

Chapter 2

Accelerated Motion in Special Relativity

Let a particle of mass m be acted upon by a force F^i in some inertial frame. Then

$$F^i = \frac{dp^i}{dt} = m \frac{\ddot{z}^i}{\dot{t}} = m \frac{\ddot{z}^i}{\gamma}, \quad t = z^0,$$

where p^i is the three-vector portion of the energy-momentum four-vector of the particle in that frame:

$$p^\mu = m \dot{z}^\mu,$$

the world line of the particle being represented in the parametric form $z^\mu(\tau)$ and the dot denoting differentiation with respect to the proper time τ :

$$-1 = \dot{z}^2 = -(\dot{z}^0)^2 + \dot{z}^i \dot{z}^i = -\dot{t}^2 \gamma^{-2},$$

and

$$\gamma = \left(1 - \frac{dz^i dz^i}{dt dt} \right)^{-1/2}.$$

The force that the particle actually ‘feels’ in its own instantaneous rest frame has magnitude given by

$$F_R = ma,$$

where a is the *absolute acceleration* of the particle:

$$a^2 = \ddot{z}^2.$$

In general, $F_R \neq F$, where $F^2 = F^i F^i$. However, when the three-acceleration is parallel (or antiparallel) to the three-velocity, the two magnitudes coincide, for we then have

$$0 = \dot{z} \cdot \ddot{z} = -\dot{z}^0 \ddot{z}^0 + \dot{z}^i \ddot{z}^i = -\dot{t} \left(\ddot{z}^0 \pm \left| \frac{dz}{dt} \right| |\ddot{z}| \right),$$

$$a^2 = -(\ddot{z}^0)^2 + \ddot{z}^i \ddot{z}^i = \left[1 - \left(\frac{dz}{dt} \right)^2 \right] \ddot{z}^2 = \frac{\ddot{z}^2}{\gamma^2} = \frac{F^2}{m^2}.$$

Therefore, a particle which starts from rest under the action of a constant force will experience a constant absolute acceleration.

Let us determine the motion of such a particle under the initial conditions

$$z = 0, \quad \dot{z} = 0, \quad t = 0 \quad \text{at } \tau = 0.$$

Since the motion is in a straight line, we may retain only two coordinates, t and z . We have

$$\begin{aligned} -1 &= -\dot{t}^2 + \dot{z}^2, & \dot{t} &= \sqrt{1 + \dot{z}^2}, \\ 0 &= -\dot{t} \ddot{t} + \dot{z} \ddot{z}, & \ddot{t} &= \frac{\dot{z} \ddot{z}}{\dot{t}}, \\ a^2 &= -\ddot{t}^2 + \ddot{z}^2 = \left(1 - \frac{\dot{z}^2}{\dot{t}^2} \right) \ddot{z}^2 = \frac{\ddot{z}^2}{1 + \dot{z}^2}. \end{aligned}$$

Let $u = \dot{z}$. Then $\ddot{z} = u du/dz$ and

$$a dz = \frac{u du}{\sqrt{1 + u^2}},$$

assuming motion in the positive z direction, whence

$$\begin{aligned} az &= \sqrt{1 + u^2} - 1, & u^2 &= (1 + az)^2 - 1, \\ d\tau &= \frac{dz}{\sqrt{(1 + az)^2 - 1}} = \frac{1}{a} \frac{d(1 + az)}{\sqrt{(1 + az)^2 - 1}}, \\ \tau &= \frac{1}{a} \cosh^{-1}(1 + az), \\ z &= \frac{1}{a} (\cosh a\tau - 1), \\ \dot{z} &= \sinh a\tau, \\ \dot{t} &= \cosh a\tau, \\ t &= \frac{1}{a} \sinh a\tau, \\ z &= \frac{1}{a} \left(\sqrt{1 + a^2 t^2} - 1 \right) \longrightarrow \begin{cases} \frac{1}{2} a t^2 & \text{as } t \rightarrow 0, \\ t & \text{as } t \rightarrow \infty, \end{cases} \\ v \equiv \frac{dz}{dt} &= \frac{at}{\sqrt{1 + a^2 t^2}} \longrightarrow \begin{cases} at & \text{as } t \rightarrow 0, \\ 1 & \text{as } t \rightarrow \infty. \end{cases} \end{aligned}$$

Table 2.1 Comparing proper time and inertial time at different distances during the voyage

τ (year)	z (ly)	t (year)
1	0.0628	1.01
5	1.777	6.41
10	10.26	24.2
15	40.5	85.0
20	146	297
30	1806	3 616
40	22 024	44 052
50	268 000	536 000
60	3 270 000	6 540 000

Problem 4 A cosmic spaceship departs from earth at a constant absolute acceleration of 950 cm/s^2 (slightly less than the acceleration due to gravity at the earth's surface). It maintains this acceleration for $\tau/4$ years of proper time, after which it decelerates at the same rate and in the same direction for another $\tau/4$ years of proper time. At the end of this time it is at rest with respect to the earth, but at a distance of z light years. Its crew at this point executes a certain assigned mission on a nearby planet, which takes a negligible amount of time compared to τ , and then returns to earth by an acceleration–deceleration procedure identical with that of the outward journey. The total voyage has required τ years of proper time. Let t be the number of years that have elapsed on earth since departure. Obtain expressions for z and t in terms of τ , and construct a table giving z and t for selected values of τ ranging from 1 to 60 years. (Hint: express the acceleration in light years per year and use symmetry arguments to simplify the problem Table 2.1)

Solution 4

$$1 \text{ ly/year}^2 = \frac{3 \times 10^{10}}{3.16 \times 10^7} = 950 \text{ cm/s}^2,$$

so $a = 1$. By symmetry, we have

$$z = 2 \left(\cosh \frac{\tau}{4} - 1 \right), \quad t = 4 \sinh \frac{\tau}{4}.$$

Problem 5 Suppose the spaceship of Problem 4 did not attempt to return to earth but merely executed a single acceleration–deceleration maneuver. How far would it have traveled in 50 years of proper flight time, and how much time would have elapsed back on earth?

Solution 5 We have

$$z = 2 \left(\cosh \frac{\tau}{2} - 1 \right), \quad t = 2 \sinh \frac{\tau}{2}.$$

For $\tau = 50$ year, we have

$$t \approx z \approx e^{\tau/2} = e^{25} = 72 \times 10^9 \begin{cases} \text{years,} \\ \text{light years.} \end{cases}$$

Problem 6 A cosmic spaceship makes use of the following propulsion mechanism. During an interval of proper time $d\tau$ the rest mass of the ship decreases by an amount $-dm$. This mass decrement is used in the following way. A fraction ξ ($0 < \xi \leq 1$) is converted into kinetic energy (relative to the ship) of the remaining fraction. This remaining fraction is ejected from the ship in a constant (backward) direction, with the relative velocity v_e corresponding to the kinetic energy it has acquired. Express v_e as a function of ξ . What is the proper impulse dp imparted to the ship during the proper time $d\tau$ as a result of the ejection of the ‘propellant’? (Express it as a function of v_e and dm .) What is the absolute acceleration a experienced by the ship as a result of this impulse?

Suppose the ship starts from rest (relative to some inertial frame) with an initial mass m_0 , and suppose ξ (and hence v_e) remains constant in time. Obtain an expression for the velocity v of the ship at any instant as a function of v_e , m_0 and the mass m remaining at that instant. (Do not assume constant absolute acceleration.) For what value of v_e is the propulsion process most efficient, i.e., what physically allowable value of v_e yields the maximum value of v for a given m and m_0 ? To what value of ξ does this correspond? (To obtain the most efficient propulsion it will be necessary for the ship to carry antimatter as fuel.)

