Large Coupling Convergence: Overview and New Results

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Abstract. In this paper we present a couple of old and new results related to the problem of large coupling convergence. Several aspects of convergence are discussed, namely norm resolvent convergence as well as convergence within Schatten-von Neumann classes. We also discuss the rate of convergence with a special emphasis on the optimal rate of convergence, for which we give necessary and sufficient conditions. The collected results are then used for the case of Dirichlet operators. Our method is purely analytical and is supported by a wide variety of examples.

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1. Introduction

For non-negative potentials V, convergence of Schrödinger operators $-\Delta + bV$ as the coupling constant b goes to infinity has been studied for a long time, cf. [9], [11], [12], and the references therein. Motivated by the fact that there has been created a rich theory of point interactions described in detail in the monograph [1], one has recently made an attempt to include singular, measure-valued potentials in these investigations. In addition, it turned out that perturbations by differential operators of the same order are important in a variety of applications in engineering, cf. [14], [15].

All the mentioned families $(H_b)_{b>0}$ of operators are of the following form: One is given a non-negative self-adjoint operator H in a Hilbert space $\mathcal{H}$. Set

$$D(\mathcal{E}) := D(\sqrt{H}),$$
$$\mathcal{E}(u,v) := (\sqrt{H}u, \sqrt{H}v) \quad \forall\, u,v \in D(\mathcal{E}).$$

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$\mathcal{E}$ is a form in $\mathcal{H}$, i.e., a semi-scalar product on a linear subspace of $\mathcal{H}$. Hence

$$\mathcal{E}_1(u,v) := \mathcal{E}(u,v) + (u,v) \quad \forall\, u,v \in D(\mathcal{E})$$

defines a scalar product on $D(\mathcal{E})$. The form $\mathcal{E}$ is closed, i.e., $(D(\mathcal{E}),\mathcal{E}_1)$ is a Hilbert space. Moreover, it is densely defined, i.e., $D(\mathcal{E})$ is dense in $\mathcal{H}$. In addition, one is given a form $\mathcal{P}$ in $\mathcal{H}$ such that for every $b > 0$ the form $\mathcal{E} + b\mathcal{P}$, defined by

$$\begin{aligned} D(\mathcal{E}+b\mathcal{P}) &:= D(\mathcal{E}) \cap D(\mathcal{P}),\\ (\mathcal{E}+b\mathcal{P})(u,v) &:= \mathcal{E}(u,v) + b\,\mathcal{P}(u,v) \quad \forall\, u,v \in D(\mathcal{E}+b\mathcal{P}), \end{aligned}$$

is densely defined and closed. Then, by Kato's representation theorem, for every $b > 0$ there exists a unique non-negative self-adjoint operator H_b in $\mathcal{H}$ such that

$$\begin{aligned} D(\sqrt{H_b}) &= D(\mathcal{E}+b\mathcal{P}),\\ \left\|\sqrt{H_b}u\right\|^2 &= (\mathcal{E}+b\mathcal{P})(u,u) \quad \forall\, u \in D(\mathcal{E}+b\mathcal{P}). \end{aligned}$$

H_b is called the self-adjoint operator associated with $\mathcal{E}+b\mathcal{P}$. By Kato's monotone convergence theorem, the operators $(H_b+1)^{-1}$ converge strongly as b goes to infinity. In a wide variety of applications it turns out that it is more easy to analyze the limit than the approximants $(H_b+1)^{-1}$. For this reason one might use the following strategy for the investigation of the operator H_b for large b: One studies the limit of the operators $(H_b+1)^{-1}$ and estimates the error one produces by replacing $(H_b+1)^{-1}$ by the limit. This leads to the question about how fast the operators $(H_b+1)^{-1}$ converge. It is also important to find out which kind of convergence takes place. For instance, convergence with respect to the operator norm admits much stronger conclusions about the spectral properties than strong convergence, cf., e.g., the discussion of this point in [22, Chap. VIII.7].

One has achieved a variety of results within the general framework described above. One has discovered that there exists a universal upper bound for the rate of convergence (Corollary 2.8), and one has derived a criterion for convergence with maximal rate (Theorem 2.7). In general, only strong convergence takes place. However, one has found a variety of conditions which are sufficient for locally uniform convergence (Theorem 2.6, Theorem 2.7, and Proposition 2.9), and in certain cases one has even arrived at estimates for the rate of convergence (Theorem 2.7 and Proposition 2.9).

One has even found conditions which are sufficient for convergence within a Schatten (-von Neumann) class of finite order, cf. Sections 2.5 and 2.6.2. This admits strong conclusions about the spectral properties. For instance, if H and H_0 are non-negative self-adjoint operators and $(H+1)^{-1} - (H_0+1)^{-1}$ belongs to the trace class, then, by the Birman-Kuroda theorem, the absolutely continuous spectral parts of H and H_0 are unitarily equivalent and, in particular, H and H_0 have the same absolutely continuous spectrum. Often, $(H+1)^{-1}-(H_0+1)^{-1}$ does not belong to the trace class, but $(H+1)^{-k}-(H_0+1)^{-k}$ for some sufficiently large k does and, again the Birman-Kuroda Theorem, this implies that the absolutely continuous parts of H and H_0 are unitarily equivalent. This note also contains some new results on the convergence of powers of resolvents, cf. Section 2.8. These results are based on a generalization of the celebrated Dynkin's formula in Section 2.7.

One has introduced the concept of the trace of a Dirichlet form in order to study time changed Markov processes. The generator of the time changed process plays also an important role in the investigation of large coupling convergence for the Dirichlet operators, cf. Section 3.2. If one perturbs a Dirichlet operator by an equilibrium measure times a coupling constant b and let b go to infinity, then one gets, at least in the conservative case, large coupling convergence with maximal rate, cf. Theorem 3.16. A simple domination principle described in Section 3.3 makes it possible to use results on the perturbation by one measure in order to derive results on perturbations by other measures.

In this note we concentrate on non-negative perturbations. If one studies large coupling convergence of magnetic Schrödinger operators, then one needs different techniques. We refer to [17] and the references therein for results in this direction.

In addition to new results we have collected material which can be found at the following places (we do not claim that these are the original sources in all cases):

[3]: Lemma 3.7
[4]: Lemmas 2.2 and 2.4, Theorems 2.6 and 2.7, Corollary 2.8, Proposition 2.9 a), Sections 2.5 and 3.4
[6]: Lemma 2.3, Lemma 2.15
[7]: Section 2.6.1, Examples 2.1 and 3.19, and Eqs. (3.20) and (3.22)
[8]: Section 2.7
[13]: Section 2.4 up to Lemma 2.15 and the examples, Section 3.1, and Theorem 3.5, cf. also [20]
[16]: Eq. (3.21)
[23]: Eq. (2.10)
[25]: Lemma 2.5

2. Non-negative form perturbations

2.1. Notation and general hypotheses

Let $\mathcal{E}$ denote a densely defined closed form in the Hilbert space $(\mathcal{H}, (\,\cdot,\cdot\,))$ and H be the self-adjoint operator associated with $\mathcal{E}$. Let $\mathcal{P}$ denote a form in $\mathcal{H}$ such that $\mathcal{E}+\mathcal{P}$ is a densely defined and closed form in $\mathcal{H}$. Note that we do not require $\mathcal{P}$ be closable, i.e., we do not only admit regular, but also singular form perturbations of H.

Example 2.1. Let J be a closed operator from the Hilbert space $(D(\mathcal{E}), \mathcal{E}_1)$ to an auxiliary Hilbert space $\mathcal{H}_{\mathrm{aux}}$. Let

$$\begin{aligned} D(\mathcal{P}) &:= D(J), \\ \mathcal{P}(u,v) &:= (Ju, Jv)_{\mathrm{aux}} \quad \forall\, u, v \in D(J). \end{aligned}$$

Then $\mathcal{E} + b\mathcal{P}$ is a closed form in $\mathcal{H}$ for every $b > 0$. If $D(J)$ is dense in $(D(\mathcal{E}), \mathcal{E}_1)$ and, in addition, $\mathrm{ran}(J)$ is dense in $\mathcal{H}_{\mathrm{aux}}$, then JJ^* is an invertible non-negative self-adjoint operator in $\mathcal{H}_{\mathrm{aux}}$.

Proof. Let (u_n) be a sequence in $D(\mathcal{E} + b\mathcal{P}) = D(J)$ such that

$$\begin{aligned}(\mathcal{E} + b\mathcal{P})\,(u_n - u_m, u_n - u_m) &+ \|u_n - u_m\|^2\\ &= \mathcal{E}_1(u_n - u_m, u_n - u_m) + b\|Ju_n - Ju_m\|^2_{\mathrm{aux}} \to 0 \quad \text{as } n, m \to \infty. \qquad (2.1)\end{aligned}$$

In order to prove that $\mathcal{E} + b\mathcal{P}$ is closed we only have to show that there exists a $u \in D(J)$ such that

$$\begin{aligned}(\mathcal{E} + b\mathcal{P})(u_n - u, u_n - u) &+ \|u_n - u\|^2\\ &= \mathcal{E}_1(u_n - u, u_n - u) + b\,\|Ju_n - Ju\|^2_{\mathrm{aux}} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Since $\mathcal{E}_1$ is non-negative and $b > 0$, it follows from (2.1) that

$$\mathcal{E}_1(u_n - u_m, u_n - u_m) \to 0 \quad \text{as } n, m \to \infty.$$

Since $\mathcal{E}$ is closed, this implies that there exists a $u \in D(\mathcal{E})$ such that

$$\mathcal{E}_1(u_n - u, u_n - u) \to 0 \quad \text{as } n \to \infty. \qquad (2.2)$$

Since $\mathcal{E}_1$ is non-negative and $b > 0$, it also follows from (2.1) that

$$\|Ju_n - Ju_m\|^2_{\mathrm{aux}} \to 0 \quad \text{as } n, m \to \infty$$

and hence the sequence (Ju_n) in $\mathcal{H}_{\mathrm{aux}}$ is convergent. Since J is a closed operator from the Hilbert space $(D(\mathcal{E}), \mathcal{E}_1)$ to the Hilbert space $\mathcal{H}_{\mathrm{aux}}$ and (Ju_n) is convergent in $\mathcal{H}_{\mathrm{aux}}$, (2.2) implies that $u \in D(J)$ and $\|Ju_n - Ju\|_{\mathrm{aux}} \to 0$. Thus $\mathcal{E} + b\mathcal{P}$ is closed.

Suppose now, in addition, that $D(J)$ is dense in $(D(\mathcal{E}), \mathcal{E}_1)$ and $\mathrm{ran}(J)$ is dense in $\mathcal{H}_{\mathrm{aux}}$. Since J is closed, the domain $D(J^*)$ of the adjoint J^* of J is dense in $\mathcal{H}_{\mathrm{aux}}$ and $J = J^{**}$. Hence JJ^* is a non-negative self-adjoint operator in $\mathcal{H}_{\mathrm{aux}}$. If $JJ^*u = 0$, then $\mathcal{E}_1(J^*u, J^*u) = (u, JJ^*u)_{\mathrm{aux}} = 0$ and hence $u \in \ker(J^*) = \mathrm{ran}(J)^\perp$. $\mathrm{ran}(J)^\perp = \{0\}$, since $\mathrm{ran}(J)$ is dense in $\mathcal{H}_{\mathrm{aux}}$. Thus all assertions in the example are proven. □

Indeed, Example 2.1 covers the most general non-negative form perturbation of H:

Lemma 2.2. *There exist an auxiliary Hilbert space $\mathcal{H}_{\mathrm{aux}}$ and a closed operator J from the Hilbert space $(D(\mathcal{E}), \mathcal{E}_1)$ to $\mathcal{H}_{\mathrm{aux}}$ such that*

$$D(J) = D(\mathcal{E} + \mathcal{P}),$$
$$(Ju, Jv)_{\mathrm{aux}} = \mathcal{P}(u, v) \quad \forall\, u, v \in D(J),$$

and $\mathrm{ran}(J)$ *is dense in $\mathcal{H}_{\mathrm{aux}}$. Thus, in particular, $\mathcal{E} + b\mathcal{P}$ is closed for every $b > 0$.*

Proof. We define an equivalence relation $\sim$ on $D(\mathcal{E})\cap D(\mathcal{P})$ as follows: $f \sim g$ if and only if $\mathcal{P}(f-g, f-g) = 0$. For every $f \in D(\mathcal{E}) \cap D(\mathcal{P})$ let $[f]$ be the equivalence class with respect to this equivalence relation and denote by $\mathcal{H}_{\text{aux}}$ the completion of the quotient space $(D(\mathcal{E}) \cap D(\mathcal{P}), \mathcal{P})/\sim$, with respect to the norm

$$|||[f]||| = \mathcal{P}(f,f),\ \forall\,[f] \in \big(D(\mathcal{E}) \cap D(\mathcal{P})\big)/\sim .$$

Then it easily follows from the hypothesis that $\mathcal{E}+\mathcal{P}$ is closed that

$$\begin{aligned} D(J) &:= D(\mathcal{E}) \cap D(\mathcal{P}), \\ Jf &:= [f] \quad \forall\, f \in D(J), \end{aligned}$$

defines a closed operator from $(D(\mathcal{E}), \mathcal{E}_1)$ to $\mathcal{H}_{\text{aux}}$ with the required properties. □

In the following, we choose an auxiliary Hilbert space $\mathcal{H}_{\text{aux}}$ and a closed operator J from $(D(\mathcal{E}), \mathcal{E}_1)$ to $\mathcal{H}_{\text{aux}}$ as in the previous lemma, i.e., such that

$$\begin{aligned} D(J) &= D(\mathcal{E}) \cap D(\mathcal{P}), \\ (Ju, Jv)_{\text{aux}} &= \mathcal{P}(u,v) \quad \forall\, u,v \in D(J), \end{aligned} \tag{2.3}$$

and set

$$\mathcal{E}^J := \mathcal{E} + \mathcal{P}. \tag{2.4}$$

For every $b > 0$, we denote by H_b^J (or simply by H_b if J is clear from the context) the self-adjoint operator in $\mathcal{H}$ associated with $\mathcal{E} + b\mathcal{P}$.

If not stated otherwise, we assume, in addition, that

$$D(J) \supset D(H). \tag{2.5}$$

This hypothesis is less restrictive than it might seem at a first glance. In fact, J may also be regarded as an operator from $\big(D(\mathcal{E}^J), \mathcal{E}_1^J\big)$ to $\mathcal{H}_{\text{aux}}$ and then J is a bounded, everywhere defined operator and, in particular, it is closed. Thus, if necessary, we may replace $\mathcal{E}$ and H by $\mathcal{E}^J$ and H_1, respectively, and then the hypothesis (2.5) is satisfied (with H_1 in place of H). Moreover, we have

$$\begin{aligned} H_{b+1} &= (H_1)_b \quad \forall\, b > 0, \\ \lim_{b\to\infty} (H_b + 1)^{-1} &= \lim_{b\to\infty} ((H_1)_b + 1)^{-1}. \end{aligned} \tag{2.6}$$

Under the hypothesis (2.5), $D(J)$ is dense in $(D(\mathcal{E}), \mathcal{E}_1)$, and we set

$$\check{H} := (JJ^*)^{-1}. \tag{2.7}$$

Note that $\check{H}$ is an invertible non-negative self-adjoint operator in $\mathcal{H}_{\text{aux}}$.

Let

$$\begin{aligned} D(\mathcal{E}_\infty^J) &:= \{u \in D(\mathcal{E}+\mathcal{P}) \colon \mathcal{P}(u,u) = 0\}, \\ \mathcal{E}_\infty^J(u,v) &:= \mathcal{E}(u,v) \quad \forall\, u,v \in D(\mathcal{E}_\infty), \end{aligned} \tag{2.8}$$

where J and $\mathcal{P}$ are related via (2.3) (often we shall omit J in the notation). Let

$$\mathcal{H}_\infty^J := \overline{\{u \in D(\mathcal{E}+\mathcal{P}) \colon \mathcal{P}(u,u) = 0\}}, \tag{2.9}$$

i.e., let $\mathcal{H}_\infty^J$ be the closure of the kernel of J in the Hilbert space $\mathcal{H}$. By Kato's monotone convergence theorem, $\mathcal{E}_\infty^J$ is a densely defined closed form in the Hilbert space $\mathcal{H}_\infty^J$ and

$$(H_b+1)^{-1} \to (H_\infty+1)^{-1} \oplus 0 \text{ strongly as } b \to \infty, \tag{2.10}$$

where H_∞ denotes the self-adjoint operator in $\mathcal{H}_\infty^J$ associated to $\mathcal{E}_\infty^J$. We shall abuse notation and write $(H_\infty+1)^{-1}$ instead of $(H_\infty+1)^{-1} \oplus 0$.

We set

$$L(H,P) := \liminf_{b\to\infty} b\,\|(H_b+1)^{-1} - (H_\infty+1)^{-1}\|.$$

We shall also use the following abbreviations:

$$D_b := (H+1)^{-1} - (H_b+1)^{-1}, \quad D_\infty := (H+1)^{-1} - (H_\infty+1)^{-1},$$
$$G := (H+1)^{-1}. \tag{2.11}$$

2.2. A resolvent formula

We have an explicit expression for the resolvents of the self-adjoint operators H_b. This fact will play a key role throughout this note.

Lemma 2.3. *Let J be a closed operator from $(D(\mathcal{E}),\mathcal{E}_1)$ to an auxiliary Hilbert space $\mathcal{H}_{\rm aux}$ such that*

$$D(J) \supset D(H).$$

Let $b>0$ and let H_b be the self-adjoint operator in $\mathcal{H}$ associated with the closed form $\mathcal{E}^{bJ}$ in $\mathcal{H}$ defined as follows:

$$D(\mathcal{E}^{bJ}) := D(J),$$
$$\mathcal{E}^{bJ}(u,v) := \mathcal{E}(u,v) + b(Ju,Jv)_{\rm aux} \quad \forall\, u,v \in D(J).$$

Then, with $G := (H+1)^{-1}$, the following resolvent formula holds:

$$(H+1)^{-1} - (H_b+1)^{-1} = (JG)^*\left(\frac{1}{b} + JJ^*\right)^{-1} JG. \tag{2.12}$$

Proof. Replacing J by $\sqrt{b}J$, if necessary, we may assume that $b=1$. On the other hand the following identity holds true: for all $u \in \mathcal{H}$ and $v \in D(J^*)$

$$(J^*v,u) = \mathcal{E}_1(J^*v,Gu) = (v,JGu)_{\rm aux} = ((JG)^*v,u). \tag{2.13}$$

Let $u \in \mathcal{H}$. Since JJ^* is a non-negative self-adjoint operator in $\mathcal{H}_{\rm aux}$, the operator $1+JJ^*$ in $\mathcal{H}_{\rm aux}$ is bounded, self-adjoint, and invertible, and

$$D((1+JJ^*)^{-1}) = \mathcal{H}_{\rm aux}.$$

Since $\operatorname{ran}(1+JJ^*)^{-1} = D(JJ^*)$, we obtain that $u \in D(J^*(1+JJ^*)^{-1}JG)$ and $J^*(1+JJ^*)^{-1}JGu \in D(J) = D(\mathcal{E}^J)$.

