The Browder Spectrum of an Elementary Operator

Derek Kitson

Abstract. We relate the ascent and descent of n-tuples of multiplication operators $M_{a,b}(u) = aub$ to that of the coefficient Hilbert space operators a, b. For example, if $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b}^* = (b_1^*, \ldots, b_m^*)$ have finite non-zero ascent and descent s and t, respectively, then the $(n+m)$-tuple $(L_{\mathbf{a}}, R_{\mathbf{b}})$ of left and right multiplication operators has finite ascent and descent $s+t-1$. Using these results we obtain a description of the Browder joint spectrum of $(L_{\mathbf{a}}, R_{\mathbf{b}})$ and provide formulae for the Browder spectrum of an elementary operator acting on $B(H)$ or on a norm ideal of $B(H)$.

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Introduction

The Browder spectrum of a Banach space operator is obtained by removing from the ordinary spectrum all eigenvalues of finite multiplicity which are poles of the resolvent. Let H be a complex Hilbert space and $B(H)$ the collection of bounded operators on H. An elementary operator on $B(H)$ is an operator of the form $\mathcal{E} : B(H) \to B(H)$, $u \mapsto a_1ub_1 + \cdots + a_nub_n$ where $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in B(H)^n$. Spectral properties of elementary operators have been considered by various authors (see [2, 3, 5, 6, 7, 9, 10]). In this article we obtain formulae for the Browder spectrum of $\mathcal{E}$. These formulae may have applications to Weyl and Browder type theorems for elementary operators.

The ascent of a linear mapping a acting on a vector space X is usually described as the length of the increasing chain of null spaces

$$\{0\} \subseteq \ker a \subseteq \ker a^2 \subseteq \ker a^3 \subseteq \cdots$$

This value is also the smallest non-negative integer r for which $\ker a \cap \operatorname{ran} a^r = \{0\}$. The descent of a is the length of the decreasing chain of range spaces

$$X \supseteq \operatorname{ran} a \supseteq \operatorname{ran} a^2 \supseteq \operatorname{ran} a^3 \supseteq \cdots$$

and equals the smallest r for which $\ker a^r + \operatorname{ran} a = X$. Let $\mathbf{a} = (a_1, \ldots, a_n)$ be an n-tuple of linear mappings on a vector space X. We write $N(\mathbf{a}) = \bigcap_{j=1}^n \ker a_j$ and $R(\mathbf{a}) = \sum_{j=1}^n \operatorname{ran} a_j$. We denote by $\mathbf{a}^r$ the lexicographically ordered n^r-tuple of operators on X consisting of all products $a_{j_1} \ldots a_{j_r}$. For example if $n = 2$ then

$$\begin{aligned} \mathbf{a} &= (a_1, a_2) \\ \mathbf{a}^2 &= (a_1^2,\, a_1a_2,\, a_2a_1,\, a_2^2) \\ \mathbf{a}^3 &= (a_1^3,\, a_1^2a_2,\, a_1a_2a_1,\, a_1a_2^2,\, a_2a_1^2,\, a_2a_1a_2,\, a_2^2a_1,\, a_2^3) \quad \text{etc.} \end{aligned}$$

In the terminology of [8], the ascent of $\mathbf{a} = (a_1, \ldots, a_n)$ on X is the smallest r such that $N(\mathbf{a}) \cap R(\mathbf{a}^r) = \{0\}$. The descent of $\mathbf{a} = (a_1, \ldots, a_n)$ on X is the smallest r such that $N(\mathbf{a}^r) + R(\mathbf{a}) = X$. If the ascent and descent of $\mathbf{a} = (a_1, \ldots, a_n)$ are both finite then they are equal ([8, Proposition 1.9]). We will denote the ascent of an n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ by $\alpha(\mathbf{a}, X)$ or simply $\alpha(\mathbf{a})$ if the space X is understood. Similarly we write $\delta(\mathbf{a}, X)$ or simply $\delta(\mathbf{a})$ for the descent of $\mathbf{a} = (a_1, \ldots, a_n)$.

For operators $a, b \in B(H)$ define the multiplication operator $M_{a,b} : B(H) \to B(H)$, $u \mapsto aub$. With I the identity operator on H, the left multiplication operator is $L_a = M_{a,I}$ and the right multiplication operator is $R_b = M_{I,b}$. Given an n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ of operators on H we write $L_{\mathbf{a}} = (L_{a_1}, \ldots, L_{a_n})$ and $R_{\mathbf{a}} = (R_{a_1}, \ldots, R_{a_n})$. More generally, we will consider multiplication operators acting on ideals $\mathcal{I}$ of $B(H)$ such as the compact operators, the trace-class operators, the Hilbert-Schmidt operators and all p-Schatten classes of operators.

