Chapter 4
Boundary Element Methods

In Chap. 3 we transformed strongly elliptic boundary value problems of second order in domains $\Omega \subset \mathbb{R}^3$ into boundary integral equations. These integral equations were formulated as variational problems on a Hilbert space H:

$$\text{Find } u \in H: \qquad b(u, v) = F(v) \qquad \forall v \in H, \tag{4.1}$$

which, in the simplest cases, was chosen as one of the Sobolev spaces $H^s(\Gamma)$, $s = -1/2, 0, 1/2$. The functional $F \in H'$ denotes the given right-hand side, which, in the case of the direct method (see Sect. 3.4.2), may again contain integral operators. The sesquilinear form $b(\cdot, \cdot)$ has the abstract form

$$b(u, v) = (Bu, v)_{L^2(\Gamma)}$$

with the integral operator

$$(Bu)(\mathbf{x}) = \lambda_1(\mathbf{x})\, u(\mathbf{x}) + \lambda_2(\mathbf{x}) \int_\Gamma k(\mathbf{x}, \mathbf{y}, \mathbf{y} - \mathbf{x})\, u(\mathbf{y})\, ds_{\mathbf{y}} \qquad \mathbf{x} \in \Gamma \text{ a.e.} \tag{4.2}$$

Convention 4.0.1. *The inner product* $(\cdot, \cdot)_{L^2(\Gamma)}$ *is again identified with the continuous extension on* $H^{-s}(\Gamma) \times H^s(\Gamma)$.

The coefficients λ_1, λ_2 are bounded. For $\lambda_1 = 0$, a.e., one speaks of an integral operator of the first kind, otherwise of the second kind. In some applications the kernel function is not improperly integrable, and the integral is defined by means of a suitable regularization (see Theorem 3.3.22).

The sesquilinear form in (4.1) associated with the boundary integral operator in (4.2) satisfies a Gårding inequality: There exist a $\gamma > 0$ and a compact operator $T : H \to H'$ such that

$$\forall u \in H : |b(u, u) + \langle Tu, u \rangle_{H' \times H}| \geq \gamma \|u\|_H^2 . \tag{4.3}$$

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The variational formulation (4.1) of the integral equations forms the basis of the numerical solution thereof, by means of finite element methods on the boundary $\Gamma = \partial\Omega$, the so-called boundary element methods. They are abbreviated by “BEM”.

Note: Readers who are familiar with the concept of finite element methods will recognize it here. One essential conceptual difference between the BEM and the finite element method is the fact that, in the BEM, the resulting finite element meshes *usually* consist of curved elements and therefore, in general, *no affine* parametrization over a reference element can be found.

Primarily, we consider the Galerkin BEM, which is the most natural method for the variational formulation (4.1) of the boundary integral equation. In Sect. 4.1 we will describe the Galerkin BEM for the boundary value problems of the Laplace equation with Dirichlet, Neumann and mixed boundary conditions, all of which lead to boundary integral equations of the first kind with positive definite bilinear forms. We obtain quasi-optimal approximations and prove asymptotic convergence rates for the Galerkin BEM. In Sect. 4.2 we will then study Galerkin methods in an abstract form for operators that are only positive with a compact perturbation. We will also present a general framework for the convergence analysis of Galerkin methods. In Sect. 4.3 we will finally prove the approximation properties of the boundary element spaces.

4.1 Boundary Elements for the Potential Equation in $\mathbb{R}^3$

We will first introduce the Galerkin BEM for integral equations of the classical potential problem in $\mathbb{R}^3$ and derive relevant error estimates for the simplest boundary elements.

4.1.1 Model Problem 1: Dirichlet Problem

Let $\Omega^- \subset \mathbb{R}^3$ be a bounded polyhedral domain, the boundary $\Gamma = \partial\Omega^-$ of which consists of finitely many, disjoint, plane faces Γ^j, $j = 1, \ldots, J$: $\Gamma = \bigcup_{j=1}^{J} \overline{\Gamma^j}$.

In the exterior $\Omega^+ = \mathbb{R}^3 \backslash \overline{\Omega^-}$ we consider the Dirichlet problem

$$\Delta u = 0 \text{ in } \Omega^+, \tag{4.4a}$$

$$u = g_D \text{ on } \Gamma, \tag{4.4b}$$

$$|u(\mathbf{x})| = O(\|\mathbf{x}\|^{-1}) \text{ for } \|\mathbf{x}\| \to \infty. \tag{4.4c}$$

In Chap. 2 (Theorem 3.5.3) we have shown the unique solvability of Problem (4.4).

Proposition 4.1.1. *For all* $g_D \in H^{1/2}(\Gamma)$ *Problem (4.4) has a unique solution* $u \in H^1(L, \Omega^+)$ *with* $L = -\Delta$.

Proof. Theorem 2.10.11 implies the unique solvability of the variational formulation associated with (4.4) in $H^1(L, \Omega^+)$ with $L = -\Delta$. In Sect. 2.9.3 we have shown that the solution also solves (4.4a) and (4.4b) almost everywhere.

Decay Condition: Theorem 3.5.3 provides us with the unique solvability of the boundary integral equation that results from (4.4) (with the single layer ansatz) in $H^{-1/2}(\Gamma)$. The associated single layer potential is in $H^1(L, \Omega^+)$ (see Exercise 3.1.14) and, thus, is the unique solution.

Finally, in (3.22) we have shown that the single layer potential satisfies the decay condition (4.4c). □

We will now reduce (4.4) to a boundary integral equation of the first kind. We ensure that (4.4a), (4.4c) are satisfied by means of the single layer ansatz (see Chap. 3)

$$u(\mathbf{x}) = (S\varphi)(\mathbf{x}) = \int_\Gamma \frac{\varphi(\mathbf{y})}{4\pi \|\mathbf{x}-\mathbf{y}\|} \, ds_{\mathbf{y}}, \qquad \mathbf{x} \in \Omega^+. \tag{4.5}$$

The unknown density φ from (4.5) is the solution of the boundary integral equation

$$V\varphi = g_D \qquad \text{on } \Gamma \tag{4.6}$$

with the single layer operator

$$(V\varphi)(\mathbf{x}) := \int_\Gamma \frac{\varphi(\mathbf{y})}{4\pi \|\mathbf{x}-\mathbf{y}\|} \, ds_{\mathbf{y}} \qquad \mathbf{x} \in \Gamma. \tag{4.7}$$

(4.6) defines a boundary integral equation of the first kind. The Galerkin boundary element method is based on the variational formulation of the integral equation. Instead of imposing (4.6) for all $\mathbf{x} \in \Gamma$, we multiply (4.6) by a "test function" and integrate over Γ. This gives us: Find $\varphi \in H^{-1/2}(\Gamma)$ such that

$$\begin{aligned} \int_\Gamma (V\varphi)\eta \, ds_{\mathbf{x}} &= \int_\Gamma \left(\int_\Gamma \frac{\varphi(\mathbf{y})}{4\pi \|\mathbf{x}-\mathbf{y}\|} \, ds_{\mathbf{y}} \right) \eta(\mathbf{x}) ds_{\mathbf{x}} \\ &= \int_\Gamma g_D(\mathbf{x}) \, \eta(\mathbf{x}) \, ds_{\mathbf{x}} \qquad \forall \eta \in H^{-1/2}(\Gamma). \end{aligned} \tag{4.8}$$

For the Laplace operator we only consider vector spaces over the field $\mathbb{R}$ and not over $\mathbb{C}$, so that in (4.8) there is no complex conjugation.

The "integrals" in (4.8) should be interpreted as duality pairings in $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ in the following way. For $\varphi \in H^{-1/2}(\Gamma)$ we have $V\varphi \in H^{1/2}(\Gamma)$ and, by Convention 4.0.1, we can write (4.8) as

$$\text{Find } \varphi \in H^{-1/2}(\Gamma): \ (V\varphi, \eta)_{L^2(\Gamma)} = (g_D, \eta)_{L^2(\Gamma)} \qquad \forall \eta \in H^{-1/2}(\Gamma). \tag{4.9}$$

The left-hand side in (4.9) defines a bilinear form $b(\cdot,\cdot)$ on the Hilbert space $H = H^{-1/2}(\Gamma)$ with

$$b(\varphi,\eta) := (V\varphi,\eta)_{L^2(\Gamma)}, \tag{4.10}$$

and the right-hand side defines a linear functional on $H^{-1/2}(\Gamma)$:

$$F(\eta) := (g_D,\eta)_{L^2(\Gamma)}. \tag{4.11}$$

Keeping the duality of $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ in mind, it follows from

$$|F(\eta)| \leq \left(\sup_{\mu\in H^{-1/2}(\Gamma)\setminus\{0\}} \frac{|(g_D,\mu)_{L^2(\Gamma)}|}{\|\mu\|_{H^{-1/2}(\Gamma)}} \right) \|\eta\|_{H^{-1/2}(\Gamma)} = \|g_D\|_{H^{1/2}(\Gamma)} \|\eta\|_{H^{-1/2}(\Gamma)}$$

that F is continuous on $H^{-1/2}(\Gamma)$.

For sufficiently smooth functions φ,η in (4.10) we have, by virtue of Fubini's theorem,

$$b(\varphi,\eta) = \int_\Gamma \int_\Gamma \frac{\eta(\mathbf{x})\varphi(\mathbf{y})}{4\pi\,\|\mathbf{x}-\mathbf{y}\|}\, ds_{\mathbf{y}}\, ds_{\mathbf{x}} = b(\eta,\varphi) \tag{4.12}$$

and therefore the form $b(\cdot,\cdot)$ is symmetric. Furthermore, it is also $H^{-1/2}$-elliptic (see Theorem 3.5.3). According to the Lax–Milgram lemma (see Sect. 2.1.6), Problem (4.9) has a unique solution $\varphi \in H^{-1/2}(\Gamma)$ for all $g_D \in H^{1/2}(\Gamma)$. In the representational formula (4.5) this φ gives us the unique solution u of the exterior problem (4.4).

The *discretization* of the boundary integral equation consists in the approximation of the unknown density function φ in (4.6) by means of a function $\tilde{\varphi}$ which is defined by finitely many coefficients $(\alpha_i)_{i=1}^N$ in the basis representation. In the Galerkin boundary element method, this is achieved by restricting φ,η in the variational form (4.9) to finite-dimensional subspaces, the boundary element spaces, which we will now construct.

4.1.2 Surface Meshes

Almost all boundary elements are based on a surface mesh $\mathcal{G}$ of the boundary Γ. A surface mesh is the finite union of curved triangles and quadrilaterals on the boundary Γ, which satisfy suitable compatibility conditions. A general element of $\mathcal{G}$ is called a "panel".

For the definition we introduce the reference elements

$$\begin{aligned} &\textit{Unit triangle:} \quad \widehat{S}_2 := \left\{(\xi_1,\xi_2)\in\mathbb{R}^2 : 0<\xi_2<\xi_1<1\right\} \\ &\textit{Unit square:} \quad \widehat{Q}_2 := (0,1)^2. \end{aligned} \tag{4.13}$$

Our generic notation for the reference element is $\hat{\tau}$.

Definition 4.1.2. A surface mesh $\mathcal{G}$ of the boundary Γ is a decomposition of Γ into finitely relatively open, disjoint elements $\tau \subset \Gamma$ that satisfy the following conditions:

(a) $\mathcal{G}$ is a covering of Γ:

$$\Gamma = \overline{\bigcup_{\tau \in \mathcal{G}} \tau}.$$

(b) Every element $\tau \in \mathcal{G}$ is the image of a reference element $\hat{\tau}$ under a regular reference mapping χ_τ. Then χ_τ is called regular if the Jacobian $\mathbf{J}_\tau = D\chi_\tau$ satisfies the condition

$$0 < \lambda_{\min} \leq \inf_{\hat{\xi} \in \hat{\tau}} \inf_{\substack{\mathbf{v} \in \mathbb{R}^2 \\ \|\mathbf{v}\|=1}} \left\langle \mathbf{J}_\tau \left(\hat{\xi}\right) \mathbf{v}, \mathbf{J}_\tau \left(\hat{\xi}\right) \mathbf{v} \right\rangle \leq \sup_{\hat{\xi} \in \hat{\tau}} \sup_{\substack{\mathbf{v} \in \mathbb{R}^2 \\ \|\mathbf{v}\|=1}} \left\langle \mathbf{J}_\tau \left(\hat{\xi}\right) \mathbf{v}, \mathbf{J}_\tau \left(\hat{\xi}\right) \mathbf{v} \right\rangle$$
$$\leq \lambda_{\max} < \infty.$$

(c) For a plane triangle $\tau \in \mathcal{G}$ with straight edges and vertices $\mathbf{P}_0$, $\mathbf{P}_1$ and $\mathbf{P}_2$, the regular mapping χ_τ is affine:

$$\chi_\tau \left(\hat{\xi}\right) = \mathbf{P}_0 + \hat{\xi}_1 \left(\mathbf{P}_1 - \mathbf{P}_0\right) + \hat{\xi}_2 \left(\mathbf{P}_2 - \mathbf{P}_1\right). \tag{4.14}$$

For a plane quadrilateral $\tau \in \mathcal{G}$ with straight edges and vertices $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ (the numbering is counterclockwise) the mapping is bilinear:

$$\chi_\tau \left(\hat{\xi}\right) = \mathbf{P}_0 + \hat{\xi}_1 \left(\mathbf{P}_1 - \mathbf{P}_0\right) + \hat{\xi}_2 \left(\mathbf{P}_3 - \mathbf{P}_0\right) + \hat{\xi}_1 \hat{\xi}_2 \left(\mathbf{P}_2 - \mathbf{P}_3 + \mathbf{P}_0 - \mathbf{P}_1\right). \tag{4.15}$$

Figure 4.1 illustrates Definition 4.1.2 for a triangular and a quadrilateral element.

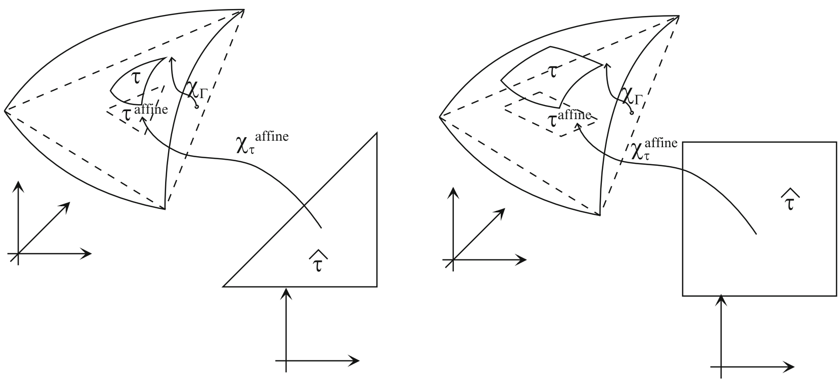


Fig. 4.1 Schematic illustration of the reference mappings; triangular panel (*left*), parallelogram (*right*)

Exercise 4.1.3. *Show the following:*

(a) The affine mapping χ_τ in (4.14) is regular if and only if $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$ are vertices of a non-degenerate (plane) triangle τ, i.e., they are not colinear. Find an estimate for the constants $\lambda_{\min}$, $\lambda_{\max}$ from Definition 4.1.2(b) in terms of the interior angles of τ.

(c) Let $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ be the vertices of a plane quadrilateral τ with straight edges. The mapping χ_τ from (4.15) is regular if all interior angles are smaller than π and larger than 0.

In some cases we will impose a compatibility condition for the intersection of two panels.

Definition 4.1.4. A surface mesh $\mathcal{G}$ of Γ is called *regular* if:

(a) The intersection of two different elements $\tau, \tau' \in \mathcal{G}$ is either empty, a common vertex or a common side.

(b) The parametrizations of the panel edges of neighboring panels coincide: For every pair of different elements $\tau, \tau' \in \mathcal{G}$ with common edge $e = \overline{\tau} \cap \overline{\tau'}$ we have

$$\chi_\tau|_{\hat{e}} = \chi_{\tau'} \circ \gamma_{\tau,\tau'}|_{\hat{e}},$$

where $\hat{e} := \chi_\tau^{-1}(e)$ and $\gamma_{\tau,\tau'} : \hat{\tau} \to \hat{\tau}$ is a suitable affine bijection.

Remark 4.1.5. *Throughout this section we assume that the boundary Γ is Lipschitz and admits a regular surface mesh in the sense of Definitions 4.1.2 and 4.1.4. This is a true restriction since not every Lipschitz surface admits a regular surface mesh.*

For later error estimates we will introduce a few geometric parameters, which represent a measure for the distortion of the panels as well as bounds for their diameters.

Assumption 4.1.6. *There exist open subsets $U, V \subset \mathbb{R}^3$ and a diffeomorphism $\chi_\Gamma : U \to V$ with the following properties:*

(a) $\Gamma \subset U$.

(b) For every $\tau \in \mathcal{G}$, there exists a regular reference mapping $\chi_\tau : \hat{\tau} \to \tau$ of the form

$$\chi_\tau = \chi_\Gamma \circ \chi_\tau^{\text{affine}} : \hat{\tau} \to \tau,$$

where $\chi_\tau^{\text{affine}} : \mathbb{R}^2 \to \mathbb{R}^3$ is a regular, affine mapping.

Example 4.1.7.

1. Let Γ be a piecewise smooth surface that has a bi-Lipschitz continuous parametrization over the polyhedral surface $\hat{\Gamma}$: $\chi_\Gamma : \hat{\Gamma} \to \Gamma$. Let $\mathcal{G}^{\text{affine}} := \left\{\tau_i^{\text{affine}} : 1 \le i \le N\right\}$ be a regular surface mesh of $\hat{\Gamma}$ with the associated reference mappings $\chi_{\tau^{\text{affine}}}^{\text{affine}} : \hat{\tau} \to \tau^{\text{affine}}$. Then $\mathcal{G} := \left\{\chi_\Gamma\left(\tau^{\text{affine}}\right) : \tau^{\text{affine}} \in \mathcal{G}^{\text{affine}}\right\}$ defines a regular surface mesh of Γ which satisfies Assumption 4.1.6.

2. *For the unit sphere* $\Gamma := \left\{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\right\}$ *one can choose the inscribed double pyramid with vertices* $(\pm 1, 0, 0)^\intercal$, $(0, \pm 1, 0)^\intercal$, $(0, 0, \pm 1)^\intercal$ *as a polyhedral surface* $\hat{\Gamma}$, *while* $\chi_\Gamma : \hat{\Gamma} \to \Gamma$ *is defined by* $\chi_\Gamma(\mathbf{x}) := \mathbf{x}/\|\mathbf{x}\|$. *By means of* χ_Γ, *regular surface meshes on* Γ *can then be generated through lifting of regular surface meshes of the polyhedral surface* $\hat{\Gamma}$.

In order to construct a sequence of refined surface meshes for Γ, in many cases the procedure is as follows.

Remark 4.1.8. *Let* Γ *be the surface of a bounded Lipschitz domain* $\Omega \subset \mathbb{R}^3$. *In the first step we construct a polyhedron* $\hat{\Gamma}$ *along a bi-Lipschitz continuous mapping* $\chi_\Gamma : \hat{\Gamma} \to \Gamma$ *(see Example 4.1.7). Let* $\mathcal{G}_0^{\text{affine}}$ *be a (very coarse) surface mesh of* $\hat{\Gamma}$. *Then* $\mathcal{G}_0 := \left\{\tau = \chi_\Gamma\left(\tau^{\text{affine}}\right) : \tau^{\text{affine}} \in \mathcal{G}_0^{\text{affine}}\right\}$ *defines a coarse surface mesh of* Γ. *We can obtain a sequence* $\left(\mathcal{G}_\ell^{\text{affine}}\right)_\ell$ *of finer surface meshes if, during each refinement, we decompose every panel in* $\mathcal{G}_0^{\text{affine}}$ *into new panels by means of a fixed refinement method. For triangular elements, for example, we interconnect the midpoints of the sides and for quadrilateral elements we connect both pairs of opposite midpoints. This gives us a sequence of surface meshes by* $\mathcal{G}_\ell := \left\{\tau = \chi_\Gamma\left(\tau^{\text{affine}}\right) : \tau^{\text{affine}} \in \mathcal{G}_\ell^{\text{affine}}\right\}$.

Convention 4.1.9. *If* τ *and* τ^{affine} *appear in the same context the relation between the two is given by* $\tau = \chi_\Gamma\left(\tau^{\text{affine}}\right)$.

The following definition is illustrated in Fig. 4.2.

Definition 4.1.10. Let Assumption 4.1.6 be satisfied. The constants $c_{\text{affine}} > 0$ ($C_{\text{affine}} > 0$) are the maximal (minimal) constants in

$$c_{\text{affine}} \|\mathbf{x} - \mathbf{y}\| \le \|\chi_\Gamma(\mathbf{x}) - \chi_\Gamma(\mathbf{y})\| \le C_{\text{affine}} \|\mathbf{x} - \mathbf{y}\| \quad \forall \mathbf{x}, \mathbf{y} \in \tau^{\text{affine}}, \forall \tau^{\text{affine}} \in \mathcal{G}^{\text{affine}}$$

and describe the distortion of curved panels τ compared to their affine pullbacks τ^{affine}.

The diameter of a panel $\tau \in \mathcal{G}$ is given by

$$h_\tau := \sup_{\mathbf{x}, \mathbf{y} \in \tau} \|\mathbf{x} - \mathbf{y}\|$$

and the inner width ρ_τ by the incircle diameter of τ^{affine}.

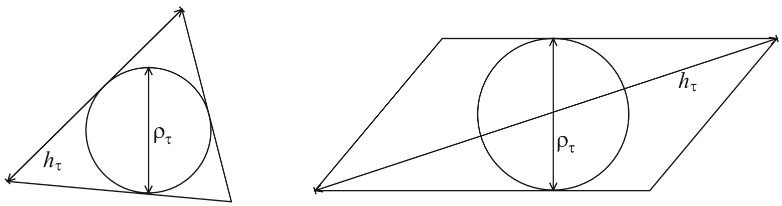


Fig. 4.2 Diameter of a panel and incircle diameter; triangular panel (*left*), parallelogram (*right*)

The *mesh width* $h_{\mathcal{G}}$ of a surface mesh $\mathcal{G}$ is given by

$$h_{\mathcal{G}} := \max\{h_\tau : \tau \in \mathcal{G}\}. \tag{4.16}$$

We write h instead of $h_{\mathcal{G}}$ if the mesh $\mathcal{G}$ is clear from the context.

Remark 4.1.11. *For plane panels τ, ρ_τ is the incircle diameter of τ.*
The diameters of τ and τ^{affine} satisfy

$$C_{\text{affine}}^{-1} h_\tau \leq \sup_{\mathbf{x},\mathbf{y}\in\tau^{\text{affine}}} \|\mathbf{x}-\mathbf{y}\| = h_{\tau^{\text{affine}}} \leq c_{\text{affine}}^{-1} h_\tau.$$

Definition 4.1.12. The shape-regularity constant $\kappa_{\mathcal{G}}$ is given by

$$\kappa_{\mathcal{G}} := \max_{\tau\in\mathcal{G}} \frac{h_\tau}{\rho_\tau}. \tag{4.17}$$

For some theorems we will assume, apart from the shape-regularity, that the diameters of all triangles are of the same order of magnitude.

Definition 4.1.13. The constant $q_{\mathcal{G}}$ that describes the quasi-uniformity is given by

$$q_{\mathcal{G}} := h_{\mathcal{G}} / \min\{h_\tau : \tau \in \mathcal{G}\}.$$

Remark 4.1.14. *In order to study the convergence of boundary element methods, we will consider sequences $(\mathcal{G}_\ell)_{\ell\in\mathbb{N}}$ of surface meshes whose mesh width $h_\ell := h_{\mathcal{G}_\ell}$ tends to zero. It is essential that the constant for the shape-regularity $\kappa_\ell := \kappa_{\mathcal{G}_\ell}$ remains uniformly bounded above:*

$$\sup_{\ell\in\mathbb{N}} \kappa_\ell \leq \kappa < \infty. \tag{4.18}$$

In a similar way the constants of quasi-uniformity $q_\ell := q_{\mathcal{G}_\ell}$ have to be bounded above in some theorems:

$$\sup_{\ell\in\mathbb{N}} q_\ell \leq q < \infty. \tag{4.19}$$

We call a mesh family $(\mathcal{G}_\ell)_{\ell\in\mathbb{N}}$ with the property (4.18) shape-regular *and with the property (4.19)* quasi-uniform.

Exercise 4.1.15. *Show the following:*

(a) If the surface mesh $\mathcal{G}_0$ is regular and if finer surface meshes $(\mathcal{G}_\ell)_\ell$ are constructed according to the method described in Remark 4.1.8 then all surface meshes $(\mathcal{G}_\ell)_\ell$ are regular.
(b) The constants concerning shape-regularity and quasi-uniformity are, under the conditions in Part (a), uniformly bounded with respect to ℓ.

4.1.3 Discontinuous Boundary Elements

The boundary element method defines an approximation of the unknown density φ in the boundary integral equation (4.6) which is described by finitely many parameters. This can, for example, be achieved by (piecewise) polynomials on the elements τ of a mesh $\mathcal{G}$.

Example 4.1.16. *(Piecewise Constant Boundary Elements)*

Let $\Gamma = \partial\Omega$ be piecewise smooth and let $\mathcal{G}$ be a – not necessarily regular – surface mesh on Γ. Then $S_{\mathcal{G}}^0$ denotes all piecewise constant functions on the mesh $\mathcal{G}$

$$S_{\mathcal{G}}^0 := \{\psi \in L^\infty(\Gamma) \mid \forall\tau \in \mathcal{G} : \psi|_\tau \text{ is constant}\}. \tag{4.20}$$

Since $\psi \in L^\infty(\Gamma)$, we only need to define ψ in the interior of an element, as the boundary $\partial\tau$, i.e., the set of edges and vertices of the panel, is a set of zero measure.

Every function $\psi \in S_{\mathcal{G}}^0$ is defined by its values ψ_τ on the elements $\tau \in \mathcal{G}$ and can be written in the form

$$\psi(\mathbf{x}) = \sum_{\tau\in\mathcal{G}} \psi_\tau b_\tau(\mathbf{x}) \tag{4.21}$$

with the characteristic function $b_\tau : \Gamma \to \mathbb{R}$ of $\tau \in \mathcal{G}$:

$$b_\tau(\mathbf{x}) := \begin{cases} 1 & \mathbf{x} \in \tau, \\ 0 & \text{otherwise}. \end{cases} \tag{4.22}$$

In particular, $S_{\mathcal{G}}^0$ is a vector space of dimension $N = \#\{\tau : \tau \in \mathcal{G}\}$ with basis $\{b_\tau : \tau \in \mathcal{G}\}$.

In many cases the piecewise constant approximation of the unknown density converges too slowly and, instead, one uses polynomials of degree $p \geq 1$. In the same way as in Example 4.1.16 this leads to the boundary element spaces $S_{\mathcal{G}}^p$. For their definition we need polynomials of total degree p on the reference element as well as the convention for multi-indices from (2.67)

$$\mathbb{P}_p^\Delta = \operatorname{span}\left\{\xi^\mu : \mu \in \mathbb{N}_0^2 \wedge |\mu| \leq p\right\}. \tag{4.23}$$

For $p = 1$ and $p = 2$, $\mathbb{P}_p^\Delta$ contains all polynomials of the form

$$\begin{array}{lll} a_{00} + a_{10}\xi_1 + a_{01}\xi_2 & \forall a_{00}, a_{10}, a_{01} \in \mathbb{R} & \text{for } p = 1, \\ a_{00} + a_{10}\xi_1 + a_{01}\xi_2 + a_{20}\xi_1^2 + a_{11}\xi_1\xi_2 + a_{02}\xi_2^2 & \forall a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02} \in \mathbb{R} & \text{for } p = 2. \end{array}$$

Definition 4.1.17. Let $\Gamma = \partial\Omega$ be piecewise smooth and let $\mathcal{G}$ be a surface mesh of Γ. Then, for $p \in \mathbb{N}_0$,

$$S_{\mathcal{G}}^p := \left\{\psi : \Gamma \to \mathbb{K} \mid \forall\tau \in \mathcal{G} : \psi \circ \chi_\tau \in \mathbb{P}_p^\Delta\right\}. \tag{4.24}$$

We simply write S^p or only S if the reference to the surface mesh $\mathcal{G}$ is obvious.

Remark 4.1.18. *Note that in (4.24) the functions $\psi \in S^p$ do not constitute polynomials on the surface Γ. Only once they have been "transported back" to the reference element $\hat{\tau}$ by means of the element mapping χ_τ (see Fig. 4.1) is this the case. The parametrizations χ_τ of the elements $\tau \in \mathcal{G}$ in Definition 4.1.2 (b,c) are thus part of the set $S^p_{\mathcal{G}}$. A change in parametrization χ_τ will lead (with the same mesh $\mathcal{G}$) to a different $S^p_{\mathcal{G}}$. Therefore for a mesh $\mathcal{G}$ we summarize the element mappings χ_τ in the mapping vector*

$$\chi := \{\chi_\tau : \tau \in \mathcal{G}\} \tag{4.25}$$

and instead of (4.24) we write $S^p_{\mathcal{G},\chi}$.

Remark 4.1.19. *Note that (4.24) also holds for meshes $\mathcal{G}$ with quadrilateral elements, i.e., with reference element $\hat{\tau} = (0,1)^2$. Since S^p does not require continuity across element boundaries, the space of polynomials $\mathbb{P}^\Delta_p$ in (4.23) can also be applied to quadrilateral meshes.*

For the realization of the boundary element spaces we need a basis for $\mathbb{P}^\Delta_p$, which we denote by $\widehat{N}_{(i,j)}(\hat{\xi}_1, \hat{\xi}_2)$ and which satisfies

$$\mathbb{P}^\Delta_p = \operatorname{span}\left\{\widehat{N}_{(i,j)} : \ 0 \le i, j \le p, \ i + j \le p\right\}. \tag{4.26}$$

For example, $\widehat{N}_{(i,j)}(\xi_1, \xi_2) := \hat{\xi}_1^i \hat{\xi}_2^j$, $0 \le i + j \le p$ as in (4.23), would be admissible basis functions.

Remark 4.1.20. *(Nesting of Spaces)*

We have $\mathbb{P}^\Delta_p \subset \mathbb{P}^\Delta_q$ for all $p \le q$. Therefore we can always choose a basis in $\mathbb{P}^\Delta_q$ which contains the basis functions from $\mathbb{P}^\Delta_p$ as a subset. The basis functions $\widehat{N}_{(i,j)}$ in (4.23) have this property.

Once we have determined a basis $\widehat{N}_{(i,j)}(\hat{\xi})$ on $\hat{\tau}$, every $\psi \in S^p_{\mathcal{G},\chi}$ on a panel $\tau \in \mathcal{G}$ can be written as

$$\psi|_\tau = \sum_{0 \le i+j \le p} \alpha_{i,j} \left(\widehat{N}_{(i,j)} \circ \chi_\tau^{-1}\right)$$

and

$$N^\tau_{(i,j)} := \widehat{N}_{(i,j)} \circ \chi_\tau^{-1} \qquad 0 \le i + j \le p$$

spans the restriction $\{\psi|_\tau : \psi \in S^p(\Gamma, \mathcal{G}, \chi)\}$. In order to give a basis of $S^p_{\mathcal{G},\chi}$ suitable indices, we define

$$\iota_p := \left\{\mu \in \mathbb{N}_0^2 : |\mu| \le p\right\}.$$

Thus we have

$$S^p_{\mathcal{G},\chi} = \operatorname{span}\left\{b_{(\mu,\tau)}(x) : (\mu, \tau) \in \iota_p \times \mathcal{G}\right\}, \tag{4.27}$$

where the global basis functions $b_I(x)$ with the multi-index $I = (\mu, \tau)$ denote the zero extension of the element function N_μ^τ to Γ: For

$$I = (\mu, \tau) \in \iota_p \times \mathcal{G} =: \mathcal{I}(\mathcal{G}, p) =: \mathcal{I} \tag{4.28}$$

we explicitly have

$$b_I(\mathbf{x}) := \begin{cases} N_\mu^\tau(\mathbf{x}), \; \mathbf{x} \in \tau, \\ 0 \qquad\quad \text{otherwise.} \end{cases} \tag{4.29}$$

Hence, every ψ can be written as a combination of the basis function $b_I(x)$:

$$\psi(\mathbf{x}) = \sum_{I \in \mathcal{I}} \psi_I \, b_I(\mathbf{x}), \qquad \mathbf{x} \in \tau, \quad \tau \in \mathcal{G}. \tag{4.30}$$

Let $|\mathcal{G}|$ be the number of elements in the mesh $\mathcal{G}$. The dimension of $S^p_{\mathcal{G},\chi}$ or the number of degrees of freedom is then given by

$$N = |\mathcal{G}| \, (p+1)(p+2)/2 = \dim(S^p_{\mathcal{G},\chi}). \tag{4.31}$$

Every function in $\psi \in S^p_{\mathcal{G},\chi}$ is then uniquely characterized by the vector $(\psi_I)_{I \in \mathcal{I}(\mathcal{G},p)}$ $\subset \mathbb{R}^N \cong \mathbb{R}^{\mathcal{I}(\mathcal{G},p)}$ as in (4.30).

4.1.4 Galerkin Boundary Element Method

The simplest boundary element method for Problem (4.6) consists in approximating the unknown density φ in (4.9) by a piecewise constant function $\varphi_S \in S^0(\Gamma, \mathcal{G})$.

Convention 4.1.21. *The boundary element functions depend on the boundary element space $S^p(\Gamma, \mathcal{G}, \chi)$; in particular, they depend on Γ, the surface mesh $\mathcal{G}$ and the polynomial degree p. We will, whenever possible, use the abbreviated notation φ_S instead of $\varphi_{S^p_{\mathcal{G},\chi}}$.*

Inserting (4.30) into (4.6) or into the variational formulation (4.8) leads to a contradiction: since, in general, we have $\varphi_S \neq \varphi$, (4.6) and (4.8) cannot be satisfied with $\varphi = \varphi_S$, which is why the statements have to be weakened. As φ_S is determined by N parameters $\left(\varphi_I^S\right)_{I \in \mathcal{I}}$ [see (4.29)–(4.31)], we are looking for N conditions to determine φ_I^S. In the Galerkin boundary element method we only let the test function η run through a basis of $S^p_{\mathcal{G}}$ in the variational formulation of the boundary integral equation (4.9). The Galerkin approximation of the integral equation (4.9) then reads:

Find $\varphi_S \in S^p_{\mathcal{G},\chi}$ such that

$$b(\varphi_S, \eta_S) = F(\eta_S) \qquad \forall \eta_S \in S^p_{\mathcal{G},\chi}, \tag{4.32}$$

with $b(\cdot,\cdot)$ and $F(\cdot)$ from (4.10) and (4.11) respectively.

