

# Spectral Theory of Pseudo-Differential Operators on $\mathbb{S}^1$

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**Abstract.** For a bounded pseudo-differential operator with the dense domain  $C^\infty(\mathbb{S}^1)$  on  $L^p(\mathbb{S}^1)$ , the minimal and maximal operator are introduced. An analogue of Agmon-Douglis-Nirenberg [1] is proved and then is used to prove the uniqueness of the closed extension of an elliptic pseudo-differential operator of symbol of positive order. We show the Fredholmness of the minimal operator. The essential spectra of pseudo-differential operators on  $\mathbb{S}^1$  are described.

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## 1. Introduction

In this paper the focus is on pseudo-differential operators on the unit circle  $\mathbb{S}^1$  centered at the origin. For  $-\infty < m < \infty$ , let  $S^m(\mathbb{S}^1 \times \mathbb{Z})$  be the set all functions  $\sigma$  in  $C^\infty(\mathbb{S}^1 \times \mathbb{Z})$  such that for all nonnegative integers  $\alpha$  and  $\beta$  there exists a positive constant  $C_{\alpha,\beta}$  for which

$$|(\partial_\theta^\alpha \partial_n^\beta \sigma)(\theta, n)| \leq C_{\alpha,\beta} (1 + |n|)^{m-\beta}, \quad \theta \in [-\pi, \pi], \quad n \in \mathbb{Z}.$$

Let  $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $-\infty < m < \infty$ . Then we define the pseudo-differential operator  $T_\sigma$  on  $L^1(\mathbb{S}^1)$  by

$$(T_\sigma f)(\theta) = \sum_{n \in \mathbb{Z}} e^{in\theta} \sigma(\theta, n) (\mathcal{F}_{\mathbb{S}^1} f)(n), \quad \theta \in [-\pi, \pi],$$

where

$$(\mathcal{F}_{\mathbb{S}^1} f)(n) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-in\theta} f(\theta) d\theta, \quad n \in \mathbb{Z}.$$

Basic properties of pseudo-differential operators with symbols in  $S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $-\infty < m < \infty$ , can be found in [2, 3, 4, 6, 10, 9]. The basic calculi for the

product and the formal adjoint of pseudo-differential operators with symbols in  $S^m(\mathbb{S}^1 \times \mathbb{Z})$  can be found in [9].

A symbol  $\sigma$  in  $S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $-\infty < m < \infty$ , is said to be elliptic if there exist positive constants  $C$  and  $R$  such that

$$|\sigma(\theta, n)| \geq C(1 + |n|)^m, \quad |n| \geq R, \quad \theta \in [-\pi, \pi].$$

The following theorem gives a parametrix for an elliptic pseudo-differential operator with symbol in  $S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $\infty < m < -\infty$ , see [9].

**Theorem 1.1.** *Let  $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $-\infty < m < \infty$  be elliptic. Then there exists a symbol  $\tau \in S^{-m}(\mathbb{S}^1 \times \mathbb{Z})$  such that*

$$T_\sigma T_\tau = I + K \quad \text{and} \quad T_\tau T_\sigma = I + R,$$

where  $K$  and  $R$  are infinitely smoothing in the sense that they are pseudo-differential operators with symbols in  $\cap_{m \in \mathbb{R}} S^m(\mathbb{S}^1 \times \mathbb{Z})$ .

Similar results for the symbol class  $S^m(\mathbb{R}^n \times \mathbb{R}^n)$  of the pseudo-differential operators on  $\mathbb{R}^n$  have been studied for example in [15].

In Section 2, we recall  $L^p$ -Sobolev spaces  $H^{s,p}$ ,  $-\infty < s < \infty$ ,  $1 \leq p \leq \infty$ , and we give some of the results in [7]. Then in Section 3, we consider bounded pseudo-differential operators  $T_\sigma$  on  $L^p(\mathbb{S}^1)$ ,  $1 < p < \infty$  with dense domain  $C^\infty(\mathbb{S}^1)$ . The smallest and largest closed extension of  $T_\sigma$  are provided. The analogue of Agmon-Douglis-Nirenberg [1], is given to prove that for an elliptic symbol  $\sigma$  of positive order  $m$ , the corresponding pseudo-differential operator has a unique closed extension with domain  $H^{m,p}$  on  $L^p(\mathbb{S}^1)$ . In Section 4, we focus on Fredholmness of pseudo-differential operator and its essential spectrum. Results on the Fredholmness of pseudo-differential operators on  $\mathbb{R}^n$  can be found in [16, 13]. By using Theorem 2.9 in [7], we see that the minimal operator of an elliptic pseudo-differential operator of positive order is Fredholm. The essential spectra of the pseudo-differential operator and the minimal (maximal) operator are then provided. Similar results for the SG Pseudo-differential operator on  $\mathbb{R}^n$  are given in [5, 8].