By Kato's representation theorem,

$$\mathcal{E}_1^J((H_1+1)^{-1}u,v) = (u,v) \quad \forall\, u \in \mathcal{H}, v \in D(\mathcal{E}^J).$$

On the other hand,

$$\begin{aligned}
&\mathcal{E}_1^J(Gu - J^*(1+JJ^*)^{-1}JGu, v) \\
&\qquad = \mathcal{E}_1(Gu, v) + (JGu, Jv)_{\mathrm{aux}} \\
&\qquad\qquad - ((1+JJ^*)^{-1}JGu, Jv)_{\mathrm{aux}} - (JJ^*(1+JJ^*)^{-1}JGu, Jv)_{\mathrm{aux}} \\
&\qquad = (u, v) \quad \forall\, u \in \mathcal{H},\, v \in D(\mathcal{E}^J).
\end{aligned}$$

Thus

$$(H_1+1)^{-1}u = Gu - J^*(1+JJ^*)^{-1}JGu \quad \forall\, u \in \mathcal{H},$$

and it only remains to show that

$$J^*v = (JG)^*v \quad \forall\, v \in D(J^*). \tag{2.14}$$

This is true by identity (2.13). □

2.3. Convergence with respect to the operator norm

If not otherwise stated, J is a closed operator from the Hilbert space $(D(\mathcal{E}), \mathcal{E}_1)$ to an auxiliary Hilbert space $\mathcal{H}_{\mathrm{aux}}$ and, in addition, $D(J) \supset D(H)$. Let

$$\begin{aligned}
D(\mathcal{P}) &:= D(J), \\
\mathcal{P}(u, v) &:= (Ju, Jv)_{\mathrm{aux}} \quad \forall\, u, v \in D(J),
\end{aligned}$$

and H_b be the self-adjoint operator in $\mathcal{H}$ associated to $\mathcal{E} + b\mathcal{P}$.

By Lemma 2.1, JJ^* is a non-negative invertible self-adjoint operator in $\mathcal{H}_{\mathrm{aux}}$. For every $h \in \mathcal{H}_{\mathrm{aux}}$ we denote by μ_h the spectral measure of h with respect to the self-adjoint operator $\check{H} := (JJ^*)^{-1}$ in $\mathcal{H}_{\mathrm{aux}}$, i.e., the unique finite positive Radon measure on $\mathbb{R}$ such that, with $(E_{\check{H}}(\lambda))_{\lambda \in \mathbb{R}}$ being the spectral family of $\check{H}$,

$$\mu_h((-\infty, \lambda]) = \|E_{\check{H}}(\lambda)h\|_{\mathrm{aux}}^2 \quad \forall \lambda \in \mathbb{R}. \tag{2.15}$$

Since $\check{H}$ is invertible and non-negative,

$$\mu_h((-\infty, 0]) = 0 \quad \forall\, h \in \mathcal{H}_{\mathrm{aux}}. \tag{2.16}$$

By (2.12), for every $b > 0$

$$D_b := (H+1)^{-1} - (H_b+1)^{-1} = (JG)^*(\frac{1}{b} + JJ^*)^{-1}JG. \tag{2.17}$$

Hence D_b is a bounded non-negative self-adjoint operator in $\mathcal{H}$ and the spectral calculus yields that

$$\begin{aligned}
(D_b f, f) &= ((JG)^*(\frac{1}{b} + JJ^*)^{-1}JGf, f) \\
&= ((\frac{1}{b} + JJ^*)^{-1}JGf, JGf)_{\mathrm{aux}} \\
&= \int \frac{1}{\frac{1}{b} + \frac{1}{\lambda}}\, d\mu_h(\lambda) \quad \forall\, f \in \mathcal{H},
\end{aligned} \tag{2.18}$$

where $h := JGf$. Thus $D_\infty := \lim_{b\to\infty} D_b = (H+1)^{-1} - (H_\infty+1)^{-1}$ is also a bounded non-negative self-adjoint operator in $\mathcal{H}$ and it follows from (2.18) in conjunction with (2.16) and the monotone convergence theorem that

$$(D_\infty f, f) = \int \lambda\, d\mu_h(\lambda) \quad \forall f \in \mathcal{H}, \tag{2.19}$$

where $h := JGf$. By (2.18) and (2.19),

$$((D_\infty - D_b)f, f) = \int \frac{\lambda^2}{b+\lambda}\, d\mu_h(\lambda) \quad \forall f \in \mathcal{H}, \tag{2.20}$$

where $h := JGf$. Thus $D_\infty - D_b = (H_b+1)^{-1} - (H_\infty+1)^{-1}$ is a bounded non-negative self-adjoint operator in $\mathcal{H}$, too.

Lemma 2.4.

a) *We have*

$$\operatorname{ran}(JG) \subset D(\check{H}^{1/2}) \textit{ and } D_\infty = (\check{H}^{1/2}JG)^*\check{H}^{1/2}JG. \tag{2.21}$$

In particular, D_∞ is compact if and only if $\check{H}^{1/2}JG$ is compact.

b) *If* $\operatorname{ran}(JG) \subset D(\check{H})$, *then*

$$D_\infty = (JG)^*\check{H}JG. \tag{2.22}$$

Proof. a) Let $f \in \mathcal{H}$ and $h := JGf$. By (2.19),

$$(D_\infty f, f) = \int \lambda\, d\mu_h(\lambda) < \infty,$$

and hence, by the spectral calculus, it follows that $h = JGf \in D(\check{H}^{1/2})$ and $\|\check{H}^{1/2}JGf\|^2_{\text{aux}} = (D_\infty f, f)$. Since D_∞ is a bounded non-negative self-adjoint operator, we have

$$\|D_\infty\| = \sup_{\|f\|=1} (D_\infty f, f).$$

Thus

$$\|\check{H}^{1/2}JG\|^2 = \|D_\infty\|. \tag{2.23}$$

Since $JGf \in D(\check{H}^{1/2})$ for every $f \in \mathcal{H}$, the spectral calculus yields

$$\left[\frac{1}{b} + \check{H}^{-1}\right]^{-1/2} JG \to \check{H}^{1/2}JG \text{ strongly as } b \to \infty,$$

and hence

$$\left(\left[\frac{1}{b} + \check{H}^{-1}\right]^{-1/2} JG\right)^* \left[\frac{1}{b} + \check{H}^{-1}\right]^{-1/2} JG \to (\check{H}^{1/2}JG)^*\check{H}^{1/2}JG \tag{2.24}$$

weakly as b goes to infinity. The operators on the left-hand side equal

$$(JG)^*(\frac{1}{b} + JJ^*)^{-1}JG = (H+1)^{-1} - (H_b+1)^{-1} = D_b$$

and converge even strongly to D_∞ as $b \to \infty$. Thus (2.21) is proved.

b) (2.22) follows from (2.21) and the fact that $(JG)^*\check{H}^{1/2} \subset (\check{H}^{1/2}JG)^*$. □

By the preceding lemma, $\check{H}^{1/2}JG$ is a bounded everywhere defined operator from $\mathcal{H}$ to $\mathcal{H}_{\mathrm{aux}}$. That does not guarantee that the resolvents $(H+b)^{-1}$ converge locally uniformly, cf. the examples 2.17 and 2.18. By Theorem 2.6 below, the stronger requirement that $\check{H}^{1/2}JG$ is compact implies convergence of the operators $(H_b+1)^{-1}$ with respect to the operator norm. We shall use the following result for the proof of Theorem 2.6.

Lemma 2.5. *Let (A_n) be a sequence of non-negative bounded self-adjoint operators converging strongly to the compact self-adjoint operator $C:\mathcal{H}\to\mathcal{H}$. Suppose that A_n is dominated by C, i.e.,*

$$(A_n f, f) \le (Cf, f) \quad \forall\, f \in \mathcal{H},$$

for every $n \in \mathbb{N}$. Then the operators A_n converge locally uniformly to C.

Proof. The operator $C - A_n$ is non-negative, bounded and self-adjoint and hence

$$\|C - A_n\| = \sup_{\|f\|=1} ((C-A_n)f, f)$$

for every n.

Let $\varepsilon > 0$. Since C is a non-negative compact self-adjoint operator and the A_n converge to C strongly, we can choose an orthonormal family $(e_j)_{j=1}^N$ and an n_0 such that

$$(Ch, h) \le \frac{\varepsilon}{2}\|h\|^2 \quad \forall\, h \in \operatorname{span}(e_1,\ldots,e_N)^\perp$$

and

$$\|(A_n - C)g\| \le \frac{\varepsilon}{6}\|g\| \quad \forall\, g \in \operatorname{span}(e_1,\ldots,e_N)\,\forall\, n \ge n_0,$$

respectively. Let $f \in \mathcal{H}$ and $\|f\| = 1$. Choose $g \in \operatorname{span}(e_1,\ldots,e_N)$ and $h \in \operatorname{span}(e_1,\ldots,e_N)^\perp$ such that $f = g + h$. For all $n \ge n_0$

$$\begin{aligned}((C-A_n)f,f) &= ((C-A_n)g,g) + 2\mathrm{Re}(((C-A_n)g,h)) + ((C-A_n)h,h)\\ &\le \|(C-A_n)g\|(\|g\| + 2\|h\|) + (Ch,h) \le \varepsilon.\end{aligned}$$

□

Theorem 2.6. *Suppose that $D(H) \subset D(J)$ and the operator $\check{H}^{1/2}JG$ from $\mathcal{H}$ to $\mathcal{H}_{\mathrm{aux}}$ is compact. Then*

$$\|(H_b+1)^{-1} - (H_\infty+1)^{-1}\| \to 0, \quad b \to \infty.$$

Proof. We only need to show that $D_\infty - D_b = (H_b+1)^{-1} - (H_\infty+1)^{-1}$ converge to zero with respect to the operator norm as b goes to infinity. By (2.17), D_b is a non-negative bounded self-adjoint operator in $\mathcal{H}$ for every $b > 0$. By (2.16) in conjunction with (2.20), $D_\infty - D_b$ is a non-negative bounded self-adjoint operator in $\mathcal{H}$, too. By definition, $D_\infty - D_b$ converge to zero strongly as b goes to infinity. By (2.21), along with $\check{H}^{1/2}JG$ also D_∞ is a compact operator.

The remaining part of the proof follows now from the preceding lemma: The operators D_b are non-negative self-adjoint operators and, by (2.16) in conjunction with (2.20), are dominated by the compact self-adjoint operator D_∞, and they converge to D_∞ strongly as b goes to infinity. Hence $\lim_{b\to\infty}\|D_\infty - D_b\| = 0$. □

Of course, one is not only interested in the question whether norm convergence takes place but one also wants to derive estimates for the rate of convergence. We shall show that convergence faster than $O(1/b)$ is not possible for the operators $(H_b+1)^{-1}$, cf. Corollary 2.8 below. Under the additional assumption that the domain $D(H)$ of H is contained in the domain $D(J)$ of J we can even provide a criterion for convergence with maximal rate $O(1/b)$:

Theorem 2.7. *Suppose that*

$$D(H) \subset D(J)$$

and $Ju \neq 0$ for at least one $u \in D(J)$. Then the following holds:

a) *The mapping $b \mapsto b\,\|(H_b+1)^{-1} - (H_\infty+1)^{-1}\|$ is nondecreasing and*

$$\begin{aligned} L(H,P) &:= \liminf_{b\to\infty} b\,\|(H_b+1)^{-1} - (H_\infty+1)^{-1}\| \\ &= \limsup_{b\to\infty} b\,\|(H_b+1)^{-1} - (H_\infty+1)^{-1}\| > 0 \end{aligned}$$

b) *$L(H,P) < \infty \iff J(D(H)) \subset D(\check{H})$.*

c) *If $J(D(H)) \subset D(\check{H})$, then*

$$L(H,P) = \|\check{H}JG\|^2 < \infty. \tag{2.25}$$

Proof. Let $f \in \mathcal{H}$, $h = JGf$, and μ_h be the spectral measure of h with respect to $\check{H}$. By (2.20),

$$b((D_\infty - D_b)f, f) = \int \frac{b\lambda^2}{b+\lambda}\, d\mu_h(\lambda).$$

This implies in conjunction with (2.16) and the monotone convergence theorem (from measure theory), that the mapping $b \mapsto b((D_\infty - D_b)f, f)$ is nondecreasing and

$$\lim_{b\to\infty} b((D_\infty - D_b)f, f) = \int \lambda^2\, d\mu_h(\lambda).$$

Since μ_h is the spectral measure of h with respect to the self-adjoint operator $\check{H}$, it follows that

$$\lim_{b\to\infty} b((D_\infty - D_b)f, f) = \|\check{H}JGf\|^2_{\mathrm{aux}} \qquad \text{if } JGf \in D(\check{H}), \tag{2.26}$$

$$\lim_{b\to\infty} b((D_\infty - D_b)f, f) = \infty \qquad \text{if } JGf \notin D(\check{H}). \tag{2.27}$$

By (2.27),

$$\liminf_{b\to\infty} b\|D_\infty - D_b\| = \infty, \tag{2.28}$$

if there exists an $f \in \mathcal{H}$ such that $JGf \notin D(\check{H})$.

Suppose now that $\mathrm{ran}(JG) \subset D(\check{H}) = \mathrm{ran}(JJ^*)$. JG is closed, since J is closed and G is bounded and closed. Since $D(JG) = \mathcal{H}$, it follows from the closed graph theorem that JG is bounded. Since $\check{H}$ is closed, this implies that $\check{H}JG$ is closed. Since $D(\check{H}JG) = \mathcal{H}$, it follows from the closed graph theorem that $\check{H}JG$ is bounded. Moreover, by (2.26),

$$\liminf_{b\to\infty} b\|D_\infty - D_b\| \geq \|\check{H}JGf\|^2_{\text{aux}},$$

if $\|f\| = 1$, and hence

$$\liminf_{b\to\infty} b\|D_\infty - D_b\| \geq \|\check{H}JG\|^2. \tag{2.29}$$

By (2.20) in conjunction with (2.16), $D_\infty - D_b$ is a non-negative self-adjoint operator in $\mathcal{H}$. Thus

$$\|D_\infty - D_b\| = \sup_{\|f\|=1} ((D_\infty - D_b)f, f). \tag{2.30}$$

(2.20) in conjunction with (2.16) also implies that for every normalized $f \in \mathcal{H}$ and $h = JGf$

$$b((D_\infty - D_b)f, f) \leq \int \lambda^2\, d\mu_h(\lambda) \leq \|\check{H}JG\|^2.$$

In conjunction with (2.30), this implies that

$$b\|D_\infty - D_b\| \leq \|\check{H}JG\|^2 \quad \forall b > 0. \tag{2.31}$$

By (2.28), (2.29), (2.31), part b) and c) of the theorem are proved. In addition, we have shown that the mapping

$$b \mapsto b\,\|D_b - D_\infty\| = b\,\|(H_b+1)^{-1} - (H_\infty+1)^{-1}\|$$

is nondecreasing and hence

$$\begin{aligned} L(H,P) &:= \liminf_{b\to\infty} b\,\|(H_b+1)^{-1} - (H_\infty+1)^{-1}\| \\ &= \limsup_{b\to\infty} b\,\|(H_b+1)^{-1} - (H_\infty+1)^{-1}\|. \end{aligned} \tag{2.32}$$

It remains to prove that $L(H,P) > 0$. We conduct the proof by contradiction. If $L(H,P)$ were equal to zero, then, by c), we would have $JG = 0$. Thus the kernel of J would contain $\operatorname{ran}(G) = D(H)$ and hence it would be dense in $(D(\mathcal{E}), \mathcal{E}_1)$. Since the kernel of a closed operator is closed it would follow that $J = 0$, which contradicts the fact that the range of J is dense in $\mathcal{H}_{\text{aux}}$. Thus $L(H,P) > 0$. $\square$

Part a) of the preceding theorem in conjunction with formula (2.6) yields the following corollary where we do not require that $D(J) \supset D(H)$.

Corollary 2.8. *Let $\mathcal{P}$ be a form in $\mathcal{H}$ such that $\mathcal{E} + \mathcal{P}$ is a densely defined closed form in $\mathcal{H}$. Let $\mathcal{P}(u,u) \neq 0$ for at least one $u \in D(\mathcal{E}+\mathcal{P})$. For every $b > 0$ let H_b be the self-adjoint operator in $\mathcal{H}$ associated to $\mathcal{E} + b\mathcal{P}$. Then*

$$\begin{aligned} L(H,P) &:= \liminf_{b\to\infty} b\,\|(H_b+1)^{-1} - (H_\infty+1)^{-1}\| \\ &= \limsup_{b\to\infty} b\,\|(H_b+1)^{-1} - (H_\infty+1)^{-1}\| > 0. \end{aligned}$$

Trivially, we get large coupling convergence with maximal rate, i.e., as fast as $O(1/b)$, if the auxiliary Hilbert space $\mathcal{H}_{\text{aux}}$ is finite-dimensional. We shall also give a variety of nontrivial examples. On the other hand, there are other examples, where $\|(H_b+1)^{-1}-(H_\infty+1)^{-1}\|$ converge to zero as c/b^r for some strictly positive finite constant c and some $r \in (0,1)$. Let $0<r<1$. It is an open problem to find a criterion for convergence with rate $O(1/b^r)$ to take place. In part a) of the following proposition we give a sufficient condition and in part b) we show that this condition is "almost necessary".

Proposition 2.9. *Let $0<r<1$ and $s_0 = \frac{1}{2}+\frac{r}{2}$. Suppose that $D(H) \subset D(J)$.*

a) *If $J(D(H)) \subset D(\check{H}^{s_0})$, then*

$$\|(H_b+1)^{-1}-(H_\infty+1)^{-1}\| \le (1-r)^{1-r} r^r \|\check{H}^{1/2+r/2} JG\|^2 \frac{1}{b^r} \quad \forall\, b>0.$$

b) *Let $u \in \mathcal{H}$. If*

$$\|(H_b+1)^{-1}u-(H_\infty+1)^{-1}u\| \le \frac{c}{b^r} \quad \forall\, b>0,$$

for some finite constant c, then $JGu \in D(\check{H}^s)$ for every $s<s_0$.