Solution 6 The kinetic energy of the fraction $1 - \xi$ is equated with the rest energy of the fraction ξ to give

$$(1 - \xi)(-dm)(\gamma_e - 1) = \xi(-dm),$$

whence

$$(1 - \xi)(\gamma_e - 1) = \xi, \quad \gamma_e - 1 = \frac{\xi}{1 - \xi}, \quad \gamma_e = \frac{1}{1 - \xi},$$

$$v_e^2 = 1 - \frac{1}{\gamma_e^2} = 1 - (1 - \xi)^2 = \xi(2 - \xi).$$

Therefore,

$$\boxed{v_e = \sqrt{\xi(2 - \xi)}}$$

The proper impulse imparted to the ship during the proper time $d\tau$ is

$$\boxed{dp = (1 - \xi)(-dm)\gamma_e v = (-dm)v_e}$$

The absolute acceleration experienced by the ship as a result of this impulse is then determined from

$$ma = \frac{dp}{d\tau} = -v_e \frac{dm}{d\tau},$$

so

$$a = -v_e \frac{d}{d\tau} \log m$$

We remember that

$$\begin{aligned} a &= \frac{\ddot{z}}{\dot{t}} = \frac{d}{dt} \dot{z} = \frac{d}{dt} (i v) = \frac{d}{dt} (\gamma v) \\ &= \gamma^3 v^2 \frac{dv}{dt} + \gamma \frac{dv}{dt} = \gamma^3 \left(v^2 + \frac{1}{\gamma^2} \right) \frac{dv}{dt} \\ &= \gamma^3 \frac{dv}{dt} = \gamma^2 \frac{dv}{d\tau}, \end{aligned}$$

so that

$$\frac{dv}{1 - v^2} = a d\tau = -v_e d \log m,$$

and finally,

$$\tanh^{-1} v = v_e \log \frac{m_0}{m}.$$

The result is

$$v = \tanh \log \left(\frac{m_0}{m} \right)^{v_e} = \frac{\left(\frac{m_0}{m} \right)^{v_e} - \left(\frac{m}{m_0} \right)^{v_e}}{\left(\frac{m_0}{m} \right)^{v_e} + \left(\frac{m}{m_0} \right)^{v_e}}$$

For the most efficient propulsion, $v_e = 1$, $\zeta = 1$, in which case we have

$$v = \frac{m_0^2 - m^2}{m_0^2 + m^2}$$

Problem 7 Suppose in Problem 6 that v_e is chosen for most efficient propulsion, and suppose fuel is used at such a rate as to maintain constant absolute acceleration. Obtain an expression for the mass m remaining at the proper time τ ($\tau = 0$ when $m = m_0$). Obtain the corresponding expression for m as a function of the time t in the inertial frame in which the spaceship is at rest when $m = m_0$. (Choose the origin of time so that $t = 0$ at this instant.)

Suppose the spaceship executes a single acceleration–deceleration maneuver as in Problem 6, with $a = 950 \text{ cm/s}^2$. Let the total proper time elapsed from the beginning to the end of the maneuver be τ , and let the final mass of the ship (i.e., the *payload*) be m_1 . Construct a table showing the values of z , t , and m_0/m for selected values of τ ranging from 1 to 50 years. Here z and t are respectively the total distance covered and the total time elapsed for the complete voyage in the frame in which the spaceship is at rest when $m = m_0$. Include also a column in your table giving the kinetic energy, in electron volts, of the interstellar hydrogen

Table 2.2 Parameters for Problem 7

τ (year)	z (ly)	t (year)	m_0/m_1	$K_{\max}$ (eV)
1	0.255	1.04	2.718	1.20×10^8
3	2.70	4.26	20.09	1.27×10^9
5	10.26	12.1	148.4	4.81×10^9
10	146	148	22 026	6.86×10^{10}
20	22 024	22 026	4.85×10^8	1.03×10^{13}
30	3 270 000	3 270 000	1.07×10^{13}	1.53×10^{15}
40	485 000 000	485 000 000	2.36×10^{17}	2.27×10^{17}
50	72 000 000 000	72 000 000 000	5.20×10^{21}	3.37×10^{19}

nuclei (protons) as seen from the ship at the midpoint of the journey when the relative velocity of ship and nuclei is a maximum. (Assume the hydrogen to be at rest in the original rest frame.) This is the bombardment energy against which the crew of the ship will have to be shielded (Table 2.2).

Solution 7 We have

$$v_e = 1, \quad a = -\frac{d}{d\tau} \log m, \quad a\tau = \log \frac{m_0}{m},$$

whence

$$m = m_0 e^{-a\tau}$$

Now

$$m = m_0(\cosh a\tau - \sinh a\tau),$$

so

$$m = \begin{cases} m_0(\sqrt{1 + a^2 t^2} - at) & \xrightarrow{t \rightarrow 0} m_0(1 - at), \\ m_0 at \left(\sqrt{1 + \frac{1}{a^2 t^2}} - 1 \right) & \xrightarrow{t \rightarrow \infty} \frac{m_0}{2at}. \end{cases}$$

For the acceleration–deceleration maneuver with $a = 1 \text{ ly/year}^2$, we have (see solution to Problem 6)

$$z = 2 \left(\cosh \frac{\tau}{2} - 1 \right), \quad t = 2 \sinh \frac{\tau}{2}, \quad \frac{m_0}{m_1} = e^\tau,$$

$$\gamma_{\max} = \dot{t}_{\max} = \cosh \frac{\tau}{2},$$

$$K_{\max} = m_p(\gamma_{\max} - 1) = m_p \left(\cosh \frac{\tau}{2} - 1 \right), \quad \text{where } m_p = 0.938 \times 10^9 \text{ eV}.$$

Problem 8 [Taken from *Tau Zero* by Paul Anderson, Doubleday, Garden City, New York (1970).] A spaceship is traveling between galaxies at a velocity v with

respect to the intergalactic gas. (This gas is presumably mainly hydrogen, although it may consist of many other elements as well, including antihydrogen.) The ship is equipped with a scoop of cross-sectional area A with which it traps the gas in its path. The trapped gas is passed through a nuclear furnace which transmutes it (e.g., binding deuterium nuclei into helium, annihilating proton–antiproton pairs, etc.). The reaction products are then ejected, with no loss of total energy (relative to the ship), out the ‘back’ end of the ship. Let dm be the mass of gas trapped by the ship in a proper time interval $d\tau$. This mass arrives with total energy γdm ($\gamma = 1/\sqrt{1-v^2}$) relative to the ship, and the reaction products leave with total energy $(1-\xi)\gamma' dm$ ($\gamma' = 1/\sqrt{1-v'^2}$), where ξ is the fractional decrease of rest mass under the transmutation and v' is the ejection velocity relative to the ship. By equating the two energies, obtain a relation between γ , γ' and ξ , and also an expression for the total proper impulse transmitted to the ship as a result of the transmutation of the mass dm .

If the ship is traveling sufficiently fast, the above process may be used as a kind of ram-jet process whereby the ship is propelled without having to carry its own fuel supply, so that its mass M remains constant. Obtain an expression for the absolute acceleration a imparted to the ship by the ram-jet process as a function of v (or γ), A , ξ , M and the density ρ of the intergalactic gas (in its own rest frame). Obtain the limiting form of this expression as $v \rightarrow 1$, $\gamma \rightarrow \infty$. How big will a be in this limiting case if $A = 10^3 \text{ m}^2$, $M = 10^4 \text{ kg}$, $\xi = 0.01$, and $\rho = 10^{-26} \text{ kg/m}^3$? What implications does your answer have for the feasibility of such a ship? If the origin of proper time is chosen so that the ship’s velocity is v_0 when $\tau = 0$, obtain an expression for v as a function of τ in the special case $\xi = 1$ (total conversion of matter). (Note: in solving this problem do not forget to take into account the compression of the gas, as seen from the ship’s frame, resulting from the Lorentz contraction. This is crucial!)

