction

In addition to the skeleton, the *dense set*,

$$\text{Dns } L = \{a \mid a^* = 0\},$$

is another significant subset of a Stone algebra. The elements of $\text{Dns } L$ are called *dense*.

We can easily check that $\text{Dns } L$ is a filter of L and $1 \in \text{Dns } L$; thus $\text{Dns } L$ is a distributive lattice with unit. Since $a \vee a^* \in \text{Dns } L$, for every $a \in L$, we can interpret the identity

$$a = a^{**} \wedge (a \vee a^*)$$

to mean that every $a \in L$ can be represented in the form

$$a = b \wedge c, \quad b \in \text{Skel } L, \quad c \in \text{Dns } L.$$

Such an interpretation correctly suggests that if we know $\text{Skel } L$ and $\text{Dns } L$ and the relationships between elements of $\text{Skel } L$ and $\text{Dns } L$, then we can describe L . The relationship is expressed by the homomorphism

$$\varphi_L: \text{Skel } L \rightarrow \text{Fil}(\text{Dns } L)$$

defined by

$$\varphi_L: a \mapsto \{x \in \text{Dns } L \mid x \geq a^*\}.$$

Theorem 214. *Let L be a Stone algebra. Then $\text{Skel } L$ is a boolean algebra, $\text{Dns } L$ is a distributive lattice with unit, and φ_L is a $\{0, 1\}$ -homomorphism of $\text{Skel } L$ into $\text{Fil}(\text{Dns } L)$. The triple*

$$(\text{Skel } L, \text{Dns } L, \varphi_L)$$

characterizes L up to isomorphism.

Proof. The first statement is easily verified. To get the characterization result, for $a \in \text{Skel } L$, set

$$F_a = \{x \mid x^{**} = a\}.$$

The sets $\{F_a \mid a \in \text{Skel } L\}$ form a partition of L ; for a small example, see [Figure 40](#). Obviously, $F_0 = \{0\}$ and $F_1 = \text{Dns } L$. The map $x \mapsto x \vee a^*$ sends F_a into $F_1 = \text{Dns } L$; in fact, the map is an isomorphism between the filters F_a and $\varphi_L(a) \subseteq \text{Dns } L$. Thus $x \in F_a$ is completely determined by a and $x \vee a^* \in \varphi_L(a)$, that is, by a pair (a, z) , where $a \in \text{Skel } L$ and $z \in \varphi_L(a)$, and every such pair determines one and only one element of L . To complete our proof, we have to show how the ordering on L can be determined by such pairs.

Let $x \in F_a$ and $y \in F_b$. Then $x \leq y$ implies that $x^{**} \leq y^{**}$, that is, $a \leq b$. Since $x \leq y$ iff