In Section 1 we show that if $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b}^* = (b_1^*, \ldots, b_m^*)$ have finite non-zero ascent and descent s and t respectively then the $(n+m)$-tuple $(L_{\mathbf{a}}, R_{\mathbf{b}}) = (L_{a_1}, \ldots, L_{a_n}, R_{b_1}, \ldots, R_{b_m})$ has finite ascent and descent $s + t - 1$. We also obtain bounds for the ascent and descent of $M_{\mathbf{a},\mathbf{b}} = (M_{a_i,b_j})_{i=1,j=1}^{n,\,m}$. In [8] a Browder joint spectrum σ_b was introduced for n-tuples of Banach space operators. In Section 2 we obtain a description of the Browder joint spectrum of $(L_{\mathbf{a}}, R_{\mathbf{b}})$ and collect formulae for the Browder spectrum of an elementary operator acting on $B(H)$ or on a norm ideal of $B(H)$,

$$\begin{aligned} \sigma_b(\mathcal{E}) &= \big(\sigma_l(\mathbf{a}) \circ \sigma_b^-(\mathbf{b})\big) \cup \big(\sigma_b^+(\mathbf{a}) \circ \sigma_r(\mathbf{b})\big) \cup \big(\sigma_r(\mathbf{a}) \circ \sigma_b^+(\mathbf{b})\big) \cup \big(\sigma_b^-(\mathbf{a}) \circ \sigma_l(\mathbf{b})\big) \\ &= (\sigma_H(\mathbf{a}) \circ \sigma_b(\mathbf{b})) \cup (\sigma_b(\mathbf{a}) \circ \sigma_H(\mathbf{b})) \\ &= (\sigma_T(\mathbf{a}) \circ \sigma_{Tb}(\mathbf{b})) \cup (\sigma_{Tb}(\mathbf{a}) \circ \sigma_T(\mathbf{b})) \end{aligned}$$

Here σ_l and σ_r are the left and right spectra, σ_b^+ and σ_b^- are analogues of the semi-Browder spectra of an operator, σ_H is the Harte spectrum, σ_T is the Taylor spectrum and σ_{Tb} is the Taylor–Browder spectrum.

1. Ascent and descent

Theorem 1.1. *Let* $\mathbf{a} = (a_1, \ldots, a_n)$ *be an* n*-tuple of operators on a Hilbert space* H*. Let* $\mathcal{I}$ *be a left ideal of* $B(H)$*. Then*

(i) $\alpha(L_{\mathbf{a}}, \mathcal{I}) = \alpha(\mathbf{a}, H)$
(ii) $\delta(L_{\mathbf{a}}, \mathcal{I}) = \delta(\mathbf{a}, H)$

Proof. (i) If $u \in N(L_{\mathbf{a}}) \cap R((L_{\mathbf{a}})^r)$ then $\operatorname{ran} u \subseteq N(\mathbf{a}) \cap R(\mathbf{a}^r)$. We conclude that $\alpha(L_{\mathbf{a}}) \leq \alpha(\mathbf{a})$. For the reverse inequality, suppose $x \in N(\mathbf{a}) \cap R(\mathbf{a}^r)$. Then $x = \sum a_{i_1} \ldots a_{i_r}(x_{i_1 \ldots i_r})$ for some $x_{i_1 \ldots i_r} \in H$. Choose $y \in H$ such that $\langle x, y\rangle = 1$. For each $x_{i_1 \ldots i_r}$ consider the rank one operator $p_{i_1 \ldots i_r} \in \mathcal{I}$ given by $p_{i_1 \ldots i_r}(z) = \langle z, y\rangle x_{i_1 \ldots i_r}$ for all $z \in H$. Let $p = \sum a_{i_1} \ldots a_{i_r} p_{i_1 \ldots i_r}$. Then $p \in N(L_{\mathbf{a}}) \cap R((L_{\mathbf{a}})^r)$ and $x = p(x)$. We conclude that $\alpha(\mathbf{a}) \leq \alpha(L_{\mathbf{a}})$.

(ii) Suppose $N((L_{\mathbf{a}})^r) + R(L_{\mathbf{a}}) = \mathcal{I}$. Let $x \in H$ and let $p \in \mathcal{I}$ be a rank one operator with $p(x) = x$. Then $p = u + v$ for some $u \in N((L_{\mathbf{a}})^r)$ and some $v \in R(L_{\mathbf{a}})$. We have $v = L_{a_1}(v_1) + \cdots + L_{a_n}(v_n)$ for some $v_1, \ldots, v_n \in \mathcal{I}$. Now $x = (u+v)(x) = u(x) + (a_1 v_1 + \cdots + a_n v_n)(x) \in N(\mathbf{a}^r) + R(\mathbf{a})$. Hence $N(\mathbf{a}^r) + R(\mathbf{a}) = H$ and so $\delta(\mathbf{a}) \leq \delta(L_{\mathbf{a}})$.

For the reverse inequality, suppose $N(\mathbf{a}^r) + R(\mathbf{a}) = H$. Let $q \in B(H)$ be the orthogonal projection onto $N(\mathbf{a}^r)$. Define $T : H^{n+1} \to H$ by $T(x, y_1, \ldots, y_n) = q(x) + a_1(y_1) + \cdots + a_n(y_n)$. Note that T is surjective and so there exists a right inverse operator $C : H \to H^{n+1}$. Write $C(z) = (f(z), g_1(z), \ldots, g_n(z))$ where $f, g_1, \ldots, g_n \in B(H)$. Then for all $u \in \mathcal{I}$ we have

$$u = (T \circ C)u = (q \circ f)u + (a_1 \circ g_1)u + \cdots + (a_n \circ g_n)u \in N((L_{\mathbf{a}})^r) + R(L_{\mathbf{a}})$$

We conclude that $N((L_{\mathbf{a}})^r) + R(L_{\mathbf{a}}) = \mathcal{I}$ and so $\delta(L_{\mathbf{a}}) \leq \delta(\mathbf{a})$. □

By appealing to adjoints we can obtain further equalities. Given an n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ we denote by $\mathbf{a}^* = (a_1^*, \ldots, a_n^*)$ the n-tuple of adjoint operators.