Solution 8 We have

$$\boxed{(1-\xi)\gamma' = \gamma}$$

and

$$\gamma^2 v'^2 = \gamma'^2 - 1 = \frac{\gamma^2}{(1-\xi)^2} - 1 = \frac{\gamma^2 - 1 + \xi(2-\xi)}{(1-\xi)^2}.$$

Hence,

$$\boxed{\begin{aligned} dp &= [(1-\xi)\gamma'v' - \gamma v] dm \\ &= \left[\sqrt{\gamma^2 - 1 + \xi(2-\xi)} - \sqrt{\gamma^2 - 1} \right] dm \end{aligned}}$$

Since

$$\frac{dm}{d\tau} = A\rho\gamma v = A\rho\sqrt{\gamma^2 - 1},$$

we have

$$\boxed{a = \frac{1}{M} \frac{dp}{d\tau} = \frac{A\rho}{M} (\gamma^2 - 1) \left[\sqrt{1 + \frac{\xi(2 - \xi)}{\gamma^2 - 1}} - 1 \right]}$$

$$\xrightarrow{\gamma \rightarrow \infty} \frac{1}{2} \xi(2 - \xi) \frac{A\rho}{M}$$

In MKS units,

$$a_{\text{lim}} = \frac{1}{2} \xi(2 - \xi) \frac{A\rho c^2}{M} = \frac{1}{2} \times 0.01 \times 1.99 \times \frac{10^3 \times 10^{-26} \times 9 \times 10^{16}}{10^4} \text{ m/s}^2,$$

whence

$$\boxed{a_{\text{lim}} = 9 \times 10^{-13} \text{ m/s}^2 !!}$$

Now

$$\gamma^2 - 1 = \frac{1}{1 - v^2} - 1 = \frac{v^2}{1 - v^2}, \quad 1 + \frac{1}{\gamma^2 - 1} = 1 + \frac{1 - v^2}{v^2} = \frac{1}{v^2}.$$

When $\xi = 1$, we have (see also the solution to Problem 6)

$$\frac{dv}{1 - v^2} = a d\tau = \frac{A\rho}{M} \frac{v^2}{1 - v^2} \left(\frac{1}{v} - 1 \right) d\tau,$$

$$dv = \frac{A\rho}{M} v(1 - v) d\tau,$$

$$\left(\frac{1}{v} + \frac{1}{1 - v} \right) dv = \frac{A\rho}{M} d\tau,$$

$$\log \frac{v}{v_0} - \log \frac{1 - v}{1 - v_0} = \frac{A\rho}{M} \tau, \quad \frac{v}{1 - v} = \frac{v_0}{1 - v_0} e^{A\rho\tau/M},$$

$$\left(1 + \frac{v_0}{1 - v_0} e^{A\rho\tau/M} \right) v = \frac{v_0}{1 - v_0} e^{A\rho\tau/M},$$

$$\boxed{v = \frac{v_0 e^{A\rho\tau/M}}{1 + v_0 (e^{A\rho\tau/M} - 1)} = \frac{1}{1 + \frac{1 - v_0}{v_0} e^{-A\rho\tau/M}}}$$

2.1 Accelerated Meter Stick

Let a meter stick be idealized as a line parallel to the x^1 -axis in a certain Lorentz frame characterized by coordinates x^μ . The points of the meter stick may be labeled by a single parameter ξ . Let $x^i(\xi, t)$ be the coordinates of the point ξ at the time $t(=x^0)$. Suppose that

$$\begin{aligned}
x^1(\xi, t) &= \xi, & 0 \leq \xi \leq 1 & \text{ (range of meter stick),} \\
x^2(\xi, t) &= f(t), & \text{for all } \xi, & \text{where } |f'(t)| < 1 \text{ for all } t, \\
x^3(\xi, t) &= 0, & \text{for all } \xi & \text{and } t.
\end{aligned}$$

Under these conditions, the meter stick always appears to be straight and parallel to the x^1 -axis and to move in the (x^1, x^2) plane in the x^2 direction according to a law of motion given by the arbitrary function $f(t)$. At least that is how it appears in the present Lorentz frame! Note that all points of the meter stick appear to move in unison in the x^2 direction in this frame. Because the concept of simultaneity is frame-dependent, we may expect it to behave in a different fashion in some other Lorentz frame. Let us see how it behaves in a Lorentz frame that moves with velocity $v(<1)$ in the x^1 direction relative to the present frame. The relevant Lorentz transformation is

$$t = \frac{\bar{t} + v\bar{x}^1}{\sqrt{1 - v^2}}, \quad x^1 = \frac{v\bar{t} + \bar{x}^1}{\sqrt{1 - v^2}}, \quad x^2 = \bar{x}^2, \quad x^3 = \bar{x}^3,$$

which yields

$$\begin{aligned}
\bar{x}^1(\xi, \bar{t}) &= \sqrt{1 - v^2}x^1(\xi, t) - v\bar{t} = \sqrt{1 - v^2}\xi - v\bar{t}, \\
\bar{x}^2(\xi, \bar{t}) &= x^2(\xi, t) = f\left(\frac{\bar{t} + v\bar{x}^1(\xi, \bar{t})}{\sqrt{1 - v^2}}\right) = f\left(\sqrt{1 - v^2}\bar{t} + v\xi\right), \\
\bar{x}^3(\xi, \bar{t}) &= x^3(\xi, t) = 0.
\end{aligned}$$

The first of these equations shows the meter stick moving in the $\bar{x}^1$ direction with velocity $-v$ and suffering a Lorentz contraction in that direction. The second equation shows the meter stick also moving in the $\bar{x}^2$ direction, at a rate reduced by the time dilation factor. This equation shows, moreover, that the motion is now not in unison. The points having the greater ξ values lead the others. Although the third equation shows that the motion continues to be in a plane, it is not possible to express the new appearance of the meter stick in terms of a simple tilt in this plane. This would be possible only if the function $f(t)$ were linear. More generally, *the meter stick now ceases to appear as a straight line*. But meter sticks do not bend just because we choose to look at them in a new reference frame! Or do they? In order to examine this question, we must study the general problem of rigidity.

2.2 Rigid Motions in Special Relativity

We shall study first the general motion of an arbitrary continuous medium in spacetime. We shall have occasion to consider continuous media several times in

these lectures, and therefore the formalism developed here will have a utility extending beyond the present context.

Let the component particles of the medium be labeled by three parameters ζ^i , $i = 1, 2, 3$, and let the world line of particle ζ be given by four functions $x^\mu(\zeta, \tau)$, $\mu = 0, 1, 2, 3$, where τ is its proper time. In the general theory of relativity the x^μ may be arbitrary coordinates in curved spacetime, but here we may assume them to be standard coordinates of some Lorentz frame.

Let $\zeta^i + \delta\zeta^i$ be the labels of a neighboring particle. Its world line is given by the functions

$$x^\mu(\zeta + \delta\zeta, \tau) = x^\mu(\zeta, \tau) + x^\mu_{,i}(\zeta, \tau)\delta\zeta^i,$$

where the comma followed by a Latin index denotes partial differentiation with respect to the corresponding ζ . The four-vector $x^\mu_{,i}(\zeta, \tau)\delta\zeta^i$, representing the difference between the two sets of world-line functions, is not generally orthogonal to the world line of ζ . To get such a vector it is necessary to apply the projection tensor on the instantaneous hyperplane of simultaneity:

$$\delta x^\mu \equiv P^\mu_\nu x^\nu_{,i} \delta\zeta^i, \quad P^{\mu\nu} = \eta^{\mu\nu} + \dot{x}^\mu \dot{x}^\nu,$$

where the dot denotes partial differentiation with respect to τ , and we note that in general relativity the projection tensor will take the form $P^{\mu\nu} = g^{\mu\nu} + \dot{x}^\mu \dot{x}^\nu$, with $g^{\mu\nu}$ the metric tensor of spacetime. It is easy to verify that application of the projection tensor corresponds to a simple proper-time shift of amount

$$\delta\tau = \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu_{,i} \delta\zeta^i,$$

so that

$$\delta x^\mu = x^\mu(\zeta + \delta\zeta, \tau + \delta\tau) - x^\mu(\zeta, \tau).$$

The two particles ζ and $\zeta + \delta\zeta$ appear, in the instantaneous rest frame of either, to be separated by a distance δs given by

$$(\delta s)^2 = (\delta x)^2 = \gamma_{ij} \delta\zeta^i \delta\zeta^j,$$

where

$$\gamma_{ij} = P_{\mu\nu} x^\mu_{,i} x^\nu_{,j}.$$

The quantity γ_{ij} is called the *proper metric* of the medium. The medium undergoes *rigid motion* if and only if its proper metric is independent of τ . Under rigid motion the instantaneous separation distance between any pair of neighbouring particles is constant in time.