Remark 4.1.22. *(i) The Galerkin discretization (4.32) of (4.8) is achieved by restricting the trial and test functions φ, η to the subspace $S^p_{\mathcal{G},\chi} \subset H^{-1/2}(\Gamma)$ in the variational formulation (4.8).*

(ii) The boundary element solution φ_S in (4.32) is independent of the basis chosen for the subspace.

The *computation* of the approximation φ_S requires that we choose a concrete basis for the subspace. Therefore, [see (4.29)–(4.31)] for a fixed $p \in \mathbb{N}_0$, we choose the basis

$$(b_I : I \in \mathcal{I}(\mathcal{G}, p)) \tag{4.33}$$

for $S^p_{\mathcal{G},\chi}$. Then (4.32) is equivalent to the linear system of equations:
Find $\varphi \in \mathbb{R}^N$ such that

$$\mathbf{B}\varphi = \mathbf{F}. \tag{4.34}$$

Here the *system matrix* $\mathbf{B} = (B_{I,J})_{I,J\in\mathcal{I}(\mathcal{G},p)}$ and the right-hand side $\mathbf{F} = (F_J)_{J\in\mathcal{I}(\mathcal{G},p)} \in \mathbb{R}^N$ with $I = (\mu, \tau)$ and $J = (\nu, t)$ are given by

$$\begin{aligned} B_{I,J} &:= b(b_I, b_J) \qquad\qquad (4.35)\\ &= \int_\Gamma \int_\Gamma \frac{b_J(\mathbf{x})\, b_I(\mathbf{y})}{4\pi\,\|\mathbf{x}-\mathbf{y}\|} ds_{\mathbf{y}}\, ds_{\mathbf{x}} = \int_t \int_\tau \frac{N^t_\nu(\mathbf{x})\, N^\tau_\mu(\mathbf{y})}{4\pi\,\|\mathbf{x}-\mathbf{y}\|} ds_{\mathbf{y}}\, ds_{\mathbf{x}} \end{aligned}$$

$$F_J := F(b_J) = \int_\Gamma g_D(\mathbf{x}) b_J(\mathbf{x})\, ds_{\mathbf{x}} = \int_t g_D(\mathbf{x}) N^t_\nu(\mathbf{x})\, ds_{\mathbf{x}}. \tag{4.36}$$

Remark 4.1.23. *The matrix $\mathbf{B}$ in (4.34) is dense because of (4.35), which means that all entries $B_{I,J}$ are, in general, not equal to zero. Furthermore, the twofold surface integral in (4.35) can very often not be computed exactly, even for polyhedrons, and requires numerical integration methods for its approximation. The influence of this additional approximation will be discussed in Chap. 5. In this chapter we will always assume that the matrix $\mathbf{B}$ can be determined exactly.*

Proposition 4.1.24. *The system matrix $\mathbf{B}$ in (4.34) is symmetric and positive definite.*

Proof. From the symmetry of $b(\varphi, \eta) = b(\eta, \varphi)$ we immediately have

$$B_{I,J} = b(b_I, b_J) = b(b_J, b_I) = B_{J,I},$$

and subsequently $\mathbf{B} = \mathbf{B}^\intercal$. Now let $\varphi \in \mathbb{R}^N$ be arbitrary. Then we have

$$\begin{aligned}\varphi^{\mathsf{T}}\mathbf{B}\varphi &= \sum_{I,J\in\mathcal{I}(\mathcal{G},p)} \varphi_J\varphi_I B_{I,J} = \sum_{I,J}\varphi_J\varphi_I b(b_I,b_J) = b\left(\sum_I \varphi_I b_I, \sum_J \varphi_J b_J\right)\\ &= b(\varphi_S,\varphi_S) \geq \gamma\|\varphi_S\|^2_{H^{-1/2}(\Gamma)} > 0\end{aligned}$$

if and only if $\varphi_S \neq 0$. Since $\{b_I : I \in \mathcal{I}\}$ is a basis of S^p, we have $\varphi_S \neq 0$ if and only if $\varphi \neq \mathbf{0} \in \mathbb{R}^N$. Therefore $\mathbf{B}$ is positive definite. □

Thus the discrete problem (4.32) or (4.34) has a unique solution $\varphi_S \in S^p_{\mathcal{G}}$.
The following proposition supplies us with an estimate for the error $\varphi - \varphi_S$.

Proposition 4.1.25. *Let φ be the exact solution of (4.9). The Galerkin solution φ_S of (4.32) converges quasi-optimally*

$$\|\varphi - \varphi_S\|_{H^{-1/2}(\Gamma)} \leq \frac{\|b\|}{\gamma}\min_{\eta_S\in S^p}\|\varphi - \eta_S\|_{H^{-1/2}(\Gamma)}. \tag{4.37}$$

The error satisfies the Galerkin orthogonality

$$b(\varphi - \varphi_S, \eta_S) = 0 \qquad \forall \eta_S \in S^p. \tag{4.38}$$

Proof. We will first prove the statement in (4.38). If we only consider (4.10) for test functions from S^p we can subtract (4.32) and obtain

$$b(\varphi - \varphi_S, \eta_S) = b(\varphi,\eta_S) - b(\varphi_S,\eta_S) = F(\eta_S) - F(\eta_S) = 0 \qquad \forall \eta_S \in S^p.$$

Next we prove (4.37). For the error $e_S = \varphi - \varphi_S$ we have by the ellipticity and the continuity of the boundary integral operator V and (4.38)

$$\begin{aligned}\gamma\|\varphi - \varphi_S\|^2_{H^{-1/2}(\Gamma)} &\leq b(e_S,e_S) = b(e_S,\varphi-\varphi_S)\\ &= b(e_S,\varphi) - b(e_S,\varphi_S) = b(e_S,\varphi) - b(e_S,\eta_S) = b(e_S,\varphi-\eta_S)\\ &\leq \|b\|\|e_S\|_{H^{-1/2}(\Gamma)}\|\varphi-\eta_S\|_{H^{-1/2}(\Gamma)}\end{aligned}$$

for all $\eta_S \in S^p$.

If we cancel $\|e_S\|_{H^{-1/2}(\Gamma)}$ and minimize over $\eta_S \in S^p$ we obtain the assertion (4.37). □

The inequality in (4.37) shows that the Galerkin error $\|\varphi - \varphi_S\|_{H^{-1/2}(\Gamma)}$ coincides with the error of the best approximation of φ in S^p up to a multiplicative constant. This is where the term *quasi-optimality* for the a priori error estimate (4.37) originates.

Remark 4.1.26 (Collocation). *We obtained the Galerkin discretization (4.32) from (4.8) by restricting the trial and test functions φ, η to the subspace $S^p \subset S$. Alternatively, one can insert φ_S into (4.6) and impose the equation*

$$(V\varphi_S)(\mathbf{x}_J) = g_D(\mathbf{x}_J) \qquad J \in \mathcal{I}(\mathcal{G}, p) \tag{4.39}$$

only in N collocation points $\{\mathbf{x}_J : J \in \mathcal{I}\}$. The solvability of (4.39) depends strongly on the choice of collocation points $\{\mathbf{x}_J : J \in \mathcal{I}\}$. Equation (4.39) is also equivalent to a linear system of equations, where the entries of the system matrix $\mathbf{B}^{coll}$ are defined by

$$B^{coll}_{I,J} := \int_\tau \frac{b_J(\mathbf{y})}{4\pi \, \|\mathbf{x}_I - \mathbf{y}\|} ds_{\mathbf{y}}. \tag{4.40}$$

Note that $\mathbf{B}^{coll}$ is again dense, but not symmetric.

The collocation method (4.39) is widespread in the field of engineering, because the computation of the matrix entries (4.40) only requires the evaluation of one integral over the surface Γ, instead of, as with the Galerkin method, a twofold integration over Γ. However, the stability and convergence of collocation methods on polyhedral surfaces is still an open question, especially with integral equations of the first kind. For integral operators of zero order or equations of the second kind we only have stability results in some special cases. For a detailed discussion on collocation methods we refer to, e.g., [6, 8, 87, 187, 207, 215] and the references contained therein.

We now return to the Galerkin method.

Remark 4.1.27 (Stability of the Galerkin Projection). *The Galerkin method (4.32) defines a mapping*

$$\Pi^p_S : H^{-1/2}(\Gamma) \to S^p_{\mathcal{G},\chi} : \qquad \Pi^p_S \varphi := \varphi_S,$$

which is called the Galerkin projection. Clearly, Π^p_S is linear and because of the ellipticity of the boundary integral operator V we have

$$\begin{aligned} \gamma \|\Pi^p_S \varphi\|^2_{H^{-1/2}(\Gamma)} = \gamma \|\varphi_S\|^2_{H^{-1/2}(\Gamma)} &\le b(\varphi_S, \varphi_S) = b(\varphi, \varphi_S) \\ &\le \|b\| \|\varphi\|_{H^{-1/2}(\Gamma)} \|\Pi^p_S \varphi\|_{H^{-1/2}(\Gamma)}, \end{aligned}$$

from which we have, after canceling, the boundedness of the Galerkin projection $\Pi^p_S : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ independent of the mesh $\mathcal{G}$:

$$\|\Pi^p_S \varphi\|_{H^{-1/2}(\Gamma)} \le \frac{\|b\|}{\gamma} \|\varphi\|_{H^{-1/2}(\Gamma)}. \tag{4.41}$$

The quasi-optimality (4.37) and the boundedness of the Galerkin projection combined with the following corollary give us the convergence of the Galerkin BEM.

Corollary 4.1.28. *Let $(\mathcal{G}_\ell)_{\ell \in \mathbb{N}}$ be a sequence of meshes on Γ with a mesh width $h_\ell = h_{\mathcal{G}_\ell}$ and let $h_\ell \to 0$ for $\ell \to \infty$. Then the sequence $(\varphi_\ell)_{\ell \in \mathbb{N}}$ of boundary element solutions (4.32) in $S_\ell = S^p_{\mathcal{G}_\ell}$ converges to φ for every fixed $p \in \mathbb{N}_0$.*

Proof. Since $S_\ell^0 \subseteq S_\ell^p$ for all $p \in \mathbb{N}_0$, we will only consider the case $p = 0$. S_ℓ^0 are step functions on meshes whose mesh width converges to zero. The density follows from the construction of the Lebesgue spaces

$$\overline{\bigcup_{\ell \in \mathbb{N}} S_\ell^0}^{\|\cdot\|_{L^2(\Gamma)}} = L^2(\Gamma)$$

and from Proposition 2.5.2 we have the dense embedding $L^2(\Gamma) \subset H^{-1/2}(\Gamma)$.

For $\varphi \in H^{-1/2}(\Gamma)$ and an arbitrary $\varepsilon > 0$ we can therefore choose a $\tilde{\varphi}$ from $L^2(\Gamma)$ and an $\ell \in \mathbb{N}$, combined so that $\tilde{\varphi}_\ell \in S_\ell^0$, such that

$$\|\varphi - \tilde{\varphi}\|_{H^{-1/2}(\Gamma)} \le \varepsilon/2 \quad \text{and} \quad \|\tilde{\varphi} - \tilde{\varphi}_\ell\|_{L^2(\Gamma)} \le \varepsilon/2.$$

From this we have

$$\|\varphi - \tilde{\varphi}_\ell\|_{H^{-1/2}(\Gamma)} \le \|\varphi - \tilde{\varphi}\|_{H^{-1/2}(\Gamma)} + \|\tilde{\varphi} - \tilde{\varphi}_\ell\|_{H^{-1/2}(\Gamma)} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \le \varepsilon.$$

The quasi-optimality of the Galerkin method gives us

$$\|\varphi - \varphi_\ell\|_{H^{-1/2}(\Gamma)} \le \frac{\|b\|}{\gamma} \|\varphi - \tilde{\varphi}_\ell\|_{H^{-1/2}(\Gamma)} \le \varepsilon \frac{\|b\|}{\gamma}.$$

As $\varepsilon > 0$ is arbitrary, we have the assertion for $\ell \to \infty$. $\square$

4.1.5 Convergence Rate of Discontinuous Boundary Elements

We have seen in Proposition 4.1.25 that the approximations $\varphi_S \in S$ from the Galerkin boundary element method approximate the exact solution φ of the equation of the first kind (4.9) quasi-optimally: the error $\varphi - \varphi_S$, which is measured in the "natural" $H^{-1/2}(\Gamma)$-norm, is – up to a multiplicative constant – just as large as

$$\min \left\{ \|\varphi - \psi_S\|_{H^{-1/2}(\Gamma)} : \psi_S \in S \right\} \tag{4.42}$$

which is the error of the best approximation in the space S. The convergence rate of the BEM indicates how fast the error converges to zero in relation to an increase in the degrees of freedom N. Here we will only prove the convergence rate for $p = 0$, while the general case will be treated in Sect. 4.3. We begin with the second Poincaré inequality on the reference element $\hat{\tau}$.

Convention 4.1.29. *Variables on the reference element are always marked by a "ˆ". If the variables $\mathbf{x} \in \tau$ and $\hat{\mathbf{x}} \in \hat{\tau}$ appear in the same context this should always be understood in terms of the relation $\mathbf{x} = \chi_\tau(\hat{\mathbf{x}})$. Derivatives with respect to variables in the reference element are also marked by a "ˆ". We will write, for example, $\widehat{\nabla}$ as an abbreviation for $\nabla_{\hat{\mathbf{x}}}$. Should the functions $u : \tau \to \mathbb{K}$ and $\hat{u} : \hat{\tau} \to \mathbb{K}$ appear in the same context, they are connected by the relation $u \circ \chi_\tau = \hat{u}$.*

Proposition 4.1.30. *Let $\hat{\tau} \subset \mathbb{R}^2$ be the reference element, $\hat{\varphi} \in H^1(\hat{\tau})$ and $\hat{\varphi}_0 := \frac{1}{|\hat{\tau}|} \int_{\hat{\tau}} \hat{\varphi} \, d\hat{\mathbf{x}}$. Then there exists some $\hat{c} > 0$ such that*

$$\|\hat{\varphi} - \hat{\varphi}_0\|_{L^2(\hat{\tau})} \leq \hat{c} \|\widehat{\nabla} \hat{\varphi}\|_{L^2(\hat{\tau})}, \tag{4.43}$$

where $\hat{c}$ depends only on $\hat{\tau}$.

Proof. The assertion follows directly from the proof of Corollary 2.5.10. □

In the following we will derive error estimates for a simplified situation. We will discuss the general case in Sect. 4.3. Here we let Γ be a plane manifold in $\mathbb{R}^3$ with a polygonal boundary. As integrals are invariant under rotation and translation, we assume without loss of generality that

$$\Gamma \text{ is a two-dimensional polygonal domain,} \tag{4.44}$$

i.e., we restrict ourselves to the two-dimensional approximation problem in the plane.

Furthermore, let $\mathcal{G} = \{\tau_i : 1 \leq i \leq N\}$ be a surface mesh on Γ of shape-regular triangles with straight edges and with mesh width $h > 0$. Then the triangles $\tau \in \mathcal{G}$ are affinely equivalent to the reference element $\hat{\tau}$ via the transformation (4.14):

$$\tau \ni \mathbf{x} = \chi_\tau(\hat{\mathbf{x}}) = \mathbf{P}_0 + \mathbf{J}\hat{\mathbf{x}}, \qquad \hat{\mathbf{x}} \in \hat{\tau}, \tag{4.45}$$

where $\mathbf{J}$ is the matrix with the columns $\mathbf{P}_1 - \mathbf{P}_0$ and $\mathbf{P}_2 - \mathbf{P}_1$ (see Fig. 4.1). With (4.45) and the chain rule

$$\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial \hat{x}_1} \frac{\partial \hat{x}_1}{\partial x_\alpha} + \frac{\partial}{\partial \hat{x}_2} \frac{\partial \hat{x}_2}{\partial x_\alpha} \qquad \alpha = 1, 2,$$

the relation

$$\nabla = \left(\mathbf{J}^{-1}\right)^{\mathsf{T}} \widehat{\nabla}, \qquad d\mathbf{x} = (\det \mathbf{J}) \, d\hat{\mathbf{x}} = 2 \, |\tau| \, d\hat{\mathbf{x}} \tag{4.46}$$

follows. This leads to the transformation formula for Sobolev norms

$$\begin{aligned} \|\widehat{\nabla} \hat{\varphi}\|^2_{L^2(\hat{\tau})} &= \int_{\hat{\tau}} |\widehat{\nabla} \hat{\varphi}|^2 \, d\hat{\mathbf{x}} = \frac{|\hat{\tau}|}{|\tau|} \int_\tau (\nabla \varphi)^\top \mathbf{J}\mathbf{J}^\top (\nabla \varphi) d\mathbf{x} \\ &\leq \frac{|\hat{\tau}|}{|\tau|} \lambda_\tau \int_\tau \|\nabla \varphi\|^2 \, d\mathbf{x}, \end{aligned} \tag{4.47}$$

where λ_τ denotes the largest eigenvalue of $\mathbf{J}\mathbf{J}^{\mathsf{T}} \in \mathbb{R}^{2\times 2}$. Furthermore, we have for the left-hand side of (4.43)

$$\|\hat{\varphi} - \hat{\varphi}_0\|^2_{L^2(\hat{\tau})} = \frac{|\hat{\tau}|}{|\tau|} \|\varphi - \varphi_0\|^2_{L^2(\tau)} \tag{4.48}$$

with $\varphi_0 := \frac{1}{|\tau|}\int_\tau \varphi d\mathbf{x}$. If we combine (4.48) with (4.43) and (4.47) we obtain

$$\|\varphi - \varphi_0\|^2_{L^2(\tau)} = \frac{|\tau|}{|\hat{\tau}|}\|\hat{\varphi} - \hat{\varphi}_0\|^2_{L^2(\hat{\tau})} \le \hat{c}^2\frac{|\tau|}{|\hat{\tau}|}\|\widehat{\nabla}\hat{\varphi}\|^2_{L^2(\hat{\tau})} \le \hat{c}^2\lambda_\tau \|\nabla\varphi\|^2_{L^2(\tau)} \quad \forall\tau \in \mathcal{G}. \tag{4.49}$$

Exercise 4.1.32 shows that

$$\lambda_\tau \le \|\mathbf{P}_1 - \mathbf{P}_0\|^2 + \|\mathbf{P}_2 - \mathbf{P}_1\|^2 \le 2h_\tau^2. \tag{4.50}$$

From this we have

$$\|\varphi - \varphi_0\|_{L^2(\tau)} \le \sqrt{2}\hat{c}h_\tau|\varphi|_{H^1(\tau)}. \tag{4.51}$$

Squaring and then summing over all $\tau \in \mathcal{G}$ leads to the following error estimate.

Proposition 4.1.31. *Let (4.44) hold. Let $\mathcal{G}$ be a surface mesh of Γ. Let $\varphi \in L^2(\Gamma)$ with $\varphi|_\tau \in H^1(\tau)$ for all $\tau \in \mathcal{G}$. Then we have the error estimate*

$$\min_{\psi\in S^0_{\mathcal{G}}} \|\varphi - \psi\|_{L^2(\Gamma)} \le \sqrt{2}\hat{c}\left(\sum_{\tau\in\mathcal{G}} h_\tau^2|\varphi|^2_{H^1(\tau)}\right)^{1/2}. \tag{4.52}$$

For $\varphi \in H^1(\Gamma)$ the error estimate can be simplified to

$$\min_{\psi\in S^0_{\mathcal{G}}} \|\varphi - \psi\|_{L^2(\Gamma)} \le \sqrt{2}\hat{c}h_{\mathcal{G}}|\varphi|_{H^1(\Gamma)}. \tag{4.53}$$

Exercise 4.1.32. *Let τ be a plane triangle with straight edges in $\mathbb{R}^2$ with vertices $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2$. Let the matrix $\mathbf{J}$ and the eigenvalue λ_τ be defined as in (4.45) and (4.47) respectively. Show that*

$$\lambda_\tau \le \|\mathbf{P}_1 - \mathbf{P}_0\|^2 + \|\mathbf{P}_2 - \mathbf{P}_1\|^2 .$$

From the approximation property we will now derive an error estimate for the Galerkin solution.

Theorem 4.1.33. *Let Γ be the surface of a polyhedron. Let the surface mesh $\mathcal{G}$ consist of triangles with straight edges.*

For the solution φ of the integral equation of the first kind (4.6) we assume that for an $0 \le s \le 1$ we have

$$\varphi \in H^s(\Gamma). \tag{4.54}$$

Then the Galerkin approximation $\varphi_S \in S^0_{\mathcal{G}}$ satisfies the error estimate

$$\|\varphi - \varphi_S\|_{H^{-1/2}(\Gamma)} \le C\, h^{s+1/2}\|\varphi\|_{H^s(\Gamma)}. \tag{4.55}$$

Proof. The conditions of the theorem allow us to apply Proposition 4.1.31. With (4.37) we obtain for the Galerkin solution φ_S the error estimate

$$\|\varphi - \varphi_S\|_{H^{-1/2}(\Gamma)} = \|\varphi - \Pi_S^0 \varphi\|_{H^{-1/2}(\Gamma)} \leq \frac{\|b\|}{\gamma} \min_{\psi_S \in S_{\mathcal{G}}^0} \|\varphi - \psi_S\|_{H^{-1/2}(\Gamma)}.$$

The definition of the $H^{-1/2}(\Gamma)$-norm gives us

$$\|\varphi - \psi_S\|_{H^{-1/2}(\Gamma)} = \sup_{\eta \in H^{1/2}(\Gamma)\backslash\{0\}} \frac{(\varphi - \psi_S, \eta)_{L^2(\Gamma)}}{\|\eta\|_{H^{1/2}(\Gamma)}}. \tag{4.56}$$

We will first consider the case $\varphi \in H^1(\Gamma)$ and choose ψ_S elementwise as the mean value of φ

$$P\varphi := \psi_S \quad \text{with} \quad \psi_S|_\tau := \frac{1}{|\tau|} \int_\tau \varphi \, d\mathbf{x}, \qquad \tau \in \mathcal{G},$$

i.e., P is the L^2-orthogonal projection onto $S_{\mathcal{G}}^0$. Hence it follows from Proposition 4.1.31 that

$$\|\psi_S\|_{L^2(\Gamma)} \leq \|\varphi\|_{L^2(\Gamma)}, \;\; \|\varphi - \psi_S\|_{L^2(\Gamma)} \leq 2\|\varphi\|_{L^2(\Gamma)}, \;\; \|\varphi - \psi_S\|_{L^2(\Gamma)} \leq ch\|\varphi\|_{H^1(\Gamma)}. \tag{4.57}$$

If in Proposition 2.1.62 we choose $T = I - P$ we have $T : L^2(\Gamma) \to L^2(\Gamma)$ and $T : H^1(\Gamma) \to L^2(\Gamma)$. For the norms we have, by (4.57), the estimates

$$\|T\|_{L^2(\Gamma) \leftarrow L^2(\Gamma)} \leq 2 \quad \text{and} \quad \|T\|_{L^2(\Gamma) \leftarrow H^1(\Gamma)} \leq ch.$$

Proposition 2.1.62 implies that $T : H^s(\Gamma) \to L^2(\Gamma)$ for all $0 \leq s \leq 1$ and that

$$\|T\|_{L^2(\Gamma) \leftarrow H^s(\Gamma)} \leq ch^s.$$

This is equivalent to the error estimate

$$\|\varphi - \psi_S\|_{L^2(\Gamma)} \leq c\, h^s \|\varphi\|_{H^s(\Gamma)}. \tag{4.58}$$

In order to derive an error estimate for the $H^{-1/2}(\Gamma)$-norm, we use (4.56) and note that the equality

$$|\,(\varphi - \psi_S, \eta)_{L^2(\Gamma)}\,| = |\,(\varphi - \psi_S, \eta - \eta_S)_{L^2(\Gamma)}\,|$$

holds for an arbitrary $\eta_S \in S_{\mathcal{G}}^0$. By using $\varphi \in H^s(\Gamma)$, $\eta \in H^{1/2}(\Gamma)$ and (4.58) and by choosing η_S elementwise as the integral mean value of η, we obtain the estimate

$$\begin{aligned}\left|(\varphi - \psi_S, \eta)_{L^2(\Gamma)}\right| = \left|(\varphi - \psi_S, \eta - \eta_S)_{L^2(\Gamma)}\right| &\leq \|\varphi - \psi_S\|_{L^2(\Gamma)} \, \|\eta - \eta_S\|_{L^2(\Gamma)} \\ &\leq ch^s \|\varphi\|_{H^s(\Gamma)} h^{1/2} \|\eta\|_{H^{1/2}(\Gamma)}.\end{aligned}$$

$\square$

The error estimate (4.55) shows that the convergence rate $h^{s+1/2}$ of the BEM depends on the regularity of the solution φ. In Sect. 3.2 we stated the regularity – the maximal $s > 0$ such that $\varphi \in H^{-1/2+s}(\Gamma)$ – without knowing the exact solution φ explicitly. Ideally, φ is smooth on the entire surface ($s = \infty$) or at least on every panel. The convergence rate would then be bounded by the polynomial order p of the boundary elements, due to the fact that the following generalization of Theorem 4.1.33 holds.

Corollary 4.1.34. *Let the exact solution of (4.9) satisfy $\varphi \in H^s(\Gamma)$ for an $s \geq 0$. Then the boundary element solution $\varphi_S \in S_{\mathcal{G}}^p$ satisfies the error estimate*

$$\|\varphi - \varphi_S\|_{H^{-1/2}(\Gamma)} \leq c h_{\mathcal{G}}^{1/2+\min(s,p+1)} \|\varphi\|_{H^s(\Gamma)}, \tag{4.59}$$

for a surface mesh $\mathcal{G}$ of the boundary Γ, which consists of triangles with straight edges. Here the constant c depends on p and the shape-regularity of the surface mesh.

The proof of Corollary 4.1.34 will be completed in Sect. 4.3.4 (see Remark 4.3.21).

4.1.6 Model Problem 2: Neumann Problem

Let $\Omega^- \subset \mathbb{R}^3$ be a bounded interior domain with boundary Γ and $\Omega^+ := \mathbb{R}^3 \backslash \overline{\Omega^-}$. For $g_N \in H^{-1/2}(\Gamma)$ we consider the Neumann problem

$$\Delta u = 0 \qquad \text{in } \Omega^+, \tag{4.60}$$

$$\gamma_1 u = g_N \qquad \text{on } \Gamma, \tag{4.61}$$

$$|u(\mathbf{x})| \leq C \|\mathbf{x}\|^{-1} \qquad \text{for } \|\mathbf{x}\| \to \infty. \tag{4.62}$$

The exterior problem (4.60)–(4.62) has a unique solution u, which can be represented as a double layer potential

$$u(\mathbf{x}) = \frac{1}{4\pi} \int_\Gamma \varphi(\mathbf{y}) \frac{\partial}{\partial \mathbf{n}_\mathbf{y}} \frac{1}{\|\mathbf{x} - \mathbf{y}\|} \, ds_\mathbf{y}, \qquad \mathbf{x} \in \Omega^+. \tag{4.63}$$

Thanks to the jump relations (see Corollary 3.3.12)

$$\frac{1}{4\pi} \int_\Gamma \frac{\partial}{\partial \mathbf{n}_\mathbf{y}} \frac{1}{\|\mathbf{x} - \mathbf{y}\|} \, ds_\mathbf{y} = \begin{cases} -1 & \mathbf{x} \in \Omega^-, \\ -\frac{1}{2} & \mathbf{x} \in \Gamma \text{ and } \Gamma \text{ is smooth in } \mathbf{x} \\ 0 & \mathbf{x} \in \Omega^+ \end{cases}$$

$u(\mathbf{x})$ in (4.63) does not change if a constant is added to φ. If we put (4.63) into the boundary condition (4.61) we obtain the equation

$$-W\varphi = \frac{\partial}{\partial \mathbf{n_x}}\left(\frac{1}{4\pi}\int_\Gamma \varphi(\mathbf{y})\frac{\partial}{\partial \mathbf{n_y}}\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\,ds_\mathbf{y}\right) = g_N(\mathbf{x}), \qquad \mathbf{x}\in\Gamma. \tag{4.64}$$

The following remark shows that the derivative $\partial/\partial\mathbf{n_x}$ and the integral do not commute.

Remark 4.1.35. *The normal derivative $\partial/\partial\mathbf{n_x}$, applied to the kernel in (4.64), yields*

$$\frac{\partial^2}{\partial\mathbf{n_x}\partial\mathbf{n_y}}\frac{1}{\|\mathbf{x}-\mathbf{y}\|} = \frac{\langle \mathbf{n_x},\mathbf{n_y}\rangle}{\|\mathbf{x}-\mathbf{y}\|^3} - 3\frac{\langle \mathbf{n_x},\mathbf{x}-\mathbf{y}\rangle\langle \mathbf{n_y},\mathbf{x}-\mathbf{y}\rangle}{\|\mathbf{x}-\mathbf{y}\|^5}.$$

Therefore the kernel of the associated hypersingular integral operator is not integrable.

There are three possibilities of representing the integral operator $W\varphi$ on the surface: (a) by extending the definition of an integral to strongly singular kernel functions (see [201, 211]), (b) by integration by parts (see Sect. 3.3.4) and (c) by introducing suitable differences of test and trial functions (see [117, Sect. 8.3]). In this section we will consider option (b). The notation and theorems from Sect. 3.3.4 can be simplified for the Laplace problem, so that they read

$$\begin{aligned}
\operatorname{curl}_\Gamma \varphi &:= \gamma_0\left(\operatorname{grad} Z_-\varphi\right)\times\mathbf{n},\\
b(\varphi,\eta) &= \int_\Gamma\int_\Gamma \frac{\langle \operatorname{curl}_\Gamma \varphi\,(\mathbf{y}), \operatorname{curl}_\Gamma \eta\,(\mathbf{x})\rangle}{4\pi\,\|\mathbf{x}-\mathbf{y}\|}\,ds_\mathbf{y}ds_\mathbf{x},
\end{aligned}$$

where $Z_- : H^{1/2}(\Gamma) \to H^1(\Omega^-)$ is an arbitrary extension operator (see Theorem 2.6.11 and Exercise 3.3.25).

The variational formulation of the boundary integral equation is given by (see Theorem 3.3.22): Find $\varphi \in H^{1/2}(\Gamma)/\mathbb{K}$ such that

$$b(\varphi,\eta) = -\left(g_N,\eta\right)_{L^2(\Gamma)} \qquad \forall\eta\in H^{1/2}(\Gamma)/\mathbb{K}. \tag{4.65}$$

In Theorem 3.5.3 we have already shown that the density φ in (4.63) is the unique solution of the boundary integral equation (4.65). The proof was based on the fact that the bilinear form $b(\cdot,\cdot)$ is symmetric, continuous and $H^{1/2}(\Gamma)/\mathbb{K}$-elliptic.

4.1.7 Continuous Boundary Elements

The Galerkin method is based on the concept of replacing the infinite-dimensional Hilbert space by a finite-dimensional *sub*space. The bilinear form that is associated with the hypersingular integral operator is defined on the Sobolev space

$H^{1/2}(\Gamma)/\mathbb{K}$. As the discontinuous boundary element functions from Example 4.1.16 and Definition 4.1.17 are not contained in $H^{1/2}(\Gamma)/\mathbb{K}$ (see Exercise 2.4.4), we will introduce *continuous* boundary element spaces for the Neumann problem.

We again start with a mesh $\mathcal{G}$ on the boundary Γ. In order to define continuous boundary elements, we assume (see Definition 4.1.4):

$$\text{The surface mesh } \mathcal{G} \text{ is regular.} \tag{4.66}$$

This means that the intersection $\overline{\tau} \cap \overline{\tau}'$ of two different panels is either empty, a vertex or an entire edge. Furthermore, the boundary elements are either triangles or quadrilaterals and are images of the reference triangle or quadrilateral $\hat{\tau}$ respectively (see Fig. 4.1). Note that the boundary edges of the panels "have the same parametrization on both sides" in the case of continuous boundary elements (see Definition 4.1.4).

We assume that the boundary Γ is piecewise smooth (see Definition 2.2.10 and Fig. 4.1) so that the reference mappings $\chi_\tau : \hat{\tau} \to \tau$ can be chosen as smooth diffeomorphisms. As in the case for discontinuous boundary elements, the continuous boundary elements are also piecewise polynomials on the surface Γ. When using discontinuous elements, a boundary element function φ_S is locally a polynomial of degree p in *each* element $\tau \in \mathcal{G}$:

$$\forall \tau \in \mathcal{G}: \qquad \varphi_S \circ \chi_\tau \in \mathbb{P}_p^\Delta(\hat{\tau}).$$

With continuous elements we have for $\tau \in \mathcal{G}$:

$$\varphi_S \circ \chi_\tau \in \mathbb{P}_p^\tau := \begin{cases} \mathbb{P}_p^\Delta & \text{if } \tau \text{ is a triangular element,} \\ \mathbb{P}_p^\square & \text{if } \tau \text{ is a quadrilateral element,} \end{cases} \tag{4.67}$$

where for $p \geq 1$ the polynomial space $\mathbb{P}_p^\Delta$ is defined as in (4.23) and

$$\mathbb{P}_p^\square := \operatorname{span}\{\hat{\xi}_1^i \hat{\xi}_2^j : 0 \leq i, j \leq p\}.$$

Now we come to the definition of continuous boundary element functions of degree $p \geq 1$.

Definition 4.1.36. Let Γ be a piecewise smooth surface, $\mathcal{G}$ a regular surface mesh of Γ and $\chi = \{\chi_\tau : \tau \in \mathcal{G}\}$ the mapping vector. Then the space of continuous boundary elements of degree $p \geq 1$ is given by

$$S_{\mathcal{G},\chi}^{p,0} := \{\varphi \in C^0(\Gamma) \mid \forall \tau \in \mathcal{G} : \varphi|_\tau \circ \chi_\tau \in \mathbb{P}_p^\tau\}.$$

In order to make the distinction between continuous and discontinuous boundary elements of degree p we will from now on denote discontinuous elements by $S_{\mathcal{G},\chi}^{p,-1}$.

