Preface

The discoveries of the past decade have opened new perspectives for the old field of Hamiltonian systems and led to the creation of a new field: symplectic topology. Surprising rigidity phenomena demonstrate that the nature of symplectic mappings is very different from that of volume preserving mappings which raised new questions, many of them still unanswered. On the other hand, due to the analysis of an old variational principle in classical mechanics, global periodic phenomena in Hamiltonian systems have been established. As it turns out, these seemingly different phenomena are mysteriously related. One of the links is a class of symplectic invariants, called symplectic capacities. These invariants are the main theme of this book which grew out of lectures given by the authors at Rutgers University, the RUB Bochum and at the ETH Zürich (1991) and also at the Borel Seminar in Bern 1992. Since the lectures did not require any previous knowledge, only a few and rather elementary topics were selected and proved in detail. Moreover, our selection has been prompted by a single principle: the action principle of mechanics. The action functional for loops in the phase space, given by

$$F(\gamma) \;=\; \int_{\gamma} pdq - \int_{0}^{1} H\Big(t, \gamma(t)\Big)dt \;,$$

differs from the old Hamiltonian principle in the configuration space defined by a Lagrangian. The critical points of F are those loops γ which solve the Hamiltonian equations associated with the Hamiltonian H and hence are the periodic orbits. This variational principle is sometimes called the least action principle. However, there is no minimum for F. Indeed, the action principle is very degenerate. All its critical points are saddle points of infinite Morse index, and at first sight, the principle appears quite useless for existence proofs. But surprisingly it is very effective. This will be demonstrated using several variational techniques starting from minimax arguments due to P. Rabinowitz and ending with A. Floer's homology. The book includes the following subjects:

The introductory chapter presents in a rather unsystematic way some background material. We give the definitions of symplectic manifolds and symplectic mappings and briefly recall the Hamiltonian formalism. For convenience, Cartan's calculus is used. The classification of 2-dimensional symplectic manifolds by the Euler-characteristic and the total volume is proved. Some questions dealt with later on in detail are raised and discussed in special examples. We illustrate the so-called direct method of the calculus of variations in order to establish a periodic orbit on a convex energy surface of a Hamiltonian system in $\mathbb{R}^{2n}$. The Birkhoff invariants are introduced in order to describe without proofs the intricate orbit structure of a Hamiltonian system near an equilibrium point or near a periodic solution. These local results are quite in contrast to the global questions dealt with in the following chapters.

In a systematic way the symplectic invariants, called symplectic capacities, are introduced axiomatically in Chapter 2. Considering the family of all symplectic manifolds of fixed dimension $2n$, a capacity c is a map associating with every symplectic manifold (M, ω) a positive number $c(M, \omega)$ or ∞ satisfying these axioms: a monotonicity axiom for symplectic embeddings, a conformality axiom for the symplectic structure, and a normalization axiom which rules out the volume in higher dimensions. For subsets of $\mathbb{R}^{2n}$, the capacity extends a familiar linear symplectic invariant for positive quadratic forms to nonlinear symplectic mappings. If M and N are symplectically diffeomorphic then $c(M, \omega) = c(N, \tau)$. In view of its monotonicity property a capacity represents, in particular, an obstruction to certain symplectic embeddings and it will be used in order to explain rigidity phenomena for symplectic embeddings, discovered by Ya. Eliashberg and M. Gromov. In particular, Gromov's squeezing theorem is deduced using capacities as well as Eliashberg's C^0-stability of symplectic diffeomorphisms. We introduce a notion of a symplectic homeomorphism, a concept which raises many questions. There are many different capacity functions. For example, the size of the largest ball in $\mathbb{R}^{2n}$ which can be symplectically embedded into a symplectic manifold (M, ω) leads to a special capacity called the Gromov width. It is the smallest capacity function originally introduced by M. Gromov. There are many other "embedding" capacities.