Proof. a) By (2.16) in conjunction with (2.20), $(H_b+1)^{-1}-(H_\infty+1)^{-1}$ is a non-negative bounded self-adjoint operator in $\mathcal{H}$ and hence

$$\|(H_b+1)^{-1}-(H_\infty+1)^{-1}\| = \sup_{\|f\|=1} ((D_\infty - D_b)f, f).$$

By (2.20), this implies that

$$\|(H_b+1)^{-1}-(H_\infty+1)^{-1}\| = \sup_{\|f\|=1} \int \frac{\lambda^2}{\lambda+b}\, d\mu_h(\lambda),$$

where f and h are related via $h = JGf$ and μ_h denotes the spectral measure of h with respect to $\check{H}$. Moreover,

$$\int \frac{\lambda^2}{\lambda+b}\, d\mu_h(\lambda) \le \max_{\lambda\in(0,\infty)} \frac{\lambda^{1-r}}{\lambda+b} \int |\lambda^{1/2+r/2}|^2\, d\mu_h(\lambda).$$

By elementary calculus,

$$\max_{\lambda\in(0,\infty)} \frac{\lambda^{1-r}}{\lambda+b} = \frac{(1-r)^{1-r}\, r^r}{b^r}.$$

By the spectral calculus,

$$\int |\lambda^{1/2+r/2}|^2\, d\mu_h(\lambda) = \|\check{H}^{1/2+r/2} h\|^2_{\text{aux}}.$$

If $h = JGf$ and $\|f\|=1$, then

$$\|\check{H}^{1/2+r/2} h\|_{\text{aux}} \le \|\check{H}^{1/2+r/2} JG\|,$$

and part a) of the Proposition is proved.

b) Conversely let $f \in \mathcal{H}$ and assume that

$$\|(H_b+1)^{-1} - (H_\infty+1)^{-1}\| \le \frac{c}{b^r} \quad \forall\, b > 0,$$

for some finite constant c. Let $h = JGf$. We may assume that $\|f\| = 1$. Let $1/2 < s < s_1 < s_0 := r/2 + 1/2$. Then

$$\begin{aligned} c &\ge b^r \,\|D_\infty f - D_b f\| \ge b^r \,(D_\infty f - D_b f, f) \\ &= b^r \int \frac{\lambda^2}{\lambda+b} \mu_h(d\lambda) = \int \lambda^{2s_1} \frac{b^r\,\lambda^{2-2s_1}}{\lambda+b}\, d\mu_h(\lambda) \quad \forall\, b > 0. \end{aligned} \tag{2.33}$$

In the second step we have used (2.20). Since $2s_0 - 1 = r$, we have

$$t := \frac{r}{2s_1 - 1} > \frac{r}{2s_0 - 1} = 1.$$

For all $b \ge 1$ and $\lambda \in [b, b^t]$, we have

$$\frac{b^r\,\lambda^{2-2s_1}}{\lambda+b} \ge \frac{1}{2}\,\lambda^{1-2s_1} b^r \ge \frac{1}{2}\,(b^t)^{1-2s_1} b^r = \frac{1}{2}.$$

By (2.33), this implies

$$\int_{[b,b^t]} \lambda^{2s_1}\, \frac{1}{2}\, d\mu_h(\lambda) \le c \quad \forall\, b \ge 1.$$

Thus

$$\begin{aligned} \int_{[2,\infty)} \lambda^{2s}\, d\mu_h(\lambda) &\le \sum_{n=0}^{\infty} \int_{[2^{t^n}, 2^{t^{n+1}})} \lambda^{2s_1}\, \frac{1}{(2^{t^n})^{2s_1-2s}}\, d\mu_h(\lambda) \\ &\le 2c \sum_{n=0}^{\infty} \left(\frac{1}{2^{2s_1-2s}} \right)^{t^n} < \infty \end{aligned}$$

and hence $h = JGf \in D(\check{H}^s)$. Thus the assertion b) of Proposition 2.9 is also proved. □

2.4. Schrödinger operators

In this section we illustrate above general definitions and results with the aid of Schrödinger operators with regular and singular potentials.

We denote by $\mathbb{D}$ the classical Dirichlet form, i.e., the form in $L^2(\mathbb{R}^d) := L^2(\mathbb{R}^d, dx)$ defined as follows:

$$\begin{aligned} D(\mathbb{D}) &:= H^1(\mathbb{R}^d), \\ \mathbb{D}(u,v) &:= \int \nabla \bar{u} \cdot \nabla v dx \quad \forall\, u, v \in H^1(\mathbb{R}^d). \end{aligned} \tag{2.34}$$

Here dx denotes the Lebesgue measure and $H^1(\mathbb{R}^d)$ the Sobolev space of order one. $\mathbb{D}$ is a densely defined closed form in $L^2(\mathbb{R}^d)$. We shall denote by $-\Delta$ the self-adjoint operator in $L^2(\mathbb{R}^d)$ associated to $\mathbb{D}$.

The capacity of a compact subset K of $\mathbb{R}^d$ and an arbitrary subset B of $\mathbb{R}^d$ is defined as follows:

$$\begin{aligned}\operatorname{cap}(K) &:= \inf\{\mathbb{D}_1(u,u)\colon u \in C_0^\infty(\mathbb{R}^d),\ u \geq 1 \text{ on } K\},\\ \operatorname{cap}(B) &:= \sup\{\operatorname{cap}(K)\colon K \subset B,\ K \text{ is compact}\},\end{aligned} \tag{2.35}$$

respectively. A function $u : \mathbb{R}^d \to \mathbb{C}$ is quasi-continuous if and only if for every $\varepsilon > 0$ there exists an open set G_ε such that

$$\operatorname{cap}(G_\varepsilon) < \varepsilon \tag{2.36}$$

and the restriction $u \restriction \mathbb{R}^d \setminus G_\varepsilon$ of u to $\mathbb{R}^d \setminus G_\varepsilon$ is continuous. We shall use the following elementary results:

Lemma 2.10.

a) *Every $u \in H^1(\mathbb{R}^d)$ has a quasi-continuous representative.*
b) *If $\tilde{u}$ and u° are quasi-continuous and $\tilde{u} = u^\circ$ dx-a.e., then $\tilde{u} = u^\circ$ q.e. (quasi-everywhere), i.e.,*

$$\operatorname{cap}(\{x \in \mathbb{R}^d\colon \tilde{u}(x) \neq u^\circ(x)\}) = 0. \tag{2.37}$$

c) *If (u_n) is a sequence in $H^1(\mathbb{R}^d)$, $u \in H^1(\mathbb{R}^d)$ and $\mathbb{D}_1(u_n - u, u_n - u) \to 0$ as $n \to \infty$, then there exists a subsequence (u_{n_j}) of (u_n) such that*

$$\tilde{u}_{n_j} \to \tilde{u} \ \text{q.e.}, \tag{2.38}$$

i.e., $\operatorname{cap}(\{x \in \mathbb{R}^d\colon \tilde{u}_{n_j}(x) \not\to \tilde{u}(x)\}) = 0$. *Here $\tilde{u}_{n_j}$ and $\tilde{u}$ denote any quasi-continuous representative of u_{n_j} and u, respectively.*

The proof of the latter lemma can be found in [13].

In the following we shall denote by u both an element of $H^1(\mathbb{R}^d)$ and any quasi-continuous representative of u. It will not matter which quasi-continuous representative is chosen and it will always be clear from the context what is meant.

Remark 2.11. In the one-dimensional case $\operatorname{cap}(\{a\}) = 2$ for every $a \in \mathbb{R}$ and hence a function is quasi-continuous if and only if it is continuous. Thus, in the one-dimensional case, it makes sense to write $u(a)$ if $u \in H^1(\mathbb{R})$ and $a \in \mathbb{R}$. Here $u(a)$ is just the value of the unique continuous representative of u at the point a.

Definition 2.12. Let μ be a positive Radon measure on $\mathbb{R}^d$ charging no set with capacity zero.

a) We define the form $\mathcal{P}_\mu$ in $L^2(\mathbb{R}^d)$ as follows:

$$\begin{aligned} D(\mathcal{P}_\mu) &:= \{u \in H^1(\mathbb{R}^d) : \int |u|^2\, d\mu < \infty\},\\ \mathcal{P}_\mu(u,v) &:= \int \bar{u}\, v\, d\mu \quad \forall\, u,v \in D(\mathcal{P}_\mu).\end{aligned} \tag{2.39}$$

b) We define the operator J^μ from $H^1(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d,\mu)$ as follows:

$$D(J^\mu) := \{u \in H^1(\mathbb{R}^d) : \int |u|^2\, d\mu < \infty\},$$
$$J^\mu u := u \quad \mu\text{-a.e. } \forall\, u \in D(J^\mu). \tag{2.40}$$

Lemma 2.13. *Let μ be a positive Radon measure on $\mathbb{R}^d$ charging no set with capacity zero. Then the operator J^μ is closed and $\mathbb{D} + b\mathcal{P}_\mu$ is a non-negative densely defined closed form in $L^2(\mathbb{R}^d)$ for any $b > 0$.*

Proof. Let (u_n) be a sequence in $D(J^\mu)$, $u \in H^1(\mathbb{R}^d)$ and $v \in L^2(\mathbb{R}^d,\mu)$ satisfying $\mathbb{D}_1(u_n - u, u_n - u) \to 0$ as $n \to \infty$, and $J^\mu u_n \to v$ as $n \to \infty$. By Lemma 2.10 c), a suitably chosen subsequence of (u_n) converges to u q.e. and hence μ-a.e. Thus $u = v$ μ-a.e. and hence $u \in D(J^\mu)$ and $J^\mu u_n \to u$ as $n \to \infty$. Thus the operator J^μ is closed, and, by Lemma 2.1, it follows that $\mathbb{D} + b\mathcal{P}_\mu$ is also closed. □

Definition 2.14. Let μ be a positive Radon measure on $\mathbb{R}^d$ charging no set with capacity zero. We denote by $-\Delta + \mu$ the non-negative self-adjoint operator in $L^2(\mathbb{R}^d)$ associated to $\mathbb{D} + \mathcal{P}_\mu$ and put

$$(-\Delta + \infty\mu + 1)^{-1} := \lim_{b\to\infty} (-\Delta + b\mu + 1)^{-1}.$$

In the absolutely continuous case, i.e., if $d\mu = V dx$ for some function V, we also write V instead of $V dx$.

In a wide variety of applications one is interested in the question whether the operator J^μ is compact. There exists a rich literature on this topic. Here we shall only need the following result.

Lemma 2.15. *Suppose that $D(J^\mu) = H^1(\mathbb{R})$ and*

$$\mu(\{y \in \mathbb{R} :\, |x - y| < 1\}) \to 0, \quad |x| \to \infty. \tag{2.41}$$

Then the operator J^μ from $H^1(\mathbb{R})$ to $L^2(\mathbb{R},\mu)$ is compact.

The proof of this lemma can be found in [6].

Example 2.16. Let $(x_n)_{\in\mathbb{Z}}$ and $(a_n)_{\in\mathbb{Z}}$ be families of real numbers satisfying

$$d := \inf_{n\in\mathbb{Z}} (x_{n+1} - x_n) > 0 \text{ and } a_n > 0 \quad \forall\, n \in \mathbb{Z}. \tag{2.42}$$

Let $\Gamma := \{x_n :\, n \in \mathbb{Z}\}$ and $-\Delta_D^\Gamma$ the Laplacian in $L^2(\mathbb{R})$ with Dirichlet boundary conditions at every point of Γ, i.e., let $-\Delta_D^\Gamma$ be the non-negative self-adjoint operator in $L^2(\mathbb{R})$ associated to the form $\mathbb{D}_\infty$ in $L^2(\mathbb{R})$ defined as follows:

$$D(\mathbb{D}_\infty) := \{u \in H^1(\mathbb{R})\colon u = 0 \text{ on } \Gamma\},$$
$$\mathbb{D}_\infty(u,v) := \mathbb{D}(u,v) \quad \forall\, u,v \in D(\mathbb{D}_\infty). \tag{2.43}$$

Then the operators $-\Delta + b\sum_{n\in\mathbb{Z}} a_n\delta_{x_n}$ converge in the strong resolvent sense to $-\Delta_D^\Gamma$. Here δ_x denotes the Dirac measure with unit mass at x.

Proof. $-\Delta + b\sum_{n\in\mathbb{Z}} a_n\delta_{x_n}$ is the self-adjoint operator associated to $\mathbb{D} + b\mathcal{P}_\mu$ with $\mu := \sum_{n\in\mathbb{Z}} a_n\delta_{x_n}$ and we may replace in formula (2.8) $\mathcal{E}$ and $\mathcal{P}$ by $\mathbb{D}$ and $\mathcal{P}_\mu$, respectively. Then the assertion on strong resolvent convergence follows from Kato's monotone convergence theorem, cf. (2.10). □

Different choices of the weights a_n in the last example lead to extremely different convergence results. If the a_n go to zero as $n \to \pm\infty$, then the operators $-\Delta + b\sum_{n\in\mathbb{Z}} a_n\delta_{x_n}$ do not converge in the norm resolvent sense, cf. the next example. On the other hand, if $\inf_{n\in\mathbb{Z}} a_n > 0$, then these operators converge in the norm resolvent with maximal rate of convergence, i.e., as fast as $O(1/b)$, cf. Example 3.8 below.

Example 2.17 (Continuation of Example 2.16). We choose $(x_n)_{n\in\mathbb{Z}}$, $(a_n)_{n\in\mathbb{Z}}$, d, Γ, $-\Delta_D^\Gamma$, and μ as in the previous example. Assume, in addition, that

$$\lim_{|n|\to\infty} a_n = 0 \text{ and } D := \sup_{n\in\mathbb{Z}}(x_{n+1} - x_n) < \infty. \tag{2.44}$$

Then the operators $-\Delta + b\sum_{n\in\mathbb{Z}} a_n\delta_{x_n}$ do not converge in the norm resolvent sense.

Proof. The hypothesis (2.44) implies that $\mathcal{P}_\mu$ is an infinitesimal small form perturbation of $\mathbb{D}$, cf. [5], and hence, in particular, $D(J^\mu) = H^1(\mathbb{R})$. In conjunction with Lemma 2.15 and the hypotheses (2.42) and (2.44) this implies that the operator J^μ is compact. In Lemma 2.3 we may replace H, H_b, G and J by $-\Delta$, $-\Delta + b\sum_{n\in\mathbb{Z}} a_n\delta_{x_n}$, $(-\Delta+1)^{-1}$ and J^μ, respectively. Then the resolvent formula (2.12) yields that $(-\Delta+1)^{-1} - (-\Delta + b\sum_{n\in\mathbb{Z}} a_n\delta_{x_n} + 1)^{-1}$ is compact, too. By Weyl's essential spectrum theorem, this implies that

$$\sigma_{\mathrm{ess}}\left(\left(-\Delta + b\sum_{n\in\mathbb{Z}} a_n\delta_{x_n} + 1\right)^{-1}\right) = \sigma_{\mathrm{ess}}((-\Delta+1)^{-1}) = [0,1]. \tag{2.45}$$

Moreover,

$$-\Delta_D^\Gamma \geq \frac{\pi^2}{D^2}$$

and hence

$$\sup \sigma((-\Delta_D^\Gamma + 1)^{-1})) \leq \frac{1}{1 + \pi^2/D^2}. \tag{2.46}$$

If the operators $-\Delta + b\sum_{n\in\mathbb{Z}} a_n\delta_{x_n}$ converged in the norm resolvent sense to the Dirichlet Laplacian $-\Delta_D^\Gamma$, then, by (2.45), we would have $\sigma(-\Delta_D^\Gamma + 1)^{-1}) \supset [0,1]$, which contradicts (2.46). Thus the operators $-\Delta + b\sum_{n\in\mathbb{Z}} a_n\delta_{x_n}$ do not converge in the norm resolvent sense. □

In Example 2.17 the operators $(-\Delta + b\mu + 1)^{-1}$ do not converge locally uniformly. In this example μ is a so-called δ-potential and, in particular, singular. In the regular case we can also have absence of convergence with respect to the operator norm, as it is shown by the next example. That the operators $(-\Delta +$

$bV+1)^{-1}$ in the next example do not converge locally uniformly can be shown by mimicking the proof in Example 2.17.

Example 2.18. Let $(a_n)_{n\in\mathbb{Z}}$ and $(b_n)_{n\in\mathbb{Z}}$ be families of real numbers with the following properties:

$$\begin{aligned} a_n < b_n < a_{n+1} \quad \forall\, n \in \mathbb{Z}, \quad D := \sup_{n\in\mathbb{Z}}(a_{n+1}-b_n) < \infty, \\ d := \inf_{n\in\mathbb{Z}}(a_{n+1}-b_n) > 0, \qquad \lim_{|n|\to\infty}(b_n-a_n) = 0. \end{aligned} \tag{2.47}$$

Let $V := \sum_{n\in\mathbb{Z}} 1_{[a_n,b_n]}$. Then the operators $(-\Delta+bV+1)^{-1}$ converge strongly as b goes to infinity, but do not converge locally uniformly.

2.5. Convergence within a Schatten-von Neumann class

Let $p\in[1,\infty)$. Let $\mathcal{H}_i$ be Hilbert spaces with scalar products $(\cdot,\cdot)_i$, $i=0,1,2,\ldots$ Let C be a compact operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then $\mathcal{H}_2$ has an orthonormal basis $\{e_i\}_{i\in I}$ such that, with $|C| := \sqrt{CC^*}$,

$$|C|e_i = \lambda_i e_i \quad \forall i \in I$$

for some suitably chosen family $(\lambda_i)_{i\in I}$ in $[0,\infty)$ which is unique up to permutations. One sets

$$\|C\|_{S_p} := \left(\sum_{i\in I}\lambda_i^p\right)^{1/p}.$$

$S_p(\mathcal{H}_1,\mathcal{H}_2)$ (short S_p) denotes the set of compact operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that $\|C\|_{S_p}<\infty$. It is called the Schatten-von Neumann class of order p. S_p is a linear space and $\|\cdot\|_{S_p}$ a norm on it. If $C\colon \mathcal{H}_1\to\mathcal{H}_2$ belongs to the class $S_p(\mathcal{H}_1,\mathcal{H}_2)$ and $A\colon\mathcal{H}_0\to\mathcal{H}_1$ and $B\colon\mathcal{H}_2\to\mathcal{H}_3$ are linear and bounded, then $CA\in S_p(\mathcal{H}_0,\mathcal{H}_2)$ and $BC\in S_p(\mathcal{H}_1,\mathcal{H}_3)$ and

$$\|CA\|_{S_p} \le \|C\|_{S_p}\|A\|, \quad \|BC\|_{S_p} \le \|C\|_{S_p}\|B\|. \tag{2.48}$$

Moreover,

$$\|C\|_{S_p} = \|C^*\|_{S_p} = \||C|\|_{S_p} \tag{2.49}$$

for every compact operator C.