Just like the space $S^{p,-1}$ of discontinuous boundary elements, the space $S^{p,0}$ is also finite-dimensional. In the following we will introduce a basis $\{\varphi_I : I \in \mathcal{I}\}$ of $S^{p,0}$. In contrast to $S^{p,-1}$, the support of the basis functions in general consists of more than one panel and the basis functions are defined piecewise on those panels. We begin with the simplest case, $p = 1$.

Example 4.1.37. *(Linear and Bilinear, Continuous Boundary Elements)*
The shape functions $\widehat{N}(\hat{\mathbf{x}})$, $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)$ on the reference element $\hat{\tau}$ are:

- *In the case of the unit triangle with vertices $\mathbf{P}_0 = (0,0)^\intercal$, $\mathbf{P}_1 = (1,0)^\intercal$, $\mathbf{P}_2 = (1,1)^\intercal$ [see (4.13)], given by*

$$\begin{aligned} \widehat{N}_0(\hat{\mathbf{x}}) &= 1 - \hat{x}_1, \\ \widehat{N}_1(\hat{\mathbf{x}}) &= \hat{x}_1 - \hat{x}_2, \\ \widehat{N}_2(\hat{\mathbf{x}}) &= \hat{x}_2 \end{aligned} \tag{4.68}$$

and

- *In the case of the unit square with vertices $\mathbf{P}_0 = (0,0)^\intercal$, $\mathbf{P}_1 = (1,0)^\intercal$, $\mathbf{P}_2 = (1,1)^\intercal$, $\mathbf{P}_3 = (0,1)^\intercal$, given by*

$$\begin{aligned} \widehat{N}_0(\hat{\mathbf{x}}) &= (1 - \hat{x}_1)(1 - \hat{x}_2), \\ \widehat{N}_1(\hat{\mathbf{x}}) &= \hat{x}_1(1 - \hat{x}_2), \\ \widehat{N}_2(\hat{\mathbf{x}}) &= (1 - \hat{x}_1)\,\hat{x}_2, \\ \widehat{N}_3(\hat{\mathbf{x}}) &= \hat{x}_1\hat{x}_2. \end{aligned} \tag{4.69}$$

We notice that the shape function $\widehat{N}_i$ is equal to 1 *at the vertex $\mathbf{P}_i$ of the reference element* 1 *and vanishes at all other vertices (see Fig. 4.3).*
It holds $\mathbb{P}_1^{\Delta}(\hat{\tau}) = \operatorname{span}\{\widehat{N}_i : i = 0, 1, 2\}$ and $\mathbb{P}_1^{\Box}(\hat{\tau}) = \operatorname{span}\{\widehat{N}_i : i = 0, \ldots 3\}$.

For the definition of the boundary element spaces of polynomial degree p we have to distinguish between quadrilateral elements and triangular elements. For the reference element $\hat{\tau} \in \mathcal{G}$ and $p \in \mathbb{N}_0$ we define the index set

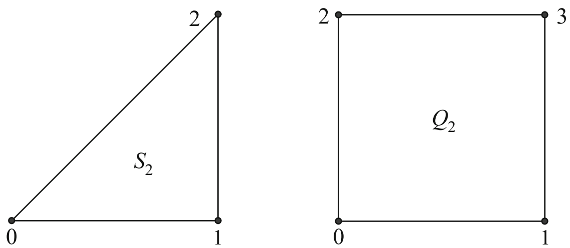
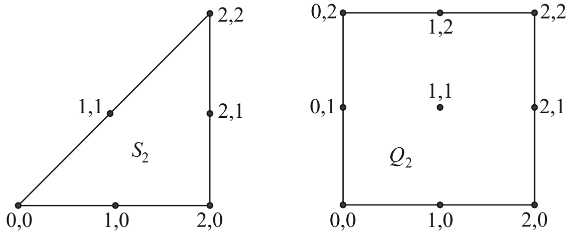


Fig. 4.3 Reference elements $\hat{\tau} = S_2$ (*left*) and $\hat{\tau} = Q_2$ (*right*) and nodal points for $\mathbb{P}_1^{\hat{\tau}}$

Fig. 4.4 Nodal points $\hat{\mathbf{P}}_{i,j}^{(2)}$ for the reference triangle (*left*) and for the unit square (*right*)



$$\iota_p^{\hat{\tau}} := \begin{cases} \left\{(i,j) \in \mathbb{N}_0^2 : 0 \le j \le i \le p\right\} & \text{in the case of the unit triangle,} \\ \left\{(i,j) \in \mathbb{N}_0^2 : 0 \le i, j \le p\right\} & \text{in the case of the unit square.} \end{cases} \tag{4.70}$$

We will omit the index $\hat{\tau}$ in $\iota_p^{\hat{\tau}}$ if the reference element is clear from the context.

Example 4.1.38 (Boundary elements of degree $p > 1$**).** *The trial spaces* $\mathbb{P}_p^{\Delta}$, $\mathbb{P}_p^{\square}$ *in (4.67) are spanned by the functions* $\widehat{N}_{(i,j)}^{(p)} \in \mathbb{P}_p^{\hat{\tau}}$ *which will be defined next. The nodal points for the reference element* $\hat{\tau}$ *are given by*

$$\widehat{P}_{(i,j)}^{(p)} := \left(\frac{i}{p}, \frac{j}{p}\right)^{\intercal}, \qquad \forall (i,j) \in \iota_p^{\hat{\tau}} \tag{4.71}$$

(see Fig. 4.4).

For $(i,j) \in \iota_p^{\hat{\tau}}$ *the shape function* $\widehat{N}_{(i,j)}^{(p)}$ *is characterized by*

$$\widehat{N}_{(i,j)}^{(p)} \in \mathbb{P}_p^{\hat{\tau}} \quad \textit{and} \quad \widehat{N}_{(i,j)}^{(p)}(\widehat{P}_{(k,\ell)}^{(p)}) = \begin{cases} 1 & (k,\ell) = (i,j), \\ 0 & (k,\ell) \in \iota_p^{\hat{\tau}} \backslash \{(i,j)\} \end{cases}$$

(see Theorem 4.1.39).

Theorem 4.1.39. *Let* $k \in \mathbb{N}$. *Then every* $q \in \mathbb{P}_k^{\hat{\tau}}$ *is uniquely determined by its values in* $\Sigma_k := \left\{(i/k, j/k) : (i,j) \in \iota_k^{\hat{\tau}}\right\}$.

The set Σ_k is called *unisolvent* for the polynomial space $\mathbb{P}_k^{\hat{\tau}}$ because of this property.

Proof. A simple calculation shows that

$$\dim \mathbb{P}_k^{\hat{\tau}} = \sharp \Sigma_k .$$

Therefore it suffices to prove either one of the following statements (a) or (b):

(a) For every vector $(b_{\mathbf{z}})_{\mathbf{z} \in \Sigma_k}$ there exists a $q \in \mathbb{P}_k^{\hat{\tau}}$ such that $q(\mathbf{z}) = b_{\mathbf{z}}$ for all $\mathbf{z} \in \Sigma_k$.
(b) If $q \in \mathbb{P}_k^{\hat{\tau}}$ and $q(\mathbf{z}) = 0$ for all $\mathbf{z} \in \Sigma_k$ then $q \equiv 0$.

Case 1: $\hat{\tau} = (0,1)^2$: For $\mu \in \iota_k^{\hat{\tau}}$ we define the function $\widehat{N}_\mu$ by

$$\widehat{N}_\mu(\mathbf{x}) := \prod_{j=1}^{2} \prod_{\substack{i_j=0 \\ i_j \neq \mu_j}}^{k} \frac{k x_j - i_j}{\mu_j - i_j}.$$

Then $\widehat{N}_\mu \in \mathbb{P}_k^{\hat{\tau}}$ with $\widehat{N}_\mu(\mu/k) = 1$ and $\widehat{N}_\mu\left(\frac{i_1}{k}, \frac{i_2}{k}\right) = 0$ for all $(i_1, i_2) \in \iota_k^{\hat{\tau}} \backslash \{\mu\}$. Now let $\left(b_\mu\right)_{\mu \in \iota_k^{\hat{\tau}}}$ be arbitrary. Then the polynomial $q \in \mathbb{P}_k^{\hat{\tau}}$

$$q(\mathbf{x}) = \sum_{\mu \in \iota_p^{\hat{\tau}}} b_\mu \widehat{N}_\mu(\mathbf{x})$$

satisfies property (a).

Case 2: $\hat{\tau}$ is the reference triangle. As in Example 4.1.37 we set

$$\hat{\lambda}_1(\mathbf{x}) := 1 - \hat{x}_1, \qquad \hat{\lambda}_2(\mathbf{x}) := \hat{x}_1 - \hat{x}_2, \qquad \hat{\lambda}_3(\mathbf{x}) := \hat{x}_2.$$

Clearly, these functions are in $\mathbb{P}_1^{\hat{\tau}}$ and have the Lagrange property

$$\forall 1 \leq i, j \leq 3 : \hat{\lambda}_i\left(\mathbf{A}_j\right) = \delta_{i,j} \quad \text{with} \quad \mathbf{A}_1 = (0,0)^\intercal, \mathbf{A}_2 = (1,0)^\intercal, \mathbf{A}_3 = (1,1)^\intercal.$$

1. $k = 1$: For a given $(b_i)_{i=1}^3 \in \mathbb{R}^3$, $q \in \mathbb{P}_1$:

$$q(\mathbf{x}) = \sum_{i=1}^{3} b_i \hat{\lambda}_i(\mathbf{x})$$

   clearly has the property (a).
2. $k = 2$: For $1 \leq i < j \leq 3$, $\mathbf{A}_{(i,j)} := \left(\mathbf{A}_i + \mathbf{A}_j\right)/2$ denote the midpoints of the edges of $\hat{\tau}$. We define

$$\begin{aligned} \widehat{N}_i &:= \hat{\lambda}_i\left(2\hat{\lambda}_i - 1\right) & 1 \leq i \leq 3, \\ \widehat{N}_{(i,j)} &:= 4\hat{\lambda}_i \lambda_j & 1 \leq i < j \leq 3. \end{aligned}$$

   Then we clearly have $\widehat{N}_k, \widehat{N}_{(i,j)} \in \mathbb{P}_2^{\hat{\tau}}$ and

$$\begin{aligned} &\widehat{N}_i\left(\mathbf{A}_j\right) = \delta_{i,j} \quad \widehat{N}_i\left(\mathbf{A}_{(k,\ell)}\right) = 0 && \forall i, k, \ell, \\ &\widehat{N}_{(i,j)}(\mathbf{A}_k) = 0 \quad \widehat{N}_{(i,j)}\left(\mathbf{A}_{(k,\ell)}\right) = \delta_{i,k}\delta_{j,\ell} && \forall i, j, k, \ell. \end{aligned}$$

   For a given $\{b_{\mathbf{z}} : \mathbf{z} \in \Sigma_2\} = \left\{b_i, b_{(k,\ell)}\right\}$, the polynomial $q \in \mathbb{P}_2^{\hat{\tau}}$ defined by

$$q(\mathbf{x}) := \sum_{i=1}^{3} b_i \widehat{N}_i(\mathbf{x}) + \sum_{1 \leq k < \ell \leq 3} b_{(k,\ell)} \widehat{N}_{(k,\ell)}(\mathbf{x})$$

   has the property (a).

3. $k = 3$: This case will be treated in Exercise 4.1.40.
4. $k \geq 4$: Let $q \in \mathbb{P}_k^\Delta$ with $q(\mathbf{z}) = 0$ for all $\mathbf{z} \in \Sigma_k$. Then q vanishes on all edges of $\hat{\tau}$. Therefore there exists a $\psi \in \mathbb{P}_{k-3}^\Delta$ such that

$$q = \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3 \psi \quad \text{and} \quad \forall \mathbf{z} \in \Sigma_k \cap \hat{\tau} : \psi(\mathbf{z}) = 0.$$

(Note that $\hat{\tau}$ is open.) The problem can thus be reduced to

$$\left(\psi \in \mathbb{P}_{k-3}^\Delta\right) \wedge (\forall \mathbf{z} \in \Sigma_k \cap \hat{\tau} : \psi(\mathbf{z}) = 0) \Longrightarrow \psi \equiv 0. \tag{4.72}$$

Property (b) follows by induction over k as follows.
Let $\hat{\tau}'$ be the triangle with vertices $\mathbf{A} = \left(\frac{2}{k+1}, \frac{1}{k+1}\right)^\mathsf{T}$, $\mathbf{B} = \left(\frac{k}{k+1}, \frac{1}{k+1}\right)^\mathsf{T}$, $\mathbf{C} = \left(\frac{k}{k+1}, \frac{k-1}{k+1}\right)^\mathsf{T}$. Then we have $\Sigma_k \cap \hat{\tau} =: \Sigma_k' \subset \widehat{\tau}'$. The transformation

$$T : \hat{\tau} \to \hat{\tau}' : T\xi = \mathbf{A} + \left(1 - \frac{3}{k+1}\right)\xi$$

is affine and therefore $\tilde{\psi} = \psi \circ T \in \mathbb{P}_{k-3}^\Delta$. Furthermore, we have $T^{-1}\Sigma_k' = \Sigma_{k-3}$. Hence (4.72) is equivalent to

$$\left(\tilde{\psi} \in \mathbb{P}_{k-3}^\Delta\right) \wedge \left(\forall \mathbf{z} \in \Sigma_{k-3} : \tilde{\psi}(\mathbf{z}) = 0\right) \Longrightarrow \tilde{\psi} \equiv 0.$$

This, however, is statement (b) for $k \leftarrow k - 3$. Since the induction hypothesis for $k = 1, 2, 3$ is given by steps 1–3 in the proof, the assertion follows by virtue of the equivalence of the two statements (a) and (b). □

Exercise 4.1.40. *Let $\hat{\tau}$ be the unit triangle. For $\mathbb{P}_3^{\hat{\tau}}$ construct a Lagrange basis for the set of mesh points Σ_3 (see Theorem 4.1.39).*

In combination with the polynomial space $\mathbb{P}_p^{\hat{\tau}}$ on $\hat{\tau}$ we define an interpolation operator $\widehat{I}^p$ for the set of nodal points $\Sigma_p = \left(\widehat{P}_{(i,j)}^{(p)}\right)_{(i,j)\in\iota_p}$ for continuous functions $\varphi \in C^0\left(\overline{\hat{\tau}}\right)$ by

$$\widehat{I}^p \varphi := \sum_{(i,j)\in\iota_p^{\hat{\tau}}} \varphi\left(\widehat{P}_{(i,j)}^{(p)}\right) \widehat{N}_{(i,j)}^{(p)}. \tag{4.73}$$

The Sobolev embedding theorem (Theorem 2.5.4) proves the continuity of the embedding $H^t(\hat{\tau}) \hookrightarrow C^0\left(\overline{\hat{\tau}}\right)$ thanks to $\hat{\tau} \subset \mathbb{R}^2$ for $t > 1$ and therefore $\widehat{I}^p$ is defined on $H^t(\hat{\tau})$, thus

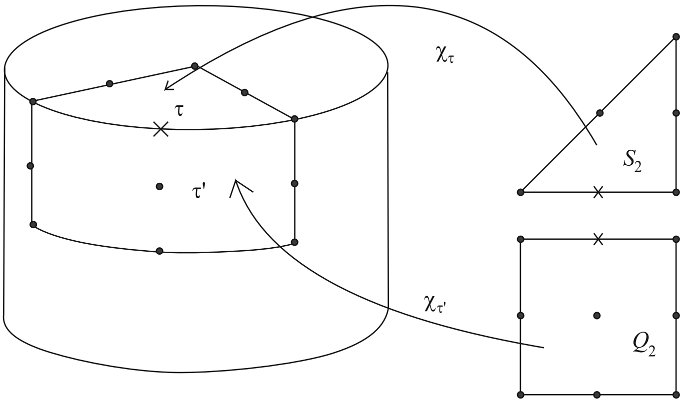


Fig. 4.5 Quadratic triangular and quadrilateral elements which share a common edge. The compatibility of the parametrizations ensures that the midpoints (*cross marks*) of the pullbacks of the common edge in the reference elements are mapped to the same surface points

$$\widehat{I}^p : H^t(\hat{\tau}) \to \mathbb{P}_p^{\hat{\tau}} \quad \text{and continuous:} \quad \left\| \widehat{I}^p \right\|_{C^0(\overline{\hat{\tau}}) \leftarrow H^t(\hat{\tau})} < \infty.$$

One obtains the set of nodal points on the surface by lifting the set of nodes on the reference element by means of the element parametrization

$$\mathcal{I} := \left\{ \chi_\tau \left(\widehat{P}_{(i,j)} \right) : \forall \tau \in \mathcal{G}, \quad \forall (i,j) \in \iota_p^{\hat{\tau}} \right\}. \tag{4.74}$$

Clearly, in a mesh $\mathcal{G}$ on Γ there will be nodal points that lie in more than one element, more precisely, that lie in their closures. As an example, consider Fig. 4.5 with two panels that have a common edge.

If the parameter representation $\chi_\tau, \chi_{\tau'}$ of the panels $\tau, \tau' \in \mathcal{G}$ is not compatible, the edge midpoint "×" on the common edge will be mapped to different points in $\hat{\tau}$, $\widehat{\tau'}$, depending on whether it is associated with τ or τ'. Thus, regular element mappings (see Definition 4.1.4) must parametrize edges $e = \overline{\tau} \cap \overline{\tau}'$ "identically from both sides". In the following we will always assume in the definition of continuous boundary elements $S_{\mathcal{G},\chi}^{p,0}$ that $\mathcal{G}$ and χ are regular.

Example 4.1.41 (*p*-Parametric Boundary Elements). *Let $\mathcal{G}$ be a regular mesh on Γ and let $q \geq 1$ be given and fixed. Then we can approximate a regular, generally non-linear, parametrization $\chi_\tau : \hat{\tau} \longrightarrow \tau \in \mathcal{G}$ by means of a p-parametric element mapping*

$$\widetilde{\chi}_\tau(\hat{\mathbf{x}}) := \sum_{(i,j) \in \iota_q^{\hat{\tau}}} \mathbf{P}_{(i,j)}^{(q)}(\tau) \widehat{N}_{(i,j)}^{(q)}(\hat{\mathbf{x}}), \qquad \hat{\mathbf{x}} \in \hat{\tau}, \tag{4.75}$$

where $\mathbf{P}_{(i,j)}^{(q)}(\tau) := \chi_\tau \left(\widehat{P}_{(i,j)}^{(q)} \right)$ denotes the lifted nodes of the reference element.

Remark 4.1.42. *In practical applications the construction (4.75) is used for $p = 1$ and $p = 2$ with the shape functions $\widehat{N}^{(p)}_{(i,j)}$ for the set of points $\widehat{P}^{(p)}_{(i,j)}$ in (4.71). In every case the approximation panel $\tilde{\tau} := \widetilde{\chi}_\tau(\hat{\tau})$ interpolates the exact panel τ at the points $\mathbf{P}^{(p)}_{(i,j)}$. It is known from interpolation theory (see Sect. 7.1.3.1) that, for the quality of the approximation, the choice of interpolation points becomes essential for high orders of approximation such as $p \geq 3$. For $p \geq 3$ the images of the Gauss–Lobatto points for the unit square represent a better choice for the set of nodes $\mathbf{P}^{(p)}_{(i,j)}$. Similar sets of points are known for the unit triangle (see [16, 130]).*

In the following we will always assume that the χ describe the surface Γ exactly. The influence of the approximation of the domain on the accuracy of the boundary element solution is discussed in Chap. 8.

We define the space of the continuous, piecewise polynomial boundary elements of degree $p \geq 1$ by a basis b_I. For this, let $\mathcal{I}$ be, as in (4.74), the set of all nodal points in the mesh $\mathcal{G}$. The basis function $b_\mathbf{P}$ for the nodal point $\mathbf{P} \in \mathcal{I}$ is characterized by the conditions

$$b_\mathbf{P} \in S^{p,0}_\mathcal{G} \quad \text{and} \quad b_\mathbf{P}(\mathbf{P}') := \begin{cases} 1 & \text{for } \mathbf{P}' = \mathbf{P}, \\ 0 & \text{for } \mathbf{P}' \neq \mathbf{P}, \quad \mathbf{P}' \in \mathcal{I}. \end{cases} \tag{4.76}$$

For a nodal point $\mathbf{P} \in \mathcal{I}$ we define a local neighborhood of triangles by $\Gamma_\mathbf{P} := \bigcup\{\overline{\tau} : \tau \in \mathcal{G},\ \mathbf{P} \in \overline{\tau}\}$. Then we have

$$\operatorname{supp}(b_\mathbf{P}) = \Gamma_\mathbf{P}. \tag{4.77}$$

In order to derive a local representation of the basis functions by element shape functions, we need a relation between global indices $\mathbf{P} \in \mathcal{I}$ and local indices $(i,j) \in \iota^{\hat{\tau}}_p$. For $\tau \in \mathcal{G}$ and $I = (i,j) \in \iota^{\hat{\tau}}_p$ we define a mapping $\operatorname{ind} : \mathcal{G} \times \iota^{\hat{\tau}}_p \to \mathcal{I}$ by

$$\operatorname{ind}(\tau, I) := \chi_\tau\left(\widehat{P}_{(i,j)}\right) \in \mathcal{I}. \tag{4.78}$$

With this we have, for $\tau \in \mathcal{G}$, $I = (i,j) \in \iota^{\hat{\tau}}_p$ and $\mathbf{P} = \operatorname{ind}(\tau, I) \in \mathcal{I}$, the relation

$$b_\mathbf{P}|_\tau = N^\tau_{(i,j)} := \widehat{N}_I \circ \chi^{-1}_\tau. \tag{4.79}$$

In the following we will show that the functions in $S^{p,0}_{\mathcal{G},\chi}$ are Lipschitz continuous and are thus contained in $H^1(\Gamma)$. In order to compare the Euclidian distance with the surface distance, we introduce the geodesic distance

$$\operatorname{dist}_\Gamma(\mathbf{x}, \mathbf{y}) := \inf\left\{\operatorname{length}\left(\gamma_{\mathbf{x},\mathbf{y}}\right) : \gamma_{\mathbf{x},\mathbf{y}} \text{ is a path in } \Gamma \text{ that connects } \mathbf{x} \text{ and } \mathbf{y}\right\}$$

and the constant g_Γ

$$g_\Gamma := \sup_{\mathbf{x},\mathbf{y}\in\Gamma} \left\{ \frac{\operatorname{dist}_\Gamma(\mathbf{x},\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|} \right\}. \tag{4.80}$$

Remark 4.1.43. *The functions $\varphi_S \in S^{p,0}_{\mathcal{G},\chi}$ are Lipschitz continuous*

$$|\varphi_S(\mathbf{x}) - \varphi_S(\mathbf{y})| \le C \|\mathbf{x}-\mathbf{y}\| \qquad \forall \mathbf{x},\mathbf{y}\in\Gamma,$$

where C depends on Γ, $\mathcal{G}$, χ and g_Γ.

Proof. The continuity of $\varphi_S \in S^{p,0}_{\mathcal{G},\chi}$ follows directly from the definition so that we only need to prove the Lipschitz continuity. Let $\mathbf{x},\mathbf{y}\in\Gamma$ and let $\gamma_{\mathbf{x},\mathbf{y}}$ be a connecting path with minimal length on Γ. Let $(\tau_j)_{j=0}^q \subset \mathcal{G}$ be a minimal subset of $\mathcal{G}$ with the property:

$$\mathbf{x}\in\overline{\tau_0}, \quad \mathbf{y}\in\overline{\tau_q}, \quad \gamma_{\mathbf{x},\mathbf{y}} \subset \bigcup_{j=1}^{q} \overline{\tau_j}$$
$$\forall 1 \le j \le q : \overline{\tau_{j-1}} \cap \overline{\tau_j} \text{ is a common edge } e_j \text{ and } e_j \cap \gamma_{\mathbf{x},\mathbf{y}} \ne \emptyset.$$

We fix the points $\mathbf{M}_j$ on $e_j \cap \gamma_{\mathbf{x},\mathbf{y}}$, $1 \le j \le q$ and set $\mathbf{M}_0 = \mathbf{x}$ and $\mathbf{M}_{q+1} = \mathbf{y}$. Without loss of generality we assume that all $(\mathbf{M}_j)_{j=0}^{q+1}$ are distinct; otherwise we simply eliminate points that appear in the sequence more than once. Then, by the continuity of φ_S, we have

$$\varphi_S(\mathbf{y}) - \varphi_S(\mathbf{x}) = \varphi_S(\mathbf{M}_{q+1}) - \varphi_S(\mathbf{M}_0) = \sum_{j=0}^{q} \left(\varphi_S(\mathbf{M}_{j+1}) - \varphi_S(\mathbf{M}_j) \right).$$

The points $\mathbf{M}_{j+1}$, $\mathbf{M}_j$ are in the panel τ_j. Since $\varphi_S|_\tau$ is the composition of a polynomial with a diffeomorphism, these restrictions are Lipschitz continuous. With

$$c_\tau := \sup_{\mathbf{x},\mathbf{y}\in\tau} \frac{|\varphi_S(\mathbf{x}) - \varphi_S(\mathbf{y})|}{\|\mathbf{x}-\mathbf{y}\|}$$

we have

$$\left|\varphi_S(\mathbf{M}_{j+1}) - \varphi_S(\mathbf{M}_j)\right| \le c_\tau \left\|\mathbf{M}_{j+1} - \mathbf{M}_j\right\| \le c_\tau L\left(\gamma_{\mathbf{M}_j,\mathbf{M}_{j+1}}\right),$$

where $L\left(\gamma_{\mathbf{M}_j,\mathbf{M}_{j+1}}\right)$ denotes the length of the shortest connecting path in Γ that connects $\mathbf{M}_j$ with $\mathbf{M}_{j+1}$. Finally, with (4.80) we have

$$|\varphi_S(\mathbf{y}) - \varphi_S(\mathbf{x})| \le \left(\max_{1\le j\le q} c_{\tau_j} \right) L(\gamma_{\mathbf{x},\mathbf{y}}) \le g_\Gamma \left(\max_{1\le j\le q} c_{\tau_j} \right) \|\mathbf{x}-\mathbf{y}\|,$$

which is the Lipschitz continuity of φ_S. □

4.1.8 Galerkin BEM with Continuous Boundary Elements

The inclusion $S_{\mathcal{G},\chi}^{p,0} \subset H^{1/2}(\Gamma)$ of the continuous boundary elements permits the Galerkin discretization of the hypersingular boundary integral equation:
Find $\varphi_S \in S_{\mathcal{G}}^{p,0}/\mathbb{K}$ such that

$$b(\varphi_S, \eta_S) = (g_N, \eta_S)_{L^2(\Gamma)} \qquad \forall \eta_S \in S_{\mathcal{G}}^{p,0}/\mathbb{K}. \tag{4.81}$$

The ellipticity (Theorem 3.5.3) implies the existence of a unique solution of Problem (4.81). The system matrix of the hypersingular integral equation has similar properties to the matrix of the single layer potential (see Proposition 4.1.24).

Proposition 4.1.44. *The system matrix* $\mathbf{W}$ *of the bilinear form* $b : S_{\mathcal{G}}^{p,0}/\mathbb{R} \times S_{\mathcal{G}}^{p,0}/\mathbb{R} \to \mathbb{R}$ *in (4.65) is symmetric and positive definite. The entries* $W_{I,J}$, $I, J \in \mathcal{I}$ *have the explicit form*

$$W_{I,J} = \int_\Gamma \int_\Gamma \frac{\langle \operatorname{curl}_\Gamma b_I(\mathbf{x}), \operatorname{curl}_\Gamma b_J(\mathbf{y})\rangle}{4\pi \|\mathbf{x}-\mathbf{y}\|} ds_{\mathbf{y}} ds_{\mathbf{x}} = W_{J,I}. \tag{4.82}$$

The integrals in (4.82) are, according to Remark 4.1.43, weakly singular and therefore the matrix entries are well defined. We can write the actual generation of the matrix by means of integrals over single panels, with the help of the index allocation (4.78). In the following we will give an algorithmic description in the form of a pseudo programming language.

procedure generate_system_matrix;
for all $\tau, t \in \mathcal{G}$ **do begin**
  for all $I = (i, i') \in \iota_p^{\hat{\tau}}$, $J = (j, j') \in \iota_{\hat{t}}^{p}$ **do begin**

$$W_{\tau,t}^{I,J} := \int_\tau \int_t G(\mathbf{x}-\mathbf{y}) \left\langle \operatorname{curl}_\Gamma \left(\widehat{N}_{(i,i')} \circ \chi_\tau^{-1}(\mathbf{x}) \right), \operatorname{curl}_\Gamma \left(\widehat{N}_{(j,j')} \circ \chi_t^{-1}(\mathbf{y}) \right) \right\rangle ds_{\mathbf{y}} ds_{\mathbf{x}};$$

$$K := \operatorname{ind}(\tau, I); \quad L := \operatorname{ind}(t, J); \quad W_{K,L} := W_{K,L} + W_{\tau,t}^{I,J}; \tag{4.83}$$

end;end;

Exercise 4.1.45. *Let* $\tau, t \in \mathcal{G}$ *be panels with reference elements* $\hat{\tau}$, $\hat{t}$ *and reference mappings* χ_τ, χ_t. *The Jacobian of the transformation is denoted by* $\mathbf{J}_\tau := \left[\hat{\partial}_1 \chi_\tau, \hat{\partial}_2 \chi_\tau\right]$ *and we set* $\widehat{\nabla}^\perp := \left(\hat{\partial}_2, -\hat{\partial}_1\right)$. *For sufficiently smooth functions* $u : \tau \to \mathbb{R}$ *prove the relation*

$$g_\tau \operatorname{curl}_\Gamma u \circ \chi_\tau = \mathbf{J}_\tau \widehat{\nabla}^\perp \hat{u},$$

where $g_\tau := \sqrt{\det\left(\mathbf{J}_\tau^\intercal \mathbf{J}_\tau\right)}$ *and* $\hat{u} := u \circ \chi_\tau$.

For the local system matrix $W_{\tau,t}^{I,J}$ in (4.83) we have the representation

$$\int_{\hat{\tau}}\int_{\hat{t}} \frac{\left\langle \left(\mathbf{J}_\tau \widehat{\nabla}^\perp \widehat{N}_{(i,i')}\right)(\hat{\mathbf{x}}), \left(\mathbf{J}_t \widehat{\nabla}^\perp \widehat{N}_{(j,j')}\right)(\hat{\mathbf{y}})\right\rangle}{4\pi \left\|\chi_\tau(\hat{\mathbf{x}}) - \chi_t(\hat{\mathbf{y}})\right\|} d\hat{\mathbf{y}} d\hat{\mathbf{x}}.$$

(Hint: Use Exercise 3.3.25.)

In the same way as in Proposition 4.1.25 we obtain a quasi-optimal estimate for the Galerkin error for continuous boundary elements on a regular mesh $\mathcal{G}$.

Proposition 4.1.46. *The Galerkin approximation $\varphi_S \in S_{\mathcal{G}}^{p,0}$ of the solution φ of the hypersingular boundary integral equation converges quasi-optimally:*

$$\|\varphi - \varphi_S\|_{H^{1/2}(\Gamma)/\mathbb{K}} \le \frac{\|b\|}{\gamma} \min_{\psi_S \in S_{\mathcal{G}}^{p,0}} \|\varphi - \psi_S\|_{H^{1/2}(\Gamma)/\mathbb{K}}. \tag{4.84}$$

The Galerkin projection $\Pi_{\mathcal{G}}^{(p)} : H^{1/2}(\Gamma)/\mathbb{K} \to S_{\mathcal{G}}^{p,0}/\mathbb{K}$, given by $\Pi_{\mathcal{G}}^{(p)}\varphi = \varphi_S$, is stable:

$$\|\Pi_{\mathcal{G}}^{(p)}\|_{H^{1/2}(\Gamma)/\mathbb{K} \leftarrow H^{1/2}(\Gamma)/\mathbb{K}} \le \|b\|/\gamma, \tag{4.85}$$

where the norm of the bilinear form $b(\cdot,\cdot)$ is given by

$$\|b\| := \sup_{\varphi \in H^{1/2}(\Gamma)\setminus\{0\}} \sup_{\eta \in H^{1/2}(\Gamma)\setminus\{0\}} \frac{b(\varphi,\eta)}{\|\varphi\|_{H^{1/2}(\Gamma)/\mathbb{K}} \|\eta\|_{H^{1/2}(\Gamma)/\mathbb{K}}}$$

[see (2.29)].

Thanks to the stability result (4.85), the search for convergence rates of the Galerkin BEM is again reduced to the study of the approximation properties of the spaces $S_{\mathcal{G}}^{p,0}$.

4.1.9 Convergence Rates with Continuous Boundary Elements

In order to find convergence rates for the boundary element approximation φ_S in (4.81) of the hypersingular equation (4.65), we need approximation properties of the continuous boundary element spaces, which we will now specify. For this, let the boundary Γ be bounded and piecewise smooth in the sense of Definition 2.2.10.

Remark 4.1.47. *The partitioning of Γ which is employed in Definition 2.2.10 of piecewise smoothness is denoted here by $\mathcal{C} = \{\Gamma_i : 1 \le i \le q\}$ instead of $\mathcal{G}$ in order to distinguish the notation from the boundary element mesh $\mathcal{G}$ and its panels $\tau \in \mathcal{G}$ (cf. Definition 4.1.2). In this light, the cardinality q of $\mathcal{C}$ depends only on Γ and is, in particular, independent of the discretization parameters. However, we always assume that the boundary element mesh is compatible with $\mathcal{C}$ in the sense that, for any $\tau \in \mathcal{G}$, there exists a $\Gamma_i \in \mathcal{C}$ with $\tau \subset \Gamma_i$.*

We will prove the approximation property and the convergence rates for the Galerkin solution under the assumption that the exact solution belongs to the space $H^t_{\mathrm{pw}}(\Gamma)$ which we will define next.

Definition 4.1.48. Let Γ be piecewise smooth with partitioning $\mathcal{C} := \{\Gamma_i : 1 \leq i \leq q\}$:

(a) For $t > 1$, the space $H^t_{\mathrm{pw}}(\Gamma)$ contains all functions $\psi \in H^1(\Gamma)$ which satisfy

$$\forall \Gamma_i \in \mathcal{C} : \quad \psi|_{\Gamma_i} \in H^t(\Gamma_i)$$

and is furnished with the graph norm

$$\|\psi\|_{H^t_{\mathrm{pw}}(\Gamma)} := \left(\sum_{\Gamma_i \in \mathcal{C}} \|\psi\|^2_{H^t(\Gamma_i)} \right)^{1/2}. \tag{4.86}$$

(b) For $0 \leq t \leq 1$, the space $H^t_{\mathrm{pw}}(\Gamma)$ equals $H^t(\Gamma)$ and the norm $\|\cdot\|_{H^t_{\mathrm{pw}}(\Gamma)}$ is the usual $H^t(\Gamma)$-norm.