Chapter 3 is devoted entirely to a very detailed construction of a distinguished symplectic capacity c_0. It is dynamically defined by means of Hamiltonian systems. It measures the minimal C^0-oscillation of a Hamiltonian function $H : M \to \mathbb{R}$ which allows to conclude the existence of a fast periodic solution of the corresponding Hamiltonian vector field X_H on M. In the special case of a connected 2-dimensional symplectic manifold, the capacity c_0 agrees with the total area. The existence proof is based on the above action principle which is introduced from scratch in its proper functional analytic framework. The interesting aspect of this principle is that it is bounded neither from below nor from above so that standard variational techniques do not apply directly. Techniques going back to P. Rabinowitz permit us to establish effectively distinguished saddle points of the functional representing special periodic solutions of the system. In the special case of a convex, bounded and smooth domain $U \subset \mathbb{R}^{2n}$, the capacity is represented by a distinguished closed characteristic of its boundary ∂U: it has minimal reduced action equal to $c_0(U)$. But, in general, it is rather difficult to compute the invariant c_0. Some of the recent computations based on more advanced techniques of first order elliptic systems and Fredholm theory are presented without proofs. With the construction of the capacity c_0, the proofs of the rigidity phenomena described in Chapter 2 are complete. Due to its special properties this invariant turns out to be useful also for the dynamics of Hamiltonian systems.

In Chapter 4 the dynamical capacity c_0 is applied to an old question of the qualitative theory of Hamiltonian systems originating in celestial mechanics: does a compact energy surface carry a periodic orbit? We shall demonstrate that many well-known global existence results previously obtained by technically intricate proofs emerge immediately from this invariant. The phenomenon is simply this: if a compact hypersurface in a symplectic manifold possesses a neighborhood of finite capacity c_0, then there are always uncountably many closed characteristics nearby. If one poses, in addition, symplectically invariant restrictions, such as of "contact type", then the hypersurface itself carries a closed characteristic. We shall prove, in particular, the seminal solution of the Weinstein conjecture in $\mathbb{R}^{2n}$ due to C. Viterbo. A nonstandard symplectic torus shows that, in contrast to the Gromov width mentioned above, not every compact symplectic manifold is of finite capacity c_0. Our special example is related to M. Herman's celebrated counterexample to the closing lemma which answers a longstanding open question in dynamical systems. M. Herman's "non-closing-lemma" is proved at the end of the chapter.

In Chapter 5 we study the subgroup $\mathcal{D}$ of symplectic diffeomorphisms of $\mathbb{R}^{2n}$ which are generated by time dependent Hamiltonian vector fields of compact support. The distance from the identity map or the energy $E(\varphi)$ of such a symplectic diffeomorphism φ will be measured by means of the oscillation of its generating Hamiltonian function. This will lead to a surprising bi-invariant metric on $\mathcal{D}$ called the Hofer metric and defined by $d(\varphi, \psi) = E(\varphi^{-1} \circ \psi)$. The definition does not involve derivatives of the Hamiltonian and is of C^0-nature. The verification of the metric property requiring that $d(\varphi, \psi) = 0$ if and only if $\varphi = \psi$ is the difficult aspect. It is based on more refined minimax arguments for the action functional valid simultaneously for a large class of Hamiltonians. We shall investigate the relations of this distinguished metric to the dynamical symplectic invariant c_0 introduced in Chapter 3 and also to another symplectic invariant which is defined for subsets of $\mathbb{R}^{2n}$ and called the displacement energy. The displacement energy of a subset U measures the minimal energy $E(\varphi)$ needed in order to dislocate a given set U from itself in the sense that $U \cap \varphi(U) = \emptyset$. The bi-invariant metric will also be compared with the standard sup-metric. Geodesic arcs associated with the metric will be defined and described in detail. A special example of a geodesic arc is the flow generated by an autonomous Hamiltonian. An important role in our approach is played by the action spectrum of a Hamiltonian mapping $\varphi \in \mathcal{D}$, which turns out to be a nowhere dense subset of the real numbers. Our minimax principle singles out a nontrivial continuous section of the action spectrum bundle over $\mathcal{D}$ called the γ-invariant. This invariant is the main technical tool in this chapter. It allows the characterization of the geodesics and is used also in the existence proof of infinitely many nontrivial periodic points for compactly supported Hamiltonian mappings.