Let $B\colon\mathcal{H}_1\to\mathcal{H}_2$ be linear and bounded, Q_1 be an orthogonal projection in $\mathcal{H}_1$, and Q_2 be an orthogonal projection in $\mathcal{H}_2$ such that the dimension N of the range of Q_2 is finite. Then $|Q_2BQ_1|^2 = Q_2BQ_1B^*Q_2$ and hence $|Q_2BQ_1|$ is compact and

$$\||Q_2BQ_1|\|_{S_p} = \||Q_2BQ_1|\restriction \operatorname{ran}(Q_2)\|_{S_p}. \tag{2.50}$$

Since $|Q_2BQ_1|\restriction\operatorname{ran}(Q_2)$ belongs to the finite-dimensional space of all linear mappings from $\operatorname{ran}(Q_2)$ into itself and all norms on a finite-dimensional space are equivalent, there exists a finite constant c, depending only on p and N such that

$$\||Q_2BQ_1|\restriction\operatorname{ran}(Q_2)\|_{S_p} \le c\,\||Q_2BQ_1|\restriction\operatorname{ran}(Q_2)\| \le c\,\|B\|. \tag{2.51}$$

By (2.49) to (2.51),

$$\|Q_2 B Q_1\|_{S_p} \le c\|B\| \tag{2.52}$$

for some finite constant c, depending only on p and $N < \infty$, provided the range of Q_1 or the range of Q_2 is at most N-dimensional.

If A is a non-negative bounded self-adjoint operator and dominated by the compact self-adjoint operator B, then A and $B - A$ are also compact and it follows easily from the min-max principle for compact operators that

$$\|A\|_{S_p} \le \|B\|_{S_p} \text{ and } \|B - A\|_{S_p} \le \|B\|_{S_p}. \tag{2.53}$$

In the proof of Theorem 2.6 we have used that strong convergence of non-negative self-adjoint operators dominated by a compact self-adjoint operator implies operator-norm convergence. Similarly, strong convergence of non-negative self-adjoint operators dominated by a self-adjoint operator in S_p implies convergence in S_p:

Lemma 2.19. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative bounded self-adjoint operators in the Hilbert space $\mathcal{H}$ dominated by the non-negative bounded self-adjoint operator A. Let $1 \le p < \infty$. If $A \in S_p$ and $\lim_{n\to\infty} \|Au - A_n u\| = 0$ for all $u \in \mathcal{H}$, then*

$$\lim_{n\to\infty} \|A - A_n\|_{S_p} = 0. \tag{2.54}$$

Proof. By Lemma 2.5, $\lim_{n\to\infty} \|A_n - A\| = 0$.

A admits the representation

$$A = \sum_{i\in I} \lambda_i \,(e_i, \cdot)\, e_i$$

for some orthonormal system $(e_i)_{i\in I}$ and some family $(\lambda_i)_{i\in I}$ of non-negative real numbers satisfying

$$\sum_{i\in I} \lambda_i^p = \|A\|_{S_p}^p.$$

Let $\varepsilon > 0$. We choose a finite subset I_0 of I such that

$$\sum_{i\in I\setminus I_0} \lambda_i^p \le \varepsilon^p$$

and denote by Q the orthogonal projection onto the orthogonal complement of the finite-dimensional space spanned by $\{e_i : i \in I_0\}$. Then

$$QAQ = \sum_{i\in I\setminus I_0} \lambda_i \,(e_i, \cdot)\, e_i$$

and, in particular,

$$\|QAQ\|_{S_p}^p = \sum_{i\in I\setminus I_0} \lambda_i^p \le \varepsilon^p.$$

Since $Q(A-A_n)Q$ is dominated by QAQ, it follows that

$$\|Q(A-A_n)Q\|_{S_p} \le \varepsilon \quad \forall n \in \mathbb{N}. \tag{2.55}$$

Since the range of the orthogonal projection $1-Q$ is finite-dimensional and $\lim_{n\to\infty} \|A-A_n\| = 0$, it follows from (2.52), that

$$\begin{aligned}\lim_{n\to\infty} \|(1-Q)(A-A_n)Q\|_{S_p} &= \lim_{n\to\infty} \|(1-Q)(A-A_n)(1-Q)\|_{S_p} \\ &= \lim_{n\to\infty} \|Q(A-A_n)(1-Q)\|_{S_p} = 0.\end{aligned}$$

Since $A-A_n = Q(A-A_n)Q + (1-Q)(A-A_n)Q + Q(A-A_n)(1-Q) + (1-Q)(A-A_n)(1-Q)$, this implies in conjunction with (2.55), that

$$\limsup_{n\to\infty} \|A-A_n\|_{S_p} \le \varepsilon,$$

and the lemma is proved. □

Corollary 2.20. *Let $1 \le p < \infty$. Let $D(J) \supset D(H)$ and suppose that the operator $(H+1)^{-1} - (H_\infty+1)^{-1}$ belongs to the Schatten-von Neumann ideal of order p. Then $D_b \in S_p(\mathcal{H},\mathcal{H})$ and*

$$\|D_\infty - D_b\|_{S_p} \le \|D_\infty\|_{S_p} \text{ and } \|D_b\|_{S_p} \le \|D_\infty\|_{S_p} \tag{2.56}$$

for all $b \in (0,\infty)$. Moreover,

$$\lim_{b\to\infty} \|D_\infty - D_b\|_{S_p} = 0. \tag{2.57}$$

Proof. It holds $\lim_{b\to\infty} \|D_\infty u - D_b u\| = 0$ for all $u \in \mathcal{H}$. Hence (2.57) follows from Lemma 2.19.

By (2.16) in conjunction with (2.20), D_b is a non-negative bounded self-adjoint operator dominated by the self-adjoint operator D_∞. Hence (2.56) follows from (2.53). □

The following corollary gives a sufficient condition that the operator $D_\infty = (H+1)^{-1} - (H_\infty+1)^{-1}$ belongs to a Schatten-von Neumann ideal of finite order and gives an upper bound for the corresponding Schatten-von Neumann norm.

Corollary 2.21. *Let $D(J) \supset D(H)$ and $L(H,P) < \infty$.*

a) *Let $1 \le p < \infty$. If $JG \in S_p(\mathcal{H},\mathcal{H}_{\mathrm{aux}})$, then $D_b \in S_p(\mathcal{H},\mathcal{H})$ and*

$$\|D_\infty\|_{S_p} \le \sqrt{L(H,P)}\, \|JG\|_{S_p}. \tag{2.58}$$

b) *Let $t \in (3/2,\infty)$. If JJ^* is bounded and JG^t belongs to the Hilbert-Schmidt class $S_2(\mathcal{H},\mathcal{H}_{\mathrm{aux}})$, then*

$$\|D_\infty\|_{S_{4t-2}} \le \sqrt{L(H,P)} \left(\|JJ^*\|^{2t-2}\, \|JG^t\|_{S_2}^2\right)^{\frac{1}{4t-2}}. \tag{2.59}$$

Proof. By Theorem 2.7 and since $L(H,P) < \infty$, we have that $\operatorname{ran}(JG) \subset D(\check{H})$, $\|\check{H}JG\| = \sqrt{L(H,P)}$ and $\lim_{b\to\infty} \|D_\infty - D_b\| = 0$. By Lemma 2.4 b), this implies that

$$D_\infty = (JG)^* \check{H} JG,$$

hence (2.58) follows from (2.48) in conjunction with (2.49).

Suppose, in addition, that JJ^* is bounded. For all $h \in \mathcal{H}_{\text{aux}}$ and $f \in D(\mathcal{E})$

$$(f, (JG)^*h) = (JGf, h)_{\text{aux}} = \mathcal{E}_1(Gf, J^*h) = (f, J^*h).$$

Thus $J^*h = (JG)^*h$ for all $h \in \mathcal{H}_{\text{aux}}$. Thus $JJ^* = JG^{1/2}(JG^{1/2})^*$ and hence

$$\|JJ^*\| = \|JG^{1/2}\|^2.$$

In conjunction with the hypothesis $JG^t \in S_2$ this implies, by [6, Lemma 2], that

$$\|JG\|_{S_{4t-2}}^{4t-2} \leq \|JJ^*\|^{2t-2} \|JG^t\|_{S_2}^2,$$

hence (2.59) follows now from (2.58). □

2.6. Compact perturbations

2.6.1. Expansions. We get stronger assertions provided the operator J is compact. Let us assume that J is a compact operator from $(D(\mathcal{E}), \mathcal{E}_1)$ into $\mathcal{H}_{\text{aux}}$, that the domain of J equals $D(\mathcal{E})$, and that the range of J is dense in $\mathcal{H}_{\text{aux}}$.

Since $J\colon D(\mathcal{E}) \to \mathcal{H}_{\text{aux}}$ is compact and $G^{1/2}$ is a unitary mapping from the Hilbert space $\mathcal{H}$ onto the Hilbert space $(D(\mathcal{E}), \mathcal{E}_1)$, the operator $JG^{1/2}\colon \mathcal{H} \to \mathcal{H}_{\text{aux}}$ is also compact and there exist a family $(\lambda_k)_{k\in I}$ in $(0,\infty)$, an orthonormal system $(e_k)_{k\in I}$ in $\mathcal{H}$, and an orthonormal system $(g_k)_{k\in I}$ in $\mathcal{H}_{\text{aux}}$ with the following properties:

(i) I has only finitely many elements or $I = \mathbb{N}$ and

$$\lambda_k \to 0, \quad k \to \infty.$$

(ii)
$$JG^{1/2}f = \sum_{k\in I} \lambda_k (e_k, f) g_k \quad \forall\, f \in \mathcal{H}. \tag{2.60}$$

We shall call the latter expansion the *canonical expansion* of the operator $JG^{1/2}$ and refer the reader to [24, p. 4], for more details.

It follows that

$$(JG^{1/2})^*h = \sum_{k\in I} \lambda_k (g_k, h)_{\text{aux}}\, e_k \quad \forall\, h \in \mathcal{H}_{\text{aux}}, \tag{2.61}$$

and, in particular,

$$(JG^{1/2})^* g_k = \lambda_k e_k \quad \forall\, k \in I. \tag{2.62}$$

By (2.60) and (2.61),

$$JG^{1/2}(JG^{1/2})^*h = \sum_{k\in I} \lambda_k^2 (g_k, h)_{\text{aux}} g_k \quad \forall\, h \in \mathcal{H}_{\text{aux}}. \tag{2.63}$$

In particular,

$$JG^{1/2}(JG^{1/2})^* g_k = \lambda_k^2 g_k \quad \forall\, k \in \mathbb{N}. \tag{2.64}$$

Furthermore, $\ker((JG^{1/2})^*) = (\text{ran}(JG^{1/2}))^\perp = \{0\}$, since $\text{ran}(J)$ is dense in $\mathcal{H}_{\text{aux}}$. Thus the compact operator $JG^{1/2}(JG^{1/2})^*$ in $\mathcal{H}_{\text{aux}}$ is invertible. Therefore, (2.63) implies that $(\lambda_k^2)_{k\in I}$ is the family of eigenvalues of $JG^{1/2}(JG^{1/2})^*$ counted repeatedly according to their multiplicity, that, for any $k \in I$, the vector g_k is an eigenvector of $JG^{1/2}(JG^{1/2})^*$ corresponding to the eigenvalue λ_k^2, and that $(g_k)_{k\in I}$ is an orthonormal basis of $\mathcal{H}_{\text{aux}}$. (2.63) implies now that

$$\{1/b + JG^{1/2}(JG^{1/2})^*\}^{-1}h = \sum_{k\in I} \frac{1}{\lambda_k^2 + 1/b}\,(g_k, h)_{\text{aux}}\, g_k \quad \forall\, h \in \mathcal{H}_{\text{aux}}. \tag{2.65}$$

By (2.12), (2.60), (2.61), and (2.65),

$$D_b f := ((H+1)^{-1} - (H_b+1)^{-1})f = G^{1/2}\sum_{k\in I}\frac{\lambda_k^2}{\lambda_k^2 + 1/b}(e_k, G^{1/2}f)e_k \quad \forall\, f \in \mathcal{H}.$$

Since $G^{1/2}$ is self-adjoint and bounded, it follows that

$$\begin{aligned} D_b f &= \sum_{k\in I}\frac{\lambda_k^2}{\lambda_k^2 + 1/b}(G^{1/2}e_k, f)G^{1/2}e_k \\ &= \sum_{k\in I}\frac{\lambda_k^2}{\lambda_k^2 + 1/b}\mathcal{E}_1(G^{1/2}e_k, Gf)G^{1/2}e_k \quad \forall\, f \in \mathcal{H}. \end{aligned} \tag{2.66}$$

$(G^{1/2}e_k)_{k\in I}$ is an orthonormal system in $(D(\mathcal{E}), \mathcal{E}_1)$, since $(e_k)_{k\in I}$ is an orthonormal system in $\mathcal{H}$ and the operator $G^{1/2}$ from $\mathcal{H}$ into $(D(\mathcal{E}), \mathcal{E}_1)$ is unitary. Thus the series $\sum_{k\in I}\mathcal{E}_1(G^{1/2}e_k, Gf)G^{1/2}e_k$ converges in $(D(\mathcal{E}), \mathcal{E}_1)$ (and, therefore, also in $\mathcal{H}$),

$$\sum_{k\in I}|\mathcal{E}_1(G^{1/2}e_k, Gf)|^2 \le \mathcal{E}_1(Gf, Gf) < \infty,$$

and

$$\begin{aligned} &\mathcal{E}_1\left(\sum_{k\in I}\mathcal{E}_1(G^{1/2}e_k, Gf)G^{1/2}e_k - D_b f, \sum_{k\in I}\mathcal{E}_1(G^{1/2}e_k, Gf)G^{1/2}e_k - D_b f\right) \\ &\qquad = \sum_{k\in I}\left|\frac{1}{1+b\lambda_k^2}\right|^2 |\mathcal{E}_1(G^{1/2}e_k, Gf)|^2 \to 0, \quad b\to\infty, \end{aligned} \tag{2.67}$$

for all $f \in \mathcal{H}$. Since convergence in $(D(\mathcal{E}), \mathcal{E}_1)$ implies convergence in $\mathcal{H}$ and the operators D_b strongly converge in $\mathcal{H}$ to D_∞, (2.67) implies that

$$D_\infty f = \sum_{k\in I}\mathcal{E}_1(G^{1/2}e_k, Gf)G^{1/2}e_k = \sum_{k\in I}(G^{1/2}e_k, f)G^{1/2}e_k \quad \forall\, f \in \mathcal{H}. \tag{2.68}$$

Thus we have proved the following theorem.

Theorem 2.22. *Suppose that $D(J) = D(\mathcal{E})$ and that J is compact. Then, with $(\lambda_k)_{k\in I}$ and $(e_k)_{k\in I}$ as in the canonical expansion of $JG^{1/2}$,*

$$((H+1)^{-1} - (H_b+1)^{-1})f = \sum_{k\in I} \frac{\lambda_k^2}{\lambda_k^2 + 1/b} (G^{1/2}e_k, f)\, G^{1/2}e_k \quad \forall\, f \in \mathcal{H}, \tag{2.69}$$

$$((H+1)^{-1} - (H_\infty+1)^{-1})f = \sum_{k\in I} (G^{1/2}e_k, f)\, G^{1/2}e_k \quad \forall\, f \in \mathcal{H}, \tag{2.70}$$

$$\|(H_b+1)^{-1} - (H_\infty+1)^{-1}\| = \sup_{\|f\|=1} \sum_{k\in I} \frac{1}{1+b\lambda_k^2} \,|\, (G^{1/2}e_k, f)\,|^2. \tag{2.71}$$

Remark 2.23. The technique of regularizing the singular problem through the use of the canonical expansion is also typical for the theory of generalized pseudo inverses like presented in [21]. In this context the large coupling limits are sometimes called the limits of the large penalty. They are used in numerical analysis to regularize the 'jumping coefficients' differential equations by penalization. A good survey on regularization can be found in [18] and its use in the theory of saddle-point problems can be found in [19].

In Sections 2.3 and 2.5 the operator $\check{H} = (JJ^*)^{-1}$ has played an important role, but did occur neither in the discussion of Schrödinger operators nor in this section. Actually $\check{H}$ is useful in these contexts, too. To begin with let us mention that we can express the singular values λ_k with the aid of $\check{H}$. By (2.14), $JJ^* = J(JG)^* = JG^{1/2}(JG^{1/2})^*$. Thus the orthonormal basis $(g_k)_{k\in I}$ of $\mathcal{H}_{\rm aux}$ is contained in the domain of $\check{H}$ and

$$\check{H} g_k = \frac{1}{\lambda_k^2}\, g_k \quad \forall\, k \in I. \tag{2.72}$$

In addition, we have, by (2.62), that

$$(JG)^* g_k = G^{1/2}(JG^{1/2})^* g_k = \lambda_k G^{1/2} e_k \quad \forall\, k \in I. \tag{2.73}$$

In many applications, one can use this formula in order to describe the vectors e_k with the aid of the eigenvectors g_k of $\check{H}$. We demonstrate this in a simple case:

Let $\mathcal{E} = \mathbb{D}$ be the classical Dirichlet form in $L^2(\mathbb{R})$ and μ be a positive Radon measure on $\mathbb{R}$ such that $\mathrm{supp}(\mu) = [0,1]$. The operator $G := (-\Delta+1)^{-1}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an integral operator with kernel $g(x-y)$, where $g(x) := \frac{1}{2}\exp(-|x|)$ for all $x \in \mathbb{R}$. Since the function $\int g(\cdot - y) f(y) dy$ is continuous for all $f \in L^2(\mathbb{R})$, the mapping $J^\mu G : L^2(\mathbb{R}) \to L^2(\mathbb{R},\mu)$ is also an integral operator with the same kernel $g(x-y)$. Thus $(J^\mu G)^* : L^2(\mathbb{R},\mu) \to L^2(\mathbb{R})$ is an integral operator with kernel $\overline{g(y-x)} = g(x-y)$. Since the function $\int g(\cdot - y) h(y) \mu(dy)$ is continuous for all $h \in L^2(\mathbb{R},\mu)$, we finally obtain that also $J^\mu (J^\mu G)^* = J^\mu J^{\mu *} : L^2(\mathbb{R},\mu) \to L^2(\mathbb{R},\mu)$ is an integral operator with kernel $g(x-y)$.