Some properties of the $H^t_{\mathrm{pw}}(\Gamma)$- and the $H^t(\Gamma)$-norms are stated in the next lemma.

Lemma 4.1.49. *(a) Let* $t \geq 1$. *For any* $\psi \in H^t(\Gamma)$, *we have*

$$\|\psi\|_{H^t_{\mathrm{pw}}(\Gamma)} \leq \|\psi\|_{H^t(\Gamma)}.$$

(b) Let $s \geq 0$. *Let* ι *denote a finite index set and let* $\{v_i : i \in \iota\}$ *be a set of functions in* $H^s(\Gamma)$. *If the supports* $\omega_i := \operatorname{supp} v_i$ *satisfy*

$$\left|\omega_i \cap \omega_j\right| = 0 \qquad \forall i, j \in \iota \text{ with } i \neq j,$$

then

$$\left\| \sum_{i \in \iota} v_i \right\|^2_{H^s(\Gamma)} \leq \frac{5}{2} \sum_{i \in \iota} \|v_i\|^2_{H^s(\Gamma)}.$$

Proof. Part a: Let $t \in \mathbb{N}_0$. Then

$$\|\psi\|^2_{H^t(\Gamma)} = \sum_{\Gamma_i \in \mathcal{C}} \|\psi\|^2_{H^t(\Gamma_i)} = \|\psi\|^2_{H^t_{\mathrm{pw}}(\Gamma)}.$$

For $t \in \mathbb{R}_{\geq 0} \backslash \mathbb{N}_0$, let $t = \lfloor t \rfloor + \lambda$ with $\lambda \in]0, 1[$. We employ (2.85) to obtain

$$\|\psi\|^2_{H^t(\Gamma)} = \sum_{|\alpha| \leq \lfloor t \rfloor} |\psi_\alpha|^2_{L^2(\Gamma)} + \sum_{|\alpha| \leq \lfloor t \rfloor} \int_{\Gamma \times \Gamma} \frac{|\psi_\alpha(\mathbf{x}) - \psi_\alpha(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^{2+2\lambda}} ds_{\mathbf{x}} ds_{\mathbf{y}}$$

$$\geq \sum_{\Gamma_i \in \mathcal{C}} \left\{ \sum_{|\alpha| \leq \lfloor t \rfloor} \|\psi_\alpha\|^2_{L^2(\Gamma_i)} + \sum_{|\alpha| \leq \lfloor t \rfloor} \int_{\Gamma_i \times \Gamma_i} \frac{|\psi_\alpha(\mathbf{x}) - \psi_\alpha(\mathbf{y})|^2}{\|\mathbf{x}-\mathbf{y}\|^{2+2\lambda}} ds_{\mathbf{x}} ds_{\mathbf{y}} \right\}$$
$$= \sum_{\Gamma_i \in \mathcal{C}} \|\psi\|^2_{H^t(\Gamma_i)} .$$

Part b: The proof of Part b is as in [91, Satz 3.26]. First, we will consider the case $s \in]0,1[$. We write

$$v = \sum_{i \in \iota} v_i, \qquad D_i := \operatorname{supp} v_i, \qquad D := \bigcup_{i \in \iota} D_i = \operatorname{supp} v$$

and introduce the shorthand

$$\int_{\Gamma'} \int_{\Gamma''} [w]_s^2 := \int_{\Gamma'} \int_{\Gamma''} \frac{|w(\mathbf{x}) - w(\mathbf{y})|^2}{\|\mathbf{x}-\mathbf{y}\|^{2+2s}} ds_{\mathbf{x}} ds_{\mathbf{y}}$$

for any measurable subsets $\Gamma', \Gamma'' \subset \Gamma$ and $w \in H^s(\Gamma)$.

For any $i \in \iota$, we get

$$\int_\Gamma \int_\Gamma [v_i]_s^2 = \int_{D_i} \int_{D_i} [v_i]_s^2 + 2 \int_{D_i} \int_{\Gamma \backslash D_i} [v_i]_s^2 + \underbrace{\int_{\Gamma \backslash D_i} \int_{\Gamma \backslash D_i} [v_i]_s^2}_{=0}$$
$$= \int_{D_i} \int_{D_i} [v_i]_s^2 + 2 \int_{D_i} |v_i(\mathbf{x})|^2 \int_{\Gamma \backslash D_i} \|\mathbf{x}-\mathbf{y}\|^{-2-2s} ds_{\mathbf{y}} ds_{\mathbf{x}}. \quad (4.87)$$

On the other hand,

$$\int_\Gamma \int_\Gamma [v]_s^2 = \int_D \int_\Gamma [v]_s^2 + \int_{\Gamma \backslash D} \int_D [v]_s^2 + \underbrace{\int_{\Gamma \backslash D} \int_{\Gamma \backslash D} [v]_s^2}_{=0}$$
$$= \sum_{i \in \iota} \int_{D_i} \int_{D_i} \underbrace{[v]_s^2}_{=[v_i]_s^2} + \sum_{i \in \iota} \int_{D_i} \int_{\Gamma \backslash D_i} [v]_s^2 + \int_D \int_{\Gamma \backslash D} [v]_s^2 \quad (4.88)$$

and

$$\int_{D_i} \int_{\Gamma \backslash D_i} [v]_s^2 = \int_{D_i} \int_{\Gamma \backslash D_i} \frac{|v(\mathbf{x}) - v(\mathbf{y})|^2}{\|\mathbf{x}-\mathbf{y}\|^{2+2s}} ds_{\mathbf{x}} ds_{\mathbf{y}}$$
$$\leq 2 \underbrace{\int_{\Gamma \backslash D_i} |v(\mathbf{x})|^2 \left(\int_{D_i} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2+2s}} ds_{\mathbf{y}} \right) ds_{\mathbf{x}}}_{=:J_i}$$

$$+\ 2\int_{D_i}\underbrace{|v(\mathbf{y})|^2}_{=|v_i(\mathbf{y})|^2}\int_{\Gamma\backslash D_i}\frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2+2s}}ds_{\mathbf{x}}ds_{\mathbf{y}}$$
$$\overset{(4.87)}{=}\int_\Gamma\int_\Gamma [v_i]_s^2-\int_{D_i}\int_{D_i}[v_i]_s^2+2J_i.$$

Inserting this into (4.88) results in

$$\int_\Gamma\int_\Gamma [v]_s^2\le\sum_{i\in\iota}\left(\int_\Gamma\int_\Gamma [v_i]_s^2+2J_i\right)+\int_D\int_{\Gamma\backslash D}[v]_s^2. \tag{4.89}$$

Next, we will investigate the sum over the quantities J_i. Let χ_i denote the characteristic function for $\Gamma\backslash D_i$. Then

$$\begin{aligned}\sum_{i\in\iota}J_i&=\sum_{i\in\iota}\int_{\Gamma\backslash D_i}|v(\mathbf{x})|^2\left(\int_{D_i}\frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2+2s}}ds_{\mathbf{y}}\right)ds_{\mathbf{x}}\\&=\sum_{i\in\iota}\int_\Gamma\chi_i(\mathbf{x})\,|v(\mathbf{x})|^2\left(\int_{D_i}\frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2+2s}}ds_{\mathbf{y}}\right)ds_{\mathbf{x}}\\&=\int_\Gamma|v(\mathbf{x})|^2\underbrace{\left(\sum_{i\in\iota}\chi_i(\mathbf{x})\int_{D_i}\frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2+2s}}ds_{\mathbf{y}}\right)}_{=:f(\mathbf{x})}ds_{\mathbf{x}}.\end{aligned} \tag{4.90}$$

Let $j\in\iota$ and let $\mathbf{x}$ be an interior point of D_j, i.e., $\mathbf{x}\in\overset{\circ}{D}_j$. For any $i\in\iota$, we have

$$\chi_i(\mathbf{x}):=\left\{\begin{matrix}1 \text{ if } \mathbf{x}\in\Gamma\backslash D_i\\ 0\quad\text{if } \mathbf{x}\in D_i\end{matrix}\right\}=\left(1-\delta_{i,j}\right).$$

For $\mathbf{x}\in\overset{\circ}{D}_j$ we have

$$f(\mathbf{x})=\sum_{i\in\iota\backslash\{j\}}\int_{D_i}\frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2+2s}}ds_{\mathbf{y}}=\int_{D\backslash D_j}\frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2+2s}}ds_{\mathbf{y}}.$$

Inserting this into (4.90) results in

$$\begin{aligned}2\sum_{i\in\iota}J_i&=\sum_{j\in\iota}2\int_{D_j}\underbrace{|v(\mathbf{x})|^2}_{|v_j(\mathbf{x})|^2}\left(\int_{D\backslash D_j}\frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2+2s}}ds_{\mathbf{y}}\right)ds_{\mathbf{x}}\\&\overset{(4.87)}{\le}\sum_{j\in\iota}\int_\Gamma\int_\Gamma\left[v_j\right]_s^2.\end{aligned} \tag{4.91}$$

It remains to estimate the second term in (4.89). We have

$$\begin{aligned}
\int_D \int_{\Gamma\setminus D} [v]_s^2 &= \int_D |v(\mathbf{x})|^2 \left(\int_{\Gamma\setminus D} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2+2s}} ds_{\mathbf{y}} \right) ds_{\mathbf{x}} \\
&= \sum_{i\in\iota} \int_{D_i} \underbrace{|v(\mathbf{x})|^2}_{=|v_i(\mathbf{x})|^2} \left(\int_{\Gamma\setminus D} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2+2s}} ds_{\mathbf{y}} \right) ds_{\mathbf{x}} \\
&\le \sum_{i\in\iota} \int_{D_i} |v_i(\mathbf{x})|^2 \left(\int_{\Gamma\setminus D_i} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^{2+2s}} ds_{\mathbf{y}} \right) ds_{\mathbf{x}} \\
&\overset{(4.87)}{\le} \frac{1}{2} \sum_{i\in\iota} \int_\Gamma \int_\Gamma [v_i]_s^2 . \qquad (4.92)
\end{aligned}$$

The combination of (4.89), (4.91), and (4.92) leads to

$$\int_\Gamma \int_\Gamma [v]_s^2 \le \frac{5}{2} \sum_{i\in\iota} \int_\Gamma \int_\Gamma [v_i]_s^2 .$$

Because the $L^2(\Gamma)$-norm is additive we obtain

$$\begin{aligned}
\left\| \sum_{i\in\iota} v_i \right\|_{H^s(\Gamma)}^2 &= \|v\|_{L^2(\Gamma)}^2 + \int_\Gamma \int_\Gamma [v]_s^2 \le \sum_{i\in\iota} \|v_i\|_{L^2(\Gamma)}^2 + \frac{5}{2} \sum_{i\in\iota} \int_\Gamma \int_\Gamma [v_i]_s^2 \\
&\le \frac{5}{2} \sum_{i\in\iota} \|v_i\|_{H^s(\Gamma)}^2 .
\end{aligned}$$

The proof for $s \in \mathbb{R}_{>1}\setminus\mathbb{N}$ can be carried out in the same way. Note that the expression $[v]_s$ has to be replaced by $[v_\alpha]_s$, where v_α is defined as in (2.86). □

Proposition 4.1.50. *Let Γ be piecewise smooth and let $\mathcal{G}$ be a surface mesh of Γ:*

(a) Let[1] $\varphi \in H^t_{\mathrm{pw}}(\Gamma)$ for some $t > 1$. Then there exists a continuous interpolation $I^p_{\mathcal{G}} \varphi \in S^{p,0}_{\mathcal{G}}$ with

$$\|\varphi - I^p_{\mathcal{G}} \varphi\|_{H^s(\Gamma)} \le C\, h_{\mathcal{G}}^{\min\{t,p+1\}-s} \|\varphi\|_{H^t_{\mathrm{pw}}(\Gamma)}, \qquad s \in \{0,1\}, \qquad (4.93)$$

where the constant C depends only on p and on the constant $\kappa_{\mathcal{G}}$ from Definition 4.1.12, which describes the shape-regularity of the mesh.

(b) Let $0 \le s \le t \le 1$. Then there exists a continuous operator $Q_{\mathcal{G}} : H^t(\Gamma) \to S^{p,0}_{\mathcal{G}}$ such that, for every $\varphi \in H^t(\Gamma)$, we have

[1] In Sect. 4.3.3, we will prove the continuous embedding $H^t_{pw}(\Gamma) \hookrightarrow C^0(\Gamma)$ for $t > 1$ and piecewise smooth Lipschitz surfaces.

$$\|\varphi - Q_{\mathcal{G}}\varphi\|_{H^s(\Gamma)} \leq C h_{\mathcal{G}}^{t-s} \|\varphi\|_{H^t(\Gamma)} .$$

The operator $Q_{\mathcal{G}}$ is stable for $0 \leq s \leq 1$

$$\|Q_{\mathcal{G}}\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)} \leq C.$$

The proof of Proposition 4.1.50 is postponed to Sect. 4.3.5.

With Proposition 4.1.50 we can now derive quantitative error estimates from the quasi-optimality (4.84) of the Galerkin solution φ_S.

Theorem 4.1.51. *Let Γ be a piecewise smooth Lipschitz surface. Furthermore, let $\mathcal{G}$ be a regular surface mesh on Γ. Let $\varphi \in H^t_{\text{pw}}(\Gamma)$ with $t \geq 1/2$. Then we have for the Galerkin approximation $\varphi_S \in S_{\mathcal{G}}^{p,0}$ of (4.65) the error estimate*

$$\|\varphi - \varphi_S\|_{H^{1/2}(\Gamma)/\mathbb{K}} \leq C h^{\min(t,p+1)-1/2} \|\varphi\|_{H^t_{\text{pw}}(\Gamma)}, \tag{4.94}$$

where the constant C depends only on p and, via the constant $\kappa_{\mathcal{G}}$ from Definition 4.1.12, on the shape-regularity of the mesh.

Proof.

Case 1: $t = 1/2$.

For $\varphi \in H^{1/2}(\Gamma)/\mathbb{K}$ it follows from (4.84) that by choosing $\psi_S = 0$ we obtain the boundedness of the error $\|\varphi - \varphi_S\|_{H^{1/2}(\Gamma)/\mathbb{K}}$ by $(\|b\|/\gamma)\|\varphi\|_{H^{1/2}(\Gamma)/\mathbb{K}}$. This yields (4.94) for $t = 1/2$.

Case 2: $t > 1$.

Now let $\varphi \in H^t_{\text{pw}}(\Gamma)$ with $t > 1$. Let $T_{\mathcal{G}}^p : H^t_{\text{pw}}(\Gamma) \to S_{\mathcal{G}}^{p,0}$ be defined by

$$T_{\mathcal{G}}^p := \begin{cases} Q_{\mathcal{G}} & \text{if } t = 1, \\ I_{\mathcal{G}}^p & \text{if } t > 1. \end{cases}$$

Proposition 4.1.50 implies that $T_{\mathcal{G}}^p$ is continuous. The estimate

$$\|\varphi - \varphi_S\|_{H^{1/2}(\Gamma)/\mathbb{K}} \leq \frac{\|b\|}{\gamma}\|\varphi - T_{\mathcal{G}}^p\varphi\|_{H^{1/2}(\Gamma)/\mathbb{K}} \leq \frac{\|b\|}{\gamma}\|\varphi - T_{\mathcal{G}}^p\varphi\|_{H^{1/2}(\Gamma)}$$

follows from the quasi-optimality (4.84), and we have used $\|\varphi\|_{H^{1/2}(\Gamma)/\mathbb{K}} = \min\limits_{c\in\mathbb{R}} \|\varphi - c\|_{H^{1/2}(\Gamma)} \leq \|\varphi\|_{H^{1/2}(\Gamma)}$.

If we apply Proposition 2.1.65 with $X_0 = L^2(\Gamma)$, $X_1 = H^1(\Gamma)$ and $\theta = 1/2$ we obtain the interpolation inequality

$$\|\varphi\|^2_{H^{1/2}(\Gamma)} \leq \|\varphi\|_{L^2(\Gamma)} \|\varphi\|_{H^1(\Gamma)} .$$

With this and with Proposition 4.1.50 it follows for $t \geq 1$ that

$$\|\varphi - T_{\mathcal{G}}^{p}\,\varphi\|_{H^{1/2}(\Gamma)}^{2} \le C\,\|\varphi - T_{\mathcal{G}}^{p}\varphi\|_{L^2(\Gamma)}\|\varphi - T_{\mathcal{G}}^{p}\,\varphi\|_{H^1(\Gamma)}$$
$$\le C\,h^{2\min(t,p+1)-1}\|\varphi\|_{H_{\mathrm{pw}}^{t}(\Gamma)}^{2} \tag{4.95}$$

and therefore we have (4.94) for $t > 1$.
Case 3: $1/2 < t \le 1$.
In this case we prove (4.94) by interpolation. We have for the operator $I - Q_{\mathcal{G}}$ the estimate [cf. Proposition 4.1.50(b)]

$$\|I - Q_{\mathcal{G}}\|_{H^{1/2}(\Gamma)\leftarrow H^{1/2}(\Gamma)} \le C, \qquad \|I - Q_{\mathcal{G}}\|_{H^{1/2}(\Gamma)\leftarrow H^{1}(\Gamma)} \le C\,h^{1/2}.$$

As in the proof of Theorem 4.1.33, the estimate

$$\|(I - Q_{\mathcal{G}})\varphi\|_{H^{1/2}(\Gamma)} \le C h^{t-\frac{1}{2}}\|\varphi\|_{H^{t}(\Gamma)}.$$

follows for $1/2 \le t \le 1$ by interpolation of the linear operator $I - Q_{\mathcal{G}}$: $H^{t}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ (see Proposition 2.1.62). □

4.1.10 Model Problem 3: Mixed Boundary Value Problem*

We consider the mixed boundary value problem for the Laplace operator:

$$\Delta u = 0 \quad \text{in } \Omega^{-}, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \partial u/\partial \mathbf{n} = g_N \quad \text{on } \Gamma_N \tag{4.96}$$

for given boundary data $g_D \in H^{1/2}(\Gamma_D)$, $g_N \in H^{-1/2}(\Gamma_2)$. For the associated variational formulation we refer to Sect. 2.9.2.3. The approach that allows the discretization of mixed boundary value problems by means of the Galerkin boundary element method is due to [220, 239]. For the treatment of problems with more general transmission conditions we refer to [233].

The problem can be reduced to an integral equation for the pair of densities $(\varphi, \sigma) \in \mathbf{H} = \widetilde{H}^{-1/2}(\Gamma_D) \times \widetilde{H}^{1/2}(\Gamma_N)$. The solution of (4.96) can be represented with the help of Green's representation formula

$$u(\mathbf{x}) = (S\sigma)(\mathbf{x}) - (D\varphi)(\mathbf{x}), \qquad \mathbf{x} \in \Omega^{-}.$$

The variational formulation of the boundary integral equation reads [see (3.89)]: Find $(\varphi, \sigma) \in \mathbf{H}$ such that

$$b_{mixed}\left(\begin{pmatrix}\varphi\\ \sigma\end{pmatrix}, \begin{pmatrix}\eta\\ \kappa\end{pmatrix}\right) = (g_D, \eta)_{L^2(\Gamma_D)} + (g_N, \kappa)_{L^2(\Gamma_N)} \qquad \forall\,(\eta, \kappa) \in \mathbf{H} \tag{4.97}$$

* This section should be read as a complement to the core material of this book.

with

$$b_{mixed}\left(\begin{pmatrix}\varphi\\ \sigma\end{pmatrix},\begin{pmatrix}\eta\\ \kappa\end{pmatrix}\right) = (V_{DD}\varphi,\eta)_{L^2(\Gamma_D)} - (K_{DN}\sigma,\eta)_{L^2(\Gamma_D)} + \left(K'_{ND}\varphi,\kappa\right)_{L^2(\Gamma_N)} + (W_{NN}\sigma,\kappa)_{L^2(\Gamma_N)}\,.$$

The *boundary element discretization* is achieved by a combination of different boundary element spaces on the pieces Γ_D, Γ_N. For this let $\mathcal{G}_D$, $\mathcal{G}_N$ be surface meshes of Γ_D, Γ_N, while we assume that $\mathcal{G}_N$ is regular (see Definition 4.1.4). We use discontinuous boundary elements of order $p_1 \geq 0$ on Γ_D. The inclusion

$$S^{p_1,-1}_{\mathcal{G}_D} \subset \widetilde{H}^{-1/2}(\Gamma_D), \tag{4.98}$$

results, because the zero extension $\psi^\star$ of every function $\psi \in S^{p_1,-1}_{\mathcal{G}_D}$ satisfies the inclusion $\psi^\star \in L^2(\Gamma) \subset H^{-1/2}(\Gamma)$ and thus we have $\psi \in \widetilde{H}^{-1/2}(\Gamma_D)$.

For the approximation of $\sigma \in \widetilde{H}^{1/2}(\Gamma_N)$ we define for $p_2 \geq 1$

$$S^{p_2,0}_{\mathcal{G}_N,0} = \left\{\eta \in S^{p_2,0}_{\mathcal{G}_N} : \eta|_{\partial\Gamma_N} = 0\right\} \tag{4.99}$$

and therefore the boundary values of the functions $\eta \in S^{p_2,0}_{\mathcal{G}_N,0}$ vanish on $\partial\Gamma_N$.

Remark 4.1.52. *The zero extension $\sigma^\star$ of functions $\sigma \in S^{p,0}_{\mathcal{G}_N,0}$ satisfies $\sigma^\star \in S^{p,0}_{\mathcal{G}} \subset H^{1/2}(\Gamma)$, where we have set $\mathcal{G} := \mathcal{G}_D \cup \mathcal{G}_N$.*

With these spaces we can finally formulate the boundary element discretization of (4.97). In the following we will summarize the polynomial orders $p_1 \geq 0$ and $p_2 \geq 1$ in the vector $\mathbf{p} = (p_1, p_2)$.

Find $(\varphi_S, \sigma_S) \in S^{\mathbf{p}} := S^{p_1,-1}_{\mathcal{G}_D} \times S^{p_2,0}_{\mathcal{G}_N,0}$ such that

$$b_{mixed}\left(\begin{pmatrix}\varphi_S\\ \sigma_S\end{pmatrix},\begin{pmatrix}\eta_S\\ \kappa_S\end{pmatrix}\right) = (g_D,\eta_S)_{L^2(\Gamma_D)} + (g_N,\kappa_S)_{L^2(\Gamma_N)} \quad \forall(\eta_S,\kappa_S) \in S^{\mathbf{p}}. \tag{4.100}$$

The norm for functions $(\varphi,\sigma) \in \mathbf{H}$ is given by $\|(\varphi,\sigma)\|_{\mathbf{H}} := \|\varphi\|_{\tilde{H}^{-1/2}(\Gamma_D)} + \|\sigma\|_{\tilde{H}^{1/2}(\Gamma_N)}$. Once more the unique solvability of the boundary element discretization of the integral equation follows from the $\mathbf{H}$-ellipticity (3.112) of the bilinear form b_{mixed}, and from the Galerkin orthogonality of the error, we have the quasi-optimality.

Theorem 4.1.53. *Let $(\varphi,\sigma) \in \mathbf{H}$ be the exact solution of (4.97). The discretization (4.100) has a unique solution $(\varphi_S,\sigma_S) \in S^{\mathbf{p}}$, $\mathbf{p} = (p_1,p_2)$, which converges quasi-optimally:*

$$\|(\varphi,\sigma) - (\varphi_S,\sigma_S)\|_{\mathbf{H}} \leq C_1 \min_{(\eta,\kappa)\in S^{\mathbf{p}}} \|(\varphi,\sigma) - (\eta,\kappa)\|_{\mathbf{H}}\,. \tag{4.101a}$$

If the exact solution satisfies $(\varphi, \sigma) \in H^s_{\mathrm{pw}}(\Gamma_D) \times H^t_{\mathrm{pw}}(\Gamma_N)$ *for* $s, t \geq 0$ *we have the quantitative estimate*

$$\|(\varphi, \sigma) - (\varphi_S, \sigma_S)\|_{\mathbf{H}} \leq C_2 \Big(h^{\min\{s, p_1+1\}+\frac{1}{2}} \|\varphi\|_{H^s_{\mathrm{pw}}(\Gamma_D)} + h^{\min\{t, p_2+1\}-\frac{1}{2}} \|\sigma\|_{H^t_{\mathrm{pw}}(\Gamma_N)} \Big). \tag{4.101b}$$

Here the constant C_2 *depends only on* C_1 *in (4.101a), the shape-regularity (see Definition 4.1.12) of the surface meshes* $\mathcal{G}_D$, $\mathcal{G}_N$ *and the polynomial degrees* p_1 *and* p_2.

Proof. For the proof we only need to show the approximation property on the boundary pieces Γ_D and Γ_N. Here we use (4.59) on Γ_D and (4.93) on Γ_N for a sufficiently large $t > 1$. Hence the interpolation $I^p_{\mathcal{G}}\varphi$ in (4.93) is well defined and we have $\varphi|_{\partial\Gamma_N} = I^p_{\mathcal{G}}\varphi\big|_{\partial\Gamma_N} = 0$. Therefore the zero extension of the difference function satisfies $\left(\varphi - I^p_{\mathcal{G}}\varphi\right)^\star \in H^{1/2}(\Gamma)$ and from (4.93) with $s = 0, 1$ we have:

$$\begin{aligned} \|\left(\varphi - I^p_{\mathcal{G}}\varphi\right)^\star\|_{L^2(\Gamma)} &= \|\varphi - I^p_{\mathcal{G}}\varphi\|_{L^2(\Gamma_N)} \leq C h^{\min(t,p+1)} \|\varphi\|_{H^t_{\mathrm{pw}}(\Gamma_N)}, \\ \|\left(\varphi - I^p_{\mathcal{G}}\varphi\right)^\star\|_{H^1(\Gamma)} &= \|\varphi - I^p_{\mathcal{G}}\varphi\|_{H^1(\Gamma_N)} \leq C h^{\min(t,p+1)-1} \|\varphi\|_{H^t_{\mathrm{pw}}(\Gamma_N)}. \end{aligned} \tag{4.102}$$

Then, by interpolation as in the proof of Theorem 4.1.51 and by the boundedness of the Galerkin projection (see Remark 4.1.27), (4.101b) follows. □

*4.1.11 Model Problem 4: Screen Problems**

In this section we will discuss the Galerkin boundary element method for the screen problem from Sect. 3.5.3, which is due to [219].

Hence we again assume that an open manifold Γ_0 is given, which can be extended to a closed Lipschitz surface Γ in $\mathbb{R}^3$ in such a way that we have for $\Gamma_0^c = \Gamma \backslash \overline{\Gamma}_0$

$$\Gamma = \Gamma_0 \cup \Gamma_0^c.$$

In order to avoid technical difficulties, we require that Γ_0 and Γ_0^c be simply connected. We have already introduced the integral equations for the Dirichlet and Neumann screen problems in Sect. 3.5.3:

Dirichlet Screen Problem: For a given $g_D \in H^{1/2}(\Gamma_0)$ find $\varphi \in \widetilde{H}^{-1/2}(\Gamma_0)$ such that

$$(V\varphi, \eta)_{L^2(\Gamma_0)} = (g_D, \eta)_{L^2(\Gamma_0)} \qquad \forall \eta \in \widetilde{H}^{-1/2}(\Gamma_0). \tag{4.103}$$

* This section should be read as a complement to the core material of this book.

Neumann Screen Problem: For a given $g_N \in H^{-1/2}(\Gamma_0)$ find $\sigma \in \widetilde{H}^{1/2}(\Gamma_0)$ such that

$$(W\sigma, \kappa)_{L^2(\Gamma_0)} = (g_N, \kappa)_{L^2(\Gamma_0)} \qquad \forall \kappa \in \widetilde{H}^{1/2}(\Gamma_0). \tag{4.104}$$

The Galerkin BEM for (4.103) and (4.104) are based on a regular mesh $\mathcal{G}$ of Γ_0 and a boundary element space of polynomial degree $p_1 \geq 0$ for the Dirichlet problem (4.103) and $p_2 \geq 1$ for the Neumann problem (4.104).
Dirichlet Screen Problem: For a given $g_D \in H^{1/2}(\Gamma_0)$ find $\varphi_S \in S_{\mathcal{G}}^{p_1,-1}$ such that

$$(V\psi_S, \eta_S)_{L^2(\Gamma_0)} = (g_D, \eta_S)_{L^2(\Gamma_0)} \qquad \forall \eta_S \in S_{\mathcal{G}}^{p_1,-1}. \tag{4.105}$$

Neumann Screen Problem: For a given $g_N \in H^{-1/2}(\Gamma_0)$ find $\sigma_S \in S_{\mathcal{G},0}^{p_2,0}$ such that

$$(W\sigma_S, \kappa_S)_{L^2(\Gamma_0)} = (g, \kappa)_{L^2(\Gamma_0)} \qquad \forall \kappa \in S_{\mathcal{G},0}^{p_2,0}. \tag{4.106}$$

Note that in $S_0^{p_2,0}$ the boundary data of σ_S on $\partial\Gamma_0$ is set to zero (see Remark 4.1.52). With the ellipticity from Theorem 3.5.9 we immediately have the quasi-optimality of the discretization.

Theorem 4.1.54. *Equations (3.116), (3.117) as well as (4.105), (4.106) have a unique solution and the Galerkin solutions converge quasi-optimally:*

$$\|\psi - \psi_S\|_{\tilde{H}^{-1/2}(\Gamma_0)} \leq C \min_{\eta_S \in S_{\mathcal{G}}^{p_1,-1}} \|\psi - \eta_S\|_{\tilde{H}^{-1/2}(\Gamma_0)}, \tag{4.107a}$$

$$\|\sigma - \sigma_S\|_{\tilde{H}^{1/2}(\Gamma_0)} \leq C \min_{\kappa_S \in S_{\mathcal{G},0}^{p_2,0}} \|\sigma - \kappa_S\|_{\tilde{H}^{1/2}(\Gamma_0)}. \tag{4.107b}$$

If the exact solution of the Dirichlet problem (3.116) is contained in $H^s_{\mathrm{pw}}(\Gamma_0)$ for an $s \geq 0$ we have

$$\|\psi - \psi_S\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_0)} \leq C_1\, h^{\min(s,p_1+1)+\frac{1}{2}} \|\psi\|_{H^s_{\mathrm{pw}}(\Gamma_0)}. \tag{4.108a}$$

If the exact solution of the Neumann problem is contained in $H^t_{\mathrm{pw}}(\Gamma_0)$ for a $t > 1/2$ we have

$$\|\sigma - \sigma_S\|_{\tilde{H}^{1/2}(\Gamma_0)} \leq C_2\, h^{\min(t,p_2+1)-\frac{1}{2}} \|\sigma\|_{H^t_{\mathrm{pw}}(\Gamma_0)}. \tag{4.108b}$$

Here the constants C_1, C_2 depend only on the respective constant C in (4.107), the shape-regularity (see Definition 4.1.12) of the mesh and the polynomial degrees p_1 and p_2.

Remark 4.1.55. *In general, the exact solutions of the screen problems have edge singularities and therefore they do not have a very high order of regularity s or t in (4.108). Therefore the convergence rates of the Galerkin solutions in (4.108)* *are low, even for higher order discretizations. This problem can be overcome by an anisotropic mesh refinement near $\partial\Gamma_0$. For details we refer to [221].*

4.2 Convergence of Abstract Galerkin Methods

All boundary integral operators in Chap. 4.1 were elliptic, which allowed the use of the Lax–Milgram lemma to prove existence and uniqueness. As we have already seen with the Helmholtz problem, however, in certain practical cases we encounter indefinite boundary integral operators. Here we will show for very general subspaces and especially for non-symmetric and non-elliptic sesquilinear forms, under which circumstances the Galerkin solution $u_S \in S$ exists and the error converges quasi-optimally. An early study on this subject can be found in [223]. For a study on the convergence of general boundary element methods we refer to [215].

4.2.1 Abstract Variational Problem

We would first like to recall the abstract framework from Sect. 2.1.6 and, again, refer, e.g., to [9, Chap. 5], [151, 166, 174] as standard references and additional material.

Let H_1, H_2 be Hilbert spaces and $a(\cdot,\cdot) : H_1 \times H_2 \to \mathbb{C}$ a continuous sesquilinear form:

$$\|a\| = \sup_{u\in H_1\setminus\{0\}} \sup_{v\in H_2\setminus\{0\}} \frac{|a(u,v)|}{\|u\|_{H_1}\|v\|_{H_2}} < \infty, \tag{4.109}$$

and let the (continuous) inf–sup conditions hold: There exists a constant $\gamma > 0$ such that

$$\inf_{u\in H_1\setminus\{0\}} \sup_{v\in H_2\setminus\{0\}} \frac{|a(u,v)|}{\|u\|_{H_1}\|v\|_{H_2}} \geq \gamma > 0, \tag{4.110a}$$

and we have

$$\forall v \in H_2\setminus\{0\} : \sup_{u\in H_1} |a(u,v)| > 0. \tag{4.110b}$$

Then for every functional $F \in H_2'$ the problem

$$\text{Find } u \in H_1 : \qquad a(u,v) = F(v) \qquad \forall v \in H_2 \tag{4.111}$$

has a unique solution, which satisfies

$$\|u\|_{H_1} \leq \frac{1}{\gamma}\|F\|_{H_2'}. \tag{4.112}$$

4.2.2 Galerkin Approximation

We require the following construction of approximating subspaces for the definition of the Galerkin method, which we use to solve (4.111).

For $i = 1, 2$, let $\left(S_\ell^i\right)_{\ell\in\mathbb{N}}$ be given sequences of finite-dimensional, nested subspaces of H_i whose union is dense in H_i

$$\forall \ell \geq 0 : S_\ell^i \subset S_{\ell+1}^i, \quad \dim S_\ell^i < \infty \quad \text{and} \quad \overline{\bigcup_{\ell\in\mathbb{N}} S_\ell^i}^{\|\cdot\|_{H_i}} = H_i, \qquad i = 1, 2 \tag{4.113}$$

and whose respective dimensions satisfy the conditions

$$\begin{aligned} &N_\ell := \dim S_\ell^1 = \dim S_\ell^2 < \infty, \ \ \forall \ell \in \mathbb{N} : N_\ell < N_{\ell+1}, \\ &N_\ell \to \infty \quad \text{for } \ell \to \infty. \end{aligned} \tag{4.114}$$

Since the dimensions of S_ℓ^1 and S_ℓ^2 are equal, it follows that the system matrix for the boundary element method is square.