It is sometimes convenient to express the rigid motion condition $\dot{\gamma}_{ij} = 0$ in terms of derivatives with respect to the coordinates x^μ . Just as the x^μ are functions

of the ζ^i and τ , so, inversely, may the ζ^i and τ be regarded as functions of the x^μ , at least in the domain of spacetime occupied by the medium. We shall write

$$u^\mu \equiv \dot{x}^\mu, \quad u^2 = -1, \quad P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu.$$

If f is an arbitrary function over the domain occupied by the medium then

$$f_{,\mu} = f_{,i}\zeta^i_{,\mu} + \dot{f}\tau_{,\mu},$$

where the comma followed by a Greek index denotes partial differentiation with respect to the corresponding x . We also have

$$\dot{x} \cdot \ddot{x} = 0 \quad \text{or} \quad u \cdot \dot{u} = 0,$$

$$u_\mu u^\mu_{,v} = 0, \quad \dot{u}_\mu = u_{\mu,v} u^v, \quad u_\mu u^\mu_{,i} = 0,$$

$$x^\mu_{,i}\zeta^i_{,v} + \dot{x}^\mu \tau_{,v} = \delta^\mu_v,$$

$$\zeta^i_{,\mu} x^\mu_j = \delta^i_j, \quad \zeta^i_{,\mu} \dot{x}^\mu = 0,$$

$$\tau_{,\mu} x^\mu_{,i} = 0, \quad \tau_{,\mu} \dot{x}^\mu = 1,$$

$$P_{\mu\nu} \dot{x}^v_{,i} = P_{\mu\nu} u^v_{,i} = u_{\mu,i}.$$

We now define the *rate-of-strain tensor* for the medium:

$$\begin{aligned} r_{\mu\nu} &\equiv \dot{\gamma}_{ij} \zeta^i_{,\mu} \zeta^j_{,\nu} \\ &= \left(\dot{P}_{\sigma\tau} x^\sigma_{,i} x^\tau_{,j} + P_{\sigma\tau} \dot{x}^\sigma_{,i} x^\tau_{,j} + P_{\sigma\tau} x^\sigma_{,i} \dot{x}^\tau_{,j} \right) \zeta^i_{,\mu} \zeta^j_{,\nu} \\ &= (\dot{u}_\sigma u_\tau + u_\sigma \dot{u}_\tau) (\delta^\sigma_\mu - u^\sigma \tau_{,\mu}) (\delta^\tau_\nu - u^\tau \tau_{,\nu}) \\ &\quad + u_{\tau,i} \zeta^i_{,\mu} (\delta^\tau_\nu - u^\tau \tau_{,\nu}) + (\delta^\sigma_\mu - u^\sigma \tau_{,\mu}) u_{\sigma,j} \zeta^j_{,\nu} \\ &= \dot{u}_\mu u_\nu + u_\mu \dot{u}_\nu + \dot{u}_\mu \tau_{,\nu} + \tau_{,\mu} \dot{u}_\nu \\ &\quad + u_{\nu,\mu} - \dot{u}_\nu \tau_{,\mu} + u_{\mu,\nu} - \dot{u}_\mu \tau_{,\nu} \\ &= u_{\mu,\sigma} u^\sigma_\nu + u_\mu u^\sigma_{,\nu} + u_{\nu,\sigma} + u_{\mu,\nu} \\ &= P^\sigma_\mu P^\tau_\nu (u_{\sigma,\tau} + u_{\tau,\sigma}). \end{aligned} \tag{2.1}$$

The rate-of-strain tensor is seen to lie completely in the instantaneous hyperplane of simultaneity. It is the relativistic generalization of the nonrelativistic rate-of-strain tensor

$$r_{ij} = v_{i,j} + v_{j,i},$$

where v_i is a three-velocity field and the differentiation is with respect to ordinary Cartesian coordinates. Let us look for a moment at this tensor. The nonrelativistic condition for rigid motion is

$$r_{ij} = 0 \quad \text{everywhere.}$$

This equation implies

$$0 = r_{ij,k} = v_{i,jk} + v_{j,ik}, \quad (2.2)$$

$$0 = r_{jk,i} = v_{j,ki} + v_{k,ji}. \quad (2.3)$$

Subtracting (2.3) from (2.2) and making use of the commutativity of partial differentiation, we find

$$v_{i,jk} - v_{k,ji} = 0, \quad (2.4)$$

which, upon permutation of the indices j and k , yields also

$$v_{i,kj} - v_{j,ki} = 0. \quad (2.5)$$

Adding (2.2) and (2.5) we finally get

$$v_{i,jk} = 0,$$

which has the general solution

$$v_i = -\omega_{ij}x_j + \beta_i, \quad (2.6)$$

where ω_{ij} and β_i are functions of time only. The condition $r_{ij} = 0$ constrains ω_{ij} to be antisymmetric, i.e.,

$$\omega_{ij} = -\omega_{ji},$$

and nonrelativistic rigid motion is seen to be, at each instant, a uniform rotation with angular velocity

$$\omega_i = \frac{1}{2}\epsilon_{ijk}\omega_{jk}$$

about the coordinate origin, superimposed upon a uniform translation with velocity β_i . Because the coordinate origin may be located arbitrarily at each instant, rigid motion may alternatively be described as one in which an arbitrary particle in the medium moves in an arbitrary fashion while at the same time the medium as a whole rotates about this point in an arbitrary (but uniform) fashion. Such a motion has six degrees of freedom.

It turns out that relativistic rigid motion, which is characterized by the condition

$$r_{\mu\nu} = 0 \quad \text{or} \quad \dot{\gamma}_{ij} = 0,$$

has only *three* degrees of freedom! Pick an arbitrary particle in the medium and let it be the origin of the labels ξ^i . Let its world line $x^\mu(0, \tau)$ be arbitrary (but timelike). Introduce a local rest frame for the particle, characterized by an orthonormal triad $n_i^\mu(\tau)$:

$$n_i \cdot n_j = \delta_{ij}, \quad n_i \cdot u_0 = 0, \quad u_0^2 = -1, \quad u_0^\mu \equiv \dot{x}^\mu(0, \tau).$$

Then let the world lines of all the other particles of the medium be given by

$$x^\mu(\xi, \tau) = x^\mu(0, \sigma) + \xi^i n_i^\mu(\sigma), \quad (2.7)$$

where σ is a certain function of the ξ^i and τ . To determine this function, write

$$u^\mu = \dot{x}^\mu(\xi, \tau) = (u_0^\mu + \xi^i \dot{n}_i^\mu) \dot{\sigma},$$

all arguments being suppressed in the final expression. Here and in what follows, it is to be understood that dots over u_0 and the n_i denote differentiation with respect to σ , while the dot over σ denotes differentiation with respect to τ . It will be convenient to expand $\dot{n}_i$ in terms of the orthonormal tetrad u_0, n_i :

$$\dot{n}_i^\mu = a_{0i} u_0^\mu + \Omega_{ij} n_j^\mu.$$

The coefficients a_{0i} are determined, from the identity

$$\dot{n}_i \cdot u_0 + n_i \cdot \dot{u}_0 = 0,$$

to be just the components of the absolute acceleration of the particle $\xi = 0$ in its local rest frame:

$$a_{0i} = n_i \cdot \dot{u}_0,$$

and the identity

$$\dot{n}_i \cdot n_j + n_i \cdot \dot{n}_j = 0$$

tells us that Ω_{ij} is antisymmetric:

$$\Omega_{ij} = -\Omega_{ji}.$$

We now have

$$u^\mu = \left[(1 + \xi^i a_{0i}) u_0^\mu + \xi^i \Omega_{ij} n_j^\mu \right] \dot{\sigma}.$$

But

$$-1 = u^2 = - \left[(1 + \xi^i a_{0i})^2 - \xi^i \xi^j \Omega_{ik} \Omega_{jk} \right] \dot{\sigma}^2,$$

whence

$$\dot{\sigma} = \left[(1 + \xi^i a_{0i})^2 - \xi^i \xi^j \Omega_{ik} \Omega_{jk} \right]^{-1/2}. \quad (2.8)$$

The right hand side of this equation is a function solely of σ and the ξ^i . Therefore the equation may be integrated along each world line $\xi = \text{const.}$, subject, say, to the boundary condition

$$\sigma(\xi, 0) = 0.$$

We shall, in particular, have the necessary condition

$$\sigma(0, \tau) = \tau.$$

We note that the medium must be confined to regions where

$$(1 + \xi^i a_{0i})^2 > \xi^i \Omega_{ik} \xi^j \Omega_{jk} \quad (\geq 0).$$

Otherwise, some of its component particles will be moving faster than light.