## 2. $L^p$ -Sobolev spaces

For  $-\infty < s < \infty$ , let  $J_s$  be the pseudo-differential operator with symbol  $\sigma_s$  given by

$$\sigma_s(n) = (1 + |n|^2)^{-s/2}, \quad n \in \mathbb{Z}.$$

$J_s$  is called the Bessel potential of order  $s$ .

Now, for  $-\infty < s < \infty$  and  $1 \leq p \leq \infty$ , we define the  $L^p$ -Sobolev space  $H^{s,p}$  to be the set of all tempered distributions  $u$  for which  $J_{-s}u$  is a function in  $L^p(\mathbb{S}^1)$ . Then  $H^{s,p}$  is a Banach space in which the norm  $\|\cdot\|_{s,p}$  is given by

$$\|u\|_{s,p} = \|J_{-s}u\|_{L^p(\mathbb{S}^1)}, \quad u \in H^{s,p}.$$

It is easy to show that for  $-\infty < s, t < \infty$ ,  $J_t$  is an isometry of  $H^{s,p}$  onto  $H^{s+t,p}$ .

The following theorem is known as Sobolev embedding theorem.

**Theorem 2.1.** *Let  $1 < p < \infty$  and  $s \leq t$ . Then  $H^{t,p} \subseteq H^{s,p}$  and*

$$\|u\|_{s,p} \leq \|u\|_{t,p}, \quad u \in H^{t,p}.$$

**Proposition 2.2.** *Let  $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $-\infty < m < \infty$ . Then  $T_\sigma : H^{s,p} \rightarrow H^{s-m,p}$  is a bounded linear operator for  $1 < p < \infty$ .*

**Proposition 2.3.** *Let  $s < t$ . Then the inclusion operator  $i : H^{t,p} \hookrightarrow H^{s,p}$  is compact for  $1 \leq p \leq \infty$ .*

The results above can be found in [7].

### 3. Minimal and maximal operators

Let  $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $m \in \mathbb{R}$ . Then the formal adjoint of  $T_\sigma$ , denoted  $T_\sigma^*$  is a linear operator on  $C^\infty(\mathbb{S}^1)$  such that

$$(T_\sigma \varphi, \psi) = (\varphi, T_\sigma^* \psi), \quad \varphi, \psi \in C^\infty(\mathbb{S}^1).$$

It can be proved that the formal adjoint of  $T_\sigma$  is a pseudo-differential operator of symbol of order  $-m$  (see [10]). The following proposition guarantee that the minimal operator of  $T_\sigma$  exists.

**Proposition 3.1.** *Let  $S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $-\infty < m < \infty$ . Then  $T_\sigma : L^p(\mathbb{S}^1) \rightarrow L^p(\mathbb{S}^1)$  is closable with dense domain  $C^\infty(\mathbb{S}^1)$  for  $1 < p < \infty$ .*

*Proof.* Let  $\{\varphi_k\}_{k=1}^\infty$  be a sequence in  $C^\infty(\mathbb{S}^1)$  such that  $\varphi_k \rightarrow 0$  and  $T_\sigma \varphi_k \rightarrow f$  for some  $f$  in  $L^p(\mathbb{S}^1)$  as  $k \rightarrow \infty$ . We only need to show that  $f = 0$ . We have

$$(T_\sigma \varphi_k, \psi) = (\varphi_k, T_\sigma^* \psi), \quad \psi \in C^\infty(\mathbb{S}^1), \quad k = 1, 2, \dots$$

Let  $k \rightarrow \infty$ , then  $(f, \psi) = 0$  for all  $\psi \in C^\infty(\mathbb{S}^1)$ . By the density of  $C^\infty(\mathbb{S}^1)$  in  $L^p(\mathbb{S}^1)$ , it follows that  $f = 0$ .  $\square$