$$a \vee x \leq a \vee y \quad \text{and} \quad x \vee a^* \leq y \vee a^*,$$

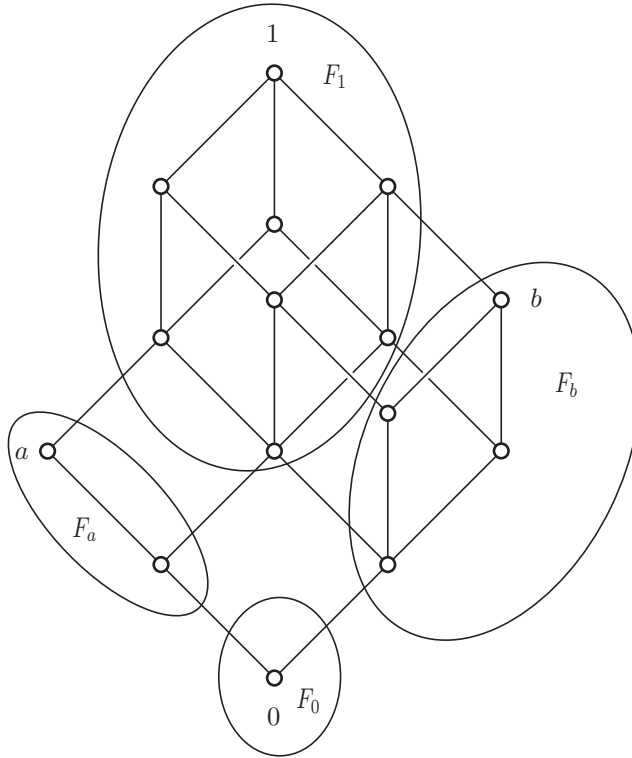


Figure 40. Decomposing a Stone algebra

and since the first of these two conditions is trivial, we obtain that

$$x \leq y \quad \text{iff} \quad a \leq b \text{ and } x \vee a^* \leq y \vee a^*.$$

Identifying x with $(x \vee a^*, a)$ and y with $(y \vee b^*, b)$, we see that the preceding conditions are stated in terms of the components of the ordered pairs, except that $y \vee a^*$ will have to be expressed by the triple.

The map φ_L is a $\{0, 1\}$ -homomorphism and a is the complement of a^* , so we conclude that $\varphi_L(a)$ and $\varphi_L(a^*)$ are complementary filters of $\text{Dns } L$. Thus every $z \in \text{Dns } L$ can be written in a unique fashion in the form $z = \varrho_a(z) \wedge z_1$, where $\varrho_a(z) \in \varphi_L(a)$ and $z_1 \in \varphi_L(a^*)$. Observe that the map ϱ_a is expressed in terms of the triple. Finally,

$$y \vee a^* = y \vee b^* \vee a^* = \varrho_a(y \vee b^*).$$

Thus

$$(u, a) \leq (v, b) \quad \text{iff} \quad a \leq b \text{ and } u \leq \varrho_a(v)$$

holds for $u \in \varphi_L(a)$ and $v \in \varphi_L(b)$. □

This result shows that a Stone algebra is characterized by its triple, see C. C. Chen and G. Grätzer [89] and [90]; these papers also provides a characterization theorem for triples, see Exercises 6.17–6.31.

Theorem 214 shows that the behavior of the skeleton and the dense set is decisive for Stone algebras. This conclusion leads us to formulate the goal of research for Stone algebras:

A problem for Stone algebras is considered solved if it can be reduced to two problems: one for boolean algebras and one for distributive lattices with unit.

6.4 A characterization theorem for Stone algebras

By applying Zorn's Lemma to prime filters of a lattice with zero, we obtain that every prime filter is contained in a maximal prime filter, or, equivalently, we get that every prime ideal contains a *minimal prime ideal* P , that is, a prime ideal P such that $Q \subset P$ for no prime ideal Q (see Exercise 1.34). Minimal prime ideals play an important role in the theory of distributive lattices with pseudocomplementation, as illustrated by the following result in G. Grätzer and E. T. Schmidt [331]:

Theorem 215. *Let L be a distributive lattice with pseudocomplementation. Then L is a Stone algebra iff*

$$P \vee Q = L,$$

for all distinct minimal prime ideals P and Q .

Proof. Let L be a Stone algebra and let P and Q be distinct minimal prime ideals. Note that $P \not\subseteq Q$, since Q is minimal; also, $Q \neq P$, hence $P - Q \neq \emptyset$. So we can choose $a \in P - Q$. Since $a \wedge a^* = 0 \in Q$, utilizing that $a \notin Q$ and Q is prime, we obtain that $a^* \in Q$.

$L - P$ is a maximal dual prime ideal, hence by the dual of Corollary 118, it is a maximal filter of L . Thus $(L - P) \vee \text{fil}(a) = L$ and so $0 = a \wedge x$ for some $x \in L - P$. Therefore, $a^* \geq x \in L - P$ and so $a^* \notin P$. Hence $a^* \in Q - P$. Similarly, $a^{**} \in P - Q$, which implies that

$$1 = a^* \vee a^{**} \in P \vee Q,$$

yielding that $P \vee Q = L$.