By Lemma 2.15, $J^\mu\colon H^1(\mathbb{R}) \to L^2(\mathbb{R},\mu)$ is compact. Thus we can choose an orthonormal system $(e_k)_{k\in\mathbb{N}}$ in $L^2(\mathbb{R})$, an orthonormal basis $(g_k)_{k\in\mathbb{N}}$ of $L^2(\mathbb{R},\mu)$, and a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of strictly positive real numbers such that

$$J^\mu G^{1/2} = \sum_{k=1}^{\infty} \lambda_k (e_k, \cdot) g_k.$$

Of course, the λ_k, e_k and g_k depend on μ, but we suppress this dependence in our notation.

Let $k \in \mathbb{N}$. The function $u_k := \int g(\cdot - y) g_k(y)\mu(dy)$ is continuous and square integrable, and, for $\operatorname{supp}(\mu) = [0,1]$, satisfies the differential equation $-y'' + y = 0$ on $\mathbb{R} \setminus [0,1]$. Thus

$$u_k(x) = \begin{cases} u_k(0)e^x, & x \le 0, \\ u_k(1)e^{1-x}, & x \ge 1. \end{cases}$$

Since u_k is the continuous representative of $\lambda_k G^{1/2} e_k = (J^\mu G)^* g_k$ and $J^\mu (J^\mu G)^* g_k = \lambda_k^2 g_k$ it follows, for the continuous representative $G^{1/2}e_k$ of $G^{1/2}e_k$, that

$$G^{1/2} e_k(x) = \lambda_k \begin{cases} g_k(0)e^x, & x \le 0, \\ g_k(x), & 0 < x < 1, \\ g_k(1)e^{1-x}, & x \ge 1. \end{cases} \tag{2.74}$$

Set

$$\alpha_k(f) := \left| \int_{-\infty}^{0} g_k(0)e^x f(x)dx + \int_0^1 g_k(x) f(x)dx + \int_1^\infty g_k(1)e^{1-x} f(x)dx \right|^2. \tag{2.75}$$

By (2.71) and (2.74), we can express the distances between the operators $(-\Delta + b\mu + 1)^{-1}$ and their limit with the aid of the self-adjoint operator $-\check{\Delta}^\mu = (J^\mu J^{\mu *})^{-1}$ in $L^2(\mathbb{R}, \mu)$. Let $b \in (0, \infty)$. Then

$$\|(-\Delta + b\mu + 1)^{-1} - (-\Delta + \infty\mu + 1)^{-1}\| = \sup_{\|f\|=1} \sum_{k=1}^{\infty} \frac{\alpha_k(f)}{E_k + b}, \tag{2.76}$$

where $-\check{\Delta}^\mu g_k = E_k g_k$ for all $k \in \mathbb{N}$, $(g_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}, \mu)$.

2.6.2. Schatten-von Neumann classes. We can use Theorem 2.22 in order to derive estimates for the rate of convergence with respect to S_p-norms.

Lemma 2.24. *Suppose that $D(J) = D(\mathcal{E})$ and J is compact. Let $1 \le p < \infty$. Then with λ_k and e_k as in the canonical expansion of $JG^{1/2}$ the following holds.*

a) *The operator $D_\infty = (H+1)^{-1} - (H_\infty + 1)^{-1}$ belongs to the Schatten-von Neumann class of order p if and only if*

$$\sum_{k\in I} \|D_\infty^{\frac{p-1}{2}} G^{1/2} e_k\|^2 < \infty. \tag{2.77}$$

If this is the case, then

$$\|D_\infty\|_{S_p}^p = \sum_{k\in I} \|D_\infty^{\frac{p-1}{2}} G^{1/2} e_k\|^2. \tag{2.78}$$

b) *Let* $0 < b < \infty$. *The operator* $D_\infty - D_b = (H_b + 1)^{-1} - (H_\infty + 1)^{-1}$ *belongs to the Schatten-von Neumann class of order* p *if and only if*

$$\sum_{k\in I} \frac{1}{1 + b\lambda_k^2} \|(D_\infty - D_b)^{\frac{p-1}{2}} G^{1/2} e_k\|^2 < \infty. \tag{2.79}$$

If this is the case, then

$$\|D_\infty - D_b\|_{S_p}^p = \sum_{k\in I} \frac{1}{1 + b\lambda_k^2} \|(D_\infty - D_b)^{\frac{p-1}{2}} G^{1/2} e_k\|^2. \tag{2.80}$$

Proof. a) Let $(f_j)_{j\in I'}$ be an orthonormal basis for $\mathcal{H}$. Since D_∞ is a non-negative self-adjoint operator, we obtain

$$\begin{aligned}
\|D_\infty\|_{S_p}^p &= \operatorname{tr}(D_\infty^p) = \sum_{j\in I'} (D_\infty^p f_j, f_j) = \sum_{j\in I'} (D_\infty D_\infty^{\frac{p-1}{2}} f_j, D_\infty^{\frac{p-1}{2}} f_j) \\
&= \sum_{j\in I', k\in I} |(G^{1/2} e_k, D_\infty^{\frac{p-1}{2}} f_j)|^2 = \sum_{k\in I} \|D_\infty^{\frac{p-1}{2}} G^{1/2} e_k\|^2. \qquad (2.81)
\end{aligned}$$

b) The proof of b) is quite similar, so we omit it. □

Theorem 2.25. *Let* $p \in \{1, 2\}$. *Suppose that* $JG^{1/2}$ *is compact. Then the following two assertions are equivalent:*

a) $\|(H_b + 1) - (H_\infty + 1)^{-1}\|_{S_p} \to 0$ *as* $b \to \infty$.
b) $(H + 1)^{-1} - (H_\infty + 1)^{-1}$ *belongs to* $S_p(\mathcal{H}, \mathcal{H})$.

Proof. It is always true that $\|(H_b + 1) - (H_\infty + 1)^{-1}\|_{S_p} \to 0$ as $b \to \infty$ if $D_\infty = (H + 1)^{-1} - (H_\infty + 1)^{-1}$ belongs to $S_p(\mathcal{H}, \mathcal{H})$, cf. Corollary 2.20.

Conversely, let first $p = 2$ and assume that

$$\lim_{b\to\infty} \|(H_b + 1) - (H_\infty + 1)^{-1}\|_{S_2} = 0. \tag{2.82}$$

Then, by Lemma 2.24,

$$\begin{aligned}
\|D_\infty - D_b\|_{S_2}^2 &= \sum_{k\in I} \frac{1}{1 + b\lambda_k^2} \|(D_\infty - D_b)^{1/2} G^{1/2} e_k\|^2 \\
&= \sum_{k\in I} \frac{1}{1 + b\lambda_k^2} ((D_\infty - D_b) G^{1/2} e_k, G^{1/2} e_k) \\
&= \sum_{k\in I} \frac{1}{1 + b\lambda_k^2} \sum_{j\in I} \frac{1}{1 + b\lambda_j^2} |(G^{1/2} e_j, G^{1/2} e_k)|^2. \qquad (2.83)
\end{aligned}$$

Similarly, we obtain

$$\sum_{k\in I} \|D_\infty^{\frac{1}{2}} G^{1/2} e_k\|^2 = \sum_{j,k\in I} |(G^{1/2} e_j, G^{1/2} e_k)|^2. \tag{2.84}$$

By (2.82) in conjunction with (2.83), we get for sufficiently large b that

$$1 \geq \|D_\infty - D_b\|_{S_2}^2 = \sum_{j,k\in I} \frac{1}{1+b\lambda_k^2}\,\frac{1}{1+b\lambda_j^2}\,|(G^{1/2}e_j, G^{1/2}e_k)|^2$$

$$\geq \frac{1}{1+b^2} \sum_{\lambda_j,\lambda_k<1} |(G^{1/2}e_j, G^{1/2}e_k)|^2 \tag{2.85}$$

and hence

$$\sum_{k\in I} \|D_\infty^{\frac{1}{2}} G^{1/2} e_k\|^2 = \sum_{j,k\in I} |(G^{1/2}e_j, G^{1/2}e_k)|^2$$

$$\leq (1+b)^2 + \sum_{\lambda_k\geq 1}\sum_{j\in I} |(G^{1/2}e_j, G^{1/2}e_k)|^2 + \sum_{\lambda_k<1}\sum_{\lambda_j\geq 1} |(G^{1/2}e_j, G^{1/2}e_k)|^2$$

$$\leq (1+b)^2 + 2\sum_{\lambda_k\geq 1} \|Ge_k\|^2 < \infty. \tag{2.86}$$

Thus, by Lemma 2.24 a), the proof is complete for the case $p = 2$. The case $p = 1$ can be treated in a similar way. □

As in the previous subsection we can express the distances between the operators $(-\Delta + b\mu + 1)^{-1}$ and their limit with the aid of the operator $-\check{\Delta}^\mu$.

Lemma 2.26. *Let μ be a positive Radon measure on $\mathbb{R}$ and suppose that* $\mathrm{supp}(\mu) = [0,1]$. *Let (g_k) be an orthonormal basis of $L^2(\mathbb{R},\mu)$ such that, with the operator $-\check{\Delta}^\mu = (J^\mu J^{\mu *})^{-1}$, the following holds:*

$$-\check{\Delta}^\mu g_k = E_k g_k \quad \forall\, k \in \mathbb{N}.$$

Then

$$\|(-\Delta + b\mu + 1)^{-1} - (-\Delta + \infty\mu + 1)^{-1}\|_{S_1} = \sum_{k=1}^{\infty} \frac{\beta_k}{E_k + b} \quad \forall\, b > 0, \tag{2.87}$$

where

$$\beta_k = \frac{1}{2}|g_k(0)|^2 + \frac{1}{2}|g_k(1)|^2 + \int_0^1 |g_k(x)|^2 dx \quad \forall\, k \in \mathbb{N}. \tag{2.88}$$

Proof. Since $E_k = 1/\lambda_k^2$ for every $k \in \mathbb{N}$, the lemma follows from (2.80) in conjunction with (2.74). □

2.7. Dynkin's formula

We can use (2.70) in order to derive an abstract version of the celebrated Dynkin's formula.

To begin with let us assume that $D(J) = D(\mathcal{E})$ and J is compact. Choose an orthonormal system $(e_k)_{k\in I}$ in $\mathcal{H}$, an orthonormal basis $(g_k)_{k\in I}$ in $\mathcal{H}_{\mathrm{aux}}$, and a family $(\lambda_k)_{k\in I}$ of non-negative real numbers as in (2.60), i.e., such that $JG^{1/2}f = \sum_{k\in I} \lambda_k (e_k, f) g_k$ for all $f \in \mathcal{H}$. Then $JG^{1/2}f = 0$ if and only if $(e_k, f) = 0$ for all $k \in I$.

$G^{1/2}$ is a unitary operator from $\mathcal{H}$ to $(D(\mathcal{E}), \mathcal{E}_1)$. Thus $(G^{1/2}e_k)_{k\in I}$ is an orthonormal system in the Hilbert space $(D(\mathcal{E}), \mathcal{E}_1)$. Moreover, $(e_k, f) = 0$ for all $k \in I$ if and only if $\mathcal{E}_1(G^{1/2}e_k, G^{1/2}f) = 0$ for all $k \in I$. Thus $(G^{1/2}e_k)_{k\in I}$ is an orthonormal basis of $\ker(J)^{\perp}$; here $\perp$ means orthogonal with respect to the scalar product $\mathcal{E}_1$ on $D(\mathcal{E})$ and "orthonormal" means "orthonormal with respect to $\mathcal{E}_1$". Thus the first equality in (2.68) yields that

$$D_\infty f = P_J G f \quad \forall\, f \in \mathcal{H}, \tag{2.89}$$

where P_J denotes the orthogonal projection in $(D(\mathcal{E}), \mathcal{E}_1)$ onto $\ker(J)^{\perp}$.

(2.89) holds true under much weaker assumptions on the operator J. It is easy to understand this fact: Let J_1 and J_2 be densely defined closed operators from $(D(\mathcal{E}), \mathcal{E}_1)$ to $\mathcal{H}_{\mathrm{aux}}$. For $i = 1, 2$ denote by $H_b^{J_i}$ the self-adjoint operator in $\mathcal{H}$ associated to $\mathcal{E}^{bJ_i}$ and put

$$D_\infty^{J_i} := (H+1)^{-1} - \lim_{b\to\infty} (H_b^{J_i} + 1)^{-1}.$$

By Kato's monotone convergence theorem,

$$\lim_{b\to\infty} (H_b^{J_1} + 1)^{-1} = \lim_{b\to\infty} (H_b^{J_2} + 1)^{-1}$$

provided $\ker(J_1) = \ker(J_2)$, cf. (2.10). Trivially, we also have $P_{J_1} = P_{J_2}$ in this case and (2.89) holds true for J_1 if and only if it holds true for J_2. Thus in order to prove (2.89) for a given operator J_1 we only have to choose a compact operator J_2 such that $\ker(J_2) = \ker(J_1)$ and $\mathrm{ran}(J_2)$ is dense in $\mathcal{H}_{\mathrm{aux}}$. Hence the next theorem follows from Lemma 2.29 below.

Theorem 2.27. *Suppose that $D(J)$ is dense in the Hilbert space $(D(\mathcal{E}), \mathcal{E}_1)$ and the auxiliary Hilbert space $\mathcal{H}_{\mathrm{aux}}$ is separable. Let P_J be the orthogonal projection in the Hilbert space $(D(\mathcal{E}), \mathcal{E}_1)$ onto the kernel* $\ker J$ *of J. Then the following* abstract Dynkin's formula *holds true*

$$(H+1)^{-1} - (H_\infty + 1)^{-1} = P_J G. \tag{2.90}$$

Remark 2.28. Since we choose $\mathcal{H}_{\mathrm{aux}}$ in such a way that $\mathrm{ran}(J)$ is dense in $\mathcal{H}_{\mathrm{aux}}$, the hypothesis that $\mathcal{H}_{\mathrm{aux}}$ be separable is, in particular, satisfied in the case when $D(J) = D(\mathcal{E})$ and J is compact.

Lemma 2.29. *Let J be a densely defined closed operator from the Hilbert space $(\mathcal{H}_1, (\cdot,\cdot)_1)$ into the separable Hilbert space $(\mathcal{H}_2, (\cdot,\cdot)_2))$. Suppose that* $\mathrm{ran}(J)$ *is dense in $\mathcal{H}_2$. Then there exists a compact operator J_2 from $\mathcal{H}_1$ into $\mathcal{H}_2$ such that $D(J_2) = \mathcal{H}_1$, the range of J_2 is dense in $\mathcal{H}_2$, and*

$$\ker(J_2) = \ker(J).$$

Proof. J^* is a closed operator from the separable Hilbert space $\mathcal{H}_2$ to the Hilbert space $\mathcal{H}_1$. Hence the Hilbert space $(D(J^*), (\cdot,\cdot)_{J^*})$ is separable, where $(u, v)_{J^*} := (u, v)_2 + (J^*u, J^*v)_1$.

Since $(D(J^*), (\cdot,\cdot)_{J^*})$ is separable, we can choose a sequence $(f_n)_{n\in\mathbb{N}}$ such that the set $\{f_n : n \in \mathbb{N}\}$ is dense in $(D(J^*), (\cdot,\cdot)_{J^*})$. Selecting a linearly independent subsequence $(g_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ and applying Gram-Schmidt orthogonalization, we get an orthonormal system $(e_n)_{n\in\mathbb{N}}$ in $\mathcal{H}_2$ with

$$\operatorname{span}\{e_n \colon n \in \mathbb{N}\} = \operatorname{span}\{g_n : n \in \mathbb{N}\}$$

and $\operatorname{span}\{e_n \colon n \in \mathbb{N}\}$ is dense in $(D(J^*), (\cdot,\cdot)_{J^*})$.

$D(J^*)$ is dense in $\mathcal{H}_2$, since J is closed. Thus $\operatorname{span}\{e_n \colon n \in \mathbb{N}\}$ is also dense in $\mathcal{H}_2$ and hence an orthonormal basis of $\mathcal{H}_2$. With this basis, we are able to define the compact operator J_2.

Set

$$\lambda_k := 2^{-k}\,\frac{1}{1+\|J^*e_k\|_1} \quad \forall\, k \in \mathbb{N}.$$

Define an operator J_0 by $D(J_0) = D(J)$ and

$$J_0 f := \sum_{k=1}^{\infty} \lambda_k\,(e_k, Jf)_2\, e_k \quad \forall\, f \in D(J_0).$$

J_0 is a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ and densely defined. Hence its closure J_2 is a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ and $D(J_2) = \mathcal{H}_2$.

J_2 is a Hilbert-Schmidt operator. To show that take an orthonormal basis $(h_j)_{j\in I}$ of $\mathcal{H}_1$ such that $h_j \in D(J)$ for every $j \in I$. Then

$$\begin{aligned}\sum_{j\in I}\|J_2 h_j\|_2^2 &= \sum_{j\in I}\left\|\sum_{k\in\mathbb{N}}\lambda_k (e_k, Jh_j)_2 e_k\right\|_2^2 \\ &= \sum_{k\in\mathbb{N}}\lambda_k^2\sum_{j\in I}|(J^*e_k, h_j)_1|^2 = \sum_{k\in\mathbb{N}}\lambda_k^2\|J^*e_k\|_1^2 < \infty.\end{aligned}$$

Next we show that $\ker(J) = \ker(J_2)$. If $Jf = 0$, then $J_0 f = J_2 f = 0$ and we obtain $\ker(J) \subset \ker(J_2)$. On the other hand, J is densely defined and closed. Hence $\ker(J) = \operatorname{ran}(J^*)^\perp$. Take an $f \in \ker(J_2)$. Then there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $D(J_0)$ such that $f = \lim_{n\to\infty} f_n$ and $J_2 f = \lim_{n\to\infty} J_0 f_n$. Let $(e_k)_{k\in\mathbb{N}}$ be the orthonormal basis in $\mathcal{H}_2$ introduced above. Then

$$\begin{aligned}0 = (J_2 f, e_k)_2 &= \lim_{n\to\infty}(J_0 f_n, e_k)_2 \\ &= \lim_{n\to\infty}\left(\sum_{m\in\mathbb{N}}\lambda_m (e_m, Jf_n)_2 (e_m, e_k)_2\right) \\ &= \lim_{n\to\infty}\lambda_k (e_k, Jf_n)_2 = \lambda_k (J^*e_k, f)_1.\end{aligned}$$

Therefore, f is orthogonal to J^*e_k for all $k \in \mathbb{N}$. Since $\operatorname{span}\{e_k \colon k \in \mathbb{N}\}$ is dense in $(D(J^*), (\cdot,\cdot)_{J^*})$, its image $\operatorname{span}\{J^*e_k \colon k \in \mathbb{N}\}$ is dense in $\operatorname{ran}(J^*)$. Thus $f \in \operatorname{ran}(J^*)^\perp = \ker(J)$.