The subject of Chapter 6 is the fixed point theory for Hamiltonian mappings on compact symplectic manifolds (M, ω). It differs from topological fixed point theories. A Hamiltonian map is a special symplectic map: it is homotopic to the identity and the homotopy is generated by the flow of a time dependent Hamiltonian vector field. Prompted by H. Poincaré's last geometric theorem, V.I. Arnold conjectured in the sixties that such a Hamiltonian map possesses at least as many fixed points as a real-valued function on M possesses critical points. Reformulated in terms of dynamical systems, the conjecture asks for a Ljusternik-Schnirelman theory respectively for a Morse theory of forced oscillations solving a time periodic Hamiltonian system on M. We shall first prove the conjecture for the special case of the standard torus T^{2n}. The proof is again based on the action principle. But this time the aim is to find all its critical points. Our strategy is inspired by C. Conley's topological approach to dynamical systems: we shall study the topology of the set of all bounded solutions of the regularized gradient equation belonging to the action functional defined on the set of contractible loops on the manifold M. This way the study of the gradient flow in the infinite dimensional loop space is reduced to the study of a gradient like continuous flow of a compact metric space, whose rest points are the desired critical points. Their number is then estimated by Ljusternik-Schnirelman theory presented in 6.3. A reinterpretation will then lead us to the proof of the Arnold conjecture for the larger class of symplectic manifolds satisfying $[\omega]|\pi_2(M) = 0$. In this general case there is no natural regularization and we are forced to investigate in 6.4 the set of bounded solutions of the non regularized gradient system which now are smooth solutions of a special system of first order elliptic partial differential equations of Cauchy Riemann type. These solutions are related to M. Gromov's pseudoholomorphic curves in M. The compactness of the solution set will be based on an analytical technique which is sometimes called bubbling off analysis. Following this procedure, we shall arrive at the high point of these developments: A. Floer's new approach to Morse theory and Floer homology. We shall merely outline Floer's beautiful ideas in 6.5. A combination of Floer's approach with the construction of the dynamical capacity c_0 results in a symplectic homology theory which is not yet in its final form and which will be sketched without proofs in the last section. The technical requirements of these theories are quite advanced and beyond the scope of this book. Floer's ideas and further related developments will be presented in detail in a sequel. Chapter 6 illustrates, in particular, that old problems emerging from celestial mechanics still lead to powerful new techniques useful also in other branches of mathematics. We should point out that the Arnold conjecture for a general symplectic manifold is still open in the dimensions ≥ 8.

The Appendix contains some technical topics presented for the convenience of the reader. In A.1 we show that a symplectic diffeomorphism can be locally represented in terms of a single function, the so-called generating function. This classical fact is used in Chapter 5. Appendix A.2 illustrates the generating functions in the construction of action-angle coordinates for integrable systems occurring in Chapter 4. A special Sobolev embedding theorem required in the analysis of the action functional (Chapter 5) is proved in A.3. We derive some basic estimates for the Cauchy-Riemann operator on the sphere (A.4), elliptic estimates near the boundary (A.5) and prove the generalized Carleman similarity principle (A.6); all these results for special partial differential equations are important in Chapter 6. While the analytical tools required in the first five chapters are introduced in detail, we make use of topological tools without explanations: we use the Brouwer mapping degree (Chapter 2), the Leray-Schauder degree (Chapter 3), the Smale degree (mod2) and (co-) homology theories (Chapter 6). References concerning these topological topics are given in A.7 and A.8 where we explain the Brouwer degree and the continuity property of the Alexander-Spanier cohomology. This continuity property is important to us for the proof of the Arnold conjecture in the general case.

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