Let us now compute the proper metric of the medium. We have

$$n_i \cdot u = -\Omega_{ij} \xi^j \dot{\sigma}, \quad (2.9)$$

$$\begin{aligned} x_{,i}^\mu &= n_i^\mu + (u_0^\mu + \xi^j \dot{n}_j^\mu) \sigma_{,i} = n_i^\mu + u^\mu \dot{\sigma}^{-1} \sigma_{,i}, \\ u_{\mu} x_{,i}^\mu &= -\Omega_{ij} \xi^j \dot{\sigma} - \dot{\sigma}^{-1} \sigma_{,i}, \\ \gamma_{ij} &= P_{\mu\nu} x_{,i}^\mu x_{,j}^\nu \\ &= \delta_{ij} - \Omega_{ik} \xi^k \sigma_{,j} - \Omega_{jk} \xi^k \sigma_{,i} - \dot{\sigma}^{-2} \sigma_{,i} \sigma_{,j} \\ &\quad + (\Omega_{ik} \xi^k \dot{\sigma} + \dot{\sigma}^{-1} \sigma_{,i}) (\Omega_{jl} \xi^l \dot{\sigma} + \dot{\sigma}^{-1} \sigma_{,j}) \\ &= \delta_{ij} + \dot{\sigma}^2 \Omega_{ik} \Omega_{jl} \xi^k \xi^l. \end{aligned} \quad (2.10)$$

From this expression and the expression (2.8) for $\dot{\sigma}$ on the preceding page, we see that the only way in which the motion of the medium can be rigid is either for all the Ω_{ij} to vanish or for all the Ω_{ij} , together with the a_{0i} , to be constants, independent of σ . In the latter case the motion is one of a six-parameter family (the Ω_{ij} and the a_{0i} are the parameters) of special motions known as *superhelical motions*, of which we shall study one simple example later (constant rotation about a fixed axis). For the present we concentrate on the case in which all the Ω_{ij} vanish.

2.3 Fermi–Walker Transport

When the Ω_{ij} vanish the triad n_i^μ is said to be *Fermi–Walker transported* along the world line of the particle $\xi = 0$. More generally, any tensor whose components relative to the tetrad u_0^μ , n_i^μ remain constant along the world line $\xi = 0$ is said to be Fermi–Walker transported along that world line. It is sometimes convenient to express the condition for Fermi–Walker transport without reference to the triad n_i^μ . Writing, for a vector A^μ along the world line,

$$A^\mu = A_u u_0^\mu + A_i n_i^\mu,$$

where

$$A_u = -A \cdot u_0, \quad A_i a_{0i} = A \cdot \dot{u}_0,$$

we have, if A^μ is Fermi–Walker transported,

$$\dot{A}^\mu = A_\mu \dot{u}_0^\mu + A_i a_{0i} u_0^\mu = A \cdot (\dot{u}_0 u_0^\mu - u_0 \dot{u}_0^\mu),$$

an equation that admits of immediate generalization to tensors of arbitrary rank.

It is not possible to maintain the orientation (in spacetime) of the local-rest-frame triad n_i^μ constant along a world line unless that world line is straight. Under Fermi–Walker transport, however, the triad remains as constantly oriented, or as *rotationless*, as possible. The components of the $\dot{n}_i$ all vanish in the instantaneous hyperplane of simultaneity.

For a general non-Fermi–Walker transported triad, the Ω_{ij} are the components of the angular-velocity tensor that describes the instantaneous rate of rotation of the triad in the instantaneous hyperplane of simultaneity. The general motion of the medium introduced in (2.7) on p. 25 may be described formally as one in which the particle $\xi = 0$ moves in an arbitrary fashion and the medium as a whole executes an arbitrary rotation about this particle. But only if the rotation is absent is this motion truly rigid. Rigid motion in special relativity therefore possesses only the three degrees of freedom that the particle $\xi = 0$ itself possesses. Even these three degrees of freedom are not always attainable. In the case of superhelical motion, there are no degrees of freedom at all. Once the medium gets into superhelical motion, it must remain frozen into it if it wants to stay rigid.

Problem 9 A particle undergoes acceleration dv/dt in a certain inertial frame. ‘Attached’ to this particle is a four-vector S^μ that is orthogonal to the particle world line and Fermi–Walker transported along this line. The four-vector therefore satisfies the equations

$$S \cdot u = 0, \quad \dot{S}^\mu = (S \cdot \dot{u}) u^\mu,$$

where u is the particle’s four-velocity and the dot denotes differentiation with respect to the proper time. Instead of dealing with S^μ , it is often convenient to work with the three-vector part of

$$\bar{S}^\mu = L_v^\mu S^\nu,$$

where L_v^μ is the Lorentz boost transformation to the local rest frame of the particle:

$$(L_v^\mu) = \begin{pmatrix} \gamma & -\gamma \mathbf{v} \\ -\gamma \mathbf{v} & \underline{1} + (\gamma - 1) \hat{\mathbf{v}} \hat{\mathbf{v}} \end{pmatrix},$$

where

$$\gamma = (1 - v^2)^{-1/2}, \quad \hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|, \quad \underline{1} = \text{unit dyadic}.$$

Show that the boost transformation is indeed a Lorentz transformation and that the inverse transformation $L_v^{-1\mu}$ back to the original frame is obtained from L_v^μ by making the replacement $\mathbf{v} \rightarrow -\mathbf{v}$. Show that in the boosted frame (rest frame), we have $\bar{S}^0 = 0$ and

$$\bar{u}^\mu = L_\nu^\mu u^\nu = (1, 0, 0, 0).$$

In the boosted frame, we may write

$$\dot{\bar{S}}^\mu = \dot{L}_\nu^\mu S^\nu + L_\nu^\mu \dot{S}^\nu = \dot{L}_\nu^\mu L_\sigma^{-1\nu} \bar{S}^\sigma + (S \cdot \dot{u}) \bar{u}^\mu,$$

of which the three-vector part reduces to

$$\dot{\bar{S}}^i = \dot{L}_\mu^i L_j^{-1\mu} \bar{S}^j.$$

By straightforwardly computing $\dot{L}_\mu^i L_j^{-1\mu}$, show that this equation may be rewritten in the three-vector language

$$\frac{d\bar{\mathbf{S}}}{dt} = \boldsymbol{\Omega} \times \bar{\mathbf{S}},$$

and obtain an expression for $\boldsymbol{\Omega}$ in terms of γ , $\mathbf{v}$, and $d\mathbf{v}/dt$. Suppose that the particle moves with constant angular velocity ω around a circle of radius a . Obtain an expression for the precession frequency $|\boldsymbol{\Omega}|$ of the three-vector $\bar{\mathbf{S}}$ under these circumstances, and show that the precession is retrograde.

Solution 9 Note first that

$$\begin{aligned} & \begin{pmatrix} \gamma & -\gamma \mathbf{v} \\ -\gamma \mathbf{v} & \mathbf{1} + (\gamma - 1) \hat{\mathbf{v}} \hat{\mathbf{v}} \end{pmatrix} \begin{pmatrix} \gamma & \gamma \mathbf{v} \\ \gamma \mathbf{v} & \mathbf{1} + (\gamma - 1) \hat{\mathbf{v}} \hat{\mathbf{v}} \end{pmatrix} \\ &= \begin{pmatrix} \gamma^2 - \gamma^2 v^2 & \gamma^2 \mathbf{v} - \gamma \mathbf{v} - \gamma(\gamma - 1) \mathbf{v} \\ -\gamma^2 \mathbf{v} + \gamma \mathbf{v} + \gamma(\gamma - 1) \mathbf{v} & -\gamma^2 \mathbf{v} \mathbf{v} + \mathbf{1} + 2(\gamma - 1) \hat{\mathbf{v}} \hat{\mathbf{v}} + (\gamma - 1)^2 \hat{\mathbf{v}} \hat{\mathbf{v}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \end{aligned}$$

because

$$-v^2 \gamma^2 + 2(\gamma - 1) + (\gamma - 1)^2 = (1 - v^2) \gamma^2 - 1 = 0.$$

Therefore, $L_\nu^{-1\mu}$ is indeed obtained by making the replacement $\mathbf{v} \rightarrow -\mathbf{v}$. Now the condition that $\dot{L}_\nu^\mu$ be a Lorentz transformation may be expressed in the form

$$\eta_{\mu\nu} L_\sigma^\mu L_\tau^\nu = \eta_{\sigma\tau},$$

or, dropping indices,

$$L^T \eta L = \eta, \quad L^T \eta = \eta L^{-1}, \quad L = \eta L^{-1T} \eta,$$

where the superscript T denotes transpose. But, in virtue of the form of η and the symmetry of L^{-1} in the present case, the last equation is obviously satisfied.