It remains to prove that $\operatorname{ran}(J_2)$ is dense in $\mathcal{H}_2$. Fix $k_0 \in \mathbb{N}$ and $\varepsilon > 0$. Since, by hypothesis, $\operatorname{ran}(J)$ is dense in $\mathcal{H}_2$, we can choose $f \in D(J)$ satisfying

$$\left\| Jf - \frac{e_{k_0}}{\lambda_{k_0}} \right\| < \varepsilon.$$

Thus $\|J_2 f - e_{k_0}\| < \varepsilon$, because of

$$\begin{aligned}
\|J_2 f - e_{k_0}\|_2^2 &= \left\| \sum_{k\in\mathbb{N}} \lambda_k (e_k, Jf)_2 e_k - e_{k_0} \right\| \\
&= \sum_{k\in\mathbb{N}, k\neq k_0} \lambda_k^2 \left|(e_k, Jf)_2\right|^2 + \lambda_{k_0}^2 \left|(e_{k_0}, Jf)_2 - \frac{1}{\lambda_{k_0}}\right|^2 \\
&\leq \sum_{k\in\mathbb{N}, k\neq k_0} \left|(e_k, Jf)_2\right|^2 + \left|(e_{k_0}, Jf)_2 - \frac{1}{\lambda_{k_0}}\right|^2 \\
&= \left\| \sum_{k\in\mathbb{N}} (e_k, Jf)_2 e_k - \frac{e_{k_0}}{\lambda_{k_0}} \right\|_2^2 = \left\| Jf - \frac{e_{k_0}}{\lambda_{k_0}} \right\|_2^2 < \varepsilon.
\end{aligned}$$

Thus $e_{k_0} \in \overline{\operatorname{ran}(J_2)}$. Since $\overline{\operatorname{span}\{e_k \colon k \in \mathbb{N}\}} = \mathcal{H}_2$, we have shown that $\operatorname{ran}(J_2)$ is dense in $\mathcal{H}_2$. □

2.8. Differences of powers of resolvents

In this section we shall use the generalized Dynkin's formula to derive the surprising result that

$$(H_b + 1)^{-k} - (H_\infty + 1)^{-k} = \left((H_b + 1)^{-1} - (H_\infty + 1)^{-1}\right)^k \quad \forall\, k \in \mathbb{N} \tag{2.91}$$

for a large class of operators H and form perturbations $\mathcal{P}$ of H. Let us recall that

$$(H_b + 1)^{-1} \to (H_\infty + 1)^{-1} \oplus 0, \quad b \to \infty,$$

for a suitably chosen non-negative self-adjoint operator H_∞ in a suitably chosen closed subspace $\mathcal{H}_\infty$ of $\mathcal{H}$ and that we abuse notation and write $(H_\infty + 1)^{-1}$ in place of $(H_\infty + 1)^{-1} \oplus 0$. Here we abuse notation again and simply write $(H_\infty + 1)^{-k}$ in place of $(H_\infty + 1)^{-k} \oplus 0$.

Before we derive formula (2.91), let us briefly mention some reasons why one might be interested in this result. Let A and A_0 be non-negative self-adjoint operators. A and A_0 may be differential operators so that passing to higher powers of the resolvents improves regularity. There are also many examples where the resolvent difference $(A + 1)^{-1} - (A_0 + 1)^{-1}$ does not belong to the trace class, but $(A + 1)^{-k} - (A_0 + 1)^{-k}$ is a trace class operator for sufficiently large k. This implies, by the Birman-Kuroda theorem, that the absolutely continuous spectral part A^{ac} of A is unitarily equivalent to A_0^{ac} and, in particular, A and A_0 have the same absolutely continuous spectrum. Estimates of the trace norm of $(A+1)^{-k} - (A_0 + 1)^{-k}$ can also be used to compare the eigenvalue distributions of A and A_0.

Lemma 2.30. *Suppose that* $D(J) \supset D(H)$ *and*

$$JGu = 0 \quad \forall\, u \in \ker(J). \tag{2.92}$$

Then the following holds:

a) $D_b(G - D_\infty) = 0$ *for all* $b > 0$.
b) $D_\infty(G - D_\infty) = 0$.

Proof. a) Let P_J be the orthogonal projection in $(D(\mathcal{E}), \mathcal{E}_1)$ onto the orthogonal complement of $\ker(J)$. Then $1 - P_J$ is the orthogonal projection onto the bi-orthogonal complement and hence onto the closure of $\ker(J)$. Since J is a closed operator, its kernel is closed and hence $1 - P_J$ is the orthogonal projection onto the kernel of J.

By the generalized Dynkin's formula, cf. Theorem 2.27,

$$D_\infty = P_J G.$$

In conjunction with the resolvent formula (2.12) and the hypothesis (2.92), this implies that

$$D_b(G - D_\infty) = (JG)^* \left(\frac{1}{b} + JJ^*\right)^{-1} JG(1 - P_J)G = 0.$$

b) Due to the fact that the operators D_b converge strongly to D_∞, b) follows from a). □

In the proof of the main theorem of this section we shall use the following telescope-sum formula which holds true for arbitrary everywhere defined operators A and B on $\mathcal{H}$.

$$A^k - B^k = \sum_{j=0}^{k-1} A^{k-1-j}\,(A - B)\,B^j. \tag{2.93}$$

If A and B are bounded self-adjoint operators and $AB = 0$, then

$$(BAu, v) = (u, ABv) = 0 \quad \forall\, u, v \in \mathcal{H}$$

and hence $BA = 0$.

Theorem 2.31. *Suppose that* $D(J) \supset D(H)$ and $\ker(J)$ *is* G*-invariant. Then*

$$(H_b + 1)^{-k} - (H_\infty + 1)^{-k} = \left((H_b + 1)^{-1} - (H_\infty + 1)^{-1}\right)^k \quad \forall\, k \in \mathbb{N}.$$

Proof. Let $k \in \mathbb{N}$. By formula (2.93) and having Lemma 2.30 in mind, we get

$$
\begin{aligned}
&(H_b+1)^{-k} - (H_\infty+1)^{-k} \\
&\qquad = \sum_{j=0}^{k-1} (H_\infty+1)^{-k-1-j} \left((H_\infty+1)^{-1} - (H_b+1)^{-1}\right) (H_b+1)^{-j} \\
&\qquad = \sum_{j=0}^{k-1} (G-D_\infty)^{k-1-j} \, (D_\infty - D_b) \, ((G-D_\infty) + (D_\infty - D_b))^j \\
&\qquad = \sum_{j=0}^{k-1} (G-D_\infty)^{k-1-j} \, (D_\infty - D_b)^{j+1} \\
&\qquad = (D_\infty - D_b)^k + \sum_{j=1}^{k-1} (G-D_\infty)^{k-j} \, (D_\infty - D_b)^j .
\end{aligned}
$$

Now observing that, by Lemma 2.30, we have, for all $f \in \mathcal{H}$,

$$\left(\sum_{j=1}^{k-1} (G-D_\infty)^{k-j} \, (D_\infty - D_b)^j f, f\right) = (f, (D_\infty - D_b)^j (G-D_\infty)^{k-j} f) = 0,$$

we get the result. $\square$

Corollary 2.32. *Under the hypotheses of Theorem* 2.31*, the following holds:*

$$\|(H_b+1)^{-k} - (H_\infty+1)^{-k}\| = \|(H_b+1)^{-1} - (H_\infty+1)^{-1}\|^k \quad \forall\, k \in \mathbb{N}. \tag{2.94}$$

In particular, there exists a $c > 0$ *such that*

$$
\begin{aligned}
&\liminf_{b\to\infty} b^k \|(H_b+1)^{-k} - (H_\infty+1)^{-k}\| \\
&\qquad = \limsup_{b\to\infty} b^k \|(H_b+1)^{-k} - (H_\infty+1)^{-k}\| = c^k > 0 \quad \forall k \in \mathbb{N},
\end{aligned} \tag{2.95}
$$

and, for any $k \in \mathbb{N}$*, we have the following equivalence:*

$$\lim_{b\to\infty} b^k \|(H_b+1)^{-k} - (H_\infty+1)^{-k}\| < \infty \iff J(D(H)) \subset D(\check{H}). \tag{2.96}$$

Proof. By (2.16) in conjunction with (2.20), the operator $D_\infty - D_b$ is non-negative, bounded, and self-adjoint. By the spectral calculus and Theorem 2.31, this implies formula (2.94). The assertions (2.95) and (2.96), respectively, follow from (2.94) in conjunction with Theorem 2.7. $\square$

We conclude this section with an example which shows that the condition (2.92) is not "artificial" at all.

Example 2.33. Let D be the open unit disc in $\mathbb{R}^2$ and T the unit circle. We consider the form in $L^2(T) = L^2(T, d\theta)$ defined by

$$\dot{\mathcal{F}}(f,f) := \frac{1}{16\pi}\int_0^{2\pi}\int_0^{2\pi} |f(\theta) - f(\theta')|^2 \sin^{-2}\left(\frac{\theta-\theta'}{2}\right) d\theta d\theta',$$
$$D(\dot{\mathcal{F}}) := \{f \in L^2(T) \colon \mathcal{F}(f,f) < \infty\}. \tag{2.97}$$

We define the form $\dot{\mathcal{E}}$ in $L^2(D)$ as follows:

$$\dot{\mathcal{E}}(f,f) := \frac{1}{2}\int_D |\nabla f|^2 dx,$$
$$D(\dot{\mathcal{E}}) := \{f \in L^2(D) \colon f \text{ is harmonic}, \mathcal{E}(f,f) < \infty\}. \tag{2.98}$$

We take

$$\dot{J} \colon (D(\dot{\mathcal{E}}), \dot{\mathcal{E}}) \to (D(\dot{\mathcal{F}}), \dot{\mathcal{F}}), \quad \dot{J}f := f \restriction T \quad \forall\, f \in D(\dot{\mathcal{E}}),$$

where $f \restriction T$ is the operation of taking the boundary limit of f. It is known, cf. [13, p. 12], that $\dot{J}$ is unitary and it preserves the subspace of constant functions. We define an equivalence relation on both $L^2(D)$ and $L^2(T)$ by $f \sim g \Leftrightarrow f - g$ is a constant function. Accordingly we define the forms

$$\mathcal{F}([f],[f]) := \dot{\mathcal{F}}(f,f),\ D(\mathcal{F}) = \big(D(\dot{\mathcal{F}})\big)/\!\sim, \tag{2.99}$$
$$\mathcal{E}([f],[f]) = \dot{\mathcal{E}}(f,f),\ D(\mathcal{E}) = \big(D(\dot{\mathcal{E}})\big)/\!\sim, \tag{2.100}$$

and

$$J \colon (D(\mathcal{E}), \mathcal{E}) \to (D(\mathcal{F}), \mathcal{F}), \quad J[f] := \dot{J}f \quad \forall\, [f] \in D(\mathcal{E}).$$

Then both $\mathcal{F}, \mathcal{E}$ and J are well defined and it is well known that $(D(\mathcal{E}), \mathcal{E})$ is a Hilbert space (which we take to be $\mathcal{H}$). Furthermore since $\dot{J}$ is unitary we conclude that J is unitary as well. Thus $\ker(J) = \{0\}$ and trivially the assumption (2.92) is satisfied. Since $\ker(J) = \{0\}$, also $\mathcal{H}_\infty = \{0\}$, cf. (2.8), and hence $(H_\infty + 1)^{-1} = 0$ and $D_\infty = G$. Since J_0 is unitary, $JJ^* = 1$ and, in particular, $\operatorname{ran}(JJ^*) = D(\mathcal{F})$. J is not unitary as an operator from $(D(\mathcal{E}), \mathcal{E}_1)$ onto $(D(\mathcal{F}), \mathcal{F})$, but the norms induced by $\mathcal{E}$ and $\mathcal{E}_1$ are equivalent and hence we still have $\operatorname{ran}(JJ^*) = D(\mathcal{F})$. Thus, by formula (2.96), there exists a constant $c \in (0, \infty)$ such that

$$\lim_{b\to\infty} b^k \|(H_b + 1)^{-k}\| = c^k$$

for all $k \in \mathbb{N}$.

It is also known that $\mathcal{E}$ and $\mathcal{F}$ in the previous example are Dirichlet forms and the perturbation corresponding to J is a so-called jumping term and, in particular, non-local, cf. [13, p. 12]. Moreover, obviously the operator J is not compact. In the next section we shall concentrate on Dirichlet forms and treat certain local perturbations, the so-called killing terms.

3. Dirichlet forms

We can combine our general methods with tools from the theory of Dirichlet forms in order to improve our results in the special, but very important case when $H_b = H + b\mu$ for some Dirichlet operator H and some killing measure μ. It is also possible to treat other kinds of perturbations, for instance, perturbations by jumping terms, as it was demonstrated in Example 2.33.

3.1. Notation and basic results

Throughout this section, X denotes a locally compact separable metric space, m a positive Radon measure on X such that $\operatorname{supp}(m) = X$ and $\mathcal{E}$ a (symmetric) Dirichlet form in $L^2(X, m)$, i.e., a densely defined closed form in $L^2(X, m)$ satisfying

$$\bar{f} \in D(\mathcal{E}) \quad \forall f \in D(\mathcal{E}), \tag{3.1}$$

(this condition is void in the real case) and possessing the contraction property

$$f^c \in D(\mathcal{E}) \text{ and } \mathcal{E}(f^c, f^c) \le \mathcal{E}(f, f) \tag{3.2}$$

for all real-valued $f \in D(\mathcal{E})$, where $f^c := \min(1, f^+)$ and $f^+ := \max(0, f)$. In addition, we require the Dirichlet form be regular, i.e., the following two conditions are satisfied:

a) The set of all f in the space $C_0(X)$ of continuous functions with compact support such that f is a representative of an element of $D(\mathcal{E})$ is dense in $(C_0(X), \|\cdot\|_\infty)$. We shall denote this set by $C_0(X) \cap D(\mathcal{E})$.
b) The set of all f in $D(\mathcal{E})$ with a continuous representative with compact support is dense in $(D(\mathcal{E}), \mathcal{E}_1)$. We shall denote this set by $C_0(X) \cap D(\mathcal{E})$, too.

The capacity (with respect to $\mathcal{E}$) of an open subset U of X and an arbitrary subset B of X is defined as follows:

$$\begin{aligned} \operatorname{cap}(U) &:= \inf\{\mathcal{E}_1(u, u) \colon u \ge 1\, m\text{-a.e. on U}\}, \\ \operatorname{cap}(B) &:= \inf\{\operatorname{cap}(U) \colon U \supset B,\ U \text{ is open}\}, \end{aligned} \tag{3.3}$$

respectively. The classical Dirichlet form $\mathbb{D}$, defined by (2.34), is a regular Dirichlet form in $L^2(\mathbb{R}^d)$ and the definition of capacity in Section 2.4 is equivalent to the definition of capacity for $\mathbb{D}$ in (3.3). As in the classical case, a function $u : X \to \mathbb{C}$ is called quasi-continuous (with respect to $\mathcal{E}$) if and only if for every $\varepsilon > 0$ there exists an open set U_ε such that $u \restriction X \backslash U_\varepsilon$ is continuous and $\operatorname{cap}(U_\varepsilon) < \varepsilon$. Moreover, as in the classical case, every $u \in D(\mathcal{E})$ has a quasi-continuous representative, two quasi-continuous representatives are equal q.e., i.e., everywhere up to a set with capacity zero, and every $\mathcal{E}_1$-convergent sequence has a subsequence converging q.e. For $u \in D(\mathcal{E})$ we denote by u also any quasi-continuous representative of u. We shall denote by H the non-negative self-adjoint operator associated to $\mathcal{E}$.

Remark 3.1. There exists a Markov process $\mathbb{M}$ such that $p_t(\cdot, B)$ is a quasi-continuous representative of $e^{-tH}1_B$ for every Borel set $B \in \mathcal{B}(X)$ with $m(B) < \infty$ and all $t > 0$. Here $p_t(x, B)$ is the transition function of $\mathbb{M}$ and $\mathbb{M}$ is even an m-symmetric Hunt process with state space $X \cup \{\Delta\}$, where Δ is added as an isolated point if X is compact and $X \cup \{\Delta\}$ is the one-point compactification of X otherwise. If $\mathcal{E} = \dfrac{1}{2}\mathbb{D}$, then the corresponding Markov process $\mathbb{M}$ is the standard Brownian motion.

In the following, let μ be a positive Radon measure on X charging no set with capacity zero. As in the classical case, we set

$$D(\mathcal{P}_\mu) := D(\mathcal{E}) \cap L^2(X, \mu), \tag{3.4}$$

$$\mathcal{P}_\mu(u, v) := \int \bar{u}v \, d\mu \quad \forall\, u, v \in D(\mathcal{E}) \tag{3.5}$$

and obtain that the operator J^μ from $(D(\mathcal{E}), \mathcal{E}_1)$ to $L^2(X, \mu)$, defined by

$$D(J^\mu) := D(\mathcal{P}_\mu), \quad J^\mu u := u \quad \mu\text{-a.e.} \quad \forall\, u \in D(J^\mu), \tag{3.6}$$

is closed and hence $\mathcal{E} + b\mathcal{P}_\mu$ is closed for all $b > 0$. For each $b > 0$, we set $\mathcal{E}^{b\mu} := \mathcal{E} + b\mathcal{P}_\mu$ and denote by $H + b\mu$ the non-negative self-adjoint operator associated with $\mathcal{E}^{b\mu}$. Moreover,

$$(H + \infty\mu + 1)^{-1} := \lim_{b\to\infty} (H + b\mu + 1)^{-1},$$
$$D_b^\mu := (H + 1)^{-1} - (H + b\mu + 1)^{-1} \quad \forall\, b \in [0, \infty].$$

Theorem 3.2. *$\mathcal{E}^\mu$ is a regular Dirichlet form in $L^2(X, m)$.*

$(H + 1)^{-1}$ has a Markovian kernel G, i.e., there exists a mapping

$$G : X \times \mathcal{B}(X) \to [0, 1]$$

such that $G(\cdot, B)$ is measurable for every B in the Borel-algebra $\mathcal{B}(X)$ of X, $G(x, X) \le 1$ and $G(x, \cdot)$ is a measure for every $x \in X$ and

$$x \mapsto \int f(y)G(x, dy)$$

is a quasi-continuous representative of $(H + 1)^{-1}f$ for every $f \in L^2(X, m)$. For every non-negative Borel measurable function f on X the function $Gf : X \to [0, \infty]$, $Gf(x) := \int f(y)G(x, dy)$ for $x \in X$, is well defined. G is also m-symmetric, i.e., $\int Gf\, h dm = \int f\, G h dm$ for all non-negative Borel measurable functions f and h. $Gf \ge 0$ q.e. if $f \ge 0$ m-a.e. $\mathcal{E}$, H, and G will be called conservative if $G1 = 1$ q.e. We shall abuse notation and denote not only the Markovian kernel of $(H + 1)^{-1}$, but also the operator $(H + 1)^{-1}$ by G. Moreover, we put

$$G^\mu := (H + \mu + 1)^{-1}$$

and denote by G^μ also the m-symmetric Markovian kernel of this operator.