To prove the converse (for this proof, see J. C. Varlet [689]), let us assume that L is not a Stone algebra and let $a \in L$ such that $a^* \vee a^{**} \neq 1$. Let R be a prime ideal (see Corollary 117) such that $a^* \vee a^{**} \in R$.

We claim that $(L - R) \vee \text{fil}(a^*) \neq L$. Indeed, if $(L - R) \vee \text{fil}(a^*) = L$, then there exists an $x \in L - R$ such that $x \wedge a^* = 0$. Then $a^{**} \geq x \in L - R$, hence $a^{**} \in L - R$, a contradiction. Let F be a maximal dual prime ideal containing $(L - R) \vee \text{fil}(a^*)$ and similarly, let G be a maximal dual prime ideal containing $(L - R) \vee \text{fil}(a^{**})$. We set $P = L - F$ and $Q = L - G$. Then P and Q are minimal prime ideals. Moreover, $P \neq Q$, because $a^* \in F = L - P$ and hence $a^* \notin P$; thus $a^{**} \in P$, while $a^{**} \notin Q$. Finally, $P, Q \subseteq R$, hence $P \vee Q \neq L$. \square

6.5 Two representation theorems for Stone algebras

We prove two representation theorems for Stone algebras that correspond to the two representation theorems for distributive lattices given in Section 1. The proofs we present use the Subdirect Product Representation Theorem of G. Birkhoff [67]. Direct proofs are possible but we shall present a proof that can be generalized to other varieties of distributive lattices with pseudocomplementation.

In the remainder of this section “algebra” means universal algebra, as defined in Section I.1.9. For the purpose of this book, the reader can substitute “lattice” or “lattice with pseudocomplementation” for “algebra”. Just as for orders and lattices, we write A for the algebra $\mathfrak{A} = (A; F)$ if there is no danger of confusion.

Definition 216. An algebra A is called *subdirectly irreducible* if there exist elements $u, v \in A$ such that $u \neq v$ and $u \equiv v \pmod{\alpha}$ for all congruences $\alpha > \mathbf{0}$.

In other words, A has at least two elements and

$$\text{Con } A = \{\mathbf{0}\} \cup \text{fil}(\text{con}(u, v)),$$

as illustrated in Figure 41, where the unique atom is the congruence $\text{con}(u, v)$. Intuitively, this means that if we collapse any two distinct elements of A , then this congruence spreads to collapse u and v .

The congruence $\text{con}(u, v)$ (unique!) is called the *base congruence* of A ; it is often called the *monolith* in the literature. An equivalent form of this definition is the following (see Section I.6.3 for the concept we are using).

Corollary 217. *The algebra A is subdirectly irreducible iff $\mathbf{0}$ is completely meet-irreducible in $\text{Con } A$.*

Example 218. A distributive lattice L is subdirectly irreducible iff $|L| = 2$.

Proof. If $|L| = 1$, then L is not subdirectly irreducible by definition. If $|L| = 2$, then obviously L is subdirectly irreducible.

Let $|L| > 2$. Then there exist $a, b, c \in L$ with $a < b < c$. We claim that $\text{con}(a, b) \wedge \text{con}(b, c) = \mathbf{0}$, which by Corollary 217 shows that L is not subdirectly irreducible. Let

$$x \equiv y \pmod{\text{con}(a, b) \wedge \text{con}(b, c)}.$$

By Theorem 141, this implies that $x \vee b = y \vee b$ and $x \wedge b = y \wedge b$; thus $x = y$ by Corollary 103. \square

Example 219. B_1 is the only subdirectly irreducible boolean algebra.

Proof. Let B be boolean. The statement is obvious for $|B| \leq 2$. If $|B| > 2$, then B has a direct product representation, $B = A_1 \times A_2$ with $|A_1|, |A_2| \geq 2$ (use Exercise 5.6); thus B cannot be subdirectly irreducible. \square

We shall need a simple universal algebraic lemma.