To show that $\bar{S}^0$ vanishes, we first show that

$$\begin{aligned} (\bar{u}^\mu) &= (L_v^\mu u^\nu) = \begin{pmatrix} \gamma & -\gamma \mathbf{v} \\ -\gamma \mathbf{v} & \underline{1} + (\gamma - 1) \hat{\mathbf{v}} \hat{\mathbf{v}} \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma \mathbf{v} \end{pmatrix} \\ &= \begin{pmatrix} \gamma^2(1 - v^2) \\ [-\gamma^2 + \gamma + \gamma(\gamma - 1)]\mathbf{v} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

From this it follows immediately that

$$0 = S \cdot u = \bar{S} \cdot \bar{u} = -\bar{S}^0.$$

Finally,

$$\begin{aligned} (\dot{L}_\mu^i) &= \left(-\gamma^3(\mathbf{v} \cdot \dot{\mathbf{v}})\mathbf{v} - \gamma \dot{\mathbf{v}}, \gamma^3(\mathbf{v} \cdot \dot{\mathbf{v}})\hat{\mathbf{v}}\hat{\mathbf{v}} + (\gamma - 1)\left(\frac{\dot{\mathbf{v}}\mathbf{v}}{v^2} + \frac{\mathbf{v}\dot{\mathbf{v}}}{v^2} - 2\frac{\mathbf{v} \cdot \dot{\mathbf{v}}}{v^2}\hat{\mathbf{v}}\hat{\mathbf{v}}\right) \right), \\ (L_j^{-1\mu}) &= \begin{pmatrix} \gamma \mathbf{v} \\ \underline{1} + (\gamma - 1)\hat{\mathbf{v}}\hat{\mathbf{v}} \end{pmatrix}, \\ (\dot{L}_\mu^i L_j^{-1\mu}) &= -\gamma^4(\mathbf{v} \cdot \dot{\mathbf{v}})\mathbf{v}\mathbf{v} - \gamma^2 \dot{\mathbf{v}}\mathbf{v} + \gamma^3(\mathbf{v} \cdot \dot{\mathbf{v}})\hat{\mathbf{v}}\hat{\mathbf{v}} \\ &\quad + (\gamma - 1)\left(\frac{\dot{\mathbf{v}}\mathbf{v}}{v^2} + \frac{\mathbf{v}\dot{\mathbf{v}}}{v^2} - 2\frac{\mathbf{v} \cdot \dot{\mathbf{v}}}{v^2}\hat{\mathbf{v}}\hat{\mathbf{v}}\right) + \gamma^3(\gamma - 1)(\mathbf{v} \cdot \dot{\mathbf{v}})\hat{\mathbf{v}}\hat{\mathbf{v}} \\ &\quad + (\gamma - 1)^2\left(\frac{\dot{\mathbf{v}}\mathbf{v}}{v^2} + \frac{\mathbf{v}\dot{\mathbf{v}}}{v^2}\hat{\mathbf{v}}\hat{\mathbf{v}} - 2\frac{\mathbf{v} \cdot \dot{\mathbf{v}}}{v^2}\hat{\mathbf{v}}\hat{\mathbf{v}}\right) \\ &= \left[-\gamma^4 v^2 + \gamma^3 - 2\frac{\gamma - 1}{v^2} + \gamma^3(\gamma - 1) - \frac{(\gamma - 1)^2}{v^2} \right] (\mathbf{v} \cdot \dot{\mathbf{v}})\hat{\mathbf{v}}\hat{\mathbf{v}} \\ &\quad + \left[-\gamma^2 + \frac{\gamma - 1}{v^2} + \frac{(\gamma - 1)^2}{v^2} \right] \dot{\mathbf{v}}\mathbf{v} + \frac{\gamma - 1}{v^2} \mathbf{v}\dot{\mathbf{v}} \\ &= \left(\gamma^2 - \frac{\gamma^2 - 1}{v^2} \right) (\mathbf{v} \cdot \dot{\mathbf{v}})\hat{\mathbf{v}}\hat{\mathbf{v}} + (-\gamma^2 + 1 + \gamma^2 - \gamma) \frac{\dot{\mathbf{v}}\mathbf{v}}{v^2} + \frac{\gamma - 1}{v^2} \mathbf{v}\dot{\mathbf{v}} \\ &= \frac{\gamma - 1}{v^2} (\mathbf{v}\dot{\mathbf{v}} - \dot{\mathbf{v}}\mathbf{v}), \end{aligned}$$

whence

$$\dot{\bar{S}} = \frac{\gamma - 1}{v^2} (\mathbf{v}\dot{\mathbf{v}} - \dot{\mathbf{v}}\mathbf{v}) \cdot \bar{S},$$

or

$$\frac{d\bar{S}}{dt} = \Omega \times \bar{S},$$

where

$$\boxed{\Omega = -\frac{\gamma-1}{v^2} \mathbf{v} \times \frac{d\mathbf{v}}{dt} = -(\gamma-1) \hat{\mathbf{v}} \times \frac{d\hat{\mathbf{v}}}{dt}}$$

In the case of the particle moving in a circle, we have

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \boldsymbol{\omega} \times \mathbf{v}, & \boldsymbol{\omega} \cdot \mathbf{v} &= 0, \\ \Omega &= -\frac{\gamma-1}{v^2} \mathbf{v} \times (\boldsymbol{\omega} \times \mathbf{v}) = -\frac{\gamma-1}{v^2} [v^2 \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{v}) \mathbf{v}], \end{aligned}$$

so that

$$\boxed{\Omega = -(\gamma-1) \boldsymbol{\omega}}$$

where

$$|\mathbf{v}| = a|\boldsymbol{\omega}|, \quad \boxed{\gamma = (1 - a^2 \omega^2)^{-1/2}}$$

2.4 Flat Proper Geometry

When the Ω_{ij} vanish the *proper geometry* of the medium is *flat* [see (2.10) on p. 26]:

$$\gamma_{ij} = \delta_{ij}.$$

Moreover, we have [see (2.9) on p. 26]

$$n_i \cdot u = 0,$$

so that the instantaneous hyperplane of simultaneity of the particle at $\xi = 0$ is an instantaneous hyperplane of simultaneity for all the other particles of the medium as well, and the triad n_i^μ serves to define a rotationless rest frame for the whole medium. In other words, the coordinate system defined by the parameters ξ^i may itself be regarded as being Fermi–Walker transported, and all the particles of the medium have a common designator of simultaneity in the parameter σ . Because σ is not generally equal to τ , however, it is not possible for the particles to have a common synchronization of standard clocks. The relation between σ and τ is given by (2.8) on p. 25 as

$$\dot{\sigma} = (1 + \xi^i a_{0i})^{-1},$$

which permits us to compute the absolute acceleration a_i of an arbitrary particle in terms of a_{0i} and the ξ^i :

$$\begin{aligned}
a_i &= n_i \cdot \dot{u} = n_i \cdot \frac{\partial u}{\partial \sigma} \dot{\sigma} = \dot{\sigma} n_i \cdot \frac{\partial}{\partial \sigma} [(1 + \xi^j a_{0j}) u_0 \dot{\sigma}] \\
&= \dot{\sigma}^2 (1 + \xi^j a_{0j}) n_i \cdot \dot{u}_0 \\
&= \frac{a_{0i}}{1 + \xi^j a_{0j}}.
\end{aligned}$$

We see that, although the motion is rigid and ‘rotationless’, not all parts of the medium ‘feel’ the same acceleration.

When the Ω_{ij} vanish it is sometimes convenient to make use of σ and ξ^i as coordinates of spacetime. In these coordinates, the metric tensor takes the form

$$\begin{aligned}
g_{00} &= \frac{\partial x^\mu}{\partial \sigma} \bigg|_{\xi} \frac{\partial x^\nu}{\partial \sigma} \bigg|_{\xi} \eta_{\mu\nu} = u^2 \dot{\sigma}^{-2} = -(1 + \xi^i a_{0i})^2, \\
g_{i0} &= g_{0i} = \frac{\partial x^\mu}{\partial \xi^i} \bigg|_{\sigma} \frac{\partial x^\nu}{\partial \sigma} \bigg|_{\xi} \eta_{\mu\nu} = (n_i \cdot u) \dot{\sigma}^{-1} = 0, \\
g_{ij} &= \frac{\partial x^\mu}{\partial \xi^i} \bigg|_{\sigma} \frac{\partial x^\nu}{\partial \xi^j} \bigg|_{\sigma} \eta_{\mu\nu} = n_i \cdot n_j = \delta_{ij},
\end{aligned}$$

which has a simple diagonal structure. We note that this metric becomes *static*, i.e., time-independent, with the parameter σ now playing the role of ‘time’, in the special case in which the acceleration of each particle is constant.