The Dirichlet form $\mathcal{E}$ is strongly local if and only if the following implication holds for all $u, v \in D(\mathcal{E})$:

$$\begin{array}{l}\text{supp}(um) \text{ and supp}(vm) \text{ compact and } v \text{ constant in} \\ \text{a neighborhood of supp}(um) \text{ implies that } \mathcal{E}(u,v) = 0.\end{array} \tag{3.7}$$

Example 3.3. $\mathbb{D}$ is a regular conservative strongly local Dirichlet form in $L^2(\mathbb{R}^d)$.

3.2. Trace of a Dirichlet form

In the remaining part of this note we shall assume that μ is a positive Radon measure on X charging no set with capacity zero (with respect to $\mathcal{E}$) that satisfies

$$D(H) \subset D(J^\mu). \tag{3.8}$$

Recently Chen, Fukushima, and Ying [10] have obtained deep results on the trace of a Dirichlet form and the associated Markov process. It turns out that traces of Dirichlet forms are also very useful for the investigation of large coupling convergence.

Before we give the definition of the trace of a Dirichlet form, we need some preparation. We put

$$F := \text{supp}(\mu)$$

and identify $L^2(X, \mu)$ and $L^2(F, \mu)$ in the canonical way, i.e., via the unitary transformation $u \mapsto u \upharpoonright F$. We further put

$$P_\mu := P_{J^\mu},$$

i.e., P_μ is the orthogonal projection in the Hilbert space $(D(\mathcal{E}), \mathcal{E}_1)$ onto the orthogonal complement of $\ker(J^\mu)$ (with respect to the scalar product $\mathcal{E}_1$). Obviously, the following implications hold:

$$J^\mu u = J^\mu w \implies u - w \in \ker(J^\mu) \implies P_\mu u = P_\mu w.$$

Hence, the following is correctly defined:

Definition 3.4. We define the form $\check{\mathcal{E}}_1^\mu$ in $L^2(F, \mu)$ as follows:

$$\begin{aligned} D(\check{\mathcal{E}}_1^\mu) &:= \text{ran}(J^\mu), \\ \check{\mathcal{E}}_1^\mu(J^\mu u, J^\mu v) &:= \mathcal{E}_1(P_\mu u, P_\mu v) \quad \forall\, u, v \in D(\mathcal{E}). \end{aligned} \tag{3.9}$$

$\check{\mathcal{E}}_1^\mu$ is called the trace of the Dirichlet form $\mathcal{E}_1$ with respect to the measure μ.

Theorem 3.5. *$\check{\mathcal{E}}_1^\mu$ is a regular Dirichlet form in $L^2(F, \mu)$.*

The proof of this theorem can be found in [13, Chapter 6].

Remark 3.6. In the Definition 3.4 we have essentially used that the Dirichlet form $\mathcal{E}_1$ is coercive. One can define the trace $\check{\mathcal{E}}^\mu$ of an arbitrary regular Dirichlet form $\mathcal{E}$ with respect to a measure μ in such a way that for $\mathcal{E}_1$ the Definition 3.5 above is equivalent to the general one. Even in the general case $\check{\mathcal{E}}^\mu$ is a regular Dirichlet form in $L^2(F, \mu)$. We shall not use these extensions in this note and omit the details, but refer the interested reader to [13, Chapter 6.2].

The operator

$$\check{H}^{\mu} := (J^{\mu} J^{\mu *})^{-1} \tag{3.10}$$

plays an important role in the discussion of large coupling convergence. It is remarkable that $\check{H}^{\mu}$ is the self-adjoint operator associated with the Dirichlet form $\check{\mathcal{E}}_1^{\mu}$.

Lemma 3.7. *$\check{H}^{\mu}$ is the self-adjoint operator associated with $\check{\mathcal{E}}_1^{\mu}$.*

Proof. $u - P_{\mu} u \in \ker(J^{\mu})$ for every $u \in D(\mathcal{E})$. Thus

$$P_{\mu} u \in D(J^{\mu}) \text{ and } J^{\mu} P_{\mu} u = J^{\mu} u \quad \forall\, u \in D(J^{\mu}). \tag{3.11}$$

Since the operator $\check{H}^{\mu}$ is self-adjoint, we only need to prove that it is a restriction of the self-adjoint operator associated with $\check{\mathcal{E}}_1^{\mu}$. For this it suffices to show that

$$\check{\mathcal{E}}_1^{\mu}(J^{\mu} J^{\mu *} f, h) = (f, h)_{L^2(\mu)} \quad \forall\, f \in D(J^{\mu} J^{\mu *})\, \forall\, h \in D(\check{\mathcal{E}}_1^{\mu}).$$

By Theorem 3.5, it suffices to prove this equality for all $f \in D(J^{\mu} J^{\mu *})$ and all $h \in C_0(F) \cap D(\check{\mathcal{E}}_1^{\mu})$. Let now $h \in C_0(F) \cap D(\check{\mathcal{E}}_1^{\mu})$ and choose $u \in D(\mathcal{E})$ such that $h = J^{\mu} u$. Then, by (3.11), $J^{\mu} P_{\mu} u = J^{\mu} u = h$. Let $f \in D(J^{\mu} J^{\mu *})$. Then

$$\check{\mathcal{E}}_1^{\mu}(J^{\mu} J^{\mu *} f, h) = \mathcal{E}_1(J^{\mu *} f, P_{\mu} u) = (f, J^{\mu} P_{\mu} u)_{L^2(\mu)} = (f, h)_{L^2(\mu)}.$$

Thus $\check{H}^{\mu}$ is the self-adjoint operator associated with $\check{\mathcal{E}}_1^{\mu}$. □

The following example illustrates the strength of the previous lemma for the investigation of large coupling convergence.

Example 3.8 (Continuation of Example 2.16). We choose $(x_n)_{n \in \mathbb{Z}}$, $(a_n)_{n \in \mathbb{Z}}$, d, Γ, $-\Delta_D^{\Gamma}$, and μ as in the Example 2.16. Assume, in addition, that

$$m_0 := \inf_{n \in \mathbb{Z}} a_n > 0. \tag{3.12}$$

Then the operators $-\Delta + b \sum_{n \in \mathbb{Z}} a_n \delta_{x_n}$ converge in the norm resolvent sense to $-\Delta_D^{\Gamma}$ with maximal rate of convergence, i.e.,

$$\lim_{b \to \infty} b \, \|(-\Delta + b \sum_{n \in \mathbb{Z}} a_n \delta_{x_n} + 1)^{-1} - (-\Delta_D^{\Gamma} + 1)^{-1}\| < \infty. \tag{3.13}$$

Proof. Let $\check{\mathbb{D}}_1^{\mu}$ be the trace of $\mathbb{D}$ with respect to the measure μ. Let $f \in L^2(\mathbb{R}, \mu)$. Then

$$\infty > \int |f|^2 \, d\mu = \sum_{n \in \mathbb{Z}} a_n |f(x_n)|^2 \geq m_0 \sum_{n \in \mathbb{Z}} |f(x_n)|^2.$$

Choose $\varphi \in C_0^{\infty}(\mathbb{R})$ such that $\varphi(0) = 1$ and $\varphi(x) = 0$ if $|x| \geq d/2$. Then $f(x_n) \cdot \varphi(\cdot - x_n)$, $n \in \mathbb{Z}$, are pairwise orthogonal elements of $H^1(\mathbb{R})$ and

$$\sum_{n \in \mathbb{Z}} \|f(x_n) \varphi(\cdot - x_n)\|_{H^1(\mathbb{R})}^2 = \sum_{n \in \mathbb{Z}} |f(x_n)|^2 \|\varphi\|_{H^1(\mathbb{R})}^2 < \infty.$$

Thus $u := \sum_{n \in \mathbb{Z}} f(x_n) \varphi(\cdot - x_n) \in H^1(\mathbb{R})$. Since $f = u$ μ-a.e., we obtain $f \in \operatorname{ran}(J^{\mu}) = D(\check{\mathbb{D}}_1^{\mu})$. Thus

$$D(\check{\mathbb{D}}_1^{\mu}) = L^2(\mathbb{R}, \mu).$$

By the previous lemma, $-\check{\Delta}^\mu := (J^\mu J^{\mu *})^{-1}$ is the self-adjoint operator associated with the closed form $\check{\mathbb{D}}_1^\mu$ in $L^2(\mathbb{R},\mu)$. Since the domain of the form associated to $-\check{\Delta}^\mu$ equals the whole Hilbert space $L^2(\mathbb{R},\mu)$, the domain of $D(-\check{\Delta}^\mu)$ equals $L^2(\mathbb{R},\mu)$, too. Thus, trivially,

$$J^\mu(D(-\Delta)) \subset D(-\check{\Delta}^\mu).$$

By Theorem 2.7, this implies the assertion (3.13). □

We shall demonstrate how to use traces of Dirichlet forms for the investigation of large coupling convergence by further examples. First we need some preparation.

Lemma 3.9. *Let μ be a positive Radon measure on $\mathbb{R}$ such that* $\operatorname{supp}(\mu) = [0,1]$. *Then*

$$\check{\mathbb{D}}_1^\mu(f,h) = \int_0^1 (\overline{f'}h' + \bar{f}h)dx + \overline{f(0)}h(0) + \overline{f(1)}h(1) \quad \forall\, f,h \in D(\check{\mathbb{D}}_1^\mu). \tag{3.14}$$

(*We recall that f denotes both an element of $D(\check{\mathbb{D}}_1^\mu)$ and the unique continuous representative of f.*)

Proof. By polarization, it suffices to consider the case $f = h$. Choose $u \in H^1(\mathbb{R})$ such that $f = J^\mu u$. By definition,

$$\check{\mathbb{D}}_1^\mu(f,f) = \mathbb{D}_1(P_\mu u, P_\mu u). \tag{3.15}$$

P_μ is infinitely differentiable on $\mathbb{R}\setminus[0,1]$ and

$$-(P_\mu u)'' + P_\mu u = 0 \text{ on } \mathbb{R}\setminus[0,1], \tag{3.16}$$

since $\mathbb{D}_1(P_\mu u, v) = 0$ for every $v \in C_0^\infty(\mathbb{R})$ with support in $\mathbb{R}\setminus[0,1]$. Since, by (3.11), $J^\mu P_\mu u = J^\mu u = f$, this implies

$$\begin{aligned} P_\mu u(x) &= P_\mu u(0)e^x = f(0)e^x \qquad \forall\, x \le 0,\\ P_\mu u(x) &= P_\mu u(1)e^{1-x} = f(1)e^{1-x} \quad \forall\, x \ge 1. \end{aligned} \tag{3.17}$$

Thus

$$\begin{aligned} \mathbb{D}_1(P_\mu u, P_\mu u) &= \int_{\mathbb{R}\setminus[0,1]} (|(P_\mu u)'|^2 + |(P_\mu u)|^2)dx + \int_0^1 (|f'|^2 + |f|^2)dx\\ &= |f(0)|^2 + |f(1)|^2 + \int_0^1 (|f'|^2 + |f|^2)dx. \end{aligned} \tag{3.18}$$

□

Corollary 3.10. *Let μ be a positive Radon measure on $\mathbb{R}$ such that* $\operatorname{supp}(\mu) = [0,1]$ *and $1_{(0,1)}\mu = 1_{(0,1)}\,dx$. Then each eigenvalue of the self-adjoint operator $-\check{\Delta}^\mu$ in $L^2(\mathbb{R},\mu)$ associated to the trace $\check{\mathbb{D}}_1^\mu$ of $\mathbb{D}_1$ with respect to the measure μ is strictly positive.*

Let $\eta > 0$ and $-\check{\Delta}^\mu f = (\eta^2+1)f$. Then there exist constants $c \in \mathbb{C}$ and $\theta \in [-\pi/2, \pi/2]$ such that (*the continuous representative of*) *f satisfies*

$$f(x) = c\sin(\eta x + \theta) \quad \forall\, x \in [0,1]. \tag{3.19}$$

Proof. Each eigenvalue of $-\check{\Delta}^\mu$ is strictly positive, since $-\check{\Delta}^\mu$ is an invertible non-negative self-adjoint operator.

Let $\eta > 0$ and $-\check{\Delta}^\mu f = (\eta^2 + 1)f$. By (3.14),

$$(-\check{\Delta}^\mu f, h)_{L^2(\mathbb{R},\mu)} = \int_0^1 (\overline{f'}h' + \bar{f}h)\, dx$$

for all infinitely differentiable functions with compact support in $(0, 1)$. This implies that f is infinitely differentiable on $(0, 1)$ and $-\check{\Delta}^\mu f = -f''(x) + f(x)$ for every $x \in (0, 1)$. Thus $-f''(x) = \eta^2 f(x)$ for all $x \in (0, 1)$ and hence there exist constants c and θ such that $f(x) = c \sin(\eta x+\theta)$ for all $x \in (0, 1)$ and, therefore, by continuity, for all $x \in [0, 1]$. □

We can now apply Lemma 2.26 in order to derive results on the rate of trace class convergence. We demonstrate how to do this through the following example.

Example 3.11. Let $\mu_1 := 1_{[0,1]}\, dx$ and $\mu_2 := \mu_1 + \delta_0 + \delta_1$. Then

$$\lim_{b\to\infty} \sqrt{b}\, \|(-\Delta + b\mu_1 + 1)^{-1} - (-\Delta + \infty\mu_1 + 1)^{-1}\|_{S_1} = \frac{3}{2} \tag{3.20}$$

and

$$\lim_{b\to\infty} \sqrt{b}\, \|(-\Delta + b\mu_2 + 1)^{-1} - (-\Delta + \infty\mu_2 + 1)^{-1}\|_{S_1} = \frac{1}{2}. \tag{3.21}$$

Proof. Let $\mu \in \{\mu_1, \mu_2\}$. Let $k \in \mathbb{N}$, $c_k \in \mathbb{R} \setminus \{0\}$, $\eta_k > 0$, $\theta_k \in [-\pi/2, \pi/2]$ and suppose that g_k with $g_k(x) = c_k \sin(\eta_k x + \theta_k)$ for all $x \in [0, 1]$ is a normalized eigenfunction of $-\check{\Delta}^\mu$. We have

$$\begin{aligned}\int_0^1 (g_k' h' + g_k h)\, dx + g_k(1)h(1) + g_k(0)h(0)\\ = \check{\mathbb{D}}_1^\mu(g_k, h) = (-\check{\Delta}^\mu g_k, h)_{L^2(\mu)} = (-g_k'' + g_k, h)_{L^2(\mu)} \quad \forall\, h \in D(\check{\mathbb{D}}^\mu).\end{aligned}$$

Moreover,

$$(-g_k'' + g_k, h)_{L^2(\mu_1)} = \int_0^1 (g_k' h' + g_k h)\, dx - g_k'(1)h(1) + g_k'(0)h(0),$$

and

$$\begin{aligned}(-g_k'' + g_k, h)_{L^2(\mu_2)}\\ = (-g_k'' + g_k, h)_{L^2(\mu_1)} + (-g_k''(1) + g_k(1))h(1) + (-g_k''(0) + g_k(0))h(0)\end{aligned}$$

for all $h \in D(\check{\mathbb{D}}^{\mu_1})$ and $h \in D(\check{\mathbb{D}}^{\mu_2})$, respectively. It follows that

$$g_k'(0) = g_k(0) \quad \text{and} \quad g_k'(1) = -g_k(1) \quad \text{if } \mu = \mu_1,$$

and

$$g_k''(0) = -g_k'(0) \quad \text{and} \quad g_k''(1) = g_k'(1) \quad \text{if } \mu = \mu_2.$$

It follows now by elementary calculus that

$$\lim_{k\to\infty} \theta_k = \pi/2 \quad \text{if } \mu = \mu_1,$$
$$\lim_{k\to\infty} \theta_k = 0 \quad \text{if } \mu = \mu_2,$$

$$\lim_{k\to\infty} (\eta_k - k\pi) = 0 \text{ and } \lim_{k\to\infty} c_k^2 = 2 \text{ in both cases.}$$

Hence

$$\lim_{k\to\infty} g_k^2(0) = \lim_{k\to\infty} g_k^2(1) = 2 \quad \text{if } \mu = \mu_1,$$
$$\lim_{k\to\infty} g_k^2(0) = \lim_{k\to\infty} g_k^2(1) = 0 \quad \text{if } \mu = \mu_2.$$

Inserting these results into Lemma 2.26 and taking Corollary 3.10 into account, we complete the proof by an elementary computation. □

Finally, we want to hint to an interesting fact. Again let $\mu_1 = 1_{[0,1]}\,dx$. Choose an orthonormal system $(g_k)_{k\in\mathbb{N}}$ in $L^2(\mathbb{R},\mu_1)$ and a sequence$(\eta_k)_{k\in\mathbb{N}}$ of strictly positive real numbers such that $-\check{\Delta}^\mu g_k = (1+\eta_k^2)\,g_k$ for all $k \in \mathbb{N}$. Then, by (2.76),

$$\|(-\Delta + b\mu_1 + 1)^{-1} - (-\Delta + \infty\mu_1 + 1)^{-1}\| \geq \sum_{k=1}^{\infty} \frac{\alpha_k(f)}{1+\eta_k^2+b}$$

for any normalized $f \in L^2(\mathbb{R})$, where

$$\alpha_k(f) := |\int_{-\infty}^{0} g_k(0)e^x f(x)\,dx + \int_0^1 g_k(x)f(x)\,dx + \int_1^\infty g_k(1)e^{1-x}f(x)\,dx|^2.$$

If we choose $f(x) := \sqrt{2}\,1_{(-\infty,0)}(x)e^x$ for all $x \in \mathbb{R}$, then, by the considerations of the previous example, $\lim_{k\to\infty} \alpha_k(f) = 1$ and hence

$$\lim_{b\to\infty} \sqrt{b}\,\|(-\Delta + b\mu_1 + 1)^{-1} - (-\Delta + \infty\mu_1 + 1)^{-1}\| \geq \frac{1}{2}. \tag{3.22}$$

Thus the operators $(-\Delta + b\mu_1 + 1)^{-1}$ do not converge faster than $O(1/\sqrt{b})$ with respect to the operator norm. On the other hand, the rate of convergence becomes $O(1/b)$, if we add $\varepsilon_0\delta_0 + \varepsilon_1\delta_1$ to the measure μ_1, where ε_1 and ε_2 are any strictly positive real numbers, cf. Example 3.19 below. Thus arbitrarily small changes of the measure can lead to strong changes of the rate of convergence.