Theorem 1.2. *Let* $\mathbf{a} = (a_1, \ldots, a_n)$ *be an* n*-tuple of operators on a Hilbert space* H*. Let* $\mathcal{I}$ *be a two-sided ideal of* $B(H)$*. Then*

(i) $\alpha(R_{\mathbf{a}}, \mathcal{I}) = \alpha(\mathbf{a}^*, H)$
(ii) $\delta(R_{\mathbf{a}}, \mathcal{I}) = \delta(\mathbf{a}^*, H)$

Proof. Note that the operator $R_a : \mathcal{I} \to \mathcal{I}$ is similar to the operator $L_{a^*} : \mathcal{I} \to \mathcal{I}$ where similarity is provided by the involution $\tau : \mathcal{I} \to \mathcal{I}$, $b \mapsto b^*$. Applying this similarity we have $\alpha(R_{\mathbf{a}}, \mathcal{I}) = \alpha(L_{\mathbf{a}^*}, \mathcal{I})$ and $\delta(R_{\mathbf{a}}, \mathcal{I}) = \delta(L_{\mathbf{a}^*}, \mathcal{I})$. The result now follows from Theorem 1.1. □

Remark 1.3. The ascent and descent of the right multiplication R_a behave somewhat differently to left multiplication. For example if a is the unilateral shift operator on ℓ^2 then $\delta(a, \ell^2) = \delta(L_a, B(\ell^2)) = \infty$ but $\delta(R_a, B(\ell^2)) = 0$. (This is noted in [1, Remark 2.3].) For a general Hilbert space operator, if a^{r+1} has closed range where $r = \alpha(a)$ then $\alpha(a, H) = \delta(R_a, B(H))$. The necessity of the closed range condition is illustrated by the self-adjoint operator $a(e_j) = \frac{1}{2^j} e_j$ on ℓ^2. In this case

$\alpha(a, \ell^2) = 0$ and a does not have closed range. It follows from the above theorem that $\delta(R_a, B(\ell^2)) = \infty$.

For a single operator $a \in B(H)$ with finite ascent and descent we have equalities $\alpha(a) = \delta(a) = \alpha(a^*) = \delta(a^*) = \alpha(L_a) = \delta(L_a) = \alpha(R_a) = \delta(R_a)$. The following example shows that this is not always true in the multivariable case.

Example. Let H be the Hilbert space with orthonormal basis $(e_{i,j})_{i,j=1}^{\infty}$. Define $\mathbf{a} = (a_1, a_2)$ where

$$a_1(e_{i,j}) = \begin{cases} e_{i-1,j} & \text{if } i > 1 \\ 0 & \text{if } i = 1 \end{cases} \quad \text{and} \quad a_2(e_{i,j}) = \frac{1}{2^j} e_{i,j}$$

Then $\alpha(\mathbf{a}) = \delta(\mathbf{a}) = 0$ but $\alpha(\mathbf{a}^*) = 0$ and $\delta(\mathbf{a}^*) = \infty$.

Theorem 1.4. *Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_m)$ be tuples of operators on a Hilbert space H. Let $\mathcal{I}$ be a two-sided ideal of $B(H)$.*

(i) *If $\alpha(\mathbf{a}) > 0$ and $\alpha(\mathbf{b}^*) > 0$ then $\alpha(\mathbf{a}) + \alpha(\mathbf{b}^*) - 1 \le \alpha((L_{\mathbf{a}}, R_{\mathbf{b}}), \mathcal{I})$.*
(ii) *If $\delta(\mathbf{a}) > 0$ and $\alpha(\mathbf{b}) > 0$ then $\delta(\mathbf{a}) + \alpha(\mathbf{b}) - 1 \le \delta((L_{\mathbf{a}}, R_{\mathbf{b}}), \mathcal{I})$.*