Problem 10 The Rotationless Constantly Accelerating Medium

Suppose the particle at $\xi = 0$ undergoes constant absolute acceleration from rest in the x^1 direction in some inertial frame. One may choose initial conditions in such a way that this motion takes the form

$$x^0(0, \sigma) = \frac{1}{a} \sinh a\sigma, \quad x^1(0, \sigma) = \frac{1}{a} \cosh a\sigma, \quad x^2(0, \sigma) = 0 = x^3(0, \sigma).$$

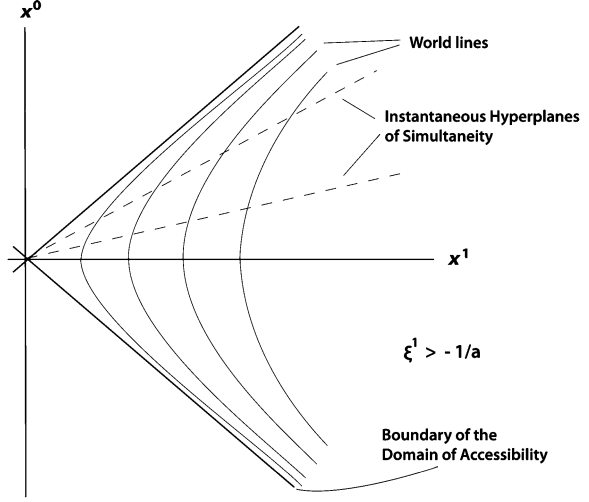
Introduce a convenient Fermi–Walker transported triad with which to define the local rest frame of the particle, and let the spacetime coordinates of the remaining particles of the medium be defined, in terms of the σ and the ξ^i , as above. Obtain σ as a function of τ under the boundary condition $\sigma = 0$ when $\tau = 0$. Obtain also explicit forms for the functions $x^\mu(\xi, \tau)$ as well as the metric of spacetime in the coordinate system σ, ξ^i . Draw a flow diagram in the (x^0, x^1) plane, showing the world lines of the particles of the medium. Draw on this diagram some instantaneous hyperplanes of simultaneity and indicate the maximum region of spacetime accessible to the medium (Fig. 2.1).

Solution 10 We have

$$u_0^0 = \cosh a\sigma, \quad u_0^1 = \sinh a\sigma, \quad u_0^2 = 0 = u_0^3.$$

We may evidently choose

Fig. 2.1 Flow diagram of a rigid medium in inertial coordinates



$n_1^0 = \sinh a\sigma$	$n_2^0 = 0$	$n_3^0 = 0$
$n_1^1 = \cosh a\sigma$	$n_2^1 = 0$	$n_3^1 = 0$
$n_1^2 = 0$	$n_2^2 = 1$	$n_3^2 = 0$
$n_1^3 = 0$	$n_2^3 = 0$	$n_3^3 = 1$

$$a_{01} = n_1 \cdot \dot{u}_0 = a, \quad a_{02} = n_2 \cdot \dot{u}_0 = 0, \quad a_{03} = n_3 \cdot \dot{u}_0 = 0,$$

$$\dot{\sigma} = (1 + a\xi^1)^{-1}, \quad \boxed{\sigma = \frac{\tau}{1 + a\xi^1}}$$

$x^0(\xi, \tau) = \frac{1}{a} \sinh a\sigma + \xi^1 \sinh a\sigma = \frac{1 + a\xi^1}{a} \sinh \frac{a\tau}{1 + a\xi^1}$ $x^1(\xi, \tau) = \frac{1}{a} \sigma + \xi^1 \cosh a\sigma = \frac{1 + a\xi^1}{a} \cosh \frac{a\tau}{1 + a\xi^1}$ $x^2(\xi, \tau) = \xi^2$ $x^3(\xi, \tau) = \xi^3$
--

$$\boxed{(g_{\mu\nu})_{\sigma, \xi} = \text{diag} \left(-(1 + a\xi^1)^2, 1, 1, 1 \right)}$$

2.5 Constant Rotation About a Fixed Axis

The simplest example of a medium undergoing rigid rotation is obtained by choosing

$$a_{0i} = 0, \quad \Omega_{12} = \omega, \quad \Omega_{23} = 0 = \Omega_{31}.$$

The world line of the particle at $\xi = 0$ is then straight, but the world lines of all the other particles are helices of constant pitch. We have

$$\dot{\sigma} = \left\{ 1 - \omega^2 \left[(\xi^1)^2 + (\xi^2)^2 \right] \right\}^{-1/2}$$

and the proper metric of the medium takes the form

$$(\gamma_{ij}) = \begin{pmatrix} 1 + (\dot{\sigma}\omega\xi^2)^2 & -(\dot{\sigma}\omega)^2\xi^1\xi^2 & 0 \\ -(\dot{\sigma}\omega)^2\xi^1\xi^2 & 1 + (\dot{\sigma}\omega\xi^1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is convenient to relabel the particles by means of three new coordinates r, θ, z given by

$$\xi^1 = r \cos \theta, \quad \xi^2 = r \sin \theta, \quad \xi^3 = z.$$

In terms of these coordinates the proper distance δs between two particles separated by displacements $\delta r, \delta \theta$, and δz takes the form

$$\delta s^2 = (\delta r)^2 + \frac{r^2}{1 - \omega^2 r^2} (\delta \theta)^2 + (\delta z)^2.$$

The second term on the right of this equation may be understood as arising from the Lorentz contraction phenomenon. The problem of the rotating medium is sometimes posed as the so-called ‘spinning disc paradox’ and stated as follows. A disc of radius r is set spinning with angular frequency ω about its axis. Radial distances are unaffected, but distances in the direction of rotation become Lorentz contracted. In particular, the circumference of the disc gets reduced to the value $2\pi R\sqrt{1 - \omega^2 R^2}$. But this contradicts the Euclidean nature of the ordinary three-space that the disc inhabits!

What in fact happens is that, when set in rotation, the disc must suffer a strain that arises for kinematic reasons quite apart from any strains it suffers on account of centrifugal forces. In particular, it must undergo a stretching of amount $(1 - \omega^2 r^2)^{-1/2}$ in the direction of rotation, to compensate the Lorentz contraction factor $(1 - \omega^2 r^2)^{1/2}$ that appears when the disc is viewed in the inertial rest frame of its axis, thereby maintaining the Euclidean nature of three-space. It is this stretching factor that appears in the proper metric of the medium.

We note that the medium must be confined to regions where $r < \omega^{-1}$ and that its motion will not be rigid if ω varies with time. We note also that the proper geometry of the medium is *not flat*.

Problem 11 Verify that the proper metric of the rotating medium takes the form

$$\text{diag}\left(1, \frac{r^2}{1 - \omega^2 r^2}, 1\right)$$

with respect to the coordinates r, θ, z .

Solution 11 We have

$$\dot{\sigma}^2 = \frac{1}{1 - \omega^2 r^2}.$$

Hence,

$$\begin{aligned} \gamma_{rr} &= \frac{\partial \xi^i}{\partial r} \frac{\partial \xi^j}{\partial r} \gamma_{ij} \\ &= \cos^2 \theta \left[1 + (\dot{\sigma})^2 \sin^2 \theta \right] - 2(\dot{\sigma})^2 \sin^2 \theta \cos^2 \theta \\ &\quad + \sin^2 \theta \left[1 + (\dot{\sigma} \omega r)^2 \cos^2 \theta \right] \\ &= 1, \end{aligned}$$

$$\begin{aligned} \gamma_{r\theta} &= \gamma_{\theta r} = \frac{\partial \xi^i}{\partial r} \frac{\partial \xi^j}{\partial \theta} \gamma_{ij} \\ &= -r \sin \theta \cos \theta \left[1 + (\dot{\sigma})^2 \sin^2 \theta \right] - r(\dot{\sigma})^2 \sin \theta \cos^3 \theta \\ &\quad + r(\dot{\sigma})^2 \sin^3 \theta \cos \theta + r \sin \theta \cos \theta \left[1 + (\dot{\sigma})^2 \cos^2 \theta \right] \\ &= 0, \end{aligned}$$

$$\gamma_{rz} = \gamma_{zr} = \frac{\partial \xi^i}{\partial r} \frac{\partial \xi^j}{\partial z} \gamma_{ij} = 0, \langle \varphi_k \rangle_\phi = \langle \Im(\ln a_k) \rangle_\phi$$