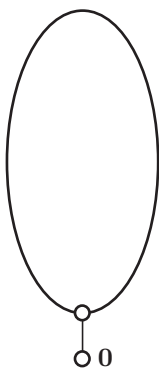


Figure 41. The congruence lattice of a subdirectly irreducible lattice; the unique atom is the base congruence

Lemma 220 (The Second Isomorphism Theorem). *Let A be an algebra and let α be a congruence relation of A . For any congruence $\beta \geq \alpha$ of A , define the relation β/α on A/α by*

$$x/\alpha \equiv y/\alpha \pmod{\beta/\alpha} \quad \text{iff} \quad x \equiv y \pmod{\beta}.$$

Then β/α is a congruence of A/α . Conversely, every congruence γ of A/α can be (uniquely) represented in the form $\gamma = \beta/\alpha$, for some congruence $\beta \geq \alpha$ of A . In particular, the congruence lattice of A/α is isomorphic with the filter $\text{fil}(\alpha)$ of the congruence lattice of A .

Proof. We have to prove that β/α is well-defined, it is an equivalence relation, and it has the Substitution Property. To represent γ , define a congruence β of A by

$$x \equiv y \pmod{\beta} \quad \text{iff} \quad x/\alpha \equiv y/\alpha \pmod{\gamma}.$$

Again, we have to verify that β is a congruence. Then $\beta/\alpha = \gamma$ follows from the definition of β . The details are trivial and left to the reader. \square

Varieties of universal algebras can be introduced by defining terms and identities, just as in the case of lattices. However, in the next theorem (see G. Birkhoff [67]), the reader can avoid the use of this terminology by substituting for “variety” the phrase “class closed under the formation of subalgebras, homomorphic images, and direct products”. (This does not make the result more general, see Theorem 469.)

Theorem 221 (Birkhoff’s Subdirect Representation Theorem). *Let \mathbf{K} be a variety of algebras. Every algebra A in \mathbf{K} can be embedded in a direct product of subdirectly irreducible algebras in \mathbf{K} .*

Proof. For $a, b \in A$ with $a \neq b$, let \mathcal{X} denote the set of all congruences α of A satisfying $a \not\equiv b \pmod{\alpha}$. Then \mathcal{X} is not empty since $\mathbf{0} \in \mathcal{X}$. Let \mathcal{C} be a chain in \mathcal{X} . Since $\alpha = \bigcup \mathcal{C}$ is a congruence and $a \not\equiv b \pmod{\alpha}$, it follows that every chain in \mathcal{X} has an upper bound. By Zorn’s Lemma, there is a maximal element $\gamma(a, b)$ of \mathcal{X} .

We claim that $A/\gamma(a, b)$ is subdirectly irreducible; in fact, the elements $u = a/\gamma(a, b)$ and $v = b/\gamma(a, b)$ satisfy the condition of Definition 216. Indeed, if α is a congruence of $A/\gamma(a, b)$ with $\alpha \neq \mathbf{0}$, then by Lemma 220, represent it as $\alpha = \beta/\gamma(a, b)$, where β is a congruence of A . Since $\alpha \neq \mathbf{0}$, we obtain that $\beta > \gamma(a, b)$, and so $a \equiv b \pmod{\beta}$. Thus $u \equiv v \pmod{\alpha}$, as claimed.

Let

$$B = \prod (A/\gamma(a, b) \mid a, b \in A, a \neq b).$$

Then B is a direct product of subdirectly irreducible algebras. We embed the algebra A into B by the map $\varphi: x \mapsto f_x$, where f_x takes on the value $x/\gamma(a, b)$ in the algebra $A/\gamma(a, b)$. Clearly, φ is a homomorphism. To show that φ is

one-to-one, assume that $f_x = f_y$. Then $x \equiv y \pmod{\gamma(a,b)}$ for all $a, b \in A$ with $a \neq b$. Therefore,

$$x \equiv y \pmod{\bigwedge (\gamma(a,b) \mid a, b \in A, a \neq b)},$$

and so $x = y$. □

We got a little bit more than claimed. If we pick $w \in A/\gamma(a,b)$, then $w = x/\gamma(a,b)$ for some $x \in A$. Thus there is an element in the representation of A whose component in $A/\gamma(a,b)$ is w ; such a representation is called a *subdirect product*. This concept is so important that we give a formal definition.