Proof. (i) Suppose there exist non-zero elements $x \in N(\mathbf{a}) \cap R(\mathbf{a}^s)$ and $y \in N(\mathbf{b}^*) \cap R((\mathbf{b}^*)^t)$. Define the rank one operator $p \in \mathcal{I}$ by $p(z) = \langle z, y\rangle x$ for all $z \in H$. Now $x = \sum a_{i_1} \dots a_{i_s}(x_{i_1 \dots i_s})$ for some $x_{i_1 \dots i_s} \in H$. Also $y = \sum b_{j_1}^* \dots b_{j_t}^*(y_{j_1 \dots j_t})$ for some $y_{j_1 \dots j_t} \in H$. For each $i_1, \dots, i_s = 1, \dots, n$ and each $j_1, \dots, j_t = 1, \dots, m$ define the rank one operator $p_{i_1 \dots i_s j_1 \dots j_t} \in \mathcal{I}$ by $p_{i_1 \dots i_s j_1 \dots j_t}(z) = \langle z, y_{j_1 \dots j_t}\rangle x_{i_1 \dots i_s}$ for all $z \in H$. Then we can verify that $p = \sum a_{i_1} \dots a_{i_s} p_{i_1 \dots i_s j_1 \dots j_t} b_{j_1} \dots b_{j_t}$. We have $p \in N(L_{\mathbf{a}}, R_{\mathbf{b}}) \cap R((L_{\mathbf{a}}, R_{\mathbf{b}})^{s+t})$ and so $s + t + 1 \le \alpha(L_{\mathbf{a}}, R_{\mathbf{b}})$. It follows that $\alpha(\mathbf{a}) + \alpha(\mathbf{b}^*) - 1 \le \alpha(L_{\mathbf{a}}, R_{\mathbf{b}})$.

(ii) Suppose there exists $x \in N(\mathbf{b}) \cap R(\mathbf{b}^t)$ with $\|x\| = 1$ and suppose there exists $y \in H$ with $y \notin N(\mathbf{a}^s) + R(\mathbf{a})$. Define the rank one operator $q \in \mathcal{I}$ by $q(z) = \langle z, x\rangle y$ for all $z \in H$. If $\delta(L_{\mathbf{a}}, R_{\mathbf{b}}) \le s + t$ then $N((L_{\mathbf{a}}, R_{\mathbf{b}})^{s+t}) + R(L_{\mathbf{a}}, R_{\mathbf{b}}) = \mathcal{I}$. Thus we can write $q = v + w$ for some $v \in N((L_{\mathbf{a}}, R_{\mathbf{b}})^{s+t})$ and some $w \in R(L_{\mathbf{a}}, R_{\mathbf{b}})$. Now $v(x) \in N(\mathbf{a}^s)$ and $w(x) \in R(\mathbf{a})$. Thus $y = q(x) = v(x) + w(x) \in N(\mathbf{a}^s) + R(\mathbf{a})$ which is a contradiction. We conclude that $s + t + 1 \le \delta(L_{\mathbf{a}}, R_{\mathbf{b}})$. It now follows that $\delta(\mathbf{a}) + \alpha(\mathbf{b}) - 1 \le \delta((L_{\mathbf{a}}, R_{\mathbf{b}}), \mathcal{I})$. $\square$

Corollary 1.5. *Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_m)$ be tuples of operators on a Hilbert space H. Let $\mathcal{I}$ be a two-sided ideal of $B(H)$. Suppose $s = \alpha(\mathbf{a}) = \delta(\mathbf{a}) < \infty$ and $t = \alpha(\mathbf{b}^*) = \delta(\mathbf{b}^*) < \infty$. Then*

(i) $\alpha((L_{\mathbf{a}}, R_{\mathbf{b}}), \mathcal{I}) = \delta((L_{\mathbf{a}}, R_{\mathbf{b}}), \mathcal{I}) = 0 \quad$ *if $s = 0$ or $t = 0$;*
(ii) $\alpha((L_{\mathbf{a}}, R_{\mathbf{b}}), \mathcal{I}) = \delta((L_{\mathbf{a}}, R_{\mathbf{b}}), \mathcal{I}) = s + t - 1 \quad$ *if $s, t > 0$.*

Proof. By Theorem 1.1 we have $\alpha(L_{\mathbf{a}}) = \delta(L_{\mathbf{a}}) = s$ and by Theorem 1.2, $\alpha(R_{\mathbf{b}}) = \delta(R_{\mathbf{b}}) = t$. (i) is clear so suppose $s, t > 0$. The operators in $L_{\mathbf{a}}$ commute with the operators in $R_{\mathbf{b}}$ and so from the argument in [8, Proposition 2.1], $(L_{\mathbf{a}}, R_{\mathbf{b}})$ has finite ascent and descent at most $s + t - 1$. Now apply Theorem 1.4. $\square$

Lemma 1.6. *Let* $\mathbf{s} = (s_1, \dots, s_n)$ *and* $\mathbf{t} = (t_1, \dots, t_m)$ *be tuples of linear mappings on a vector space* X *such that* $s_i t_j = t_j s_i$ *for all* i, j. *Let* $\mathbf{st} = (s_i t_j)_{i=1,j=1}^{n,m}$. *Then*

(i) $\alpha(\mathbf{st}) \le \max(\alpha(\mathbf{s}),\, \alpha(\mathbf{t}))$
(ii) $\delta(\mathbf{st}) \le \max(\delta(\mathbf{s}),\, \delta(\mathbf{t}))$

Proof. Suppose $r = \max(\alpha(\mathbf{s}),\, \alpha(\mathbf{t})) < \infty$. If $x \in N(\mathbf{st}) \cap R((\mathbf{st})^r)$ then $t_j(x) \in N(\mathbf{s}) \cap R(\mathbf{s}^r) = \{0\}$ for all j Hence $x \in N(\mathbf{t}) \cap R(\mathbf{t}^r) = \{0\}$. We conclude that $\alpha(\mathbf{st}) \le r$. Considering transpose operators acting on the algebraic conjugate of X (and noting [8, Proposition 3.4]) we have $\delta(\mathbf{st}) = \alpha((\mathbf{st})') = \alpha(\mathbf{s}'\mathbf{t}') \le \max(\alpha(\mathbf{s}'),\, \alpha(\mathbf{t}')) = \max(\delta(\mathbf{s}),\, \delta(\mathbf{t}))$. □