$$\begin{aligned} \gamma_{\theta\theta} &= \frac{\partial \xi^i}{\partial \theta} \frac{\partial \xi^j}{\partial \theta} \gamma_{ij} \\ &= r^2 \sin^2 \theta \left[1 + (\dot{\sigma} \omega r)^2 \sin^2 \theta \right] + 2r^2 (\dot{\sigma} \omega r)^2 \sin^2 \theta \cos^2 \theta \\ &\quad + r^2 \cos^2 \theta \left[1 + (\dot{\sigma})^2 \cos^2 \theta \right] \\ &= r^2 \left[1 + (\dot{\sigma})^2 \right] = r^2 \left(1 + \frac{\omega^2 r^2}{1 - \omega^2 r^2} \right) = \frac{r^2}{1 - \omega^2 r^2}, \end{aligned}$$

$$\gamma_{\theta z} = \gamma_{z\theta} = \frac{\partial \xi^i}{\partial \theta} \frac{\partial \xi^j}{\partial z} \gamma_{ij} = 0,$$

$$\gamma_{zz} = \frac{\partial \xi^i}{\partial z} \frac{\partial \xi^j}{\partial z} \gamma_{ij} = 1.$$

Problem 12 Show that the motion defined by the functions

$$\begin{aligned}x^0(\xi, \tau) &= \frac{1 + a\xi^3}{a} \sinh a\sigma, \\x^1(\xi, \tau) &= \xi^1 + v\sigma, \\x^2(\xi, \tau) &= \xi^2, \\x^3(\xi, \tau) &= \frac{1 + a\xi^3}{a} \cosh a\sigma,\end{aligned}$$

where a and v are constants, is rigid. Here σ is a function of τ and the ξ^i , having a form such that τ is the proper time along each world line $\xi = \text{const}$. Obtain the proper metric of the flow and state (with arguments) whether the proper geometry is flat.

Solution 12 We have

$$\begin{aligned}\dot{x}^0(\xi, \tau) &= (1 + a\xi^3)\dot{\sigma} \cosh a\sigma, \\ \dot{x}^1(\xi, \tau) &= v\dot{\sigma}, \\ \dot{x}^2(\xi, \tau) &= 0, \\ \dot{x}^3(\xi, \tau) &= (1 + a\xi^3)\dot{\sigma} \sinh a\sigma,\end{aligned}$$

whence

$$-1 = u^2 = -\left[(1 + a\xi^3)^2 - v^2\right]\dot{\sigma}^2, \quad \dot{\sigma} = \left[(1 + a\xi^3)^2 - v^2\right]^{-1/2}.$$

The medium is confined to the region

$$\xi^3 > -\frac{1-v}{a}.$$

Now

$$\begin{aligned}\left(x_{,1}^\mu\right) &= (0, 1, 0, 0), & \left(x_{,2}^\mu\right) &= (0, 0, 1, 0), \\ \left(x_{,3}^\mu\right) &= (\sinh a\sigma + (1 + a\xi^3)\sigma_{,3} \cosh a\sigma, \ v\sigma_{,3}, \ 0, \ \cosh a\sigma + (1 + a\xi^3)\sigma_{,3} \sinh a\sigma), \\ u \cdot x_{,1} &= v\dot{\sigma}, & u \cdot x_{,2} &= 0, \\ u \cdot x_{,3} &= -(1 + a\xi^3)\dot{\sigma} \sinh a\sigma \cosh a\sigma - (1 + a\xi^3)^2 \dot{\sigma} \sigma_{,3} \cosh^2 a\sigma \\ &\quad + v^2 \dot{\sigma} \sigma_{,3} + (1 + a\xi^3) \sinh a\sigma \cosh a\sigma + (1 + a\xi^3)^2 \dot{\sigma} \sigma_{,3} \sinh^2 a\sigma \\ &= -\left[(1 + a\xi^3)^2 - v^2\right]\dot{\sigma} \sigma_{,3} = -\dot{\sigma}^{-1} \sigma_{,3},\end{aligned}$$

$$\begin{aligned}\gamma_{11} &= x_{,1} \cdot x_{,1} + (u \cdot x_{,1})^2 = 1 + v^2 \dot{\sigma}^2 = 1 + \frac{v^2}{(1 + a\xi^3)^2 - v^2} \\ &= \frac{(1 + a\xi^3)^2}{(1 + a\xi^3)^2 - v^2},\end{aligned}$$

whence

$$\boxed{\gamma_{11} = (1 - \bar{v}^2)^{-1}, \quad \bar{v} \equiv \frac{v}{1 + a\xi^3}}$$

$$\gamma_{12} = \gamma_{21} = x_{,1} \cdot x_{,2} + (u \cdot x_{,1})(u \cdot x_{,2}) \equiv 0,$$

$$\gamma_{13} = \gamma_{31} = x_{,1} \cdot x_{,3} + (u_{,1})(u \cdot x_{,3}) = v\sigma_{,3} - v\sigma_{,3} \equiv 0,$$

$$\gamma_{22} = x_{,2} \cdot x_{,2} + (u \cdot x_{,2})^2 \equiv 1,$$

$$\gamma_{23} = \gamma_{32} = x_{,2} \cdot x_{,3} + (u_{,2})(u \cdot x_{,3}) \equiv 0,$$

$$\begin{aligned}\gamma_{33} &= x_{,3} \cdot x_{,3} + (u_{,3})^2 \\ &= -[\sinh a\sigma + (1 + a\xi^3)\sigma_{,3} \cosh a\sigma]^2 + (v\sigma_{,3})^2 \\ &\quad + [\cosh a\sigma + (1 + a\xi^3)\sigma_{,3}\sigma]^2 + (\dot{\sigma}^{-1}\sigma_{,3})^2 \\ &= 1 - [(1 + a\xi^3)^2 - v^2 - \dot{\sigma}^{-2}](\sigma_{,3})^2 \equiv 1,\end{aligned}$$

or

$$\boxed{(\gamma_{ij}) = \text{diag}\left(\frac{1}{1 - \bar{v}^2}, 1, 1\right)}$$

The stretching factor $1/(1 - \bar{v}^2)$, although in only one direction, prevents the proper geometry from being flat.

2.6 Irrotational Flow

When the Ω_{ij} are nonvanishing, there exists no global hypersurface of simultaneity. This is a general property of rotational motion, whether rigid or not. The motion, or flow, of a fluid medium is said to be *irrotational* if and only if there exists a family of hypersurfaces that cut the world lines of all the particles of the medium orthogonally. In order that such a family exist, one must be able to write

$$u_\mu = \lambda \phi_{,\mu} \tag{2.11}$$

for some scalar function ϕ . Here λ is a normalizing factor and the hypersurfaces $\phi = \text{const.}$ are global hypersurfaces of simultaneity. We have

$$\begin{aligned}\lambda &= \left(-\phi_{,\mu}\phi^{,\mu}\right)^{-1/2}, \\ \lambda_{,\mu} &= \left(-\phi_{,\mu}\phi^{,\mu}\right)^{-3/2} \phi^{\nu}\phi_{,\nu\mu} = \lambda^2 u^{\nu}\phi_{,\nu\mu}, \\ \phi_{,\mu\nu} &= -\lambda^{-2}u_{\mu}\lambda_{,\nu} + \lambda^{-1}u_{\mu,\nu}, \\ u_{\mu,\nu} &= \lambda P_{\mu}^{\sigma}\phi_{,\sigma\nu}, \quad P_{\mu}^{\sigma}P_{\nu}^{\tau}u_{\sigma,\tau} = \lambda P_{\mu}^{\sigma}P_{\nu}^{\tau}\phi_{,\sigma\tau}.\end{aligned}$$

Because of the commutativity of partial differentiation, we have, as the necessary and sufficient condition for the integrability of (2.11),

$$P_{\mu}^{\sigma}P_{\nu}^{\tau}(u_{\sigma,\tau} - u_{\tau,\sigma}) = 0.$$

This is the condition for irrotational flow. Its nonrelativistic analog is (see p. 23)

$$v_{i,j} - v_{j,i} = 0,$$

which implies the existence of a scalar function ϕ such that

$$v_i = \phi_{,i}.$$

A nonrelativistic motion that is both rigid and irrotational reduces simply to [see (2.6) on p. 24]

$$v_i = \beta_i,$$

where β_i is a function of time alone.

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