Consider  $T_\sigma : L^p(\mathbb{S}^1) \rightarrow L^p(\mathbb{S}^1)$  with domain  $C^\infty(\mathbb{S}^1)$ . Then by Proposition 3.1,  $T_\sigma$  has a closed extension. Let  $T_{\sigma,0}$  be the minimal operator of  $T_\sigma$  which is the smallest closed extension of  $T_\sigma$ . Then the domain  $\mathcal{D}(T_{\sigma,0})$  of  $T_{\sigma,0}$  consists of all functions  $u \in L^p(\mathbb{S}^1)$  for which there exists a sequence  $\{\varphi_k\}_{k=1}^\infty$  in  $C^\infty(\mathbb{S}^1)$  such that  $\varphi_k \rightarrow u$  in  $L^p(\mathbb{S}^1)$  and  $T_\sigma \varphi_k \rightarrow f$  for some  $f \in L^p(\mathbb{S}^1)$  in  $L^p(\mathbb{S}^1)$  as  $k \rightarrow \infty$ . It can be shown that  $f$  does not depend on the choice of  $\{\varphi_k\}_{k=1}^\infty$  in  $C^\infty(\mathbb{S}^1)$  and  $T_{\sigma,0}u = f$ .

We define the linear operator  $T_{\sigma,1}$  on  $L^p(\mathbb{S}^1)$  with domain  $\mathcal{D}(T_{\sigma,1})$  by the following. Let  $f$  and  $u$  be in  $L^p(\mathbb{S}^1)$ . Then we say that  $u \in \mathcal{D}(T_{\sigma,1})$  and  $T_{\sigma,1}u = f$  if and only if

$$(u, T_\sigma^* \varphi) = (f, \varphi), \quad \varphi \in C^\infty(\mathbb{S}^1).$$

It can be proved that  $T_{\sigma,1}$  is a closed linear operator from  $L^p(\mathbb{S}^1)$  into  $L^p(\mathbb{S}^1)$  with domain  $\mathcal{D}(T_{\sigma,1})$  containing  $C^\infty(\mathbb{S}^1)$ . In fact,  $C^\infty(\mathbb{S}^1)$  is contained in the domain  $\mathcal{D}(T_{\sigma,1}^t)$  of the true adjoint  $T_{\sigma,1}^t$  of  $T_{\sigma,1}$ . Furthermore,  $T_{\sigma,1}(u) = T_\sigma(u)$  for all  $u$  in  $\mathcal{D}(T_{\sigma,1})$ .

It is easy to see that  $T_{\sigma,1}$  is an extension of  $T_{\sigma,0}$ . In fact  $T_{\sigma,1}$  is the largest closed extension of  $T_\sigma$  in the sense that if  $B$  is any closed extension of  $T_\sigma$  such that  $C^\infty(\mathbb{S}^1) \subseteq \mathcal{D}(B^t)$ , then  $T_{\sigma,1}$  is an extension of  $B$ .  $T_{\sigma,1}$  is called the maximal operator of  $T_\sigma$ . The following theorem is an analogue of Agmon-Douglis-Nirenberg in [1].

**Proposition 3.2.** *Let  $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $m > 0$  be elliptic. Then there exist positive constants  $C$  and  $D > 0$  such that*

$$C\|u\|_{m,p} \leq \|T_\sigma u\|_{L^p(\mathbb{S}^1)} + \|u\|_{L^p(\mathbb{S}^1)} \leq D\|u\|_{m,p}, \quad u \in H^{m,p}.$$

*Proof.* By the boundedness of  $T_\sigma$  in Proposition 2.2 and the boundedness of the inclusion operator in Theorem 2.1, there exists a positive constant  $D$  such that for all  $u \in H^{m,p}$ ,

$$\|T_\sigma u\|_{L^p(\mathbb{S}^1)} + \|u\|_{L^p(\mathbb{S}^1)} \leq D\|u\|_{m,p}, \quad u \in H^{m,p}.$$