The density implies the *approximation property*

$$\forall u_i \in H_i : \qquad \lim_{\ell\to\infty} \min\{\|u_i - v\|_{H_i} : v \in S_\ell^i\} = 0. \tag{4.115}$$

Every u_i in H_i can thus be approximated by a sequence $v_\ell^i \in S_\ell^i$. In Sect. 4.1 we have already encountered the spaces $S_{\mathcal{G}}^{p,0}$ and $S_{\mathcal{G}}^{p,-1}$, and one obtains a sequence of boundary element spaces by, for example, successively refining an initially coarse mesh $\mathcal{G}_0$.

With the subspaces $\left(S_\ell^i\right)_{\ell\in\mathbb{N}} \subset H_i$ the Galerkin discretization of (4.111) is given by: Find $u_\ell \in S_\ell^1$ such that

$$a(u_\ell, v_\ell) = F(v_\ell) \qquad \forall v_\ell \in S_\ell^2. \tag{4.116}$$

A solution of (4.116) is called a *Galerkin solution*. The existence and uniqueness of the Galerkin solution is proven in the following theorem.

Theorem 4.2.1. *(i) For every functional $F \in H_2'$, (4.116) has a unique solution $u_\ell \in S_\ell^1$ if the discrete inf–sup condition*

$$\inf_{u\in S_\ell^1\backslash\{0\}} \sup_{v\in S_\ell^2\backslash\{0\}} \frac{|a(u,v)|}{\|u\|_{H_1}\|v\|_{H_2}} \geq \gamma_\ell \tag{4.117}$$

holds with a stability constant $\gamma_\ell > 0$ and if

$$\forall v \in S_\ell^2 \backslash \{0\} : \qquad \sup_{u\in S_\ell^1} |a(u,v)| > 0 \tag{4.118}$$

is satisfied.

(ii) For all ℓ let (4.118) and (4.117) be satisfied with $\gamma_\ell > 0$. Then the sequence $(u_\ell)_\ell \subset H_1$ of Galerkin solutions satisfies the error estimate

$$\|u - u_\ell\|_{H_1} \leq \left(1 + \frac{\|a\|}{\gamma_\ell}\right) \min_{v \in S_\ell^1} \|u - v\|_{H_1}. \tag{4.119}$$

Proof. Statement (i) follows from Theorem 2.1.44.

For (ii): The difference between (4.116) and (4.111) with $S_\ell^2 \subset H_2$ yields the *Galerkin orthogonality* of the error:

$$a(u - u_\ell, v) = 0 \quad \forall v \in S_\ell^2. \tag{4.120}$$

Owing to the discrete inf–sup condition (4.117) we have

$$\begin{aligned} \gamma_\ell \|u_\ell\|_{H_1} &\leq \sup_{v \in S_\ell^2 \backslash \{0\}} \frac{|a(u_\ell, v)|}{\|v\|_{H_2}} = \sup_{v \in S_\ell^2 \backslash \{0\}} \frac{|F(v)|}{\|v\|_{H_2}} \\ &\leq \sup_{v \in H_2 \backslash \{0\}} \frac{|F(v)|}{\|v\|_{H_2}} = \sup_{v \in H_2 \backslash \{0\}} \frac{|a(u, v)|}{\|v\|_{H_2}} \leq \|a\| \, \|u\|_{H_1}. \end{aligned}$$

This means that the statement $Q_\ell u := u_\ell$ defines a linear mapping $Q_\ell : H_1 \to S_\ell^1$ with $\|Q_\ell\|_{H_1 \leftarrow H_1} \leq \|a\| / \gamma_\ell$. For all $w \in S_\ell^1 \subset H_1$ it follows from (4.117) and (4.120) that we have the estimate

$$\|w - Q_\ell w\|_{H_1} \leq \frac{1}{\gamma_\ell} \sup_{v \in S_\ell^2 \backslash \{0\}} \frac{|a(w - Q_\ell w, v)|}{\|v\|_{H_2}} = 0,$$

from which we have the projection property:

$$\forall w \in S_\ell^1 : \qquad Q_\ell w = w.$$

It then follows for all $w \in S_\ell^1 \subset H_1$, that

$$\begin{aligned} \|u - u_\ell\|_{H_1} &\leq \|u - w\|_{H_1} + \|w - Q_\ell u\|_{H_1} \\ &= \|u - w\|_{H_1} + \|Q_\ell (u - w)\|_{H_1} \\ &\leq \left(1 + \frac{\|a\|}{\gamma_\ell}\right) \|u - w\|_{H_1}. \end{aligned}$$

Since $w \in S_\ell^1$ was arbitrary, we have proven (4.119). □

Remark 4.2.2. *(i) The Galerkin method (4.116) is called uniformly stable if there exists a constant $\gamma > 0$ that is independent of ℓ such that $\gamma_\ell \geq \gamma > 0$. In this case (4.119) implies the quasi-optimal convergence of the Galerkin solution.*

(ii) *The subspaces S_ℓ^1 and S_ℓ^2 contain different functions: S_ℓ^1 serves to approximate the solution and guarantees the consistency, while S_ℓ^2 guarantees the stability, because of the discrete inf–sup condition [which is equivalent to (4.117)]*

$$\forall u \in S_\ell^1 : \qquad \sup_{v \in S_\ell^2 \setminus \{0\}} \frac{|a(u,v)|}{\|v\|_{H_2}} \geq \gamma_\ell \, \|u\|_{H_1}. \tag{4.121}$$

Remark 4.2.3. *In Sect. 4.1 we have seen that for the integral equations for the Laplace problem we can always choose $S_\ell^1 = S_\ell^2$. The same property holds for the integral equation formulation of the Helmholtz equation.*

Remark 4.2.4. *Equations (4.117) and (4.118) are equivalent to the conditions*

$$\inf_{v \in S_\ell^2 \setminus \{0\}} \sup_{u \in S_\ell^1 \setminus \{0\}} \frac{|a(u,v)|}{\|u\|_{H_1} \, \|v\|_{H_2}} \geq \gamma_\ell^* \tag{4.122}$$

with $\gamma_\ell^* > 0$ and

$$\forall u \in S_\ell^1 \setminus \{0\} : \qquad \sup_{v \in S_\ell^2} |a(u,v)| > 0. \tag{4.123}$$

Remark 4.2.5. *For $H_1 = H_2 = H$ and $S_\ell^1 = S_\ell^2 = S_\ell$, (4.117) implies the condition (4.122) with $\gamma_\ell^* = \gamma_\ell$ and vice-versa.*

The Galerkin method (4.116) is equivalent to a linear system of equations. To see this we need to choose bases $\left(b_j^i\right)_{j=1}^{N_\ell}$ of S_ℓ^i, $i = 1, 2$:

$$S_\ell^1 = \operatorname{span}\{b_j^1 : j = 1, \ldots, N_\ell\}, \qquad S_\ell^2 = \operatorname{span}\{b_j^2 : j = 1, \ldots, N_\ell\}.$$

Therefore every $u \in S_\ell^1$ and $v \in S_\ell^2$ has a unique basis representation

$$u = \sum_{j=1}^{N_\ell} u_j b_j^1, \qquad v_\ell = \sum_{j=1}^{N_\ell} v_j \, b_j^2. \tag{4.124}$$

If we insert (4.124) into (4.116) we obtain:

$$\begin{aligned}
&\forall v \in S_\ell^2 : \; a(u,v) - F(v) = 0 \Longrightarrow \\
&\forall \mathbf{v} = \left(v_j\right)_{j=1}^{N_\ell} \in \mathbb{C}^{N_\ell} : \quad \sum_{j=1}^{N_\ell} \bar{v}_j \left(\left\{ \sum_{k=1}^{N_\ell} u_k \, a(b_k^1, b_j^2) \right\} - F(b_j^2) \right) = 0 \Longrightarrow \\
&\mathbf{K}_\ell \mathbf{u} = \mathbf{F}_\ell,
\end{aligned} \tag{4.125}$$

where the matrix $\mathbf{K}_\ell$ and the vectors $\mathbf{u}$, $\mathbf{F}_\ell$ are given by $\mathbf{u} = \left(u_j\right)_{j=1}^{N_\ell}$ and

$$\left.\begin{aligned}(\mathbf{K}_\ell)_{j,k} &:= a(b_k^1, b_j^2)\\ (\mathbf{F}_\ell)_j &:= F(b_j^2)\end{aligned}\right\} \qquad 1 \le j,k \le N_\ell.$$

The linear system of equations in (4.125) is the basis representation of (4.116). In engineering literature the system matrix $\mathbf{K}_\ell$ is also called the stiffness matrix of the Galerkin method (4.116) and the vector $\mathbf{F}_\ell$ on the right-hand side is called the load vector.

Proposition 4.2.6. *The stiffness matrix $\mathbf{K}_\ell$ in (4.125) is non-singular if and only if we have (4.121) with $\gamma_\ell > 0$.*

Proof. Let $\mathbf{K}_\ell$ be singular. Then there exists a vector $\mathbf{u} = \left(u_j\right)_{j=1}^{N_\ell} \in \mathbb{C}^{N_\ell}\backslash\{0\}$ with $\mathbf{K}_\ell\mathbf{u} = \mathbf{0}$. Since $\left(b_j^1\right)_{j=1}^{N_\ell}$ is a basis of S_ℓ^1 we have for the associated function $u = \sum_{j=1}^{N_\ell} u_j b_j^1 \neq 0$. It follows from (4.125) that $a(u_\ell, v_\ell) = 0$ for all $v_\ell \in S_\ell^2$. This is a contradiction to (4.121) with $\gamma_\ell > 0$.

The inverse statement is proven in the same way. □

4.2.3 Compact Perturbations

Boundary integral operators often appear in the form

$$(A+T)u = F \tag{4.126}$$

with a principal part $A \in L(H, H')$ for which the associated sesquilinear form $a(\cdot,\cdot) : H \times H \to \mathbb{C}$ satisfies the inf–sup conditions

$$\inf_{u\in H\backslash\{0\}} \sup_{v\in H\backslash\{0\}} \frac{|a(u,v)|}{\|u\|_H \; \|v\|_H} \ge \gamma > 0, \tag{4.127}$$

$$\forall v \in H\backslash\{0\} : \qquad \sup_{u\in H} |a(u,v)| > 0 \tag{4.128}$$

and a compact operator $T \in L(H, H')$. Let $t : H \times H \to \mathbb{C}$ be the sesquilinear form that is associated with T. The variational formulation:
Find $u \in H$ such that

$$a(u,v) + t(u,v) = F(v) \qquad \forall v \in H \tag{4.129}$$

is equivalent to (4.126).

The discretization of the variational problem (4.129) is based on a dense sequence of finite-dimensional subspaces $(S_\ell)_{\ell\in\mathbb{N}}$ in H:

For a given $F \in H'$ find $u_\ell \in S_\ell$ such that

$$a(u_\ell, v_\ell) + t(u_\ell, v_\ell) = F(v_\ell) \qquad \forall v_\ell \in S_\ell. \tag{4.130}$$

The following theorem states that the inf–sup condition for the principal part of the sesquilinear form together with the injectivity of the operator $A + T$ ensure well posedness of the continuous problem. Furthermore, the discrete inf–sup conditions for a dense sequence of subspaces imply (a) the well-posedness of the discrete problem, (b) the unique solvability of the continuous problem, and (c) the convergence of the Galerkin solutions to the continuous solution.

Theorem 4.2.7. *Let (4.127) and (4.128) hold, let $T \in L(H, H')$ be compact and $A + T$ injective,*

$$(A + T)u = 0 \Longrightarrow u = 0. \tag{4.131}$$

Then problem (4.126) has a unique solution $u \in H$ for every $F \in H'$.

Furthermore, let $(S_\ell)_\ell$ be a dense sequence of finite-dimensional subspaces in H and $t(\cdot,\cdot)$ the sesquilinear form associated with the compact operator T. We assume that there exist an $\ell_0 > 0$ and a $\gamma > 0$ such that for all $\ell \geq \ell_0$ the discrete inf–sup conditions

$$\inf_{u_\ell \in S_\ell \setminus \{0\}} \sup_{v_\ell \in S_\ell \setminus \{0\}} \frac{|a(u_\ell, v_\ell) + t(u_\ell, v_\ell)|}{\|u_\ell\|_H \, \|v_\ell\|_H} \geq \gamma \tag{4.132a}$$

and

$$\inf_{v_\ell \in S_\ell \setminus \{0\}} \sup_{u_\ell \in S_\ell \setminus \{0\}} \frac{|a(u_\ell, v_\ell) + t(u_\ell, v_\ell)|}{\|u_\ell\|_H \, \|v_\ell\|_H} \geq \gamma \tag{4.132b}$$

are satisfied uniformly with respect to ℓ. Then we have:

(i) For all $F \in H'$ and all $\ell \geq \ell_0$ the Galerkin equations (4.130) have a unique solution u_ℓ.

(ii) The Galerkin solutions u_ℓ converge for $\ell \to \infty$ to the unique solution $u \in H$ of the problem (4.126) and satisfy the quasi-optimal error estimate

$$\|u - u_\ell\|_H \leq C \, \min\{\|u - v_\ell\|_H : v_\ell \in S_\ell\}, \qquad \ell \geq \ell_0$$

with a constant $C > 0$ which is independent of ℓ.

Proof. As $a(\cdot,\cdot)$ satisfies the inf–sup conditions, the associated operator $A : H \to H'$ is an isomorphism with $\|A\|_{H' \leftarrow H} \leq \gamma^{-1}$ [see (2.38)]. Hence (4.126) is equivalent to the Fredholm equation

$$\left(I + A^{-1}T\right) u = A^{-1} f$$

with the compact operator $A^{-1}T : H \to H$ (see Lemma 2.1.29). By (4.131), -1 is not an eigenvalue of $A^{-1}T$ and, from the Fredholm alternative (Theorem 2.1.36), $I + A^{-1}T$ is an isomorphism $\left\|I + A^{-1}T\right\|_{H \leftarrow H} \leq C$. This yields the unique solvability of (4.126) and the continuous dependence on the data.

of (i): Theorem 2.1.44 implies both (i) and the fact that the Galerkin solution depends continuously on the data:

$$\|u_\ell\|_H \leq \frac{1}{\gamma}\|F\|_{H'}. \tag{4.133}$$

of (ii): Let

$$b(u,v) := a(u,v) + t(u,v).$$

Because of (4.133) the sequence $(u_\ell)_\ell$ of Galerkin solutions is uniformly bounded in H. Theorem 2.1.26 thus guarantees the existence of a subsequence $u_{\ell_i} \rightharpoonup u \in H$ that converges weakly in H (in the following we will again denote this sequence by u_ℓ). We will now show that, with this limit u, $b(u,v) = F(v)$ for all $v \in H$. For an arbitrary $v \in H$, $P_\ell v \in S_\ell$ denotes the orthogonal projection:

$$\forall w_\ell \in S_\ell : (v - P_\ell v, w_\ell)_H = 0.$$

Then we have

$$\begin{aligned}|b(u,v) - F(v)| \leq\ & \underbrace{|b(u,v) - b(u_\ell,v)|}_{T_1} + \underbrace{|b(u_\ell,v) - b(u_\ell,P_\ell v)|}_{T_2} \\ & + \underbrace{|b(u_\ell,P_\ell v) - F(P_\ell v)|}_{T_3} + \underbrace{|F(P_\ell v) - F(v)|}_{T_4}.\end{aligned}$$

For a fixed $v \in H$

$$b(\cdot,v) : H \to \mathbb{C}$$

defines a continuous functional in H'. The definition of weak convergence then yields the convergence of T_1 to 0 for $\ell \to \infty$.

Since $\bigcup_\ell S_\ell$ is dense in H, according to the conditions, we consequently have the consistency of the discretization sequence

$$\|u - P_\ell u\|_H = \inf_{v_\ell \in S_\ell}\|u - v_\ell\|_H \overset{\ell\to\infty}{\to} 0. \tag{4.134}$$

Thus we have for T_4

$$|T_4| = |F(v - P_\ell v)| \leq \|F\|_{H'}\|v - P_\ell v\|_H \overset{\ell\to\infty}{\to} 0.$$

Since $(u_\ell)_\ell$ is uniformly bounded, we have

$$|T_2| \leq (\|A\|_{H'\leftarrow H} + \|T\|_{H'\leftarrow H})\|u_\ell\|_H\|v - P_\ell v\|_H,$$

and the consistency again implies that $T_2 \to 0$ for $\ell \to \infty$. Finally, we have $T_3 = 0$ since $b(u_\ell, v_\ell) = F(v_\ell)$ for all $v_\ell \in S_\ell$. Therefore u is a solution of (4.126). By (4.131), u is unique.

We have thus shown the unique solvability of Problem (4.126) in H.

By (4.132), $b(\cdot,\cdot)$ satisfies the conditions of Theorem 4.2.1 for $\ell \geq \ell_0$, from which we obtain the quasi-optimality.

□

Remark 4.2.8. *Theorem 4.2.7 only holds if the discrete inf–sup conditions (4.132) are satisfied. In general, the discrete inf–sup conditions do not follow from the density of $(S_\ell)_\ell$ in H combined with (4.127) and (4.128). Instead, they have to be verified for each specific problem.*

In applications concerning boundary integral equations we often encounter the following special case of Theorem 4.2.7.

Theorem 4.2.9. *Let H be a Hilbert space and $(S_\ell)_\ell$ a dense sequence of finite-dimensional subspaces in H. We assume that for the sesquilinear forms $a(\cdot,\cdot)$ and $t(\cdot,\cdot)$ of the variational problem (4.129) we have*

(i) $a(\cdot,\cdot)$ satisfies the ellipticity condition (2.44), i.e., there exists a constant $\alpha > 0$ such that

$$\forall u \in H: \qquad |a(u,u)| \geq \alpha \|u\|_H^2. \tag{4.135}$$

(ii) The operator $T \in L(H, H')$ that is associated with the sesquilinear form $t(\cdot,\cdot): H \times H \to \mathbb{C}$ is compact.

(iii) We assume that, for $F = 0$, (4.129) only has the trivial solution:

$$\forall v \in H \backslash \{0\}: \qquad a(u,v) + t(u,v) = 0 \Longrightarrow u = 0. \tag{4.136}$$

Then the variational problem (4.129) has a unique solution $u \in H$ for every $F \in H'$.

There exists a constant $\ell_0 > 0$ such that for all $\ell \geq \ell_0$ the Galerkin equations (4.130) have a unique solution $u_\ell \in S_\ell$. The sequence $(u_\ell)_\ell$ of the Galerkin solutions converges to u and, for $\ell \geq \ell_0$, satisfies the quasi-optimal error estimate

$$\|u - u_\ell\|_H \leq C \min_{v_\ell \in S_\ell} \|u - v_\ell\|_H \tag{4.137}$$

with a constant C which is independent of ℓ.

Proof. The H-ellipticity of $a(\cdot,\cdot)$ implies the inf-sup condition (4.127), (4.128), and therefore the unique solvability of (4.129) follows from Theorem 4.2.7.

Now we will turn our attention to the Galerkin equations and prove the inf-sup condition for a sufficiently large ℓ.

We set $b(\cdot,\cdot) = a(\cdot,\cdot) + t(\cdot,\cdot)$ and define the associated operators $B: H \to H'$ and $B_\ell: S_\ell \to S_\ell'$ by

$$\forall\ u, v \in H : \langle Bu, v\rangle_{H'\times H} := b(u, v) \quad \text{and}$$
$$\forall\ u_\ell, v_\ell \in S_\ell : \langle B_\ell u_\ell, v_\ell\rangle_{S'_\ell\times S_\ell} := b(u_\ell, v_\ell).$$

The norm of $B_\ell u_\ell \in S'_\ell$ is given by

$$\|B_\ell u_\ell\|_{S'_\ell} = \sup_{v_\ell\in S_\ell\backslash\{0\}} \frac{|b(u_\ell, v_\ell)|}{\|v_\ell\|_H}$$

and the discrete inf-sup condition (4.132a) is equivalent to

$$\forall u_\ell \in S_\ell \text{ with } \|u_\ell\|_H = 1 \text{ we have: } \exists \ell_0 > 0 \text{ s.t. } \|B_\ell u_\ell\|_{S'_\ell} \geq \gamma \quad \forall \ell \geq \ell_0.$$

We will prove this statement by contradiction by using the conditions given in the theorem. For this we assume:

$$\exists\,(w_\ell)_{\ell\in\mathbb{N}} \text{ with } w_\ell \in S_\ell \text{ and } \|w_\ell\|_H = 1 \text{ such that: } \|B_\ell w_\ell\|_{S'_\ell} \to 0 \text{ for } \ell \to \infty. \tag{4.138}$$

As $(w_\ell)_\ell$ is bounded in H there exists, according to Theorem 2.1.26, a weakly convergent subsequence (which we again denote by $(w_\ell)_\ell$) such that $w_\ell \rightharpoonup w \in H$.

For all $v \in H$, $b(\cdot, v)$ defines a continuous, linear functional on H and so we have

$$\forall v \in H : b(w_\ell, v) \to b(w, v) \quad \text{for } \ell \to \infty.$$

It follows that

$$\|Bw\|_{H'} = \sup_{v\in H\backslash\{0\}} \frac{|b(w, v)|}{\|v\|_H} = \sup_{v\in H\backslash\{0\}} \lim_{\ell\to\infty} \frac{|b(w_\ell, v)|}{\|v\|_H}. \tag{4.139}$$

In the following we will estimate the numerator on the right-hand side and for this purpose we use the decomposition

$$b(w_\ell, v) = b(w_\ell, v_\ell) + b(w_\ell, v - v_\ell) \tag{4.140}$$

with the H-orthogonal projection $v_\ell = P_\ell v \in S_\ell$. From assumption (4.138) we have

$$|b(w_\ell, v_\ell)| \leq \|B_\ell w_\ell\|_{S'_\ell} \|v_\ell\|_H \leq \|B_\ell w_\ell\|_{S'_\ell} \|v\|_H \overset{\ell\to\infty}{\to} 0.$$

The fact that the spaces S_ℓ are dense in H yields for the second term in (4.140)

$$|b(w_\ell, v - v_\ell)| \leq \|b\|\,\|w_\ell\|_H\,\|v - v_\ell\|_H \leq \|b\|\,\|v - v_\ell\|_H \overset{\ell\to\infty}{\to} 0.$$

Hence for all $v \in H$ we have the convergence $\lim_{\ell\to\infty} b(w_\ell, v) = 0$ and from (4.139) we have $Bw = 0$, which, combined with the injectivity of (4.136), finally gives us $w = 0$.

We will now show the strong convergence $w_\ell \to w$ and begin with the estimate

$$\alpha \|w - w_\ell\|_H^2 \leq |a(w - w_\ell, w - w_\ell)| = |a(w - w_\ell, w) - a(w, w_\ell) + a(w_\ell, w_\ell)|. \tag{4.141}$$

Since T is compact, there exists a subsequence (which we again denote by $(w_\ell)_{\ell\in\mathbb{N}}$) such that $Tw_\ell \to Tw$ in H'. This can be written in the form

$$\sup_{\substack{v\in H\\ \|v\|_H=1}} |t(w_\ell, v) - t(w, v)| =: \delta_\ell \overset{\ell\to\infty}{\to} 0,$$

from which we deduce by using $\|w_\ell\|_H = 1$ that

$$|t(w_\ell, w_\ell) - t(w, w_\ell)| \leq \delta_\ell \|w_\ell\|_H = \delta_\ell \overset{\ell\to\infty}{\to} 0.$$

This result, combined with assumption (4.138), yields

$$0 \overset{\ell\to\infty}{\leftarrow} |b(w_\ell, w_\ell)| = |a(w_\ell, w_\ell) + t(w_\ell, w_\ell)| \leq |a(w_\ell, w_\ell) + t(w, w_\ell)| + \delta_\ell,$$

in other words:

$$a(w_\ell, w_\ell) = -t(w, w_\ell) + \tilde{\delta}_\ell \quad \text{with} \quad \lim_{\ell\to\infty} \tilde{\delta}_\ell = 0. \tag{4.142}$$

If we insert this into (4.141) we obtain

$$\alpha \|w - w_\ell\|_H^2 \leq \left| a(w - w_\ell, w) - b(w, w_\ell) + \tilde{\delta}_\ell \right|.$$

The first two terms on the right-hand side are equal to zero because of $w = 0$. We also determined $\lim_{\ell\to 0} \tilde{\delta}_\ell = 0$ in (4.142) so that we have proven $w_\ell \to w = 0$. This, however, is a contradiction to the assumption that $\|w_\ell\|_H = 1$.

Condition (4.132b) can be proven similarly.

The solvability of the Galerkin equation for $\ell \geq \ell_0$ and the error estimate (4.137) then follow from Theorem 4.2.7. □

4.2.4 Consistent Perturbations: Strang's Lemma

In this section we will consider variational formulations of boundary integral equations of abstract form:
Find $u \in H$ such that

$$b(u, v) = F(v) \qquad \forall v \in H \tag{4.143}$$

with $F \in H'$.

In general we assume that the sesquilinear form $b(\cdot,\cdot)$ is continuous and injective and that it satisfies a Gårding inequality.

Continuity:

$$\forall u, v \in H: \ |b(u,v)| \leq C_b \|u\|_H \|v\|_H. \tag{4.144}$$

Gårding Inequality:

$$\forall u \in H: |b(u,u) + (Tu,u)_{H' \times H}| \geq \alpha \|u\|_H^2 \tag{4.145}$$

with $\alpha > 0$ and a compact operator $T \in L(H, H')$.

Injectivity:

$$\forall v \in H \backslash \{0\}: \ b(u,v) = 0 \Longrightarrow u = 0. \tag{4.146}$$

Conditions (4.144)–(4.146) yield the prerequisites (i)–(iii) from Theorem 4.2.9 with $t(\cdot,\cdot) := -\langle T\cdot,\cdot\rangle_{H' \times H}$ and $a := b - t$. From Theorem 4.2.9 we derive the unique solvability of (4.143) as well as the stability (and thus the quasi-optimal convergence) of the Galerkin method as follows. For a dense sequence of finite-dimensional boundary element spaces $(S_\ell)_\ell$ in H there exists some $\ell_0 > 0$ such that for all $\ell \geq \ell_0$ the discrete inf–sup conditions

$$\begin{aligned} \inf_{u \in S_\ell \backslash \{0\}} \sup_{v \in S_\ell \backslash \{0\}} \frac{|b(u,v)|}{\|u\|_H \|v\|_H} &\geq \gamma > 0 \\ \inf_{v \in S_\ell \backslash \{0\}} \sup_{u \in S_\ell \backslash \{0\}} \frac{|b(u,v)|}{\|u\|_H \|v\|_H} &\geq \gamma > 0 \end{aligned} \tag{4.147}$$

hold, while $\gamma > 0$ is independent of ℓ. The Galerkin equations

$$\text{Find } u_\ell \in S_\ell: \qquad b(u_\ell, v) = F(v) \quad \forall v \in S_\ell \tag{4.148}$$

are, by Theorem 4.2.7, uniquely solvable for $\ell \geq \ell_0$ and we have

$$\|u - u_\ell\|_H \leq C \min_{v \in S_\ell} \|u - v\|_H. \tag{4.149}$$

In practical implementations of the Galerkin boundary element method in the form of a computer program it is usually not possible to realize the exact sesquilinear form $b(\cdot,\cdot)$. Instead, one usually uses an *approximative* sesquilinear form $b_\ell(\cdot,\cdot)$. Reasons for this are:

(a) The approximation of the system matrix by means of numerical integration
(b) The use of compressed, approximative representations of the Galerkin equations with cluster or wavelet methods,
(c) The approximation of the exact boundary Γ by means of, for example a polyhedral surface.

The perturbation of the sesquilinear form $b(\cdot,\cdot)$ as well as the functional F leads to the *perturbed Galerkin method*:

Find $\tilde{u}_\ell \in S_\ell$ such that

$$b_\ell(\tilde{u}_\ell, v) = F_\ell(v) \qquad \forall v \in S_\ell. \tag{4.150}$$

For the algorithmic realization of boundary element methods, one of the essential aims is to define the approximations (4.150) in such a way that the solutions $\tilde{u}_\ell$ exist, converge quasi-optimally and – in comparison with the computation of the exact Galerkin solution – can be calculated reasonably rapidly and with little use of computational memory. A sufficient condition in this respect is that the difference $b_\ell(\cdot,\cdot)-b(\cdot,\cdot)$ is "sufficiently small". We will specify this statement in the following.

For the Galerkin discretization we will generally assume in the following that we have chosen a dense sequence $(S_\ell)_\ell \subset H$ of subspaces of dimension $N_\ell := \dim S_\ell < \infty$ which satisfies (4.114).

Let sesquilinear forms $b_\ell : S_\ell \times S_\ell \to \mathbb{C}$ be defined for all $\ell \in \mathbb{N}$. These are *uniformly continuous* if there exists a constant $\widetilde{C}_b$ which is independent of ℓ such that

$$|b_\ell(u_\ell, v_\ell)| \le \widetilde{C}_b \|u_\ell\|_H \|v_\ell\|_H \qquad \forall u_\ell, v_\ell \in S_\ell. \tag{4.151}$$

The forms b_ℓ satisfy the *stability condition* if there exists a null sequence $(c_\ell)_{\ell\in\mathbb{N}}$ such that

$$|b(u_\ell, v_\ell) - b_\ell(u_\ell, v_\ell)| \le c_\ell \|u_\ell\|_H \|v_\ell\|_H \qquad \forall u_\ell, v_\ell \in S_\ell. \tag{4.152}$$

The stability condition will imply the existence of a unique solution of the perturbed Galerkin equations for a sufficiently large ℓ (see Theorem 4.2.11).

For the error estimate of the perturbed Galerkin solution we may measure the function u_ℓ on the right-hand side in (4.152) in a stronger norm (see Theorem 4.2.11). In this context $\|\cdot\|_U : S_\ell \to \mathbb{R}_{\ge 0}$ defines a *stronger* norm on S_ℓ if there exists a constant $C > 0$ independent of ℓ such that

$$\|u\|_H \le C \|u\|_U \qquad \forall u \in S_\ell.$$

The perturbed sesquilinear forms $b_\ell : S_\ell \times S_\ell \to \mathbb{C}$ satisfy the *consistency condition with respect to a stronger norm* $\|\cdot\|_U$ if there exists a zero sequence $(\delta_\ell)_{\ell\in\mathbb{N}}$ such that

$$|b(u_\ell, v_\ell) - b_\ell(u_\ell, v_\ell)| \le \delta_\ell \|u_\ell\|_U \|v_\ell\|_H \qquad \forall u_\ell, v_\ell \in S_\ell. \tag{4.153}$$

Remark 4.2.10. *(a) The stability condition and the continuity of $b(\cdot,\cdot)$ imply the uniform continuity of the sesquilinear form $b_\ell(\cdot,\cdot)$.*

(b) The consistency condition follows from the stability condition with $\delta_\ell = Cc_\ell$.

(c) *In many practical applications the use of the stronger norm $\|\cdot\|_U$ in (4.153) permits the use of a zero sequence $(\delta_\ell)_\ell$ which converges more rapidly than in (4.152). The convergence rate of the perturbed Galerkin solution is influenced by $(\delta_\ell)_\ell$ and not by $(c_\ell)_\ell$.*

Theorem 4.2.11. *Let the sesquilinear form $b(\cdot,\cdot)\colon H \times H \to \mathbb{C}$ be continuous, injective and let it satisfy a Gårding inequality [see (4.144)–(4.146)]. Let the stability condition (4.152) be satisfied by the approximations b_ℓ.*

Then the perturbed Galerkin method (4.150) is stable. That is, there exist $\tilde{\gamma} > 0$, $\ell_0 > 0$ such that for all $\ell \geq \ell_0$ the discrete inf–sup conditions

$$\begin{aligned} \inf_{u_\ell \in S_\ell \setminus \{0\}} \sup_{v_\ell \in S_\ell \setminus \{0\}} \frac{|b_\ell(u_\ell, v_\ell)|}{\|u_\ell\|_H \ \|v_\ell\|_H} &\geq \tilde{\gamma}, \\ \inf_{v_\ell \in S_\ell \setminus \{0\}} \sup_{u_\ell \in S_\ell \setminus \{0\}} \frac{|b_\ell(u_\ell, v_\ell)|}{\|u_\ell\|_H \ \|v_\ell\|_H} &\geq \tilde{\gamma} \end{aligned} \tag{4.154}$$

hold. The perturbed Galerkin equations (4.150) have a unique solution for $\ell \geq \ell_0$.

If in addition the approximative sesquilinear forms are uniformly continuous and satisfy the consistency condition (4.153) the solutions $\tilde{u}_\ell$ satisfy the error estimate

$$\|u - \tilde{u}_\ell\|_H \leq C \left\{ \min_{w_\ell \in S_\ell} \left(\|u - w_\ell\|_H + \delta_\ell \|w_\ell\|_U\right) + \sup_{v_\ell \in S_\ell \setminus \{0\}} \frac{|F(v_\ell) - F_\ell(v_\ell)|}{\|v_\ell\|_H} \right\}. \tag{4.155}$$

Proof. According to the assumptions, the exact sesquilinear form $b(\cdot,\cdot)$ satisfies the inf–sup conditions (4.147) as well as the stability condition (4.149). We will verify (4.154). For this let $0 \neq u_\ell \in S_\ell \subset H$ be arbitrary. Then we have

$$\begin{aligned} \sup_{v_\ell \in S_\ell \setminus \{0\}} \frac{|b_\ell(u_\ell, v_\ell)|}{\|v_\ell\|_H} &\geq \sup_{v_\ell \in S_\ell \setminus \{0\}} \left(\frac{|b(u_\ell, v_\ell)|}{\|v_\ell\|_H} - \frac{|b(u_\ell, v_\ell) - b_\ell(u_\ell, v_\ell)|}{\|v_\ell\|_H} \right) \\ &\geq \gamma \|u_\ell\|_H - \sup_{v_\ell \in S_\ell} \frac{|b(u_\ell, v_\ell) - b_\ell(u_\ell, v_\ell)|}{\|v_\ell\|_H} \\ &\geq (\gamma - c_\ell) \|u_\ell\|_H. \end{aligned} \tag{4.156}$$

If we choose $\ell_0 > 0$ so that $c_\ell < \gamma$ for all $\ell \geq \ell_0$ we have verified the first condition in (4.154). The second condition can be verified in a similar way.