Definition 222. Let the algebra B be a direct product of the algebras B_i , for $i \in I$, with the projection maps $\pi_i: B \rightarrow B_i$ for $i \in I$. A subalgebra A of B is called a *subdirect product* of the algebras B_i , for $i \in I$, if the projection map π_i maps A onto B_i for all $i \in I$.

Equivalently, an algebra $A \subseteq \prod (B_i \mid i \in I)$ is a subdirect product of the algebras B_i , for $i \in I$ if, for any $i \in I$ and for every $b \in B_i$, there is an element $a \in A$ such that $\pi_i(a) = b$.

Corollary 223. *In a variety \mathbf{K} , every algebra can be represented as a subdirect product of subdirectly irreducible algebras in \mathbf{K} .*

Observe how strong Theorem 221 is. If combined with Example 218, it yields Theorem 119; when combined with Example 219, we obtain Corollary 122.

It is interesting to observe the subtle use of the Axiom of Choice in the proof of Birkhoff's Subdirect Representation Theorem. For every $a, b \in A$ satisfying $a \neq b$, we prove that there are maximal congruences under which a and b are not congruent. We pick one such congruence, $\gamma(a,b)$. Since we pick one for every $a \neq b$, we need the Axiom of Choice for this step. In fact, G. Grätzer [264] proves that Birkhoff's Subdirect Representation Theorem is equivalent to the Axiom of Choice.

The readers should note that subdirect representations of an algebra A are in one-to-one correspondence with families $(\alpha_i \mid i \in I)$ of congruence relations of A satisfying

$$\bigwedge (\alpha_i \mid i \in I) = \mathbf{0}.$$

A subdirect representation by subdirectly irreducible algebras corresponds to families

$$(\alpha_i \mid i \in I)$$

of completely meet-irreducible congruences (see Section I.6.3 for this concept).

Thus Lemma 220 and Theorem 221 combine to yield the following statement.

Corollary 224. *Every congruence relation of an algebra is a meet of completely meet-irreducible congruences.*

Let S_1 denote the three-element chain $\{0, e, 1\}$ ($0 < e < 1$) as a distributive lattice with pseudocomplementation.

Theorem 225. *Up to isomorphism, B_1 and S_1 are the only subdirectly irreducible Stone algebras.*

Proof. B_1 and S_1 are obviously subdirectly irreducible (the congruence lattice of S_1 is a three-element chain).

Now let L be a subdirectly irreducible Stone algebra. By Lemma 212, $\text{Skel } L$ is a subalgebra of L . By definition, $|L| > 1$. If $|\text{Skel } L| > 2$, then $\text{Skel } L$ is directly decomposable and therefore, so is L . Thus $|\text{Skel } L| = 2$, that is,

$$\text{Skel } L = \{0, 1\}.$$

If $|\text{Dns } L| > 2$, then there exist congruences α and β on $\text{Dns } L$ such that $\alpha \wedge \beta = \mathbf{0}$ on $\text{Dns } L$ (by Example 218). Extend α and β to L by defining $\{0\}$ as the only additional block. We conclude that L is subdirectly reducible.

Thus $\text{Skel } L = \{0, 1\}$ and so $L = \text{Dns } L \cup \{0\}$ and $|\text{Dns } L| \leq 2$, yielding that $L \cong B_1$ or $L \cong S_1$. \square

Corollary 226. *Every Stone algebra can be embedded in a direct product of two- and three-element chains (regarded as Stone algebras).*

Proof. Combine Corollary 223 and Theorem 225. \square

See G. Grätzer [255]; a weaker form of this corollary can be found in T. P. Speed [660].