Since  $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$  is elliptic, by Theorem 1.1, there exists a symbol  $\tau \in S^{-m}(\mathbb{S}^1 \times \mathbb{Z})$  such that

$$u = T_\tau T_\sigma u - Ru, \quad u \in H^{m,p},$$

where  $R$  is an infinitely smoothing operator in the sense that  $R$  is a pseudo-differential operator with symbol in  $\cap_{m \in \mathbb{R}} S^m(\mathbb{S}^1 \times \mathbb{Z})$ . By using Proposition 2.2 again,  $T_\sigma u \in L^p(\mathbb{S}^1)$ . Therefore,  $T_\tau T_\sigma u \in H^{m,p}$ , for all  $u \in H^{m,p}$ . Moreover there exists a positive constant  $C$  such that

$$\|u\|_{m,p} \leq C(\|T_\sigma u\|_{L^p(\mathbb{S}^1)} + \|u\|_{L^p(\mathbb{S}^1)}), \quad u \in H^{m,p}. \quad \square$$

We have the following result which we use in the next theorem.

**Lemma 3.3.** *Let  $s \in \mathbb{R}$  and  $1 < p < \infty$ . Then  $C^\infty(\mathbb{S}^1)$  is dense in  $H^{s,p}$ .*

*Proof.* Let  $u \in H^{s,p}$ . Then  $J_{-s}u \in L^p(\mathbb{S}^1)$ . Since  $C^\infty(\mathbb{S}^1)$  is dense in  $L^p(\mathbb{S}^1)$ , there exists a sequence  $\{\varphi_k\}_{k=1}^\infty$  in  $C^\infty(\mathbb{S}^1)$  such that  $\varphi_k \rightarrow J_{-s}u$  in  $L^p(\mathbb{S}^1)$  as  $k \rightarrow \infty$ . Let  $\psi_k = J_s \varphi_k$ ,  $k = 1, 2, \dots$ . Then  $\psi_k \in C^\infty(\mathbb{S}^1)$ ,  $k = 1, 2, \dots$ , and

$$\begin{aligned} \|\psi_k - u\|_{s,p} &= \|J_{-s}\psi_k - J_{-s}u\|_{L^p(\mathbb{S}^1)} \\ &= \|\varphi_k - J_{-s}u\|_{L^p(\mathbb{S}^1)} \rightarrow 0, \end{aligned}$$

as  $k \rightarrow \infty$ , which completes the proof.  $\square$

The following theorem gives the domain of the minimal operator of an elliptic pseudo-differential operator with symbol of positive order.

**Theorem 3.4.** *Let  $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $m > 0$ , be elliptic. Then  $\mathcal{D}(T_{\sigma,0}) = H^{m,p}$ .*

*Proof.* Let  $u \in H^{m,p}$ . Then by using the density of  $C^\infty(\mathbb{S}^1)$  in  $H^{m,p}$ , there exists a sequence  $\{\varphi_k\}_{k=1}^\infty$  in  $C^\infty(\mathbb{S}^1)$  such that  $\varphi_k \rightarrow u$  in  $H^{m,p}$  and therefore in  $L^p(\mathbb{S}^1)$  as  $k \rightarrow \infty$ . By Proposition 3.2,  $\varphi_k$  and  $T_\sigma \varphi_k$  are Cauchy sequences in  $L^p(\mathbb{S}^1)$ . Therefore  $\varphi_k \rightarrow u$  and  $T_\sigma \varphi_k \rightarrow f$  for some  $f$  in  $L^p(\mathbb{S}^1)$  as  $k \rightarrow \infty$ . This implies that  $u \in \mathcal{D}(T_{\sigma,0})$  and  $T_{\sigma,0}u = f$ . Now assume that  $u \in \mathcal{D}(T_{\sigma,0})$ . Then there exists a sequence  $\{\varphi_k\}_{k=1}^\infty$  in  $C^\infty(\mathbb{S}^1)$  such that  $\varphi_k \rightarrow u$  in  $L^p(\mathbb{S}^1)$  and  $T_\sigma \varphi_k \rightarrow f$ , for some  $f \in L^p(\mathbb{S}^1)$  as  $k \rightarrow \infty$ . So, by Proposition 3.2,  $\{\varphi_k\}_{k=1}^\infty$  is a Cauchy sequence

in  $H^{m,p}$ . Since  $H^{m,p}$  is complete, there exists  $v \in H^{m,p}$  such that  $\varphi_k \rightarrow v$  in  $H^{m,p}$  as  $k \rightarrow \infty$ . By Sobolev embedding theorem  $\varphi_k \rightarrow v$  in  $L^p(\mathbb{S}^1)$  which implies that  $u = v \in H^{m,p}$ .  $\square$

The following theorem shows that the closed extension of an elliptic pseudo-differential operator on  $L^p(\mathbb{S}^1)$  with symbol  $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $m > 0$ , is unique and moreover by Theorem 3.4, its domain is  $H^{m,p}$ .