Combined with (4.154), it follows from Theorem 4.2.1(i) that the perturbed Galerkin equations (4.150) have a unique solution for $\ell \geq \ell_0$.

Next, we will prove the error estimate (4.155). Let $u_\ell \in S_\ell$ be the exact Galerkin solution from (4.148). For $\ell \geq \ell_0$ we have, according to (4.156), the following estimate for the perturbed Galerkin solution $\tilde{u}_\ell \in S_\ell$

$$\begin{aligned}\|u-\tilde{u}_\ell\|_H &\le \|u-u_\ell\|_H + \|u_\ell-\tilde{u}_\ell\|_H \\ &\le \|u-u_\ell\|_H + (\gamma-c_\ell)^{-1} \sup_{v_\ell\in S_\ell\setminus\{0\}} \frac{|b_\ell(u_\ell-\tilde{u}_\ell, v_\ell)|}{\|v_\ell\|_H} \\ &= \|u-u_\ell\|_H + (\gamma-c_\ell)^{-1} \sup_{v_\ell\in S_\ell\setminus\{0\}} \frac{|b_\ell(u_\ell, v_\ell)-F_\ell(v_\ell)|}{\|v_\ell\|_H} \\ &\le \|u-u_\ell\|_H + (\gamma-c_\ell)^{-1} \sup_{v_\ell\in S_\ell\setminus\{0\}} \frac{|b_\ell(u_\ell, v_\ell)-b(u_\ell,v_\ell)|+|F(v_\ell)-F_\ell(v_\ell)|}{\|v_\ell\|_H}.\end{aligned}$$

We consider the difference term $|b_\ell(u_\ell, v_\ell) - b(u_\ell, v_\ell)|$ and obtain, by using the continuity of b_ℓ and b as well as the consistency condition, for an arbitrary $w_\ell \in S_\ell$

$$\begin{aligned}|b_\ell(u_\ell, v_\ell) - b(u_\ell, v_\ell)| &\le |b_\ell(u_\ell - w_\ell, v_\ell)| + |b_\ell(w_\ell, v_\ell) - b(w_\ell, v_\ell)| \\ &\quad + |b(w_\ell - u_\ell, v_\ell)| \\ &\le \widetilde{C}_b \|u_\ell - w_\ell\|_H \|v_\ell\|_H + \delta_\ell \|w_\ell\|_U \|v_\ell\|_H \\ &\quad + C_b \|w_\ell - u_\ell\|_H \|v_\ell\|_H.\end{aligned}$$

From this we have

$$\sup_{v_\ell\in S_\ell\setminus\{0\}} \frac{|b_\ell(u_\ell, v_\ell)-b(u_\ell,v_\ell)|}{\|v_\ell\|_H} \le C \min_{w_\ell\in S_\ell} \left(\|u-w_\ell\|_H + \delta_\ell \|w_\ell\|_U\right).$$

With $c_\ell < \gamma$ and the consistency condition (4.153) we finally obtain

$$\begin{aligned}\|u-\tilde{u}_\ell\|_H \le C \min_{w_\ell\in S_\ell} \Bigg\{ \|u-w_\ell\|_H + \frac{1}{\gamma-c_\ell} \Bigg(\|u-w_\ell\|_H + \delta_\ell\|w_\ell\|_U \\ + \sup_{v_\ell\in S_\ell\setminus\{0\}} \frac{|F(v_\ell)-F_\ell(v_\ell)|}{\|v_\ell\|_H} \Bigg) \Bigg\}.\end{aligned} \tag{4.157}$$

□

Remark 4.2.12. *In connection with the boundary integral operator V for the single layer potential we have $H = H^{-1/2}(\Gamma)$. Since all the boundary element spaces we have considered so far are contained in $L^2(\Gamma)$, we can choose $\|\cdot\|_U = \|\cdot\|_{L^2(\Gamma)}$ as a stronger norm on S_ℓ. The term $\|w_\ell\|_{L^2(\Gamma)}$ on the right-hand side in (4.155) can be easily estimated if the boundary integral operator is L^2-regular, more specifically if $V^{-1} : H^1(\Gamma) \to L^2(\Gamma)$ is continuous. Let $u \in L^2(\Gamma)$ be the exact solution and $w_\ell := \Pi_\ell u$ the L^2-orthogonal projection of u onto the boundary element space S_ℓ. Then we have $\|w_\ell\|_{L^2(\Gamma)} \le \|u\|_{L^2(\Gamma)} \le C\|F\|_{H^1(\Gamma)}$ and, thus for a sufficiently large $\ell \ge \ell_0$:*

$$\begin{aligned}\|u-\tilde{u}_\ell\|_{H^{-1/2}(\Gamma)} \le C \Bigg\{ \|u-\Pi_\ell u\|_{H^{-1/2}(\Gamma)} + \delta_\ell \|F\|_{H^1(\Gamma)} \\ + \sup_{v_\ell\in S_\ell\setminus\{0\}} \frac{|F(v_\ell)-F_\ell(v_\ell)|}{\|v_\ell\|_{H^{-1/2}(\Gamma)}} \Bigg\}.\end{aligned}$$

From this we can deduce how the null sequence $(\delta_\ell)_\ell$ and the consistency of the approximation affect the right-hand side in the error estimate.

The error $\|u - \Pi_\ell u\|_{H^{-1/2}(\Gamma)}$ can be traced back to the approximation properties of S_ℓ. The choice $w_\ell = \Pi_\ell u$ yields, for an arbitrary $v_\ell \in S_\ell$

$$\begin{aligned}\|u - \Pi_\ell u\|_{H^{-1/2}(\Gamma)} &= \sup_{v \in H^{1/2}(\Gamma)\setminus\{0\}} \frac{\left|(u - \Pi_\ell u, v)_{L^2(\Gamma)}\right|}{\|v\|_{H^{1/2}(\Gamma)}} \\ &= \sup_{v \in H^{1/2}(\Gamma)\setminus\{0\}} \frac{\left|(u - \Pi_\ell u, v - v_\ell)_{L^2(\Gamma)}\right|}{\|v\|_{H^{1/2}(\Gamma)}}.\end{aligned}$$

Now we take the infimum over all $v_\ell \in S_\ell$ and obtain

$$\begin{aligned}\|u - \Pi_\ell u\|_{H^{-1/2}(\Gamma)} \leq &\left(\sup_{v \in H^{1/2}\setminus\{0\}} \inf_{v_\ell \in S_\ell\setminus\{0\}} \frac{\|v - v_\ell\|_{L^2(\Gamma)}}{\|v\|_{H^{1/2}(\Gamma)}} \right) \\ &\times \left(\inf_{w_\ell \in S_\ell} \|u - w_\ell\|_{L^2(\Gamma)} \right). \qquad (4.158)\end{aligned}$$

4.2.5 Aubin–Nitsche Duality Technique

Boundary integral equations were derived with the help of the integral equation method (direct and indirect method) for elliptic boundary value problems. In many cases our goal thus is to find the solution of the original boundary value problem by solving the boundary integral equation. The numerical solution of the boundary integral equation then only represents a part of the entire process. (Note, however, that with the direct method the boundary element method yields a quasi-optimal approximation of the unknown Cauchy data.) More importantly, the aim is to find the solution u of the original elliptic differential equation in the domain Ω. This solution can, as we will show here, be extracted from the Galerkin solution of the boundary integral equations with an increased convergence rate, a fact which stems from the representation formula.

Example 4.2.13 (Dirichlet Problem in the Interior, Ω). *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with boundary Γ and given Dirichlet data $g_D \in H^{1/2}(\Gamma)$. Find $u \in H^1(\Omega)$ such that*

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u|_\Gamma = g_D. \qquad (4.159)$$

The fundamental solution for the Laplace operator is given by $G(\mathbf{z}) := (4\pi \|\mathbf{z}\|)^{-1}$. The single layer potential $u(\mathbf{x}) = \int_\Gamma G(\mathbf{x} - \mathbf{y})\, \sigma(\mathbf{y}) ds_{\mathbf{y}}$, $\mathbf{x} \in \Omega$, leads to the boundary integral equation: Find $\sigma \in H^{-1/2}(\Gamma)$ such that

$$(V\sigma, \eta)_{L^2(\Gamma)} = (g_D, \eta)_{L^2(\Gamma)} \qquad \forall \eta \in H^{-1/2}(\Gamma), \tag{4.160}$$

where $(\cdot,\cdot)_{L^2(\Gamma)}$ *again denotes the continuous extension of the* L^2 *inner-product to the dual pairing* $\langle \cdot,\cdot \rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}$.

For a subspace $S_\ell \subset H^{-1/2}(\Gamma)$ *the Galerkin approximation* $\sigma_\ell \in S_\ell$ *is defined by: Find* $\sigma_\ell \in S_\ell$ *such that*

$$(V\sigma_\ell, \eta)_{L^2(\Gamma)} = (g_D, \eta)_{L^2(\Gamma)} \qquad \forall \eta \in S_\ell. \tag{4.161}$$

Equation (4.161) has a unique solution which satisfies the quasi-optimal error estimate

$$\|\sigma - \sigma_\ell\|_{H^{-1/2}(\Gamma)} \le C \min\left\{\|\sigma - v\|_{H^{-1/2}(\Gamma)}, v \in S_\ell\right\}. \tag{4.162}$$

We obtain the approximation of the solution $u(\mathbf{x})$ *of the boundary value problem (4.159) by*

$$u_\ell(\mathbf{x}) := \int_\Gamma G\,(\mathbf{x} - \mathbf{y})\,\sigma_\ell(\mathbf{y})\,ds_{\mathbf{y}}, \qquad \mathbf{x} \in \Omega. \tag{4.163}$$

In this section we will derive error estimates for the pointwise error $|u(\mathbf{x}) - u_\ell(\mathbf{x})|$.

4.2.5.1 Errors in Functionals of the Solution

The Aubin–Nitsche technique allows us to estimate errors in the linear functionals of the Galerkin solution. We will first introduce this method for abstract problems as discussed in Sect. 4.2.1. The abstract variational problem reads: For a given $F(\cdot) \in H'$ find a function $u \in H$ such that

$$b(u, v) = F(v) \qquad \forall v \in H. \tag{4.164}$$

Let $(S_\ell)_\ell \subset H$ be a family of dense subspaces that satisfy the discrete inf–sup conditions (4.117), (4.118). Then the Galerkin discretization of (4.164), i.e., find $u_\ell \in S_\ell$ such that

$$b(u_\ell, v_\ell) = F(v_\ell) \qquad \forall v_\ell \in S_\ell, \tag{4.165}$$

has a unique solution. The error $e_\ell = u - u_\ell$ satisfies the Galerkin orthogonality

$$b(u - u_\ell, v_\ell) = 0 \qquad \forall v_\ell \in S_\ell \tag{4.166}$$

as well as the quasi-optimal error estimate

$$\|u - u_\ell\|_H \le \frac{C}{\gamma_\ell} \min\{\|u - \varphi_\ell\|_H : \varphi_\ell \in S_\ell\}. \tag{4.167}$$

The Aubin–Nitsche argument estimates the error in *functionals* of the solution.

Theorem 4.2.14. *Let $\mathfrak{G} \in H'$ be a continuous, linear functional on the set of solutions H of Problem (4.164) which satisfies the assumptions (4.109), (4.110). Let $u_\ell \in S_\ell$ be the Galerkin approximation from (4.165) of the solution u. Furthermore, let the discrete inf–sup conditions (4.117), (4.118) be uniformly satisfied: $\gamma_\ell \geq \gamma > 0$.*

Then we have the error estimate

$$|\mathfrak{G}(u) - \mathfrak{G}(u_\ell)| \leq C\, \|u - \varphi_\ell\|_H \|w_{\mathfrak{G}} - \psi_\ell\|_H \tag{4.168}$$

for an arbitrary $\varphi_\ell \in S_\ell$, $\psi_\ell \in S_\ell$, where $w_{\mathfrak{G}}$ is the solution of the dual problem:

$$\textit{Find } w_{\mathfrak{G}} \in H: \qquad b(w, w_{\mathfrak{G}}) = \mathfrak{G}(w) \quad \forall w \in H. \tag{4.169}$$

Proof. From the continuous inf-sup conditions (4.110) Remark 2.1.45 gives us the inf-sup conditions for the adjoint problem, from which we have the existence of a unique solution.

Remark 4.2.4 shows that the discrete inf-sup conditions for $b(\cdot,\cdot)$ induce the discrete inf-sup conditions for the adjoint form $b^*(u,v) = \overline{b(v,u)}$. Therefore the adjoint problem (4.169) has a unique solution $w_{\mathfrak{G}} \in H$ for every $\mathfrak{G}(\cdot) \in H'$. By virtue of $S_\ell \subset H$ and (4.169), (4.166) it follows that

$$\begin{aligned} |\mathfrak{G}(u) - \mathfrak{G}(u_\ell)| &= |\mathfrak{G}(u - u_\ell)| = |b(u - u_\ell, w_{\mathfrak{G}})| \\ &= |b(u - u_\ell, w_{\mathfrak{G}} - v_\ell)| \qquad \forall v_\ell \in S_\ell. \end{aligned}$$

The continuity (4.109) of the form $b(\cdot,\cdot)$ and the error estimate (4.119) together yield (4.168). □

The error estimate (4.168) states that linear functionals $\mathfrak{G}(u)$ of the solution may under certain circumstances converge more rapidly than the energy error $\|u - u_\ell\|_H$. The convergence rate is superior to the rate in the energy norm by a factor $\inf\{\|w_{\mathfrak{G}} - \psi_\ell\|_H : \psi_\ell \in S_\ell\}$. The following example, for which $\mathfrak{G}(\cdot)$ represents an evaluation of the representation formula (4.163) in the domain point $\mathbf{x} \in \Omega$, makes this fact evident.

Example 4.2.15. *With the terminology used in Example 4.2.13, for the error $|u(\mathbf{x}) - u_\ell(\mathbf{x})|$ we have the estimate*

$$\begin{aligned} |u(\mathbf{x}) - u_\ell(\mathbf{x})| \leq C\, &\min\left\{\|\sigma - \varphi_\ell\|_{H^{-1/2}(\Gamma)} : \varphi_\ell \in S_\ell\right\} \\ &\times \min\left\{\|v_e - \psi_\ell\|_{H^{-1/2}(\Gamma)} : \psi_\ell \in S_\ell\right\} \end{aligned} \tag{4.170}$$

with the solution $v_e \in H^{-1/2}(\Gamma)$ of the dual problem:

Find $v_e \in H^{-\frac{1}{2}}(\Gamma)$ such that

$$(V v_e, \eta)_{L^2(\Gamma)} = (G(\mathbf{x} - \cdot), \eta)_{L^2(\Gamma)} \qquad \forall \eta \in H^{-1/2}(\Gamma). \tag{4.171}$$

With Corollary 4.1.34 we deduce the convergence rate for $S_\ell = S^{p,-1}_{\mathcal{G}_\ell}$

$$|u(\mathbf{x}) - u_\ell(\mathbf{x})| \leq C\, h_\ell^{\min(s,p+1)+\frac{1}{2}+\min(t,p+1)+\frac{1}{2}} \|\sigma\|_{H^s(\Gamma)} \|v_e\|_{H^t(\Gamma)} \tag{4.172}$$

for $s, t > -\frac{1}{2}$ *if* $\|\sigma\|_{H^s(\Gamma)}$ *and* $\|v_e\|_{H^t}$ *are bounded. If we have maximal regularity, i.e.,* $s = t = p + 1$*, the result is a doubling of the convergence rate of the Galerkin method. For example, for piecewise constant boundary elements* $p = 0$ *and (4.172) with* $s = t = 1$ *we obtain the estimate*

$$|u(\mathbf{x}) - u_\ell(\mathbf{x})| \leq C\, h_\ell^3 \|\sigma\|_{H^1(\Gamma)} \|v_e\|_{H^1(\Gamma)} \tag{4.173}$$

and, thus, third order convergence for all $\mathbf{x} \in \Omega$*. Note that the constant* C *tends to infinity for* $\operatorname{dist}(\mathbf{x}, \Gamma) \to 0$.

Remark 4.2.16 (Regularity). *Inequality (4.172) only gives a high convergence rate if the solutions* σ, v_e *are sufficiently regular. For the boundary integral operator* V *on smooth surfaces* Γ*, the property* $g_D \in H^{1/2+s}(\Gamma)$ *with* $s \geq 0$ *is sufficient so that* $\sigma \in H^{-1/2+s}(\Gamma)$*, and the property* $G(\mathbf{x} - \cdot) \in H^{1/2+t}(\Gamma)$ *with* $t \geq 0$ *is sufficient so that* $v_e \in H^{-1/2+t}(\Gamma)$ *(see Sect. 3.2). Then we have the estimates*

$$\|\sigma\|_{H^{-1/2+s}(\Gamma)} \leq C(s) \|g_D\|_{H^{1/2+s}(\Gamma)}, \quad \|v_e\|_{H^{-1/2+t}(\Gamma)} \leq C(t) \|G(\mathbf{x}, \cdot)\|_{H^{1/2+t}(\Gamma)}, \tag{4.174}$$

with a constant $C(\cdot)$ *which is independent of* g_D *and* G*. Because of the smoothness of the fundamental solution* $G(\mathbf{x} - \cdot)$ *for* $\mathbf{x} \in \Omega$, $\mathbf{y} \in \Gamma$ *we have* $G(\mathbf{x} - \cdot) \in C^\infty(\Gamma)$*. On smooth surfaces this implies the estimate (4.174) for all* $t \geq 0$*. With this (4.172) becomes*

$$|u(\mathbf{x}) - u_\ell(\mathbf{x})| \leq C_1(p)\, C_2(\mathbf{x})\, h_\ell^{2(p+1)+1}, \tag{4.175}$$

where we have $C_2(\mathbf{x}) = \|v_e\|_{H^{p+1}(\Gamma)} \leq C(p) \|G(\mathbf{x} - \cdot)\|_{H^{p+2}(\Gamma)}$.

Note that especially for elements of higher order, $C_2(\mathbf{x})$ can become very large for $\mathbf{x}$ near Γ. Formula (4.163) should therefore only be used for points $\mathbf{x}$ in the domain that are sufficiently far away from Γ. For points $\mathbf{x}$ which are very close to the boundary or even lie on Γ, a bootstrapping algorithm has been developed to extract the potentials and arbitrary Cauchy data and their derivatives near and up to the boundary (see [213]).

If a quantity which has been computed or postprocessed by using the Galerkin method converges with an order that is higher than the order of the Galerkin error in the energy norm one speaks of *superconvergence*. Similar to the superconvergence (4.168) of functionals $\mathfrak{G}(\cdot)$ of the Galerkin solution u_ℓ, one can also study the convergence of u_ℓ in norms below the energy norm.

Now let $H = H^s(\Gamma)$ be the Hilbert space for the boundary integral operator $B : H^s(\Gamma) \to H^{-s}(\Gamma)$ of order $2s$ and let $b(\cdot, \cdot)$ be the $H^s(\Gamma)$-elliptic and injective sesquilinear form associated with B:

$$b(u, v) = (Bu, v)_{L^2(\Gamma)} : H^s(\Gamma) \times H^s(\Gamma) \to \mathbb{C}.$$

Here the continuous extension of the $L^2(\Gamma)$ inner-product for the dual pairing $\langle\cdot,\cdot\rangle_{H^s(\Gamma)\times H^{-s}(\Gamma)}$ is again denoted by $(\cdot,\cdot)_{L^2(\Gamma)}$. Furthermore, let $(S_\ell)_\ell$ be a dense sequence of subspaces in $H^s(\Gamma)$and let the discrete inf–sup conditions (4.117), (4.118) hold. Then we have for $t > 0$

$$\|u - u_\ell\|_{H^{s-t}(\Gamma)} = \sup_{v\in H^{-s+t}(\Gamma)\setminus\{0\}} \frac{(v, u - u_\ell)_{L^2(\Gamma)}}{\|v\|_{H^{-s+t}(\Gamma)}}.$$

Let w_v be a solution of the adjoint problem: Find $w_v \in H^s(\Gamma)$ such that

$$b(w, w_v) = (v, w)_{L^2(\Gamma)} \qquad \forall w \in H^s(\Gamma). \tag{4.176}$$

Then with the Galerkin orthogonality (4.166) we have (transferred to the adjoint problem)

$$\begin{aligned}\|u - u_\ell\|_{H^{s-t}(\Gamma)} &= \sup_{v\in H^{-s+t}(\Gamma)\setminus\{0\}} \frac{b(u - u_\ell, w_v)}{\|v\|_{H^{-s+t}(\Gamma)}}\\ &= \sup_{v\in H^{-s+t}(\Gamma)\setminus\{0\}} \frac{b(u - u_\ell, w_v - w_\ell)}{\|v\|_{H^{-s+t}(\Gamma)}}\\ &\le C\,\|u - u_\ell\|_{H^s(\Gamma)} \sup_{v\in H^{-s+t}(\Gamma)\setminus\{0\}} \frac{\|w_v - w_\ell\|_{H^s(\Gamma)}}{\|v\|_{H^{-s+t}(\Gamma)}}.\end{aligned}$$

Since $w_\ell \in S_\ell$ was arbitrary, we obtain

$$\|u - u_\ell\|_{H^{s-t}(\Gamma)} \le C\,\|u - u_\ell\|_{H^s(\Gamma)} \sup_{v\in H^{-s+t}(\Gamma)\setminus\{0\}} \inf_{w_\ell\in S_\ell} \frac{\|w_v - w_\ell\|_{H^s(\Gamma)}}{\|v\|_{H^{-s+t}(\Gamma)}}. \tag{4.177}$$

For $t > 0$ higher convergence rates are therefore possible for u_ℓ than in the H^s-norm, assuming that the adjoint problem (4.176) has the regularity

$$v \in H^{-s+t}(\Gamma) \Longrightarrow w_v \in H^{s+t}(\Gamma), \quad \forall 0 \le t \le \bar{t}. \tag{4.178}$$

In order to obtain quantitative error estimates with respect to the mesh width h_ℓ we again consider a dense sequence of boundary element spaces $(S_\ell)_\ell$ of order p on regular meshes $\mathcal{G}_\ell$ of mesh width h_ℓ. Then the approximation property

$$\inf_{w_\ell\in S_\ell} \|w_v - w_\ell\|_{H^s(\Gamma)} \le C\, h_\ell^{\min(p+1,s+t)-s} \|w_v\|_{H^{s+t}(\Gamma)}$$

holds. These ideas are summarized in the following theorem.

Theorem 4.2.17. *Let the sesquilinear form $b(\cdot,\cdot)$ of problem (4.164) satisfy the conditions (4.109), (4.110). Let the exact solution satisfy $u \in H^r(\Gamma)$ with $r \ge s$. We assume that the adjoint problem (4.176) has the regularity (4.178) with $\bar{t} \ge 0$.*

Furthermore, let $(S_\ell)_\ell$ *be a dense sequence of boundary element spaces of order* p *in* $H^s(\Gamma)$ *on regular meshes* $\mathcal{G}_\ell$ *of mesh width* h_ℓ.

Then we have for the Galerkin solution $u_\ell \in S_\ell$ *and* $0 \le t \le \bar{t}$ *the error estimate*

$$\|u - u_\ell\|_{H^{s-t}(\Gamma)} \le C\, h_\ell^{\min(p+1,r)+\min(p+1),s+t)-2s} \|u\|_{H^r(\Gamma)}. \tag{4.179}$$

In particular, in the case of maximal regularity, i.e., for $r \ge p+1, \bar{t} \ge p+1-s$, it thus follows that we have a doubling of the convergence rate of the Galerkin method:

$$\|u - u_\ell\|_{H^{2s-p-1}} \le C\, h^{2(p+1)-2s} \|u\|_{H^{p+1}(\Gamma)}.$$

4.2.5.2 Perturbations

The efficient numerical realization of the Galerkin BEM (4.165) involves, for example, perturbations of the sesquilinear form $b(\cdot,\cdot)$ by quadrature, surface and cluster approximation of the operator or the functional $\mathfrak{G}(\cdot)$, used for the evaluation of the representation formula at a point $\mathbf{x} \in \Omega$. Instead of (4.165) one implements a perturbed boundary element method:

Find $\tilde{u}_\ell \in S_\ell$ such that

$$b_\ell(\tilde{u}_\ell, v) = F_\ell(v) \quad \forall v \in S_\ell \tag{4.180}$$

and instead of $\mathfrak{G}(u_\ell)$ one implements an approximation $\mathfrak{G}_\ell(\tilde{u}_\ell)$. Here we will study the error

$$\mathfrak{G}(u) - \mathfrak{G}_\ell(\tilde{u}_\ell) \tag{4.181}$$

of a linear functional of the solution, for example of the representation formula (see Example 4.2.13). According to Theorem 4.2.11, (4.180) has a unique solution for a sufficiently large ℓ if the exact form $b(\cdot,\cdot)$ satisfies the discrete inf–sup conditions

$$\begin{aligned} \inf_{u_\ell \in S_\ell\setminus\{0\}} \sup_{v_\ell \in S_\ell\setminus\{0\}} \frac{|b(u_\ell, v_\ell)|}{\|u_\ell\|_H \,\|v_\ell\|_H} &\ge \gamma > 0,\\ \inf_{v_\ell \in S_\ell\setminus\{0\}} \sup_{u_\ell \in S_\ell\setminus\{0\}} \frac{|b(u_\ell, v_\ell)|}{\|u_\ell\|_H \,\|v_\ell\|_H} &\ge \gamma > 0 \end{aligned} \tag{4.182}$$

on $S_\ell \times S_\ell$ and if the perturbed form $b_\ell(\cdot,\cdot)$ is uniformly continuous [see (4.151)] and at the same time satisfies the *stability and consistency conditions* (4.152), (4.153). Then for a sufficiently large ℓ we have the error estimate

$$\|u - \tilde{u}_\ell\|_H \le C \left\{ \min_{w_\ell \in S_\ell} (\|u - w_\ell\|_H + \delta_\ell \|w_\ell\|_U) + \sup_{v_\ell \in S_\ell\setminus\{0\}} \frac{|F(v_\ell) - F_\ell(v_\ell)|}{\|v_\ell\|_H} \right\}. \tag{4.183}$$

The perturbations of the right-hand side F and of the functional $\mathfrak{G}$ define the quantities

$$f_\ell := \sup_{v_\ell \in S_\ell \setminus \{0\}} \frac{|F_\ell(v_\ell) - F(v_\ell)|}{\|v_\ell\|_H} \quad \text{and} \quad g_\ell := \sup_{v_\ell \in S_\ell \setminus \{0\}} \frac{|\mathfrak{G}_\ell(v_\ell) - \mathfrak{G}(v_\ell)|}{\|v_\ell\|_H}. \tag{4.184}$$

Note that in many practical applications the perturbations F_ℓ and $\mathfrak{G}_\ell$ are not defined on H but only on S_ℓ. We assume that $(f_\ell)_\ell$ and $(g_\ell)_\ell$ are null sequences and, thus, that there exist constants C_F and C_G such that

$$\|\mathfrak{G}\|_{H'} =: C_G < \infty \quad \text{and} \quad \|F\|_{H'} + f_\ell \le C_F \qquad \forall \ell \in \mathbb{N}.$$

Theorem 4.2.18. *Let the form $b(\cdot,\cdot)$ satisfy (4.182) and let the perturbed form $b_\ell(\cdot,\cdot)$ satisfy the conditions (4.151)–(4.153). Then, for a sufficiently large ℓ, the error (4.181) has the estimate*

$$\begin{aligned}|\mathfrak{G}(u) - \mathfrak{G}_\ell(\tilde{u}_\ell)| \le{}& C\|u - u_\ell\|_H \min_{\psi_\ell \in S_\ell} \|w_{\mathfrak{G}} - \psi_\ell\|_H + \frac{C_G}{\gamma} c_\ell \|\tilde{u}_\ell - u\|_H + \frac{C_G}{\gamma} f_\ell \\ &+ \frac{C_G}{\gamma} \min_{\varphi_\ell \in S_\ell} \left(c_\ell \|u - \varphi_\ell\|_H + \delta_\ell \|\varphi_\ell\|_U\right) + \frac{C_F}{\gamma} g_\ell. \end{aligned} \tag{4.185}$$

Proof. By the definition (4.170) of $w_{\mathfrak{G}}$ and the orthogonality of the Galerkin error we have

$$\begin{aligned}|\mathfrak{G}(u) - \mathfrak{G}(\tilde{u}_\ell)| &= |b(u - \tilde{u}_\ell, w_{\mathfrak{G}})| \\ &= |b(u - u_\ell, w_{\mathfrak{G}} - \psi_\ell)| + |b(u_\ell - \tilde{u}_\ell, w_{\mathfrak{G}})|\end{aligned} \tag{4.186}$$

for an arbitrary $\psi_\ell \in S_\ell$. Furthermore, let $w_\ell^{\mathfrak{G}} \in S_\ell$ be the solution of the Galerkin equations

$$b(w_\ell, w_\ell^{\mathfrak{G}}) = b(w_\ell, w_{\mathfrak{G}}) = \mathfrak{G}(w_\ell) \qquad \forall w_\ell \in S_\ell.$$

Then, taking the Galerkin orthogonality into consideration, we have

$$\begin{aligned}|b(u_\ell - \tilde{u}_\ell, w_{\mathfrak{G}})| &= |b(u_\ell - \tilde{u}_\ell, w_\ell^{\mathfrak{G}})| = |b(u_\ell, w_\ell^{\mathfrak{G}}) - b(\tilde{u}_\ell, w_\ell^{\mathfrak{G}})| \\ &\le |F(w_\ell^{\mathfrak{G}}) - b_\ell(\tilde{u}_\ell, w_\ell^{\mathfrak{G}})| + |(b_\ell - b)(\tilde{u}_\ell, w_\ell^{\mathfrak{G}})| \\ &= |F(w_\ell^{\mathfrak{G}}) - F_\ell(w_\ell^{\mathfrak{G}})| + |(b - b_\ell)(\tilde{u}_\ell, w_\ell^{\mathfrak{G}})|.\end{aligned}$$

We consider the difference $b - b_\ell$ and with the stability and consistency conditions we obtain for an arbitrary $\varphi_\ell \in S_\ell$ the estimate

$$\begin{aligned}\left|(b - b_\ell)(\tilde{u}_\ell, w_\ell^{\mathfrak{G}})\right| &\le \left|(b - b_\ell)\left(\tilde{u}_\ell - \varphi_\ell, w_\ell^{\mathfrak{G}}\right)\right| + \left|b\left(\varphi_\ell, w_\ell^{\mathfrak{G}}\right) - b_\ell\left(\varphi_\ell, w_\ell^{\mathfrak{G}}\right)\right| \\ &\le c_\ell \|\tilde{u}_\ell - \varphi_\ell\|_H \left\|w_\ell^{\mathfrak{G}}\right\|_H + \delta_\ell \|\varphi_\ell\|_U \left\|w_\ell^{\mathfrak{G}}\right\|_H. \end{aligned} \tag{4.187}$$

With this result and with (4.186) we obtain

$$\begin{aligned}|\mathfrak{G}(u)-\mathfrak{G}_\ell(\tilde{u}_\ell)| &\le |\mathfrak{G}(u)-\mathfrak{G}(\tilde{u}_\ell)| + |\mathfrak{G}(\tilde{u}_\ell)-\mathfrak{G}_\ell(\tilde{u}_\ell)| \\ &\le |b(u-u_\ell, w_{\mathfrak{G}}-\psi_\ell)| + |(F-F_\ell)(w_\ell^{\mathfrak{G}})| \\ &\quad + |(b-b_\ell)(\tilde{u}_\ell, w_\ell^{\mathfrak{G}})| + |(\mathfrak{G}-\mathfrak{G}_\ell)(\tilde{u}_\ell)| \\ &\le C\,\|u-u_\ell\|_H\,\|w_{\mathfrak{G}}-\psi_\ell\|_H + f_\ell\left\|w_\ell^{\mathfrak{G}}\right\|_H \\ &\quad + \left\|w_\ell^{\mathfrak{G}}\right\|_H \left(c_\ell\,\|\tilde{u}_\ell-\varphi_\ell\|_H + \delta_\ell\,\|\varphi_\ell\|_U\right) + g_\ell\,\|\tilde{u}_\ell\|_H.\end{aligned} \tag{4.188}$$

According to Theorem 4.2.11, for a sufficiently large ℓ the sequence $(\tilde{u}_\ell)_\ell$ of the perturbed Galerkin solutions is stable and with ℓ_0 from Theorem 4.2.11 we have

$$\|\tilde{u}_\ell\|_H \le \frac{1}{\gamma}\left(\|F\|_{H'} + f_\ell\right) \le \frac{C_F}{\gamma} \qquad \forall \ell \ge \ell_0. \tag{4.189}$$

We use the discrete inf–sup conditions (4.182) to find a bound for the term $\left\|w_\ell^{\mathfrak{G}}\right\|_H$:

$$\gamma\|w_\ell^{\mathfrak{G}}\|_H \le \sup_{w_\ell\in S_\ell\backslash\{0\}} \frac{|b(w_\ell, w_\ell^{\mathfrak{G}})|}{\|w_\ell\|_H} = \sup_{w_\ell\in S_\ell\backslash\{0\}} \frac{|\mathfrak{G}(w_\ell)|}{\|w_\ell\|_H} \le \|\mathfrak{G}\|_{H'} = C_G$$

for $\ell \ge \ell_0$. This yields

$$\begin{aligned}|\mathfrak{G}(u)-\mathfrak{G}_\ell(\tilde{u}_\ell)| \le C\,\|u-u_\ell\|_H\,\|w_{\mathfrak{G}}-\psi_\ell\|_H + \frac{C_G}{\gamma} f_\ell \\ + \frac{C_G}{\gamma}\left(c_\ell\,\|\tilde{u}_\ell-\varphi_\ell\|_H + \delta_\ell\,\|\varphi_\ell\|_U\right) + \frac{C_F}{\gamma} g_\ell.\end{aligned}$$

The triangle inequality $\|\tilde{u}_\ell-\varphi_\ell\|_H \le \|\tilde{u}_\ell-u\|_H + \|u-\varphi_\ell\|_H$ finally yields the assertion. ☐

The inequality (4.185) can be used to bound the size of the perturbations c_ℓ, δ_ℓ, f_ℓ and g_ℓ in such a way that the functional $\mathfrak{G}_\ell\,(\tilde{u}_\ell)$ converges with the same rate as the functional $\mathfrak{G}\,(u_\ell)$ for the original Galerkin method.