Every distributive lattice can be embedded in some $\text{Pow } X$. O. Frink [206] asked whether every Stone algebra can be embedded in some $\text{Id}(\text{Pow } X)$. This problem was solved in G. Grätzer [249].

Theorem 227. *A distributive lattice with pseudocomplementation L is a Stone algebra iff it can be embedded into some $\text{Id}(\text{Pow } X)$.*

Proof. The algebra $\text{Id}(\text{Pow } X)$ is a Stone algebra, and therefore, any of its subalgebras is a Stone algebra by Corollary 211.

It is obvious that the class of Stone algebras that can be embedded into some $\text{Id}(\text{Pow } X)$ is closed under the formation of direct products and subalgebras. Hence by Corollary 226, it is sufficient to prove that B_1 and S_1 can be so embedded. For B_1 this is obvious. To embed S_1 , take an infinite set X and embed S_1 into $\text{Id}(\text{Pow } X)$ as follows:

$$\begin{aligned} 0 &\mapsto \{\emptyset\}, \\ e &\mapsto \{A \subseteq X \mid |A| < \aleph_0\}, \\ 1 &\mapsto \text{Pow } X. \end{aligned}$$

It is obvious that this is an embedding. \square

6.6 ♦ Generalizing Stone algebras

Let \mathbf{B}_n denote the variety of distributive lattices with pseudocomplementation satisfying the identity

$$(\mathbf{L}_n) \quad (x_1 \wedge \cdots \wedge x_n)^* \vee (x_1^* \wedge \cdots \wedge x_n)^* \vee \cdots \vee (x_1 \wedge \cdots \wedge x_n^*)^* = 1$$

for $n \geq 1$. Then \mathbf{B}_1 is the class of Stone algebras. K. B. Lee [500] has proved that \mathbf{B}_n , for $-1 \leq n \leq \omega$, is a complete list of varieties of distributive lattices with pseudocomplementation, where $\mathbf{B}_{-1} = \mathbf{T}$ is the trivial class, $\mathbf{B}_0 = \mathbf{B}$ is the class of boolean algebras, and \mathbf{B}_ω is the class of all distributive lattices with pseudocomplementation. Moreover,

$$\mathbf{B}_{-1} \subset \mathbf{B}_0 \subset \mathbf{B}_1 \subset \cdots \subset \mathbf{B}_n \subset \cdots \subset \mathbf{B}_\omega.$$

In H. Lakser [492] and G. Grätzer and H. Lakser [298] and [300], most of the structure theorems known for Stone algebras have been generalized to the classes \mathbf{B}_n . In these papers the amalgamation class of \mathbf{B}_n (in the sense of Section VI.4.3) is also described.

These results, and a lot more, are written up in my book G. Grätzer [257] (which was reprinted in 2008).

6.7 ♦ Background

Except for V. Glivenko's early work [233], the study of pseudocomplemented distributive lattices started only in 1956 with a solution of Problem 70 of G. Birkhoff [70] in G. Grätzer and E. T. Schmidt [331], characterizing Stone lattices by minimal prime ideals (for a simplified proof, see J. C. Varlet [689]), see Theorem 215.

The idea of a triple was conceived by the author (in 1961, while visiting O. Frink at Penn State) as a tool to prove Frink's conjecture (see O. Frink [206]). This attempt failed and as a result triples were not utilized until 1969, see C. C. Chen and G. Grätzer [89] and [90] (also Section V.1.8). Frink's conjecture was solved using the Compactness Theorem in G. Grätzer [249] (see Theorem 227). An interesting generalization can be found in H. Lakser [492].

Exercises

- 6.1. Show that every bounded chain is a pseudocomplemented distributive lattice.
- 6.2. Let L be a lattice with unit. Adjoin a new zero to L : $L_1 = \mathbf{C}_1 + L$. Show that L_1 is a pseudocomplemented lattice.