**Theorem 3.5.** *Let  $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $m > 0$ , be elliptic. Then  $T_{\sigma,0} = T_{\sigma,1}$ .*

*Proof.* Since  $T_{\sigma,1}$  is a closed extension of  $T_{\sigma,0}$ , by Theorem 3.4, it is enough to show that  $\mathcal{D}(T_{\sigma,1}) \subseteq H^{m,p}$ . Let  $u \in \mathcal{D}(T_{\sigma,1})$ . By ellipticity of  $\sigma$ , there exists  $\tau \in S^{-m}(\mathbb{S}^1 \times \mathbb{Z})$  such that

$$u = T_\tau T_\sigma u - Ru,$$

where  $R$  is an infinitely smoothing operator. Since  $T_\sigma u = T_{\sigma,1}u \in L^p(\mathbb{S}^1)$ , by Proposition 2.2, it follows that  $u \in H^{m,p}$ , which completes the proof.  $\square$

#### 4. Fredholm pseudo-differential operators

A closed linear operator  $A$  from a complex Banach space  $X$  into a complex Banach space  $Y$  with dense domain  $\mathcal{D}(A)$  is said to be Fredholm if

- the range of  $A$ ,  $R(A)$  is closed subspace of  $Y$  and
- the null space of  $A$ ,  $N(A)$  and the null space of the true adjoint of  $A$ ,  $N(A^t)$  are finite dimensional.

The index of a Fredholm operator  $A$  is defined by

$$i(A) = \dim N(A) - \dim N(A^t)$$

By Atkinson's theorem, a closed linear operator  $A : X \rightarrow Y$  with dense domain  $\mathcal{D}(A)$  is Fredholm if and only if there exists a bounded linear operator  $B : Y \rightarrow X$  such that  $K_1 = AB - I : Y \rightarrow Y$  and  $K_2 = BA - I : X \rightarrow X$  are compact operators.

Let  $A : X \rightarrow X$  be a closed linear operator with dense domain  $\mathcal{D}(A)$  in the complex Banach space  $X$ . Then the spectrum of  $A$ ,  $\Sigma(A)$  is defined by

$$\Sigma(A) = \mathbb{C} - \rho(A),$$

where  $\rho(A)$  is the resolvent set of  $A$  given by

$$\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is bijective}\}.$$

The essential spectrum  $\Sigma_w(A)$  of  $A$ , which has been defined in [14] by Wolf given by

$$\Sigma_w(A) = \mathbb{C} - \Phi_w(A), \text{ where } \Phi_w(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is Fredholm}\}.$$

Note that  $i(A - \lambda I)$  is constant for all  $\lambda$  in a connected component of  $\Phi_w(A)$ .

The essential spectrum  $\Sigma_s(A)$  of  $A$  in sense of Schechter [11] is defined by

$$\Sigma_s(A) = \mathbb{C} - \Phi_s(A), \text{ where } \Phi_s(A) = \{\lambda \in \Phi_w(A) : i(A - \lambda I) = 0\}.$$

For the properties of essential spectra see [12]. The following theorem gives a sufficient condition for  $T_\sigma : H^{s,p} \rightarrow H^{s-m,p}$  to be a Fredholm operator. The proof can be found in [7].

**Theorem 4.1.** *Let  $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $-\infty < m < \infty$  be elliptic. Then for all  $-\infty < s < \infty$  and  $1 < p < \infty$ ,  $T_\sigma : H^{s,p} \rightarrow H^{s-m,p}$  is a Fredholm operator. In particular if  $\sigma \in S^0(\mathbb{S}^1 \times \mathbb{Z})$ , then the bounded linear operator  $T_\sigma : L^p(\mathbb{S}^1) \rightarrow L^p(\mathbb{S}^1)$  is Fredholm.*

The following is an immediate corollary of Theorem 3.4 and Theorem 4.1.