Theorem 1.7. *Let* $\mathbf{a} = (a_1, \dots, a_n)$ *and* $\mathbf{b} = (b_1, \dots, b_m)$ *be tuples of operators on a Hilbert space* H. *Let* $\mathcal{I}$ *be a two-sided ideal of* $B(H)$ *and let* $M_{\mathbf{a},\mathbf{b}} = (M_{a_i,b_j})_{i=1,j=1}^{n,\,m}$. *Then*

(i) $\min(\alpha(\mathbf{a}),\, \alpha(\mathbf{b}^*)) \le \alpha(M_{\mathbf{a},\mathbf{b}},\, \mathcal{I}) \le \max(\alpha(\mathbf{a}),\, \alpha(\mathbf{b}^*))$
(ii) $\min(\delta(\mathbf{a}),\, \alpha(\mathbf{b})) \le \delta(M_{\mathbf{a},\mathbf{b}},\, \mathcal{I}) \le \max(\delta(\mathbf{a}),\, \delta(\mathbf{b}^*))$

Proof. By Lemma 1.6 we have $\alpha(M_{\mathbf{a},\mathbf{b}}) \le \max(\alpha(L_{\mathbf{a}}),\, \alpha(R_{\mathbf{b}}))$ and $\delta(M_{\mathbf{a},\mathbf{b}}) \le \max(\delta(L_{\mathbf{a}}),\, \delta(R_{\mathbf{b}}))$. Now Theorem 1.1 and Theorem 1.2 show the rightmost inequalities in (i) and (ii) hold.

Suppose there are non-zero elements $x \in N(\mathbf{a}) \cap R(\mathbf{a}^s)$ and $y \in N(\mathbf{b}^*) \cap R((\mathbf{b}^*)^t)$. Define $p \in \mathcal{I}$ by $p(z) = \langle z, y\rangle x$. Then $p \in N(M_{\mathbf{a},\mathbf{b}}) \cap R((M_{\mathbf{a},\mathbf{b}})^r)$ where $r = \min(s, t)$. It follows that $\min(\alpha(\mathbf{a}),\, \alpha(\mathbf{b}^*)) \le \alpha(M_{\mathbf{a},\mathbf{b}})$.

Suppose there exists $x \in H$ with $x \notin N(\mathbf{a}^s) + R(\mathbf{a})$ and $y \in N(\mathbf{b}) \cap R(\mathbf{b}^t)$ with $\|y\| = 1$. Define $p \in \mathcal{I}$ by $p(z) = \langle z, y\rangle x$. Let $r = \min(s, t)$ and suppose $N((M_{\mathbf{a},\mathbf{b}})^r) + R(M_{\mathbf{a},\mathbf{b}}) = \mathcal{I}$. Then $p = v + w$ for some $v \in N((M_{\mathbf{a},\mathbf{b}})^r)$ and some $w \in R(M_{\mathbf{a},\mathbf{b}})$. Now $v(y) \in N(\mathbf{a}^s)$ and $w(y) = 0$. Hence $x = p(y) = v(y) \in N(\mathbf{a}^s)$. This is a contradiction and so $\min(s, t) < \delta(M_{\mathbf{a},\mathbf{b}})$. The result follows. □

2. Browder spectrum of $(L_{\mathbf{a}},\, R_{\mathbf{b}})$

Throughout this section H is a complex Hilbert space and we consider multiplication operators acting on $B(H)$ or on a norm ideal $\mathcal{I}$ of $B(H)$. For a general reference on joint spectra we cite [11]. An n-tuple $\mathbf{a} = (a_1, \dots, a_n)$ of operators on a Banach space X is left (resp. right) invertible if there exists $b_1, \dots, b_n \in B(X)$ with $b_1 a_1 + \cdots + b_n a_n = I$ (resp. $a_1 b_1 + \cdots + a_n b_n = I$). Denote by I^{left} the collection of left invertible tuples and by I^{right} the collection of right invertible tuples. Recall the left and right spectra for a tuple of operators are

$$\sigma_l(\mathbf{a}) = \{\lambda \in \mathbb{C}^n : \mathbf{a} - \lambda \notin I^{\text{left}}\}$$
$$\sigma_r(\mathbf{a}) = \{\lambda \in \mathbb{C}^n : \mathbf{a} - \lambda \notin I^{\text{right}}\}$$

An n-tuple $\mathbf{a} = (a_1, \dots, a_n)$ is called upper semi-Fredholm if the column operator $X \to X^n$, $x \mapsto (a_1(x), \dots, a_n(x))$ has finite-dimensional kernel and closed range. An n-tuple $\mathbf{a} = (a_1, \dots, a_n)$ is lower semi-Fredholm if the row operator $X^n \to$