- 6.3. Call a lattice with zero *dense* if the element 0 is meet-irreducible. Show that every bounded dense lattice K is pseudocomplemented and that every such lattice can be constructed by the method of Exercise 6.2 with $L = \text{Dns } K$.
- 6.4. Find an example of a complete distributive lattice L that is not pseudocomplemented.
- 6.5. Prove that if L is a complete Stone lattice, then so is $\text{Id } L$. (Hint: $I^* = \text{id}(a)$, where $a = \bigwedge (x^* \mid x \in I)$.)
- 6.6. Show that a distributive pseudocomplemented lattice is a Stone lattice iff

$$(a \vee b)^{**} = a^{**} \vee b^{**}$$

for all $a, b \in L$.

- 6.7. Find a small set of identities characterizing Stone algebras.
- 6.8. Let L be a Stone algebra. Show that $\text{Skel } L$ is a *retract* of L , that is, there is a homomorphism $\varphi: L \rightarrow \text{Skel } L$ such that $\varphi(x) = x$ for all $x \in \text{Skel } L$.
- 6.9. Let L be a Stone algebra, $a, b \in \text{Skel } L$, and $a \leq b$. Prove that

$$x \mapsto (x \vee a^*) \wedge b$$

embeds F_a into F_b .

- 6.10. Let B be a boolean algebra. Define $B^{[2]} \subseteq B^2$ by $(a, b) \in B^{[2]}$ if $a \leq b$. Verify that $B^{[2]} \leq B^2$ but it is not a subalgebra of B^2 . Show that $B^{[2]}$ is a Stone lattice.
- 6.11. Let L be a pseudocomplemented distributive lattice. Show that, for all $a, b \in L$,

$$\begin{aligned}(a \vee b)^* &= a^* \wedge b^*, \\ (a \wedge b)^{**} &= a^{**} \wedge b^{**}.\end{aligned}$$

- 6.12. Prove that a prime ideal P of a Stone algebra L is minimal iff P as an ideal of L is generated by $P \cap \text{Skel } L$.
- 6.13. Show that a distributive lattice with pseudocomplementation is a Stone algebra iff every prime ideal contains exactly one minimal prime ideal (G. Grätzer and E. T. Schmidt [331]).
- *6.14. Prove that an order Q is isomorphic to the order of all prime ideals of a Stone algebra iff
- (a) every element of Q contains exactly one minimal element;
 - (b) for every minimal element m of Q , the order $\uparrow m - \{m\}$ is isomorphic to the order of all prime ideals of some distributive lattice with unit.

(See C. C. Chen and G. Grätzer [90].)

- 6.15. Give a detailed proof of the Second Isomorphism Theorem.

6.16. Prove Corollary 226 directly.

* * *

Exercises 6.17–6.31 are from C. C. Chen and G. Grätzer [89] and [90]. Let B be a boolean algebra, let D be a distributive lattice with unit, and let φ be a $\{0, 1\}$ -homomorphism of B into $\text{Fil } D$. Set

$$L = \{ (x, a) \mid a \in B, x \in \varphi(a) \},$$

and define $(x, a) \leq (y, b)$ if $a \leq b$ and $x \leq \varrho_a(y)$, where $\text{fil}(\varrho_a(y)) = \varphi(a) \wedge \text{fil}(y)$.

6.17. Verify the following formulas:

- (a) If $a \in B$ and $d \in D$, then $\varrho_a(d) = d$ iff $d \in \varphi(a)$.
- (b) $\varrho_a(d) \geq d$ for $a \in B$ and $d \in D$.
- (c) $\varrho_a(d) \wedge \varrho_{a'}(d) = d$ for $a \in B$ and $d \in D$ (where a' is the complement of a in B).
- (d) $\varrho_a \varrho_b = \varrho_{a \wedge b}$ for all $a, b \in C$.

6.18. Prove that:

- (a) $\varrho_a(d) \wedge \varrho_b(d) = \varrho_{a \vee b}(d)$ for all $a, b \in B$ and $d \in D$.
- (b) $\varrho_{a \wedge b}(d) = \varrho_a(d) \vee \varrho_b(d)$ for all $a, b \in B$ and $d \in D$.