**Corollary 4.2.** *Let  $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $m > 0$  be elliptic. Then for  $1 < p < \infty$ ,  $T_{\sigma,0}$  is a Fredholm operator on  $L^p(\mathbb{S}^1)$  with the domain  $H^{m,p}$ .*

The following theorem gives the essential spectrum of an elliptic pseudo-differential operator of positive order.

**Theorem 4.3.** *Let  $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $m > 0$  be elliptic. Then*

$$\Sigma_w(T_{\sigma,0}) = \emptyset.$$

*Proof.* Let  $\lambda \in \mathbb{C}$ . By Corollary 4.2, we need only to show that  $\sigma - \lambda$  is elliptic. The ellipticity of  $\sigma$ , implies that there exist constants  $C, R > 0$  such that

$$|\sigma(\theta, n) - \lambda| \geq C(1 + |n|)^m - |\lambda| = (1 + |n|)^m \left( C - \frac{|\lambda|}{(1 + |n|)^m} \right), \quad \theta \in [-\pi, \pi],$$

whenever  $|n| \geq R$ . Since  $(1 + |n|)^m \rightarrow \infty$  as  $|n| \rightarrow \infty$ , there exists  $M > 0$  such that

$$|\sigma(\theta, n) - \lambda| \geq \frac{C}{2}(1 + |n|)^m, \quad |n| \geq M, \quad \theta \in [-\pi, \pi],$$

which implies that  $\sigma - \lambda$  is elliptic.  $\square$

Let  $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $m \geq 0$ . Then the following theorem is a result on the essential spectra of the bounded pseudo-differential operator  $T_\sigma$  with the domain  $H^{m,p}$  on  $L^p(\mathbb{S}^1)$ .

**Theorem 4.4.** *Let  $\sigma \in S^m(\mathbb{S}^1 \times \mathbb{Z})$ ,  $m \geq 0$ . Then for  $T_\sigma$  on  $L^p(\mathbb{S}^1)$  with the domain  $H^{m,p}$ ,  $1 < p < \infty$ , we have*

$$\Sigma_w(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \geq L_i\},$$

where

$$L_i = \liminf_{|n| \rightarrow \infty} \left\{ \left( \inf_{\theta \in [-\pi, \pi]} |\sigma(\theta, n)| \right) (1 + |n|)^{-m} \right\}.$$

*Proof.* Let  $\lambda \in \mathbb{C}$  be such that  $|\lambda| < L_i$ . Then there exists  $\epsilon > 0$  such that

$$|\lambda| + \epsilon < L_i.$$

Since  $m \geq 0$ , it follows that  $|\lambda| < (L_i - \epsilon)(1 + |n|)^m$ . On the other hand, there exists a positive constant  $R$  such that

$$\inf_{|n| \geq R} \left\{ \left( \inf_{\theta \in [-\pi, \pi]} |\sigma(n, \theta)| \right) (1 + |n|)^{-m} \right\} > L_i - \frac{\epsilon}{2}.$$

So, for  $|n| \geq R$ ,

$$\begin{aligned} |\sigma(\theta, n) - \lambda| &\geq |\sigma(\theta, n)| - |\lambda| \\ &> (L_i - \frac{\epsilon}{2} - L_i + \epsilon)(1 + |n|)^m \\ &= \frac{\epsilon}{2}(1 + |n|)^m, \quad \theta \in [-\pi, \pi]. \end{aligned}$$

Therefore,  $\sigma - \lambda$  is elliptic and hence  $T_\sigma - \lambda I : L^p(\mathbb{S}^1) \rightarrow L^p(\mathbb{S}^1)$  with domain  $H^{m,p}$  is Fredholm. Thus,

$$\{\lambda \in \mathbb{C} : |\lambda| < L_i\} \subseteq \Phi_w(T_\sigma),$$

which implies that

$$\Sigma_w(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \geq L_i\}. \quad \square$$

We have the following theorem on the essential spectrum of a pseudo-differential operator of order 0 from  $L^p(\mathbb{S}^1)$  into  $L^p(\mathbb{S}^1)$ .