X, $(x_1, \ldots, x_n) \mapsto a_1(x) + \cdots + a_n(x)$ has finite codimensional range. We denote by F^+ and F^- respectively the collection of upper and lower semi-Fredholm tuples. The upper and lower Fredholm spectra are

$$\sigma_e^+(\mathbf{a}, X) = \{\lambda \in \mathbb{C}^n : \mathbf{a} - \lambda \notin F^+\}$$
$$\sigma_e^-(\mathbf{a}, X) = \{\lambda \in \mathbb{C}^n : \mathbf{a} - \lambda \notin F^-\}$$

An n-tuple is called Fredholm if it is both upper and lower semi-Fredholm. We will make use of the following formulae which can be found in [2].

$$\sigma_e^+((L_{\mathbf{a}}, R_{\mathbf{b}}), \mathcal{I}) = \big(\sigma_l(\mathbf{a}) \times \sigma_e^-(\mathbf{b})\big) \cup \big(\sigma_e^+(\mathbf{a}) \times \sigma_r(\mathbf{b})\big) \tag{2.1}$$
$$\sigma_e^-((L_{\mathbf{a}}, R_{\mathbf{b}}), \mathcal{I}) = \big(\sigma_r(\mathbf{a}) \times \sigma_e^+(\mathbf{b})\big) \cup \big(\sigma_e^-(\mathbf{a}) \times \sigma_l(\mathbf{b})\big) \tag{2.2}$$

We denote by B^+ the collection of upper semi-Fredholm tuples with finite ascent and by B^- the collection of lower semi-Fredholm tuples with finite descent. Define

$$\sigma_b^+(\mathbf{a}) = \{\lambda \in \mathbb{C}^n : \mathbf{a} - \lambda \notin B^+\}$$
$$\sigma_b^-(\mathbf{a}) = \{\lambda \in \mathbb{C}^n : \mathbf{a} - \lambda \notin B^-\}$$
$$\sigma_b(\mathbf{a}, X) = \sigma_b^+(\mathbf{a}) \cup \sigma_b^-(\mathbf{a})$$

In [8] it is shown that for commuting tuples σ_b is a compact-valued joint spectrum which satisfies a spectral mapping theorem. We note the following inclusions,

$$\sigma_e^+(\mathbf{a}) \subseteq \sigma_b^+(\mathbf{a}) \subseteq \sigma_l(\mathbf{a})$$
$$\sigma_e^-(\mathbf{a}) \subseteq \sigma_b^-(\mathbf{a}) \subseteq \sigma_r(\mathbf{a})$$

Theorem 2.1. *Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_m)$ be tuples of operators on a complex Hilbert space H and let $\mathcal{I}$ be either $B(H)$ or a norm ideal of $B(H)$. Then $\sigma_b((L_{\mathbf{a}}, R_{\mathbf{b}}), \mathcal{I}) = S_1 \cup S_2$ where*

$$S_1 = \big(\sigma_l(\mathbf{a}) \times \sigma_b^-(\mathbf{b})\big) \cup \big(\sigma_b^+(\mathbf{a}) \times \sigma_r(\mathbf{b})\big)$$
$$S_2 = \big(\sigma_r(\mathbf{a}) \times \sigma_b^+(\mathbf{b})\big) \cup \big(\sigma_b^-(\mathbf{a}) \times \sigma_l(\mathbf{b})\big)$$

Proof. First we show the inclusion $\sigma_b(L_{\mathbf{a}}, R_{\mathbf{b}}) \supseteq S_1 \cup S_2$. Suppose $\lambda \notin \sigma_b(L_{\mathbf{a}}, R_{\mathbf{b}})$. For simplicity we assume $\lambda = 0 \in \mathbb{C}^{n+m}$. Then $(L_{\mathbf{a}}, R_{\mathbf{b}})$ is Fredholm with finite ascent and finite descent. Using (2.1) and (2.2) we have four possible cases to consider:

(i) $0 \notin (\sigma_l(\mathbf{a}) \times \sigma_r(\mathbf{b}))$ and $0 \notin (\sigma_r(\mathbf{a}) \times \sigma_l(\mathbf{b}))$
(ii) $0 \notin \sigma_l(\mathbf{a}) \times \sigma_r(\mathbf{b})$ and $0 \in (\sigma_r(\mathbf{a}) \backslash \sigma_e^-(\mathbf{a})) \times (\sigma_l(\mathbf{b}) \backslash \sigma_e^+(\mathbf{b}))$
(iii) $0 \notin \sigma_r(\mathbf{a}) \times \sigma_l(\mathbf{b})$ and $0 \in (\sigma_l(\mathbf{a}) \backslash \sigma_e^+(\mathbf{a})) \times (\sigma_r(\mathbf{b}) \backslash \sigma_e^-(\mathbf{b}))$
(iv) $0 \in (\sigma_r(\mathbf{a}) \backslash \sigma_e^-(\mathbf{a})) \times (\sigma_l(\mathbf{b}) \backslash \sigma_e^+(\mathbf{b}))$ and $0 \in (\sigma_l(\mathbf{a}) \backslash \sigma_e^+(\mathbf{a})) \times (\sigma_r(\mathbf{b}) \backslash \sigma_e^-(\mathbf{b}))$