Actually, if one combines (2.76), (2.75) and the results from the previous example, then one gets via an elementary computation that

$$\lim_{b\to\infty} \sqrt{b}\,\|(-\Delta + b\mu_1 + 1)^{-1} - (-\Delta + \infty\mu_1 + 1)^{-1}\| = \frac{1}{2}. \tag{3.23}$$

3.3. A domination principle

For positive Radon measures μ on X charging no set with capacity zero let

$$\mathcal{H}^\mu_\infty := \overline{\ker(J^\mu)}$$

be the closure of $\ker(J^\mu)$ in the Hilbert space $\mathcal{H}$. We have

$$(H + \infty\mu + 1)^{-1} = (H + \infty\nu + 1)^{-1}$$

for $\mathcal{H}^\mu_\infty = \mathcal{H}^\nu_\infty$. This can be true even if the measures μ and ν are quite different; in particular, it is not necessary that the measures μ and ν are equivalent.

Intuitively one expects in the case $(H + \infty\mu + 1)^{-1} = (H + \infty\nu + 1)^{-1}$ that the operators $(H + b\mu + 1)^{-1}$ converge at least as fast as $(H + b\nu + 1)^{-1}$ if $\mu \geq \nu$. We shall prove that this is true. In this way we can use known results for one measure ν in order to derive results for another measure μ. For instance, if $(H + b\nu + 1)^{-1}$ converge with maximal rate, i.e., as fast as $O(1/b)$, and $\mu \geq \nu$ and $(H + \infty\mu + 1)^{-1} = (H + \infty\nu + 1)^{-1}$, then $(H + b\mu + 1)^{-1}$ converge with maximal rate, too.

Lemma 3.12. *Let μ and ν be positive Radon measures on X charging no set with capacity (with respect to $\mathcal{E}$) zero. Assume, in addition, that $\mu \geq \nu$. Then the operator $G^\nu - G^\mu$ is positivity preserving, i.e., it holds $(G^\nu - G^\mu)f \geq 0$ m-a.e if $f \geq 0$ m-a.e.*

Proof. Let $f, g \in L^2(X, m)$, $f \geq 0$ m-a.e., and $g \geq 0$ m-a.e. Then $G^\mu f \geq 0$ m-a.e. and $G^\nu g \geq 0$ m-a.e., since G^μ and G^ν are positivity preserving. By [13, Lemma 2.1.5], this implies that all quasi-continuous representatives of $G^\mu f$ and of $G^\nu g$ (with respect to $\mathcal{E}$) are non-negative q.e. and, therefore, also $(\mu - \nu)$-a.e.

We have, with the convention that u denotes both an element of $D(\mathcal{E})$ and any quasi-continuous representative of u, that

$$\begin{aligned}(f, G^\nu g) &= \mathcal{E}^\mu_1(G^\mu f, G^\nu g)\\ &= \mathcal{E}^\nu_1(G^\mu f, G^\nu g) + \int G^\mu f\, G^\nu g\, d(\mu - \nu)\\ &= (G^\mu f, g) + \int G^\mu f\, G^\nu g\, d\mu.\end{aligned}$$

Thus

$$\int (G^\nu f - G^\mu f) g\, dm = \int G^\mu f\, G^\nu g\, d(\mu - \nu).$$

Since the right-hand side is non-negative for every $g \in L^2(X, m)$ satisfying $g \geq 0$ m-a.e., it follows that $G^\nu f - G^\mu f \geq 0$ m-a.e. □

It holds $G = G^0$, where 0 denotes the measure which is identically equal to zero and $b'\mu \leq b\mu$ if $b' \leq b$. Hence it follows from the previous lemma that

$$G(\cdot, B) \geq G^{b'\mu}(\cdot, B) \geq G^{b\mu}(\cdot, B) \quad \forall\, B \in \mathcal{B}(X) \text{ q.e. if } 0 < b' < b. \tag{3.24}$$

Thus $(H + \infty\mu + 1)^{-1}$ has also an m-symmetric Markovian kernel $G^{\infty\mu}$ and

$$G^{b\mu}(\cdot, B) \geq G^{\infty\mu}(\cdot, B) \quad \forall B \in \mathcal{B}(X) \text{ q.e.} \tag{3.25}$$

For each $b \in [0, \infty]$, it follows that D_b^μ has an m-symmetric Markovian kernel, also denoted by D_b^μ, and that

$$D_{b'}^\mu(\cdot, B) \leq D_b^\mu(\cdot, B) \leq D_\infty^\mu(\cdot, B) \quad \forall B \in \mathcal{B}(X) \text{ q.e. if } 0 < b' < b. \tag{3.26}$$

Corollary 3.13. *Under the hypotheses of Lemma* 3.12 *and the additional assumption that*

$$D_\infty^\mu = D_\infty^\nu,$$

it holds that

$$0 \leq D_\infty^\mu f - D_b^\mu f \leq D_\infty^\nu f - D_b^\nu f \quad m\text{-a.e.} \tag{3.27}$$

for all $b > 0$ *provided that* $f \geq 0$ m-a.e. *Moreover,*

$$\|D_\infty^\mu - D_b^\mu\| \leq \|D_\infty^\nu - D_b^\nu\| \quad \forall b > 0. \tag{3.28}$$

Proof. (3.27) follows immediately from Lemma 3.12 and (3.28) follows from (3.27), since both the operators $D_\infty^\mu - D_b^\mu$ and the operators $D_\infty^\nu - D_b^\nu$ have m-symmetric Markovian kernels. □

3.4. Convergence with maximal rate and equilibrium measures

First let us recall some known facts from the potential theory of Dirichlet forms, cf. [13]. A positive Radon measure is a measure with finite energy integral (with respect to $\mathcal{E}$) if and only if there exists a constant $c > 0$ such that

$$\int |u| \, d\mu \leq c\sqrt{\mathcal{E}_1(u, u)} \quad \forall u \in C_0(X) \cap D(\mathcal{E}). \tag{3.29}$$

If μ is a measure with finite energy integral, then μ does not charge any set with capacity zero and there exists a unique element $U_1\mu$ (the 1-potential of μ) of $D(\mathcal{E})$ such that

$$\mathcal{E}_1(U_1\mu, v) = \int v \, d\mu \quad \forall v \in D(\mathcal{E}). \tag{3.30}$$

It holds that $U_1\mu \geq 0$ m-a.e. Now let μ be any positive Radon measure on X charging no set with capacity zero. Then, for all $h \in L^2(X, \mu)$ with $h \geq 0$ μ-a.e., the following holds: $h\mu$ is a measure with finite energy integral if and only if $h \in D(J^{\mu *})$. In this case $J^{\mu *}h$ equals the 1-potential $U_1(h\mu)$ of $h\mu$ and hence

$$J^{\mu *}h = U_1(h\mu) \geq 0 \text{ m-a.e. } \forall h \in D(J^{\mu *}) \text{ with } h \geq 0 \, \mu\text{-a.e.} \tag{3.31}$$

Let Γ be a closed subset of X such that the 1-capacity $\text{cap}(\Gamma)$ of Γ is finite. There exists a unique $e_\Gamma \in D(\mathcal{E})$ satisfying

$$e_\Gamma = 1 \text{ q.e. on } \Gamma \text{ and } \mathcal{E}_1(e_\Gamma, v) \geq 0 \, \forall v \in D(\mathcal{E}) \text{ with } v \geq 0 \text{ q.e. on } \Gamma. \tag{3.32}$$

Moreover, there exists a unique positive Radon measure μ_Γ on X such that μ_Γ has finite energy integral,

$$\mu_\Gamma(\Gamma) = \mu_\Gamma(X) = \text{cap}(\Gamma) \text{ and } e_\Gamma = U_1\mu_\Gamma. \tag{3.33}$$

Thus $1 \in D(J^{\mu_\Gamma *})$ and

$$J^{\mu_\Gamma} J^{\mu_\Gamma *} 1 = 1 \text{ q.e. on } \Gamma. \tag{3.34}$$

The 1-equilibrium potential e_Γ of Γ satisfies, in addition,

$$0 \le e_\Gamma \le 1 \quad m\text{-a.e.} \tag{3.35}$$

We recall that $\check{H} = (J^\mu J^{\mu *})^{-1}$ and set

$$\check{K} := J^\mu J^{\mu *} \quad \text{and} \quad \check{K}_\alpha := (\check{H} + \alpha)^{-1} \quad \forall \alpha > 0. \tag{3.36}$$

(3.34) can be used to prove that $J^{\mu_\Gamma} J^{\mu_\Gamma *}$ is a bounded operator with norm one. We prepare the proof through the following lemma.

Lemma 3.14. *Let G be a symmetric Markovian kernel and set*

$$Tf(x) := \int f(y) G(x, dy)$$

whenever the expression on the right-hand side is defined. Then

$$\|Tf\| \le (\|T1\|_\infty)^{1/2} \|f\| \quad \forall f \in L^2(X,m) \cap L^\infty(X,m)$$

and hence T extends to a bounded operator on $L^2(X,m)$ with

$$\|T\| \le (\|T1\|_\infty)^{1/2}. \tag{3.37}$$

Proof. Let $f \in L^2(X,m) \cap L^\infty(X,m)$. By Hölder's inequality,

$$|Tf|^2 \le T1 \int_X f^2(y) G(\cdot,\, dy) \le \|T1\|_\infty \int_X f^2(y) G(\cdot,\, dy). \tag{3.38}$$

This yields, by the Markov property and the symmetry of G, that $\|Tf\|^2 \le \|T1\|_\infty \|f\|^2$. □

Corollary 3.15. *Let Γ be a closed subset of X such that $0 < \operatorname{cap}(\Gamma) < \infty$. Then*

$$\|J^{\mu_\Gamma} J^{\mu_\Gamma *}\| = 1. \tag{3.39}$$

Proof. By the first resolvent equality and since the operators $\check{K}_\alpha$ are positivity preserving, the sequence $(\check{K}_{1/n} f)_{n=1}^\infty$ is pointwise non-decreasing μ_Γ-a.e. for all $f \in L^2(X, \mu_\Gamma)$ with $f \ge 0$ μ_Γ-a.e.

By (3.36) and (3.34), $1 \in D(\check{K})$ and $\check{K}1 = 1$ μ_Γ-a.e. and hence $\|\check{K}\| \ge 1$. By spectral calculus,

$$\|\check{K}_{1/n} f - \check{K} f\|_{L^2(X,\mu_\Gamma)} \to 0 \text{ as } n \to \infty \quad \forall\, f \in D(\check{K}). \tag{3.40}$$

Since the sequence $(\check{K}_{1/n} 1)_{n=1}^\infty$ is non-decreasing μ_Γ-a.e., it follows that it converges to 1 μ_Γ-a.e. and, in particular, $\check{K}_{1/n} 1 \le 1$ μ_Γ-a.e. for all $n \in \mathbb{N}$, $n \ge 1$. By Lemma 3.14, this implies that

$$\|\check{K}_{1/n}\| \le 1, \quad n = 1, 2, 3, \ldots$$

By (3.40), it follows that $\|\check{K}\| \le 1$. □

It is remarkable that the important and large class of equilibrium measures leads to large coupling convergence with maximal rate of convergence.

Theorem 3.16. *Let* Γ *be a closed subset of* X *with finite capacity and* μ_Γ *the equilibrium measure of* Γ. *Let* F *be the support of* μ_Γ. *Assume that* $(H+1)^{-1}$ *is conservative. Then*

$$\|(H+\beta\mu_\Gamma+1)^{-1}-(H+\infty\mu_\Gamma+1)^{-1}\| \leq \frac{1}{1+b} \quad \forall\, b>0. \tag{3.41}$$

Proof. By (3.26), $D_\infty^{\mu_\Gamma}-D_b^{\mu_\Gamma}$ possesses an m-symmetric Markovian kernel and, by Lemma 3.14, it suffices to prove that

$$\|(H+b\mu_\Gamma+1)^{-1}1-(H+\infty\mu_\Gamma+1)^{-1}1\|_\infty \leq \frac{1}{1+b} \quad \forall\, b>0. \tag{3.42}$$

Let $b>0$ and $(f_k)\subset C_0(X)$ such that $f_k\uparrow 1$ everywhere on X. Using the representation of G in terms of its Markovian kernel, we obtain that, by applying the monotone convergence theorem,

$$J^{\mu_\Gamma}Gf_k \to 1 \text{ in } L^2(X,\mu_\Gamma). \tag{3.43}$$

Thus observing that, by (3.34), $\left(\frac{1}{b}+\check{H}^{-1}\right)^{-1}1=\frac{b}{1+b}$, we obtain

$$D_b^{\mu_\Gamma}f_k=(I_{\mu_\Gamma}G)^*\left(\frac{1}{b}+\check{H}^{-1}\right)^{-1}J^{\mu_\Gamma}Gf_k \to \frac{b}{1+b}\,(J^{\mu_\Gamma}G)^*1. \tag{3.44}$$

By monotone convergence again, we get that $D_b^{\mu_\Gamma}f_k\uparrow D_b^{\mu_\Gamma}1$ a.e. Thus, by the latter identity and since

$$\frac{b}{1+b}\,(J^{\mu_\Gamma}G)^*1=\frac{b}{1+b}\,U_1\mu_\Gamma,$$

we arrive at $D_b^{\mu_\Gamma}1=\frac{b}{1+b}U_1\mu_\Gamma$ for all $0<b<\infty$. Since the operators $D_b^{\mu_\Gamma}$ converge to $D_\infty^{\mu_\Gamma}$ strongly, this implies that $D_\infty^{\mu_\Gamma}1=U_1\mu_\Gamma$. Thus

$$\|(H+b\mu_\Gamma+1)^{-1}1-(H+\infty\mu_\Gamma+1)^{-1}1\|_\infty \leq \frac{\|U_1\mu_\Gamma\|_\infty}{1+b} \quad \forall\, b>0. \tag{3.45}$$

Finally, the result follows from (3.33) and (3.35). □

By the previous theorem, $L(H,P_{\mu_\Gamma})\leq 1$ provided that the regular Dirichlet form $\mathcal{E}$ is conservative. For conservative strongly local regular Dirichlet forms, we can even give the exact value of $L(H,P_{\mu_\Gamma})$.

Theorem 3.17. *Suppose that the regular Dirichlet form* $\mathcal{E}$ *associated to the non-negative self-adjoint operator* H *in* $L^2(X,m)$ *has the strong local property. Let* Γ *be a closed subset of* X *with finite capacity. If the interior* Γ° *of* Γ *is not empty, then*

$$L(H,P_{\mu_\Gamma})\geq 1. \tag{3.46}$$

If, in addition, the operator $(H+1)^{-1}$ *is conservative, then*

$$L(H,P_{\mu_\Gamma})=1. \tag{3.47}$$

Proof. (3.47) follows from (3.46) and Theorem 3.16. Thus we only need to prove (3.46).

Since $U_1\mu_\Gamma = 1$ q.e. on Γ and by the strong locality of $\mathcal{E}$,

$$\int u\,dm = (U_1\mu_\Gamma, u) = \mathcal{E}_1(U_1\mu_\Gamma, u) = \int u\,d\mu_\Gamma$$

for all $u \in C_0(\Gamma^\circ) \cap D(\mathcal{E})$. Since $C_0(\Gamma^\circ) \cap D(\mathcal{E})$ is dense in $C_0(\Gamma^\circ)$ with respect to the supremum norm, it follows that

$$\mu_\Gamma = m \text{ on the Borel-Algebra } \mathcal{B}(\Gamma^\circ) \text{ of } B. \tag{3.48}$$

Choose $u \in C_0(\Gamma^\circ) \cap D(\mathcal{E})$ such that $\|u\| = 1$. For all $f \in D(J^{\mu_\Gamma})$

$$\mathcal{E}_1(f, Gu) = (f, u) = (J^{\mu_\Gamma} f, u)_{L^2(\mu_\Gamma)} = \mathcal{E}_1(f, J^{\mu_\Gamma *} u)$$

(in the second step we have used (3.48)). Thus $Gu = J^{\mu_\Gamma *}u$ and hence $\check{H} J^{\mu_\Gamma} G u = u$. Thus

$$\|\check{H} J^{\mu_\Gamma} H\| \geq \|u\|_{L^2(\mu_\Gamma)} = \|u\| = 1$$

(again, we have used (3.48) in the second step). By Theorem 2.7 (c), this implies (3.46). □

As a consequence of Theorem 3.16 in conjunction with Corollary 3.13, we obtain the next result.

Corollary 3.18. *Let $\mathcal{E}$ be a conservative Dirichlet form. Let Γ be a closed subset of X with finite capacity, $0 < c < \infty$, and let μ be a positive Radon measure on X charging no set with capacity zero and such that $\mu \geq c\,\mu_\Gamma$. Assume, in addition, that*

$$D^\mu_\infty = D^{\mu_\Gamma}_\infty.$$

(*In particular, this is true if μ is absolutely continuous with respect to the equilibrium measure μ_Γ.*) *Then*

$$\|D^\mu_\infty - D^\mu_b\| \leq \frac{1}{1+cb} \quad \forall\, b > 0.$$

If $\mathcal{E}$ equals the classical Dirichlet form $\mathbb{D}$ in $L^2(\mathbb{R})$, then the equilibrium measure of the interval $[0,1]$ equals $1_{[0,1]} = dx + \delta_0 + \delta_1$. Hence the result in the next example follows from the previous corollary. If one compares this result with (3.22), then one sees that the rate of convergence for the operators $(-\Delta + b\mu + 1)^{-1}$ can be changed strongly by an arbitrarily small change of the measure μ.

Example 3.19. Let $\varepsilon_i > 0$ for $i = 0, 1$. Let $\mu = 1_{[0,1]}\,dx + \varepsilon_0\delta_0 + \varepsilon_1\delta_1$. Let $c := \min(\varepsilon_0, \varepsilon_1)$. Then

$$\|(-\Delta + b\mu + 1)^{-1} - (-\Delta + \infty\mu + 1)^{-1}\| \leq \frac{1}{1+cb} \quad \forall\, b > 0.$$

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