**Theorem 4.5.** *Let  $\sigma \in S^0(\mathbb{S}^1 \times \mathbb{Z})$ . Then for  $T_\sigma : L^p(\mathbb{S}^1) \rightarrow L^p(\mathbb{S}^1)$ ,  $1 < p < \infty$ , we have*

$$\Sigma_s(T_\sigma) \subseteq \{\lambda : |\lambda| \leq L_s\},$$

where

$$L_s = \limsup_{|n| \rightarrow \infty} \left\{ \sup_{\theta \in [-\pi, \pi]} |\sigma(\theta, n)| \right\}.$$

*Proof.* Let  $\lambda \in \mathbb{C}$  such that  $|\lambda| > L_s$ . Then there exists  $\epsilon > 0$  such that

$$|\lambda| - \epsilon > L_s,$$

and there exists a positive number  $R$  such that

$$\sup_{|n| \geq R} \left\{ \sup_{\theta \in [-\pi, \pi]} |\sigma(\theta, n)| \right\} < L_s + \frac{\epsilon}{2}.$$

For all  $|n| \geq R$ ,

$$\begin{aligned} |\sigma(\theta, n) - \lambda| &\geq |\lambda| - |\sigma(\theta, n)| \\ &> L_s + \epsilon - L_s - \frac{\epsilon}{2} \\ &= \frac{\epsilon}{2}, \quad \theta \in [-\pi, \pi]. \end{aligned}$$

Hence  $\sigma - \lambda$  is elliptic and by Theorem 4.1,  $T_\sigma - \lambda I : L^p(\mathbb{S}^1) \rightarrow L^p(\mathbb{S}^1)$  is Fredholm. Thus,

$$\{\lambda \in \mathbb{C} : |\lambda| > L_s\} \subseteq \Phi_w(T_\sigma),$$

which is the same as

$$\Sigma_w(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq L_s\}.$$

Since  $\{\lambda \in \mathbb{C} : |\lambda| > L_s\}$  is a connected component of  $\Phi_w(T_\sigma)$ , it follows that  $i(T_\sigma - \lambda I)$  is a constant for all  $\lambda$  in  $\{\lambda \in \mathbb{C} : |\lambda| > L_s\}$ . On the other hand,

$$\rho(T_\sigma) \cap \{\lambda \in \mathbb{C} : |\lambda| > L_s\} \neq \emptyset.$$

Therefore,  $i(T_\sigma - \lambda I) = 0$  for all  $\{\lambda \in \mathbb{C} : |\lambda| > L_s\}$ . This implies that

$$\Sigma_s(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq L_s\}. \quad \square$$

We have the following spectral alternative for a pseudo-differential operator with symbol in  $S^0(\mathbb{S}^1 \times \mathbb{Z})$ .

**Corollary 4.6.** *Let  $\sigma \in S^0(\mathbb{S}^1 \times \mathbb{Z})$  be such that*

$$\limsup_{|n| \rightarrow \infty} \left( \sup_{\theta \in [-\pi, \pi]} |\sigma(\theta, n)| \right) = \liminf_{|n| \rightarrow \infty} \left( \inf_{\theta \in [-\pi, \pi]} |\sigma(\theta, n)| \right) = L > 0.$$

*Then*

$$\Sigma_w(T_\sigma) = \{\lambda \in \mathbb{C} : |\lambda| = L\} \quad \text{or} \quad \Sigma_s(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = L\}.$$

*Proof.* By Theorem 4.4 and Theorem 4.5,

$$\Sigma_w(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = L\}.$$

Suppose that

$$\Sigma_w(T_\sigma) \neq \{\lambda \in \mathbb{C} : |\lambda| = L\}.$$

Then there exists  $\lambda_0 \in \mathbb{C}$  such that  $|\lambda_0| = L$  and  $\lambda_0 \in \Phi_w(T_\sigma)$ . On the other hand, by Theorem 4.5,

$$\{\lambda \in \mathbb{C} : |\lambda| > L\} \subseteq \Phi_s(T_\sigma).$$

Hence using the fact that  $\Phi_w(T_\sigma)$  is an open set and the index of  $T_\sigma - \lambda I$  is constant on every connected component of  $\Phi_w(T_\sigma)$  we get  $i(T_\sigma - \lambda I) = 0$  for all  $\lambda \in \mathbb{C}$  with  $|\lambda| \neq L$ , which is the same as

$$\Sigma_s(T_\sigma) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = L\},$$

as asserted.  $\square$

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