If (i) holds then clearly $0 \notin S_1 \cup S_2$. Suppose (ii) holds. Since $0 \in \sigma_r(\mathbf{a})$ we have $\delta(\mathbf{a}) > 0$. Since $0 \in \sigma_l(\mathbf{b})$ and $\mathbf{b}$ is upper semi-Fredholm we have $\alpha(\mathbf{b}) > 0$. Thus by Theorem 1.4, $\delta(\mathbf{a}) + \alpha(\mathbf{b}) - 1 \leq \delta(L_{\mathbf{a}}, R_{\mathbf{b}}) < \infty$. From (ii) we have either $0 \notin \sigma_l(\mathbf{a})$ or $0 \notin \sigma_r(\mathbf{b})$. If $0 \notin \sigma_l(\mathbf{a})$ then $\alpha(\mathbf{a}) = 0$. But this forces $\alpha(\mathbf{a}) = \delta(\mathbf{a}) = 0$ which is a contradiction. Similarly if $0 \notin \sigma_r(\mathbf{b})$ then $\alpha(\mathbf{b}) = \delta(\mathbf{b}) = 0$ which is a contradiction. We conclude that (ii) cannot hold. A similar argument shows that

(iii) cannot hold. If (iv) holds then $\mathbf{a}$ and $\mathbf{b}$ are both Fredholm tuples with non-zero ascent and descent. Since $\mathbf{b}$ is Fredholm we have $\alpha(\mathbf{b}^*) = \delta(\mathbf{b})$ ([8, Proposition 3.5]). Applying Theorem 1.4 we see that $\mathbf{a}$ and $\mathbf{b}$ both have finite ascent and finite descent. Hence $0 \notin S_1 \cup S_2$. This shows that $\sigma_b(L_{\mathbf{a}}, R_{\mathbf{b}}) \supseteq S_1 \cup S_2$.

Next we show the inclusion $\sigma_b(L_{\mathbf{a}}, R_{\mathbf{b}}) \subseteq S_1 \cup S_2$. Suppose $\lambda \notin S_1 \cup S_2$. Again we will assume $\lambda = 0 \in \mathbb{C}^{n+m}$. Using (2.1) and (2.2) we see that $(L_{\mathbf{a}}, R_{\mathbf{b}})$ is Fredholm. To show that $(L_{\mathbf{a}}, R_{\mathbf{b}})$ has finite ascent and descent we consider four possibilities:

(i) $0 \notin (\sigma_l(\mathbf{a}) \times \sigma_r(\mathbf{b}))$ and $0 \notin (\sigma_r(\mathbf{a}) \times \sigma_l(\mathbf{b}))$
(ii) $0 \notin \sigma_l(\mathbf{a}) \times \sigma_r(\mathbf{b})$ and $0 \in (\sigma_r(\mathbf{a})\backslash\sigma_b^-(\mathbf{a})) \times (\sigma_l(\mathbf{b})\backslash\sigma_b^+(\mathbf{b}))$
(iii) $0 \notin \sigma_r(\mathbf{a}) \times \sigma_l(\mathbf{b})$ and $0 \in (\sigma_l(\mathbf{a})\backslash\sigma_b^+(\mathbf{a})) \times (\sigma_r(\mathbf{b})\backslash\sigma_b^-(\mathbf{b}))$
(iv) $0 \in (\sigma_r(\mathbf{a})\backslash\sigma_b^-(\mathbf{a})) \times (\sigma_l(\mathbf{b})\backslash\sigma_b^+(\mathbf{b}))$ and $0 \in (\sigma_l(\mathbf{a})\backslash\sigma_b^+(\mathbf{a})) \times (\sigma_r(\mathbf{b})\backslash\sigma_b^-(\mathbf{b}))$

If (i) holds then we obtain one of the four conditions

$$\alpha(\mathbf{a}) = \delta(\mathbf{a}) = 0 \qquad \alpha(\mathbf{b}^*) = \delta(\mathbf{b}^*) = 0$$
$$\alpha(\mathbf{a}) = \delta(\mathbf{b}^*) = 0 \qquad \alpha(\mathbf{b}^*) = \delta(\mathbf{a}) = 0$$

From Theorem 1.1 and Theorem 1.2, each condition gives $\alpha(L_{\mathbf{a}}, R_{\mathbf{b}}) = \delta(L_{\mathbf{a}}, R_{\mathbf{b}}) = 0$. If (ii) or (iii) holds then we have either $\alpha(\mathbf{a}) = \delta(\mathbf{a}) = 0$ or $\alpha(\mathbf{b}^*) = \delta(\mathbf{b}^*) = 0$. Again this implies $\alpha(L_{\mathbf{a}}, R_{\mathbf{b}}) = \delta(L_{\mathbf{a}}, R_{\mathbf{b}}) = 0$. If (iv) holds then both $\mathbf{a}$ and $\mathbf{b}^*$ are Fredholm tuples with finite non-zero ascent and descent. By Corollary 1.5 we have $\alpha(L_{\mathbf{a}}, R_{\mathbf{b}}) = \delta(L_{\mathbf{a}}, R_{\mathbf{b}}) < \infty$. Hence $0 \notin \sigma_b(L_{\mathbf{a}}, R_{\mathbf{b}})$. This shows $\sigma_b(L_{\mathbf{a}}, R_{\mathbf{b}}) \subseteq S_1 \cup S_2$. □

We use below the standard notation $A \circ B = \{\sum_{i=1}^n \lambda_i \mu_i : \lambda \in A,\ \mu \in B\}$ for subsets A, B of $\mathbb{C}^n$.