To illustrate this we consider $H = H^s(\Gamma)$ and a discretization with piecewise polynomials of order p. Then the optimal convergence rate of the unperturbed Galerkin method is given by $\|u-u_\ell\|_H \le Ch_\ell^{p+1-s}$.

Inequality (4.183) shows that the two conditions $\delta_\ell \le Ch_\ell^{p+1-s}$ and $f_\ell \le Ch_\ell^{p+1-s}$ imposed on the size of the perturbations guarantee that $\|u-\tilde{u}_\ell\|_H \le Ch_\ell^{p+1-s}$ converges with the same rate as the unperturbed Galerkin method. The optimal convergence rate for the dual problem is also $\left\|w_{\mathfrak{G}}-w_\ell^{\mathfrak{G}}\right\|_H \le Ch_\ell^{p+1-s}$

and it is our aim to control the size of the perturbation in such a way that the functional $\mathfrak{G}_\ell(\tilde{u}_\ell)$ converges at the rate $Ch_\ell^{2p+2-2s}$.

For this to hold, the perturbed sesquilinear forms, the right-hand sides and functionals in (4.152), (4.153) and (4.184) all have to satisfy the estimates

$$c_\ell \leq Ch_\ell^{p+1-s}, \quad \delta_\ell \leq Ch_\ell^{2p+2-2s}, \quad f_\ell \leq Ch_\ell^{2p+2-2s}, \quad g_\ell \leq Ch_\ell^{2p+2-2s}.$$

In the following theorem we will determine a bound for the effect of perturbations $b - b_\ell$ and $F - F_\ell$ on negative norms of the Galerkin error.

Theorem 4.2.19. *Let the assumptions from Theorem 4.2.18 hold for $H = H^s(\Gamma)$, $b : H^s(\Gamma) \times H^s(\Gamma) \to \mathbb{C}$. Furthermore, let the adjoint problem (4.176) satisfy the regularity assumption (4.178) for a $\bar{t} > 0$. Then for a sufficiently large ℓ we have the error estimate*

$$\begin{aligned} \|u - \tilde{u}_\ell\|_{H^{s-t}(\Gamma)} \leq C \Big\{ & d_{\ell,s,s+t} \|u - u_\ell\|_{H^s(\Gamma)} + c_\ell \|u - \tilde{u}_\ell\|_{H^s(\Gamma)} \\ & + f_\ell + \inf_{\varphi_\ell \in S_\ell} \left(c_\ell \|u - \varphi_\ell\|_{H^s(\Gamma)} + \delta_\ell \|\varphi_\ell\|_U \right) \Big\} \end{aligned} \tag{4.190}$$

for $0 \leq t \leq \bar{t}$ with

$$d_{\ell,s,s+t} := \sup_{w \in H^{s+t}(\Gamma) \setminus \{0\}} \left(\inf_{\psi_\ell \in S_\ell} \frac{\|w - \psi_\ell\|_{H^s(\Gamma)}}{\|w\|_{H^{s+t}(\Gamma)}} \right).$$

Proof. Let $v \in H^{-s+t}(\Gamma)$ be arbitrary and let w_v be the solution of the adjoint problem (4.176) with the right-hand side v. We then have

$$\begin{aligned} (v, u - \tilde{u}_\ell)_{L^2(\Gamma)} &= b(u - \tilde{u}_\ell, w_v) \\ &= b(u - u_\ell, w_v) + \underbrace{b(u_\ell - \tilde{u}_\ell, w_v)}_{(*)}. \end{aligned} \tag{4.191}$$

We consider $(*)$. Let $w_v^\ell \in S_\ell$ be the Galerkin approximation of w_v^ℓ:

$$b\left(v_\ell, w_v^\ell\right) = (w_v, v_\ell)_{L^2(\Gamma)} \qquad \forall v_\ell \in S_\ell.$$

With $v_\ell = u_\ell - \tilde{u}_\ell \in S_\ell$ it follows from the Galerkin orthogonality $b\left(v_\ell, w_v - w_v^\ell\right) = 0$ that we have the relation

$$\begin{aligned} (*) &= b(u_\ell - \tilde{u}_\ell, w_v) = b\left(u_\ell - \tilde{u}_\ell, w_v^\ell\right) \\ &= (b - b_\ell)\left(u_\ell - \tilde{u}_\ell, w_v^\ell\right) + b_\ell\left(u_\ell - \tilde{u}_\ell, w_v^\ell\right) \end{aligned}$$

$$= (b - b_\ell)\left(u_\ell - \tilde{u}_\ell, w_v^\ell\right) + b_\ell\left(u_\ell, w_v^\ell\right) - F_\ell(w_v^\ell)$$

$$= (b - b_\ell)\left(u_\ell - \tilde{u}_\ell, w_v^\ell\right) + (b_\ell - b)\left(u_\ell, w_v^\ell\right) + b\left(u_\ell, w_v^\ell\right) - F_\ell(w_v^\ell)$$

$$= (b - b_\ell)\left(-\tilde{u}_\ell, w_v^\ell\right) + F(w_v^\ell) - F_\ell(w_v^\ell).$$

With this we will estimate (4.191) by using (4.166) as follows. For every $\psi_\ell \in S_\ell$ we have

$$\begin{aligned} |\,(v, u - \tilde{u}_\ell)_{L^2(\Gamma)}\,| \le{} & |b\,(u - u_\ell, w_v - \psi_\ell)\,| \\ & + \left|(b - b_\ell)\left(\tilde{u}_\ell, w_v^\ell\right)\right| + \left|F(w_v^\ell) - F_\ell(w_v^\ell)\right|. \end{aligned} \tag{4.192}$$

As in (4.187), we use the consistency condition to prove for an arbitrary $\varphi_\ell \in S_\ell$ the estimate

$$\left|(b - b_\ell)(\tilde{u}_\ell, w_v^\ell)\right| \le \left(c_\ell \,\|\tilde{u}_\ell - \varphi_\ell\|_{H^s(\Gamma)} + \delta_\ell \,\|\varphi_\ell\|_U\right) \left\|w_v^\ell\right\|_{H^s(\Gamma)}, \tag{4.193}$$

where $\|\cdot\|_U$ again denotes a stronger norm than $H^s(\Gamma)$.

The regularity assumption (4.178) and the stability of the Galerkin approximations $\left(w_v^\ell\right)_\ell$ of the adjoint problem yield for all $0 \le t < \bar{t}$ and all $v \in H^{-s+t}(\Gamma)$ the estimate

$$\|w_v^\ell\|_{H^s(\Gamma)} \le C\,\|w_v\|_{H^s(\Gamma)} \le C\,\|v\|_{H^{-s}(\Gamma)} \le C\,\|v\|_{H^{-s+t}(\Gamma)}. \tag{4.194}$$

Therefore it follows from (4.192) and (4.193) with (4.184) that

$$\begin{aligned} \|u - \tilde{u}_\ell\|_{H^{s-t}(\Gamma)} ={} & \sup_{v \in H^{-s+t}(\Gamma)\setminus\{0\}} \frac{|\,(v, u - \tilde{u}_\ell)_{L^2(\Gamma)}\,|}{\|v\|_{H^{-s+t}(\Gamma)}} \\ & \le C\,\|u - u_\ell\|_{H^s(\Gamma)} \sup_{v \in H^{-s+t}(\Gamma)\setminus\{0\}} \left(\inf_{\psi_\ell \in S_\ell} \frac{\|w_v - \psi_\ell\|_{H^s(\Gamma)}}{\|v\|_{H^{-s+t}(\Gamma)}}\right) \\ & + C \inf_{\varphi_\ell \in S_\ell} \left(c_\ell\,\|\tilde{u}_\ell - \varphi_\ell\|_{H^s(\Gamma)} + \delta_\ell\,\|\varphi_\ell\|_U\right) + C f_\ell. \end{aligned}$$

The regularity assumption imposed upon the adjoint problem yields the estimate $\|v\|_{H^{-s+t}(\Gamma)} \ge C^{-1}\,\|w_v\|_{H^{s+t}(\Gamma)}$. Hence we have

$$\begin{aligned} & \sup_{v \in H^{-s+t}(\Gamma)\setminus\{0\}} \left(\inf_{\psi_\ell \in S_\ell} \frac{\|w_v - \psi_\ell\|_{H^s(\Gamma)}}{\|v\|_{H^{-s+t}(\Gamma)}}\right) \\ & \le C \sup_{w \in H^{s+t}(\Gamma)\setminus\{0\}} \left(\inf_{\psi_\ell \in S_\ell} \frac{\|w - \psi_\ell\|_{H^s(\Gamma)}}{\|w\|_{H^{s+t}(\Gamma)}}\right) = C d_{\ell,s,s+t}. \end{aligned}$$

Note that $d_{\ell,s,t}$ represents an approximation property of the space S_ℓ. Combining these results we have proved that

$$\|u-\tilde{u}_\ell\|_{H^{s-t}(\Gamma)} \le C\left\{d_{\ell,s,s+t}\,\|u-u_\ell\|_{H^s(\Gamma)} + c_\ell\,\|u-\tilde{u}_\ell\|_{H^s(\Gamma)} \right.$$
$$\left. + f_\ell + \inf_{\varphi_\ell\in S_\ell}\left(c_\ell\,\|u-\varphi_\ell\|_{H^s(\Gamma)} + \delta_\ell\,\|\varphi_\ell\|_U\right)\right\}.$$

□

With the help of inequality (4.190) we can determine sufficient conditions on the admissible magnitude of the perturbations c_ℓ, δ_ℓ, f_ℓ and g_ℓ so that the Galerkin error $\|u-\tilde{u}_\ell\|_{H^{s-t}(\Gamma)}$ converges with the same rate as the unperturbed Galerkin solution.

In order to illustrate this, we consider a discretization with piecewise polynomials of order p and assume that the continuous solution satisfies $u \in H^{p+1}(\Gamma)$. Then the optimal convergence rate of the unperturbed Galerkin method is given by $\|u-u_\ell\|_{H^{s-t}(\Gamma)} \le C h_\ell^{p+1-s+\min(p+1-s),t)}\|u\|_{H^{p+1}(\Gamma)}$.

Inequality (4.183) shows that the two conditions $\delta_\ell \le C h_\ell^{p+1-s}$ and $f_\ell \le C h_\ell^{p+1-s}$ imposed on the size of the perturbations guarantee that $\|u-\tilde{u}_\ell\|_{H^s(\Gamma)} \le C h_\ell^{p+1-s}$ converges with the same rate as the unperturbed Galerkin method (with respect to the H^s-norm). The optimal convergence rate of the term $d_{\ell,s,s+t}$ is $d_{\ell,s,s+t} \le C h_\ell^{\min\{p+1-s,t\}}$ and it is our goal to control the size of the perturbations in such a way that the term $\|u-\tilde{u}_\ell\|_{H^{s-t}(\Gamma)}$ converges at the rate $C h_\ell^{p+1-s+\min(p+1-s),t)}$. This leads to the following condition for the quantities $c_\ell, \delta_\ell, f_\ell$

$$C\left(h_\ell^{\min\{p+1-s,t\}+p+1-s} + c_\ell h^{p+1-s} + f_\ell + c_\ell h^{p+1-s} + \delta_\ell\right)$$
$$\le C h_\ell^{p+1-s+\min(p+1-s),t)}.$$

For this the perturbed sesquilinear form, right-hand sides and functionals in (4.152), (4.153) and (4.184) have to satisfy the estimates

$$c_\ell \le C h_\ell^{\min\{p+1-s,t\}},\quad \delta_\ell \le C h_\ell^{\min\{p+1-s,t\}+p+1-s},\quad f_\ell \le C h_\ell^{\min\{p+1-s,t\}+p+1-s}.$$

4.3 Proof of the Approximation Property

In Sects. 4.1–4.2.5 we have seen that the Galerkin boundary element method produces approximative solutions of boundary integral equations which converge quasi-optimally. Here we will present the proofs of the convergence rates (4.59) and (4.93) of discontinuous and continuous boundary elements on surface meshes $\mathcal{G}$ with mesh width $h > 0$.

In general we will assume that Assumption 4.1.6 holds, i.e., that the panel parametrizations can be decomposed into a regular, affine mapping $\chi_\tau^{\text{affine}}$ and a diffeomorphism χ_Γ, independent of τ, since $\chi_\tau = \chi_\Gamma \circ \chi_\tau^{\text{affine}}$. For $\chi_\tau^{\text{affine}}$ there exist $\mathbf{b}_\tau \in \mathbb{R}^3$ and $\mathbf{B}_\tau \in \mathbb{R}^{3\times 2}$ such that

$$\chi_\tau^{\text{affine}}(\hat{\mathbf{x}}) = \mathbf{B}_\tau \hat{\mathbf{x}} + \mathbf{b}_\tau.$$

The Gram matrix of this mapping is denoted by $\mathbf{G}_\tau := \mathbf{B}_\tau^\intercal \mathbf{B}_\tau \in \mathbb{R}^{2\times 2}$. It is symmetric and positive definite.

Note: The proof of the approximation property has the same structure as the proofs for the finite element methods (see, for example, [27,33,68,115]) and is also based on concepts such as the pullback to the reference element, the shape-regularity and the Bramble–Hilbert lemma.

4.3.1 Approximation Properties on Plane Panels

We use the same notation as in Sect. 4.1.2. Let $\hat{\Gamma}$ be a polyhedral surface with plane sides and let $\mathcal{G}^{\text{affine}}$ be a surface mesh of $\hat{\Gamma}$ which consists of plane triangles or parallelograms. The panels $\tau \in \mathcal{G}^{\text{affine}}$ are images of the reference element $\widehat{\tau}$ under a regular, affine transformation $\chi_\tau^{\text{affine}} : \hat{\tau} \to \tau$.

As in (4.23), for the reference element $\hat{\tau}$ and $p \geq 0$ we denote the space of all polynomials of total degree p by $\mathbb{P}_p^\Delta(\hat{\tau})$, while $\iota_p^{\hat{\tau}}$ denotes the index set for the associated unisolvent set of nodal points [see (4.70) and Theorem 4.1.39].

In preparation for Proposition 4.3.3 we will first prove a norm equivalence.

Lemma 4.3.1. *Let $k \in \mathbb{N}_{\geq 1}$. Then*

$$[u]_{k+1} := |u|_{k+1} + \sum_{(i,j)\in\iota_p^{\hat{\tau}}} \left| u\left(\frac{i}{p}, \frac{j}{p}\right)\right| \tag{4.195}$$

defines a norm on $H^{k+1}(\hat{\tau})$ which is equivalent to $\|\cdot\|_{k+1}$.

Proof. The continuity of the embedding $H^{k+1}(\hat{\tau}) \hookrightarrow C\left(\overline{\hat{\tau}}\right)$ follows from the Sobolev Embedding Theorem (see Theorem 2.5.4), and thus $[\cdot]_{k+1}$ is well defined. Therefore there exists a constant $c_1 \in \mathbb{R}_{>0}$ such that

$$[u]_{k+1} \leq c_1 \|u\|_{k+1} \qquad \forall u \in H^{k+1}(\hat{\tau}).$$

Therefore it remains to show that there exists a constant $c_2 \in \mathbb{R}_{>0}$ such that

$$\|u\|_{k+1} \leq c_2 [u]_{k+1} \qquad \forall u \in H^{k+1}(\hat{\tau}).$$

We prove this indirectly and for this purpose we assume that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset H^{k+1}(\hat{\tau})$ such that

$$\forall n \in \mathbb{N} : \|u_n\|_{k+1} = 1 \quad \text{and} \quad \lim_{n\to\infty} [u_n]_{k+1} = 0. \tag{4.196}$$

We deduce from Theorem 2.5.6 by induction over k that there exists a subsequence $\left(u_{n_j}\right)_{j\in\mathbb{N}}$ that converges to some $u \in H^k(\hat{\tau})$:

$$\lim_{j\to\infty} \left\|u_{n_j} - u\right\|_k = 0.$$

The second assumption in (4.196) yields

$$\lim_{j\to\infty} \left|u_{n_j} - u\right|_{k+1} = 0.$$

Hence $u \in H^{k+1}(\hat{\tau})$ with $|u|_{k+1} = 0$ and we have

$$\lim_{j\to\infty} \left\|u_{n_j} - u\right\|_{k+1} = 0.$$

Since $|u|_{k+1} = 0$, we have $u \in \mathbb{P}_k$ and the Sobolev Embedding Theorem implies the convergence in the nodal points

$$u(\mathbf{z}) = \lim_{j\to\infty} u_{n_j}(\mathbf{z}) \qquad \forall \mathbf{z} = \left(\frac{i}{p}, \frac{j}{p}\right), \quad (i,j) \in \iota_p^{\hat{\tau}}.$$

Theorem 4.1.39 therefore yields a contradiction to the first assumption in (4.196).

□

Lemma 4.3.2 (Bramble–Hilbert Lemma). *Let $k \in \mathbb{N}_0$. Then*

$$\inf_{p\in\mathbb{P}_k} \|u - p\|_{k+1} \le c_2 |u|_{k+1}$$

for all $u \in H^{k+1}(\hat{\tau})$, with c_2 from the proof of Lemma 4.3.1.

Proof. For $k = 0$ the statement follows from the Poincaré inequality (see Corollary 2.5.10).

In the following let $k \ge 1$ and $u \in H^{k+1}(\hat{\tau})$. Thanks to the Sobolev Embedding Theorem the point evaluation of u is well defined. Let $(b_{\mathbf{z}})_{\mathbf{z}\in\Sigma_k}$ be the vector that contains the values of u at the nodal points: $b_{\mathbf{z}} = u(\mathbf{z})$ for all $\mathbf{z} \in \Sigma_k$. Let $p \in \mathbb{P}_k^{\hat{\tau}}$ be the, according to Theorem 4.1.39, unique polynomial with $b_{\mathbf{z}} = p(\mathbf{z})$ for all $\mathbf{z} \in \Sigma_k$. Then, by Lemma 4.3.1,

$$\inf_{q\in\mathbb{P}_k} \|u - q\|_{k+1} \le \|u - p\|_{k+1} \le c_2 [u - p]_{k+1} = c_2 |u|_{k+1}.$$

□

Proposition 4.3.3. *Let* $\widehat{\Pi} : H^{p+1}(\hat{\tau}) \to H^s(\hat{\tau})$ *be linear and continuous for* $0 \leq s \leq p+1$ *such that*

$$\forall q \in \mathbb{P}_p^{\Delta}(\hat{\tau}) : \ \widehat{\Pi} q = q. \tag{4.197}$$

Then there exists a constant $c = c(\widehat{\Pi})$ *so that*

$$\forall v \in H^{p+1}(\hat{\tau}) : \ \|v - \widehat{\Pi} v\|_{H^s(\hat{\tau})} \leq \hat{c}\, |v|_{H^{p+1}(\hat{\tau})}. \tag{4.198}$$

Proof. Let $v \in H^{p+1}(\hat{\tau})$. Then by (4.197) for all $q \in \mathbb{P}_p^{\Delta}(\hat{\tau})$ we have

$$\begin{aligned} v - \widehat{\Pi} v &= v + q - \widehat{\Pi}(v+q) \\ \|v - \widehat{\Pi} v\|_{H^s(\hat{\tau})} &\leq \hat{c}\, \|v+q\|_{H^{p+1}(\hat{\tau})} \\ \hat{c} &:= \|I - \widehat{\Pi}\|_{H^s(\hat{\tau}) \leftarrow H^{p+1}(\hat{\tau})}, \end{aligned}$$

where I denotes the identity. Since $q \in \mathbb{P}_p^{\Delta}(\hat{\tau})$ was arbitrary, with Lemma 4.3.2 we deduce

$$\forall v \in H^{p+1}(\hat{\tau}) : \ \|v - \Pi v\|_{H^s(\widehat{\tau})} \leq \hat{c} \inf_{q \in \mathbb{P}_p^{\Delta}(\hat{\tau})} \|v+q\|_{H^{p+1}(\hat{\tau})} = \hat{c}\, |v|_{H^{p+1}(\widehat{\tau})}.$$

□

The estimate of the approximation error is proven by a transformation to the reference element.

First we will need some transformation formulas for Sobolev norms. Let $\tau \subset \mathbb{R}^2$ be a plane panel as before (triangle or parallelogram) with an affine parametrization $\chi_\tau^{\text{affine}} : \hat{\tau} \to \tau$. Tangential vectors on τ are defined by $\mathbf{b}_i := \partial \chi_\tau^{\text{affine}} / \partial \hat{x}_i$ for $i = 1, 2$. The (constant) normal vector $\mathbf{n}_\tau$ is oriented in such a way that $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{n}_\tau)$ forms a right system. For $\varepsilon > 0$ we set $I_\varepsilon = (-\varepsilon, \varepsilon)$ and define a neighborhood $U_\varepsilon \subset \mathbb{R}^3$ of τ by

$$U_\varepsilon := \left\{ \mathbf{z} \in \mathbb{R}^3 : \exists\, (\mathbf{x}, \alpha) \in \tau \times I_\varepsilon : \mathbf{z} = \mathbf{x} + \alpha \mathbf{n}_\tau \right\}. \tag{4.199}$$

A function $u \in H^{k+1}(\tau)$ can be extended as a constant on U_ε:

$$u^\star(\mathbf{x} + \alpha \mathbf{n}_\tau) = u(\mathbf{x}) \qquad \forall\, (\mathbf{x}, \alpha) \in \tau \times I_\varepsilon.$$

The surface gradient $\nabla_S u$ is defined by

$$\nabla_S u = \nabla u^\star \big|_\tau, \tag{4.200}$$

which gives us

$$|u|_{H^1(\tau)}^2 = \int_\tau \langle \nabla_S u, \nabla_S u \rangle.$$

The pullback of the function u to the reference element is denoted by $\hat{u} := u \circ \chi_\tau^{\text{affine}}$.

Lemma 4.3.4. *We have*

$$\|u\|_{L^2(\tau)}^2 = \frac{|\tau|}{|\hat{\tau}|} \int_{\hat{\tau}} |\hat{u}|^2$$

$$|u|_{H^1(\tau)}^2 = \frac{|\tau|}{|\hat{\tau}|} \int_{\hat{\tau}} \left\langle \widehat{\nabla} \hat{u}, \mathbf{G}_\tau^{-1} \widehat{\nabla} \hat{u} \right\rangle,$$

where $\widehat{\nabla}$ *denotes the two-dimensional gradient in the coordinates of the reference element.*

Proof. The transformation formula for surface integrals yields the first equation

$$\int_\tau |u|^2 = \frac{|\tau|}{|\hat{\tau}|} \int_{\hat{\tau}} |\hat{u}|^2 .$$

We define $\chi : \mathbb{R}^3 \to \mathbb{R}^3$ for $\hat{\mathbf{x}} \in \hat{\tau}$ and $x_3 \in \mathbb{R}$ by

$$\chi\left(\hat{\mathbf{x}}, \hat{x}_3\right) := \chi_\tau^{\text{affine}}\left(\hat{\mathbf{x}}\right) + \hat{x}_3 \mathbf{n}_\tau = \mathbf{B}_\tau \hat{\mathbf{x}} + \hat{x}_3 \mathbf{n}_\tau + \mathbf{b}_\tau$$

and we set $\widehat{U}_\varepsilon := \chi^{-1}\left(U_\varepsilon\right)$. With this we can define the function $\hat{u}^\star : \widehat{U}_\varepsilon \to \mathbb{K}$ by

$$\hat{u}^\star := u^\star \circ \chi$$

and it satisfies $\hat{u}^\star|_{\hat{\tau}} = \hat{u}$ in the sense of traces. The chain rule then yields

$$\left(\nabla_S u\right) \circ \chi_\tau^{\text{affine}} = \left(\nabla u^\star\right) \circ \chi_\tau^{\text{affine}} = \left(\mathbf{J}_\tau^{-1}\right)^\mathsf{T} \nabla \hat{u}^\star. \tag{4.201}$$

with the Jacobian $\mathbf{J}_\tau = [\mathbf{B}_\tau, \mathbf{n}_\tau]$ of the transformation χ. From this we have

$$\left(\nabla_S u\right) \circ \chi_\tau^{\text{affine}} = \left(\mathbf{J}_\tau^{-1}\right)^\mathsf{T} \widehat{\nabla} \hat{u}^\star \Big|_{\hat{\tau}} .$$

Elementary properties of the vector product give us

$$\mathbf{J}_\tau^{-1} \left(\mathbf{J}_\tau^{-1}\right)^\mathsf{T} = \begin{bmatrix} \mathbf{G}_\tau^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

and from this it follows that

$$\left\| \left(\nabla_S u\right) \circ \chi_\tau^{\text{affine}} \right\|^2 = \left\langle \widehat{\nabla} \hat{u}, \mathbf{G}_\tau^{-1} \widehat{\nabla} \hat{u} \right\rangle \qquad \text{on } \hat{\tau}.$$

Combined with the transformation formula for surface integrals we obtain the assertion. □

Lemma 4.3.5. *We have*

$$\|\mathbf{G}_\tau\| \le 2h_\tau^2, \qquad \|\mathbf{G}_\tau^{-1}\| \le \frac{2}{\pi^2}\left(\frac{h_\tau}{\rho_\tau}\right)^4 h_\tau^{-2}. \tag{4.202}$$

Proof. The Jacobian $\mathbf{B}_\tau$ of the affine transformation $\chi_\tau^{\text{affine}}$ has the column vectors $\mathbf{b}_1, \mathbf{b}_2$. The maximal eigenvalue of the symmetric, positive definite matrix $\mathbf{G}_\tau$ can be bounded by the row sum norm

$$\|\mathbf{G}_\tau\| \le \max_{i=1,2}\left\{\|\mathbf{b}_i\|^2 + \langle \mathbf{b}_1, \mathbf{b}_2\rangle\right\} \le 2h_\tau^2,$$

since $\mathbf{b}_i$ are edge vectors of τ (see Definition 4.1.2). For the inverse matrix we have

$$\mathbf{G}_\tau^{-1} = \frac{1}{\det \mathbf{G}_\tau}\begin{bmatrix} \|\mathbf{b}_2\|^2 & -\langle \mathbf{b}_1, \mathbf{b}_2\rangle \\ -\langle \mathbf{b}_1, \mathbf{b}_2\rangle & \|\mathbf{b}_1\|^2 \end{bmatrix} = \left(\frac{|\hat\tau|}{|\tau|}\right)^2 \begin{bmatrix} \|\mathbf{b}_2\|^2 & -\langle \mathbf{b}_1, \mathbf{b}_2\rangle \\ -\langle \mathbf{b}_1, \mathbf{b}_2\rangle & \|\mathbf{b}_1\|^2 \end{bmatrix}.$$

From this we have for the largest eigenvalue

$$\|\mathbf{G}_\tau^{-1}\| \le \left(\frac{|\hat\tau|}{\pi\rho_\tau^2}\right)^2 2h_\tau^2 \le \frac{2}{\pi^2}\left(\frac{h_\tau}{\rho_\tau}\right)^4 h_\tau^{-2}.$$

$\square$

Lemma 4.3.4 can be generalized for derivatives of higher order.

Lemma 4.3.6. *Let $\tau \in \mathcal{G}^{\text{affine}}$ be the affine image of the reference element $\hat\tau$*

$$\tau = \chi_\tau^{\text{affine}}(\hat\tau) \quad \textit{with} \quad \chi_\tau^{\text{affine}}(\hat{\mathbf{x}}) = \mathbf{B}_\tau \hat{\mathbf{x}} + \mathbf{b}_\tau.$$

Then

$$v \in H^k(\tau) \iff \hat v := v \circ \chi_\tau^{\text{affine}} \in H^k(\hat\tau), \tag{4.203}$$

which gives us for all $0 \le \ell \le k$

$$|v|_{H^\ell(\tau)} \le C_1 h_\tau^{1-\ell} |\hat v|_{H^\ell(\hat\tau)}, \tag{4.204a}$$

$$|\hat v|_{H^\ell(\hat\tau)} \le C_2 h_\tau^{\ell-1} |v|_{H^\ell(\tau)} \tag{4.204b}$$

with constants C_1, C_2 that depend only on k and the constant $\kappa_{\mathcal{G}}$, which describes the shape-regularity (see Definition 4.1.12).

Proof. The equivalence (4.203) follows from the chain rule, as the transformation is affine and therefore all derivatives of $\chi_\tau^{\text{affine}}$ are bounded. We will only prove the first inequality, the second can be treated in the same way.

Since $C^\infty(\tau) \cap H^\ell(\tau)$ is dense in $H^\ell(\tau)$ (see Proposition 2.3.10), it suffices to prove the statement for smooth functions.

Let $v^\star$, $\hat{v}$, U_ε, $\widehat{U}_\varepsilon$, $\hat{v}^\star$, χ, $\mathbf{J}_\tau$ be as in the proof of Lemma 4.3.4. In the following α will always denote a three-dimensional multi-index $\alpha \in \mathbb{N}_0^3$ and $\hat{\partial}$ denotes the derivative in the coordinates of the reference element. Then we have

$$|v|^2_{H^\ell(\tau)} = \sum_{|\alpha|=\ell} \int_\tau \left|\partial^\alpha v^\star\right|^2 = \frac{|\tau|}{|\widehat{\tau}|} \sum_{|\alpha|=\ell} \int_{\hat{\tau}} \left|\left(\partial^\alpha v^\star\right) \circ \chi\right|^2 .$$

The chain rule then yields

$$\left(\left(\partial^\alpha v^\star\right) \circ \chi\right) = \left(\left(\mathbf{J}_\tau^{-1}\right)^\intercal \widehat{\nabla}^\star\right)^\alpha \hat{v}^\star,$$

where $\widehat{\nabla}^\star$ denotes the three-dimensional gradient (while, in the following, the two-dimensional gradient will be denoted by $\widehat{\nabla}$, as before). For the (transposed) inverse of the Jacobian of χ we have $\left(\mathbf{J}_\tau^{-1}\right)^\intercal = [\mathbf{A}_\tau, \mathbf{n}_\tau]$ with

$$\mathbf{A}_\tau := [\mathbf{a}_1, \mathbf{a}_2] \in \mathbb{R}^{3\times 2}, \quad \mathbf{a}_1 := \frac{|\hat{\tau}|}{|\tau|} \left(\mathbf{b}_2 \times \mathbf{n}_\tau\right), \quad \mathbf{a}_2 := \frac{|\hat{\tau}|}{|\tau|} \left(\mathbf{n}_\tau \times \mathbf{b}_1\right).$$

Since $\hat{\partial}_3 \hat{v}^\star = 0$ we obtain

$$\left(\left(\partial^\alpha v^\star\right) \circ \chi\right)\big|_{\hat{\tau}} = \left(\mathbf{A}_\tau \widehat{\nabla}\right)^\alpha \hat{v}.$$

We use the convention

$$\sum_{\mu\le\alpha} \ldots := \sum_{\mu_1=0}^{\alpha_1} \sum_{\mu_2=0}^{\alpha_2} \sum_{\mu_3=0}^{\alpha_3} \ldots, \quad \binom{\alpha}{\mu} = \binom{\alpha_1}{\mu_1}\binom{\alpha_2}{\mu_2}\binom{\alpha_3}{\mu_3} \quad \text{and} \quad \mathbf{a}^\mu = \prod_{i=1}^{3} \mathbf{a}_i^{\mu_i}.$$

for the multi-indices $\mu, \alpha \in \mathbb{N}_0^3$. With this we have

$$\left(\mathbf{A}\widehat{\nabla}\right)^\alpha \hat{v} = \sum_{\mu\le\alpha} \binom{\alpha}{\mu} \mathbf{a}_1^\mu \mathbf{a}_2^{\alpha-\mu} \hat{\partial}_1^\mu \hat{\partial}_2^{\alpha-\mu} \hat{v}.$$

In order to estimate the absolute value, we use

$$\left|\mathbf{a}_{i,j}\right| \le \|\mathbf{a}_i\| \le \frac{h_\tau}{\pi \rho_\tau^2} \le \frac{\kappa_{\mathcal{G}}^2}{\pi} h_\tau^{-1}$$

and obtain with $|\alpha| = \ell$

$$\left|\left(\mathbf{A}\widehat{\nabla}\right)^\alpha \hat{v}\left(\hat{\mathbf{x}}\right)\right|^2 \le C h_\tau^{-2\ell} \sum_{\mu\le\alpha} \left|\hat{\partial}_1^\mu \hat{\partial}_2^{\alpha-\mu} \hat{v}\left(\hat{\mathbf{x}}\right)\right|^2 \tag{4.205}$$

with a constant C, which depends only on ℓ and the constant $\kappa_{\mathcal{G}}$. By integrating over $\hat{\tau}$ we obtain

$$\left\| \partial^{\alpha} v^{\star} \right\|_{L^2(\tau)}^2 = \frac{|\tau|}{|\hat{\tau}|} \left\| \left(\mathbf{A} \widehat{\nabla} \right)^{\alpha} \hat{v} \right\|_{L^2(\hat{\tau})}^2 \leq C h_{\tau}^{2-2\ell} \left| \hat{v} \right|_{H^{\ell}(\hat{\tau})}^2 .$$

If we sum over all α with $|\alpha| = \ell$ we obtain the assertion. $\square$

The following corollary is a consequence of (4.205).

Corollary 4.3.7. *Let $\tau \in \mathcal{G}^{\text{affine}}$ be the affine image of the reference element $\hat{\tau}$*

$$\tau = \chi_{\tau}^{\text{affine}} (\hat{\tau}) \quad \textit{with} \quad \chi_{\tau}^{\text{affine}} (\hat{\mathbf{x}}) = \mathbf{B}_{\tau} \hat{\mathbf{x}} + \mathbf{b}_{\tau}.$$

Then

$$v \in C^k(\tau) \Longleftrightarrow \hat{v} := v \circ \chi_{\tau}^{\text{affine}} \in C^k(\hat{\tau}),$$

which gives us for all $0 \leq \ell \leq k$

$$|v|_{C^{\ell}(\tau)} \leq C_1 h_{\tau}^{-\ell} \left| \hat{v} \right|_{C^{\ell}(\hat{\tau})}, \tag{4.206a}$$
$$|\hat{v}|_{C^{\ell}(\hat{\tau})} \leq C_2 h_{\tau}^{\ell} \left| v \right|_{C^{\ell}(\tau)} \tag{4.206b}$$

with constants C_1, C_2 that depend only on k and the constant $\kappa_{\mathcal{G}}$, which describes the shape-regularity (see Definition 4.1.12).