6.19. Show that L is an order under the given ordering.

6.20. For $(x, a), (y, b) \in L$, verify that

$$(x, a) \wedge (y, b) = (\varrho_b(x) \wedge \varrho_a(y), a \wedge b).$$

*6.21. Show that

$$(x, a) \vee (y, b) = ((\varrho_{b'}(x) \wedge y) \vee (x \wedge \varrho_{a'}(y)), a \vee b).$$

6.22. For $(x, a), (y, b), (z, c) \in L$, let

$$\begin{aligned} U &= ((x, a) \wedge (y, b)) \vee (z, c), \\ V &= ((x, a) \vee (z, c)) \wedge ((y, b) \vee (z, c)). \end{aligned}$$

Compute U ; show that

$$V = (d, (a \vee c) \wedge (b \vee c)),$$

where

$$\begin{aligned} d &= d_0 \vee d_1 \vee d_2 \vee d_3, \\ d_0 &= \varrho_{b \wedge c'}(x) \wedge \varrho_{a \wedge c'}(y) \wedge z, \\ d_1 &= \varrho_{b \wedge c'}(x) \wedge \varrho_{a \wedge c}(y) \wedge z, \\ d_2 &= \varrho_{b \vee c}(x) \wedge \varrho_{a \wedge c'}(y) \wedge z, \\ d_3 &= \varrho_{b \vee c}(x) \wedge \varrho_{b \vee c}(y) \wedge \varrho_{a' \vee b'}(z). \end{aligned}$$

- 6.23. Show that $d_0 \geq d_1$ and $d_0 \geq d_2$; therefore, $d = d_0 \vee d_3$.
 6.24. Show that L is distributive.
 6.25. Show that L is a Stone lattice.
 6.26. Identify $b \in B$ with $(1, b)$ and $d \in D$ with $(d, 1)$. Verify that

$$\begin{aligned}\text{Skel } L &= B, \\ \text{Dns } L &= D, \\ \varphi_L &= \varphi.\end{aligned}$$

In other words, we have proved the following theorem of C. C. Chen and G. Grätzer [89]:

Theorem 228 (Construction Theorem of Stone Algebras).
Given a boolean algebra B , a distributive lattice D with unit, and a $\{0, 1\}$ -homomorphism $\varphi: B \rightarrow \text{Fil } D$, there exists a Stone algebra L whose triple is (B, D, φ) .

- 6.27. Describe isomorphisms and homomorphisms of Stone algebras in terms of triples.
 6.28. Describe subalgebras of Stone algebras in terms of triples.
 6.29. For a given boolean algebra B with more than one element and distributive lattice D with unit, construct a Stone algebra L with $\text{Skel } L \cong B$ and $\text{Dns } L \cong D$. (That is, prove that $\text{Skel } L$ and $\text{Dns } L$ are independent.)
 6.30. Show that a Stone algebra L is complete if $\text{Skel } L$ and $\text{Dns } L$ are complete.
 *6.31. Characterize the completeness of Stone algebras in terms of triples.
 *6.32. Show that a distributive lattice with pseudocomplementation L has CEP—defined in Section I.3.8 (G. Grätzer and H. Lakser [298]).
 6.33. Let L be a lattice or a lattice with additional operations. If $a, b \in L$ and $[b, a]$ is simple, then $\gamma(a, b)$ (defined in the proof of Theorem 221) is unique.
 6.34. Use the Subdirect Product Representation Theorem of G. Birkhoff to prove that every distributive lattice is a subdirect product of copies of \mathbf{C}_2 . Relate this to Theorem 119.
 6.35. Find conditions under which a distributive lattice is a subdirect product of copies of \mathbf{C}_3 .
 6.36. Find conditions under which a distributive lattice is a subdirect product of copies of \mathbf{C}_n (F. W. Anderson and R. L. Blair [29]).



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