Corollary 2.2. *Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ be commuting n-tuples of operators on a complex Hilbert space H and let $\mathcal{I}$ be $B(H)$ or a norm ideal of $B(H)$. Let $\mathcal{E} : \mathcal{I} \to \mathcal{I}$ be the elementary operator $\mathcal{E}(u) = a_1 u b_1 + \cdots + a_n u b_n$. Then*

$$\sigma_b(\mathcal{E}, \mathcal{I}) = \big(\sigma_l(\mathbf{a}) \circ \sigma_b^-(\mathbf{b})\big) \cup \big(\sigma_b^+(\mathbf{a}) \circ \sigma_r(\mathbf{b})\big) \cup \big(\sigma_r(\mathbf{a}) \circ \sigma_b^+(\mathbf{b})\big) \cup \big(\sigma_b^-(\mathbf{a}) \circ \sigma_l(\mathbf{b})\big)$$

Proof. Write $\mathcal{E} = p(L_{\mathbf{a}}, R_{\mathbf{b}})$ where p is the polynomial $p(z_1, \ldots, z_n, w_1, \ldots, w_n) = \sum_{i=1}^n z_i w_i$. Applying the spectral mapping theorem for σ_b ([8, §4]) we obtain $\sigma_b(\mathcal{E}) = p(\sigma_b((L_{\mathbf{a}}, R_{\mathbf{b}}), \mathcal{I}))$ and so the result follows from Theorem 2.1. □

The Taylor–Browder spectrum of a commuting n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ is

$$\sigma_{Tb}(\mathbf{a}) = \operatorname{acc}(\sigma_T(\mathbf{a})) \cup \sigma_{Te}(\mathbf{a})$$

where σ_T is the Taylor spectrum, σ_{Te} is the essential Taylor spectrum and acc denotes the set of accumulation points. The Taylor–Browder spectrum of $(L_{\mathbf{a}}, R_{\mathbf{b}})$ is readily deduced from [5],

$$\sigma_{Tb}((L_{\mathbf{a}}, R_{\mathbf{b}}), \mathcal{I}) = (\sigma_T(\mathbf{a}) \times \sigma_{Tb}(\mathbf{b})) \cup (\sigma_{Tb}(\mathbf{a}) \times \sigma_T(\mathbf{b}))$$

Application of the spectral mapping theorem for σ_{Tb} ([4]) yields

$$\sigma_b(\mathcal{E}, \mathcal{I}) = (\sigma_T(\mathbf{a}) \circ \sigma_{Tb}(\mathbf{b})) \cup (\sigma_{Tb}(\mathbf{a}) \circ \sigma_T(\mathbf{b}))$$

Combining this with Corollary 2.2 and noting inclusions we obtain the formula

$$\sigma_b(\mathcal{E}, \mathcal{I}) = (\sigma_H(\mathbf{a}) \circ \sigma_b(\mathbf{b})) \cup (\sigma_b(\mathbf{a}) \circ \sigma_H(\mathbf{b}))$$

where $\sigma_H = \sigma_l \cup \sigma_r$ denotes the Harte spectrum.

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References

[1] M. Burgos, A. Kaidi, M. Mbekhta, M. Oudghiri, The descent spectrum and perturbations, *J. Operator Theory* 56(2) (2006), 259–271.

[2] A. Carrillo, C. Hernández, Spectra of constructs of a system of operators, *Proc. Amer. Math. Soc.* 91(3) (1984), 426–432.

[3] R.E. Curto, Spectral theory of elementary operators, in: M. Mathieu (ed.), *Elementary operators and applications* (*Blaubeuren* 1991), *World Sci. Publ., River Edge, NJ* (1992), 3–52.

[4] R.E. Curto, A.T. Dash, Browder spectral systems, *Proc. Amer. Math. Soc.* 103(2) (1988), 407–413.

[5] R.E. Curto, L. Fialkow, The spectral picture of $[L_A, R_B]$, *J. Funct. Anal.* 71 (1987), 371–392.

[6] L. Fialkow, Essential spectra of elementary operators, *Trans. Amer. Math. Soc.* 267(1) (1981), 157–174.

[7] R.E. Harte, Tensor products, multiplication operators and the spectral mapping theorem, *Proc. Royal Irish Acad.* 73A (1973), 285–302.

[8] D. Kitson, Ascent and descent for sets of operators, *Studia Math.* 191(2) (2009), 151–161.

[9] G. Lumer, M. Rosenblum, Linear operator equations, *Proc. Amer. Math. Soc.* 10(1) (1959), 32–41.

[10] M. Mathieu, Elementary operators on prime C^*-algebras, I, *Math. Ann.* 284 (1989), 223–244.

[11] V. Müller, *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras*, Birkhäuser Verlag, 2007.

Derek Kitson
School of Mathematics
Trinity College
Dublin 2, Ireland
e-mail: dk@maths.tcd.ie

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