Theorem 4.3.8. *Let $\tau \in \mathcal{G}^{\text{affine}}$ be the affine image of the reference element $\tau = \chi_{\tau}^{\text{affine}} (\hat{\tau})$. Let the interpolation operator $\widehat{\Pi} : H^s(\hat{\tau}) \to H^t(\hat{\tau})$ be continuous for $0 \leq t \leq s \leq k+1$ and let*

$$\forall q \in \mathbb{P}_k^{\hat{\tau}} : \quad \widehat{\Pi} q = q \tag{4.207}$$

hold. Then the operator $\Pi : H^s(\tau) \to H^t(\tau)$, which is defined by:

$$\Pi v := \left(\widehat{\Pi} \hat{v} \right) \circ \left(\chi_{\tau}^{\text{affine}} \right)^{-1} \quad \textit{with} \quad \hat{v} := v \circ \chi_{\tau}^{\text{affine}}, \tag{4.208}$$

satisfies the error estimate

$$\forall v \in H^{k+1}(\tau) : \ |v - \Pi v|_{H^t(\tau)} \leq C h_{\tau}^{s-t} |v|_{H^s(\tau)} \tag{4.209}$$

for $0 \leq t \leq s \leq k+1$. The constant C depends only on k and the shape-regularity of the surface mesh, more specifically, it depends on the constant $\kappa_{\mathcal{G}}$ in Definition 4.1.12.

Proof. According to Proposition 4.3.3, on the reference element $\hat{\tau}$ we have

$$\left\| \hat{v} - \widehat{\Pi} \hat{v} \right\|_{H^t(\hat{\tau})} \leq \hat{c} |\hat{v}|_{H^s(\hat{\tau})}.$$

We transport this estimate from $\hat{\tau}$ to $\tau = \chi_\tau^{\text{affine}}(\hat{\tau})$. With Lemma 4.3.6 we obtain the error estimate for $s = k + 1$

$$|v - \Pi v|_{H^t(\tau)} \leq C h_\tau^{1-t} \left| \hat{v} - \widehat{\Pi} \hat{v} \right|_{H^t(\hat{\tau})} \leq C h_\tau^{1-t} |\hat{v}|_{H^{k+1}(\hat{\tau})} \leq C h_\tau^{k+1-t} |v|_{H^{k+1}(\tau)}.$$

For $s < k + 1$, (4.209) follows from the continuity of $\widehat{\Pi} : H^s(\hat{\tau}) \to H^t(\hat{\tau})$ by means of interpolation (see proof of Theorem 4.1.33). □

Remark 4.3.9. *The interpolation operator $\widehat{I}^k$ from (4.73) satisfies the prerequisites of Proposition 4.3.3 with $p \leftarrow k \geq 1$ by virtue of the Sobolev Embedding Theorem.*

For $k = 0$, $\widehat{\Pi}$ can be defined as a mean value:

$$\left(\widehat{\Pi} v\right)(\mathbf{x}) = \frac{1}{|\hat{\tau}|} \int_{\hat{\tau}} v \qquad \forall \mathbf{x} \in \hat{\tau}.$$

*4.3.2 Approximation on Curved Panels**

In this section we will prove the approximation properties for curved panels that satisfy the following geometric assumptions (see Assumption 4.1.6 and Fig. 4.6).

For $\mathbf{x} \in \tau \in \mathcal{G}$, $\mathbf{n}_\tau(\mathbf{x}) \in \mathbb{S}_2$ denotes unit normal vector to τ at the point $\mathbf{x}$. The orientation is chosen as explained in Sect. 2.2.3 with respect to the chart χ_τ.

Assumption 4.3.10. *For every $\tau \in \mathcal{G}$ with the associated reference mapping $\chi_\tau : \hat{\tau} \to \tau$:*

- *There exists a regular, affine mapping $\chi_\tau^{\text{affine}} : \mathbb{R}^3 \to \mathbb{R}^3$ of the form*

$$\chi_\tau^{\text{affine}}(\hat{\mathbf{x}}, x_3) = \begin{bmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{x}_3 \end{pmatrix} + \begin{pmatrix} \mathbf{b}_\tau \\ 0 \end{pmatrix}$$

with $\mathbf{a} \in \mathbb{R}^{2\times 2}$, $(\hat{\mathbf{x}}, \hat{x}_3) \in \mathbb{R}^2 \times \mathbb{R}$, $\mathbf{b}_\tau \in \mathbb{R}^2$ and $\det \mathbf{a} > 0$.

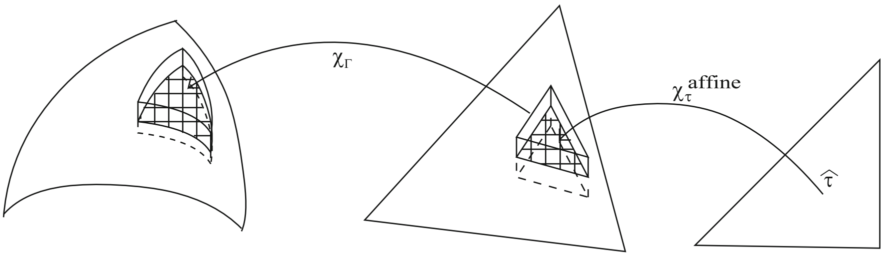


Fig. 4.6 *Left*: curved surface panel τ and three-dimensional neighborhood U_τ. *Middle*: flat surface panel τ^{affine} with neighborhood U_τ^{affine}. *Right*: reference element $\hat{\tau} \subset \mathbb{R}^2$

* This section should be read as a complement to the core material of this book.

- *There exists a C^∞-diffeomorphism $\chi : U \to V$ that is independent of $\mathcal{G}$, with open sets $U, V \subset \mathbb{R}^3$ that satisfy*

$$\begin{aligned} \tau_\varepsilon^{\text{affine}} \subset U,\ \tau_\varepsilon^{\text{affine}} &:= \left\{\chi_\tau^{\text{affine}}(\hat{\mathbf{x}}, 0) : \hat{\mathbf{x}} \in \widehat{\tau}\right\} \times (-\varepsilon, \varepsilon), \\ \tau_\varepsilon \subset V, \quad \tau_\varepsilon &:= \{\mathbf{x} + \alpha \mathbf{n}_\tau(\mathbf{x}) : \mathbf{x} \in \tau,\ \alpha \in (-\varepsilon, \varepsilon)\} \end{aligned}$$

  for an $\varepsilon > 0$ such that

$$\chi_\tau(\hat{\mathbf{x}}) = \chi \circ \chi_\tau^{\text{affine}}(\hat{\mathbf{x}}, 0).$$

- *For every function $u \in H^k(\tau)$ with a constant extension*

$$u^\star(\mathbf{x} + \alpha \mathbf{n}_\tau(\mathbf{x})) = u(\mathbf{x}) \tag{4.210}$$

  we have

$$\partial\left(u^\star \circ \chi \circ \chi_\tau^{\text{affine}}\right)/\partial \hat{x}_3 = 0. \tag{4.211}$$

  A situation of this kind was introduced in Example 4.1.7 (also see [170, Chap. 2]). First we will prove a transformation formula for composite functions.

Lemma 4.3.11. *Let $\eta : U \to V$ be a C^∞-diffeomorphism and let $U, V \subset \mathbb{R}^3$ be open sets. For a function $u \in H^k(V)$ we set $\tilde{u} = u \circ \eta$. Then $\tilde{u} \in H^k(U)$ and for all $\alpha \in \mathbb{N}_0^3$, $1 \leq |\alpha| \leq k$, we have*

$$(\partial^\alpha \tilde{u}) \circ \eta^{-1} = \sum_{|\beta|=1}^{|\alpha|} c_\beta \partial^\beta u \tag{4.212}$$

with coefficients c_β that are real linear combinations of products of the form

$$\prod_{r=1}^{|\beta|} \partial^{\mu_r} \eta_{n_r}. \tag{4.213}$$

The relevant indices for $1 \leq r \leq |\beta|$ satisfy the relations $1 \leq n_r \leq 3$, $\mu_r \in \mathbb{N}_0^3$ and $\sum_{r=1}^{|\beta|} |\mu_r| = |\alpha|$.

Proof. For the equivalence $u \in H^k(V) \iff \tilde{u} \in H^k(U)$ it suffices to prove (4.212) for smooth functions. We will prove Formula (4.212) by induction. Let $\mathbf{e}_k$ be the k-th canonical unit vector in $\mathbb{R}^3$.

Initial case: For $|\alpha| = 1$ we obtain explicitly

$$(\partial^\alpha \tilde{u}) \circ \eta^{-1} = \sum_{|\beta|=1} c_\beta \partial^\beta u, \quad \text{where for } \beta = \mathbf{e}_k \text{ we have } c_\beta = \partial^\alpha \eta_k.$$

Hypothesis: Let the statement hold for $|\alpha| \leq i - 1$.

Conclusion: Let $|\alpha| = i$, choose $k = 1, 2$, or 3, and let $\tilde{\alpha} = \alpha - \mathbf{e}_k \in \mathbb{N}_0^3$. Thus we obtain

$$(\partial^\alpha \tilde{u}) \circ \eta^{-1} = \partial_k \left(\partial^{\tilde{\alpha}} \tilde{u}\right) \circ \eta^{-1} = \left(\partial_k \sum_{|\beta|=1}^{i-1} c_\beta \left(\partial^\beta u\right) \circ \eta\right) \circ \eta^{-1}$$
$$= \sum_{|\beta|=1}^{i-1} \left(\partial_k c_\beta\right) \partial^\beta u + \sum_{|\beta|=1}^{i-1} \sum_{j=1}^{3} \left(c_\beta \partial_k \eta_j\right) \left(\partial_j \partial^\beta u\right).$$

This proves the assertion if we show that $\partial_k c_\beta$ and $c_\beta \left(\partial_k \eta_j\right)$ are of the form (4.213). With the Leibniz product rule we obtain

$$\partial_k \prod_{r=1}^{|\beta|} \partial^{\mu_r} \eta_{n_r} = \sum_{j=1}^{|\beta|} \left(\partial_k \partial^{\mu_j}\right) \eta_{n_j} \prod_{\substack{r=1 \\ r \neq j}}^{|\beta|} \partial^{\mu_r} \eta_{n_r}$$

and the expression on the right-hand side is a linear combination of terms of the form

$$\prod_{r=1}^{|\beta|} \partial^{\tilde{\mu}_r} \eta_{n_r}$$

with $\sum_{r=1}^{|\beta|} |\tilde{\mu}_r| = i$. The assertion follows analogously for the product $c_\beta \left(\partial_k \eta_j\right)$. $\square$

Corollary 4.3.12. *1. Let the conditions of Lemma 4.3.11 be satisfied. Then*

$$C_1^{-1} \|\tilde{u}\|_{L^2(U)} \leq \|u\|_{L^2(V)} \leq C_2 \|\tilde{u}\|_{L^2(U)}$$

and

$$|\tilde{u}|_{H^k(U)}^2 \leq C_1 \sum_{i=1}^{k} |u|_{H^i(V)}^2 \quad \textit{and} \quad |u|_{H^k(V)}^2 \leq C_2 \sum_{i=1}^{k} |\tilde{u}|_{H^i(U)}^2 .$$

The constants C_1, C_2 depend only on k and the derivatives of η, η^{-1} up to the order $\max\{1, k\}$.

2. Let Assumption 4.3.10 and the conditions of Lemma 4.3.11 be satisfied with $\eta \leftarrow \chi$. For $\tau \in \mathcal{G}$, $\tau^{\text{affine}} := \chi^{-1}(\tau)$ and $u \in H^k(\tau)$, $\tilde{u}(\hat{\mathbf{x}}) := u \circ \chi(\hat{\mathbf{x}}, 0)$ we have

$$C_3^{-1} \|\tilde{u}\|_{L^2(\tau^{\text{affine}})}^2 \leq \|u\|_{L^2(\tau)}^2 \leq C_4 \|\tilde{u}\|_{L^2(\tau^{\text{affine}})}^2$$

and

$$|\tilde{u}|^2_{H^k(\tau^{\text{affine}})} \le C_3 \sum_{i=1}^{k} |u|^2_{H^i(\tau)} \quad \text{and} \quad |u|^2_{H^k(\tau)} \le C_4 \sum_{i=1}^{k} |\tilde{u}|^2_{H^i(\tau^{\text{affine}})} .$$

The constants C_3, C_4 again depend only on k and the derivatives of χ, χ^{-1} up to the order k.

Proof. Statement 1 follows from the transformation formula (4.212).

For the second statement we define a constant extension of u in the direction of the normal as a function $u^\star$, according to (4.210), and note that the normal derivative of $u^\star$ vanishes, i.e., we have $|u^\star|_{H^k(\tau_\varepsilon)} = |u|_{H^k(\tau)}$.

From (4.211) we have $|u^\star \circ \chi|_{H^k(\tau_\varepsilon^{\text{affine}})} = |\tilde{u}|_{H^k(\tau^{\text{affine}})}$ and, thus, we have the assertion in Part 1. □

At the next step we will apply Lemma 4.3.11 to the composite reference mapping and study how far this depends on the panel diameter h_τ.

Lemma 4.3.13. *Let Assumption 4.3.10 and the conditions of Lemma 4.3.11 hold with $\eta \leftarrow \chi$. For $\tau \in \mathcal{G}$ and $u \in H^k(\tau)$, $\tau \subset V$, $\hat{u} := u \circ \chi_\tau$ we have*

$$v \in H^k(\tau) \Longleftrightarrow \hat{v} := v \circ \chi_\tau \in H^k(\hat{\tau}) \tag{4.214}$$

and

$$|u|^2_{H^k(\tau)} \le C_1 h_\tau^{2-2k} \sum_{i=1}^{k} |\hat{u}|^2_{H^i(\hat{\tau})} , \tag{4.215a}$$

$$|\hat{u}|^2_{H^k(\hat{\tau})} \le C_2 h_\tau^{2k-2} \sum_{i=1}^{k} |u|^2_{H^i(\tau)} . \tag{4.215b}$$

The constants C_1, C_2 depend only on k, the constant $\kappa_{\mathcal{G}}$ of the shape-regularity (see Definition 4.1.12) and the derivatives χ, χ^{-1} up to the order k.

Proof. It follows from Corollary 4.3.12 that

$$|u|^2_{H^k(\tau)} \le C \sum_{i=1}^{k} |\tilde{u}|^2_{H^i(\tau^{\text{affine}})} .$$

We can therefore apply the transformation formulas from Lemma 4.3.6, which gives us the estimates

$$|u|^2_{H^k(\tau)} \leq C_1 h_\tau^2 \sum_{i=1}^{k} h^{-2i} \, |\tilde{u}|^2_{H^i\left(\tau^{\text{affine}}\right)} \leq C_2 h_\tau^{2-2k} \sum_{i=1}^{k} |\hat{u}|^2_{H^i(\hat{\tau})},$$

$$|\hat{u}|^2_{H^k(\hat{\tau})} \leq C_3 h_\tau^{2k-2} \, |\tilde{u}|^2_{H^k\left(\tau^{\text{affine}}\right)} \leq C_4 h_\tau^{2k-2} \sum_{i=1}^{k} |u|^2_{H^i(\tau)}.$$

□

With this we obtain the analogy of Theorem 4.3.8 for curved panels.

Theorem 4.3.14. *Let Assumption 4.3.10 and the conditions from Lemma 4.3.11 hold with* $\eta \leftarrow \chi$. *Let* $\tau \in \mathcal{G}$ *be the image of the reference element* $\hat{\tau}$ *as given by* $\tau = \chi \circ \chi_\tau^{\text{affine}}(\hat{\tau})$. *Let the interpolation operator* $\widehat{\Pi} : H^s(\hat{\tau}) \to H^t(\hat{\tau})$ *satisfy the conditions from Theorem 4.3.8 for* $0 \leq t \leq s \leq k+1$.

Then we have for the operator $\Pi : H^s(\tau) \to H^t(\tau)$, *which is defined by*

$$\Pi v := \left(\widehat{\Pi}\hat{v}\right) \circ \chi_\tau^{-1} \quad \textit{with} \quad \hat{v} := v \circ \chi_\tau,$$

the error estimate for $0 \leq t \leq s \leq k+1$

$$\forall v \in H^{k+1}(\tau): \; |v - \Pi v|_{H^t(\tau)} \leq C h_\tau^{s-t} \, \|v\|_{H^s(\tau)}. \tag{4.216}$$

The constant C *depends only on* k, *the shape-regularity of the surface mesh via the constant* $\kappa_{\mathcal{G}}$ *in Definition 4.1.12 and the derivatives of* χ, χ^{-1} *up to the order* k.

Theorem 4.3.8 and Theorem 4.3.14 contain the central, local approximation properties that are combined in Sects. 4.3.4 and 4.3.5 to form error estimates for boundary elements. The easiest way of constructing a global approximation for continuous boundary elements and sufficiently smooth functions is by means of interpolation. For this the functions $u \in H^s_{\text{pw}}(\Gamma)$ need to be continuous. In the following section we will show that this is the case for $s > 1$.

4.3.3 Continuity of Functions in $H^s_{\text{pw}}(\Gamma)$ for $s > 1$

In order to avoid technical difficulties, we will generally assume in this section that we are dealing with the geometric situation from Example 4.1.7(1).

Assumption 4.3.15. Γ *is a piecewise smooth Lipschitz surface that can be parametrized bi-Lipschitz continuously over a polyhedral surface* $\hat{\Gamma}$: $\chi_\Gamma : \hat{\Gamma} \to \Gamma$.

Then the Sobolev spaces $H^s(\Gamma)$ on Γ are defined invariantly for $|s| \leq 1$, which means that they do not depend on the chosen parametrization of Γ (see Proposition 2.4.2). For a higher differentiation index $s > 1$, $H^s_{\text{pw}}(\Gamma)$ is defined as in (4.86). These spaces form a scale with

$$L^2(\Gamma) = H^0_{\mathrm{pw}}(\Gamma) \supset H^s_{\mathrm{pw}}(\Gamma) \supset H^t_{\mathrm{pw}}(\Gamma), \; 0 < s < t. \tag{4.217}$$

Lemma 4.3.16. *For $s > 1$ every $u \in H^s_{\mathrm{pw}}(\Gamma)$ is continuous on Γ, i.e., $H^s_{\mathrm{pw}}(\Gamma) \subset C^0(\Gamma)$.*

Proof. Γ is the bi-Lipschitz continuous image of a polyhedral surface: $\Gamma = \chi_\Gamma\left(\hat{\Gamma}\right)$ and therefore it suffices to prove the statement for polyhedral surfaces. Let $\hat{\Gamma}^j$, $1 \leq j \leq J$, be the plane, relatively closed polygonal faces of the polyhedron.

Let $u \in H^s_{\mathrm{pw}}\left(\hat{\Gamma}\right)$ for $s > 1$. The Sobolev Embedding Theorem implies that $u \in C^0\left(\overline{\hat{\Gamma}^j}\right)$ for all $1 \leq j \leq J$ and, thus, it suffices to prove the continuity across the common edges of the surface pieces $\hat{\Gamma}_j$. For this we consider two pieces $\hat{\Gamma}_i$ and $\hat{\Gamma}_j$ with a common edge $\widehat{E}$. Then there exists an (open) polygonal domain $U \subset \mathbb{R}^2$ and a bi-Lipschitz continuous mapping $\chi : \overline{U} \to \hat{\Gamma}_i \cup \hat{\Gamma}_j$ with the properties

$\overline{U}_1 := \chi^{-1}\left(\hat{\Gamma}_i\right), \quad \overline{U}_2 := \chi^{-1}\left(\hat{\Gamma}_j\right), \quad$ and $\quad \chi|_{U_k}$ is affine for $k = 1, 2$.

U_1, U_2 are disjoint and $\overline{U} = \overline{U_1 \cup U_2}$.

$e := \chi^{-1}\left(\widehat{E}\right) = \overline{U_1} \cap \overline{U_2}$.

We only need to show that $w := u \circ \chi$ is continuous over e. Clearly, we have $w_k := w \circ \chi_k \in H^s(U_k)$, $k = 1, 2$, and $w \in H^1(U)$. If we combine this result with the statements from Theorem 2.6.8 and Remark 2.6.10 we obtain the assertion. □

4.3.4 Approximation Properties of $S^{p,-1}_{\mathcal{G}}$

We will now prove the error estimate (4.59) for the following two geometric situations.

Assumption 4.3.17 (Polyhedral Surface). *Γ is the surface of a polyhedron. The mesh $\mathcal{G}$ on Γ consists of plane panels with straight edges with mesh width $h > 0$.*

Assumption 4.3.18 (Curved Surface). *Assumption 4.3.10 holds and the conditions from Lemma 4.3.11 are satisfied with $\eta \leftarrow \chi$.*

Theorem 4.3.19. *Let either Assumption 4.3.17 or Assumption 4.3.18 hold. Let $s \geq 0$. Then there exists an operator $I^{p,-1}_{\mathcal{G}} : H^s_{\mathrm{pw}}(\Gamma) \to S^{p,-1}_{\mathcal{G}}$ such that*

$$\left\| u - I^{p,-1}_{\mathcal{G}} u \right\|_{L^2(\Gamma)} \leq C\, h^{\min(p+1,s)} \|u\|_{H^s(\Gamma)}. \tag{4.218}$$

For a polyhedral surface the constant C depends only on p and the shape-regularity of the mesh $\mathcal{G}$ via the constant $\kappa_{\mathcal{G}}$ from Definition 4.1.12. In the case of a curved *surface it also depends on the derivatives of the global transformations χ, χ^{-1} up to the order k.*

Proof. Let $\widehat{\Pi}_{\hat{\tau}}^{p} : H^{s}(\hat{\tau}) \to \mathbb{P}_{p}^{\Delta}$ be the L^2-projection:

$$\left(\widehat{\Pi}_{\hat{\tau}}^{p} u, q\right)_{L^2(\hat{\tau})} = (u, q)_{L^2\left(\hat{\tau}\right)} \qquad \forall q \in \mathbb{P}_{p}^{\Delta}. \tag{4.219}$$

We lift this projection to the panels $\tau \in \mathcal{G}$ by means of

$$\left(\Pi_{\tau}^{p} u_{\tau}\right)(\mathbf{x}) := \left(\widehat{\Pi}_{\hat{\tau}}^{p} \hat{u}_{\tau}\right) \circ \chi_{\tau}^{-1}(\mathbf{x}) \qquad \forall \mathbf{x} \in \tau,$$

where $u_{\tau} := u|_{\tau}$ and $\hat{u}_{\tau} := u_{\tau} \circ \chi_{\tau}$. The operator $I_{\mathcal{G}}^{p,-1}$ then consists of the panelwise composition of Π_{τ}^{p}:

$$I_{\mathcal{G}}^{p,-1} u\Big|_{\tau} := \Pi_{\tau}^{p} u \qquad \forall \tau \in \mathcal{G}.$$

Obviously, this defines a mapping from $H_{\mathrm{pw}}^{s}(\Gamma)$ to $S_{\mathcal{G}}^{p,-1}$. The operator $\widehat{\Pi}_{\hat{\tau}}^{p}$ satisfies the prerequisites of Theorem 4.3.8, because we have for the orthogonal projection:

1. $$\left\|\widehat{\Pi}_{\hat{\tau}}^{p} \hat{v}\right\|_{0} \leq \|\hat{v}\|_{0} \qquad \forall \hat{v} \in L^2(\hat{\tau}).$$
   Since $\widehat{\Pi}_{\hat{\tau}}^{p} \hat{v}$ is a polynomial in a finite-dimensional space $\mathbb{P}_{p}^{\Delta}$, all norms are equivalent and there exists a constant $C_p > 0$ such that for all $0 \leq t \leq s \leq p+1$ we have
   $$\left\|\widehat{\Pi}_{\hat{\tau}}^{p} \hat{v}\right\|_{s} \leq C_p \left\|\widehat{\Pi}_{\hat{\tau}}^{p} \hat{v}\right\|_{0} \leq C_p \|\hat{v}\|_{0} \leq C_p \|\hat{v}\|_{t} \qquad \forall \hat{v} \in H^{s}(\hat{\tau}).$$
2. It follows immediately from the characterization (4.219) that
   $$\widehat{\Pi} q = q \qquad \forall q \in \mathbb{P}_{p}^{\Delta}.$$

Therefore we can apply (4.209) or (4.216) with $t = 0$ and obtain the error estimate

$$\left|v - I_{\mathcal{G}}^{p,-1} v\right|_{L^2(\tau)} \leq C h_{\tau}^{s} \|v\|_{H^{s}(\tau)} \tag{4.220}$$

for all $v \in H^{s}(\Gamma)$ with $0 \leq s \leq p+1$. If we then square and sum over all $\tau \in \mathcal{G}$ we obtain the assertion. $\square$

Theorem 4.3.19 gives us error estimates in negative norms by means of the same duality argument as in the proof of Theorem 4.1.33. This is the subject of the following theorem.

Theorem 4.3.20. *Let the assumption from Theorem 4.3.19 be satisfied. Then we have for the interpolation $I_{\mathcal{G}}^{p,-1}$ and $0 \leq t \leq s \leq p+1$ and all $u \in H_{\mathrm{pw}}^{s}(\Gamma)$ the estimate*

$$\|u - I_{\mathcal{G}}^{p,-1} u\|_{H^{-t}(\Gamma)} \leq C h^{s+t} \|u\|_{H^{s}(\Gamma)}. \tag{4.221}$$

Proof. The continuous extension of the L^2 inner-product to $H^{-t}_{\mathrm{pw}}(\Gamma) \times H^t_{\mathrm{pw}}(\Gamma)$ is again denoted by $(\cdot,\cdot)_0$. Since $I^{p,-1}_{\mathcal{G}}$ consists locally of L^2-orthogonal projections, we have for an arbitrary $\varphi_{\mathcal{G}} \in S^{p,-1}_{\mathcal{G}}$

$$\left\| u - I^{p,-1}_{\mathcal{G}} u \right\|_{H^{-t}(\Gamma)} = \sup_{\varphi \in H^t(\Gamma)\setminus\{0\}} \frac{\left| \left(u - I^{p,-1}_{\mathcal{G}} u, \varphi \right)_0 \right|}{\|\varphi\|_{H^t(\Gamma)}}$$
$$= \sup_{\varphi \in H^t(\Gamma)\setminus\{0\}} \frac{\left| \left(u - I^{p,-1}_{\mathcal{G}} u, \varphi - \varphi_{\mathcal{G}} \right)_0 \right|}{\|\varphi\|_{H^t(\Gamma)}} \tag{4.222}$$

(see proof of Theorem 4.1.33). If we choose $\varphi_{\mathcal{G}} = I^{p,-1}_{\mathcal{G}} \varphi \in S^{p,-1}_{\mathcal{G}}$, (4.221) follows by means of a twofold application of (4.218). □

Remark 4.3.21. *Corollary 4.1.34 follows from (4.221) with $t = \frac{1}{2}$.*

4.3.5 Approximation Properties of $S^{p,0}_{\mathcal{G}}$

Here we will prove approximation properties of continuous boundary elements that have already been introduced in Proposition 4.1.50.

Theorem 4.3.22. *Let Assumption 4.3.17 or Assumption 4.3.18 hold:*

(a) Then there exists an interpolation operator $I^{p,0}_{\mathcal{G}} : H^s_{\mathrm{pw}}(\Gamma) \to S^{p,0}_{\mathcal{G}}$ such that

$$\left\| u - I^{p,0}_{\mathcal{G}} u \right\|_{H^t(\Gamma)} \le C h^{s-t} \|u\|_{H^s_{\mathrm{pw}}(\Gamma)} \tag{4.223}$$

for $t = 0, 1$, $1 < s \le p + 1$ and all $u \in H^s_{\mathrm{pw}}(\Gamma)$. For a polyhedral surface the constant C depends only on p and on the shape-regularity of the mesh $\mathcal{G}$ via the constant $\kappa_{\mathcal{G}}$ from Definition 4.1.12. In the case of a curved surface it also depends on the derivatives of the global transformations χ, χ^{-1} up to the order k.

(b) Let $u \in H^s(\Gamma)$ for some $1 < s \le p + 1$. Then, for any $0 \le t \le 1$, we have

$$\left\| u - I^{p,0}_{\mathcal{G}} u \right\|_{H^t(\Gamma)} \le C h^{s-t} \|u\|_{H^s(\Gamma)} .$$

Proof. Part a: Lemma 4.3.16 implies that $u \in H^s_{\mathrm{pw}}(\Gamma) \subset C^0(\Gamma)$ for $s > 1$. We define $I^{p,0}_{\mathcal{G}} u$ on $\tau \in \mathcal{G}$ by

$$\left(I^{p,0}_{\mathcal{G}} u_\tau \right)(\mathbf{x}) := \left(\widehat{I}^p \hat{u}_\tau \right) \circ \chi_\tau^{-1}(\mathbf{x}) \qquad \forall \mathbf{x} \in \tau \tag{4.224}$$

with $u_\tau := u|_\tau$, $\hat{u}_\tau := u_\tau \circ \chi_\tau$ and the interpolation operator $\widehat{I}^p$ from (4.73) for the set of nodal points Σ_p from Theorem 4.1.39. By Theorem 4.1.39 this operator is well defined and satisfies

$$\begin{aligned} \left(\widehat{I}^p \hat{u}_\tau\right)(\mathbf{z}) &= \hat{u}_\tau(\mathbf{z}) \; \forall \mathbf{z} \in \Sigma_p, \\ \widehat{I}^p q = q & \qquad \forall q \in \mathbb{P}_p^{\hat{\tau}}. \end{aligned}$$

By Lemma 4.3.1 we have on the reference element

$$\begin{aligned} \left\|\widehat{I}^p \hat{u}_\tau\right\|_{H^t(\hat{\tau})} &\leq \left\|\widehat{I}^p \hat{u}_\tau\right\|_{H^{p+1}(\hat{\tau})} \leq c_2 \left(\left|\widehat{I}^p \hat{u}_\tau\right|_{H^{p+1}(\hat{\tau})} + \sum_{\mathbf{z}\in\Sigma^p} \left|\left(\widehat{I}^p \hat{u}_\tau\right)(\mathbf{z})\right|\right) \\ &= c_2 \sum_{\mathbf{z}\in\Sigma^p} |\hat{u}(\mathbf{z})| \\ &\leq c_2 \|\hat{u}\|_{C^0(\overline{\hat{\tau}})} \leq C c_2 \|\hat{u}\|_{H^s(\hat{\tau})}. \end{aligned}$$

Therefore Theorem 4.3.8 or Theorem 4.3.14 is applicable and for $1 < s \leq p+1$ and $t \in \{0, 1\}$ we obtain the estimate

$$\forall u \in H^s(\tau): \left|u_\tau - I_{\mathcal{G}}^{p,0} u_\tau\right|_{H^t(\tau)} \leq C h_\tau^{s-t} \|u_\tau\|_{H^s(\tau)}. \tag{4.225}$$

If we square (4.225) and sum over all $\tau \in \mathcal{G}^{\text{affine}}$ we obtain (4.223).

Part b: By using Lemma 4.1.49 we derive from Part a the estimate

$$\left\|u - I_{\mathcal{G}}^{p,0} u\right\|_{H^t(\Gamma)} \leq C h^{s-t} \|u\|_{H^s(\Gamma)} \tag{4.226}$$

for $t \in \{0, 1\}$. We apply Proposition 2.1.62 with $T = I - I_{\mathcal{G}}^{p,0}$, $Y_0 = Y_1 = H^s(\Gamma)$, $X_0 = L^2(\Gamma)$, $X_1 = H^1(\Gamma)$, and $\theta = t \in (0, 1)$ to interpolate the inequality (4.226). The result is

$$\begin{aligned} \|T\|_{H^t(\Gamma)\leftarrow H^s(\Gamma)} &\leq \|T\|_{L^2(\Gamma)\leftarrow H^s(\Gamma)}^{1-t} \|T\|_{H^1(\Gamma)\leftarrow H^s(\Gamma)}^{t} \\ &\leq \left(C h_{\mathcal{G}}^s\right)^{1-t} \left(C h_{\mathcal{G}}^{s-1}\right)^t = C h_{\mathcal{G}}^{s-t} \end{aligned}$$

and this implies the assertion of Part b. $\square$

Next we investigate the approximation property for functions in $H_{\text{pw}}^s(\Gamma)$ for $0 \leq s \leq 1$. Recall that $H_{\text{pw}}^s(\Gamma) = H^s(\Gamma)$ in this case. In general, functions in $H^s(\Gamma)$ are not continuous and the application of the pointwise interpolation $I_{\mathcal{G}}^{p,0}$ is not defined. We will introduce the *Clément interpolation operator* $Q_{\mathcal{G}} : L^1(\Gamma) \to S_{\mathcal{G}}^{1,0}$ for the approximation of functions in $H^s(\Gamma)$ if $0 \leq s \leq 1$ (cf. [69]). To avoid technicalities, we consider only the case that all panels are (possibly curved) surface triangles. Let $\mathcal{I}$ denote the set of panel vertices with corresponding continuous, piecewise linear nodal basis $(b_{\mathbf{z}})_{\mathbf{z}\in\mathcal{I}}$. For $\mathbf{z} \in \mathcal{I}$ and $\tau \in \mathcal{G}$, we introduce local meshes $\mathcal{G}_{\mathbf{z}}$ and $\mathcal{G}_{\tau}$ by

$$\mathcal{G}_{\mathbf{z}} := \{\tau \in \mathcal{G} \mid \tau \subset \operatorname{supp} b_{\mathbf{z}}\}, \quad \mathcal{G}_{\tau} := \left\{t \in \mathcal{G} \mid \bar{t} \cap \bar{\tau} \neq \emptyset\right\}.$$

The corresponding surface patches on Γ are denoted by

$$\omega_{\mathbf{z}} := \bigcup_{\tau\in\mathcal{G}_{\mathbf{z}}} \bar{\tau}, \quad \omega_{\tau} := \bigcup_{t\in\mathcal{G}_{\tau}} \bar{t}.$$

For functions $f \in L^1(\Gamma)$ and $\mathbf{z} \in \mathcal{I}$, the functional $\pi_{\mathbf{z}} : L^1(\Gamma) \to \mathbb{C}$ is defined by

$$\pi_{\mathbf{z}}(f) := \frac{1}{|\omega_{\mathbf{z}}|} \int_{\omega_{\mathbf{z}}} f(\mathbf{y})\, ds_{\mathbf{y}}.$$

Remark 4.3.23. *For $\mathbf{z} \in \mathcal{I}$, we set $h_{\mathbf{z}} := \max_{\tau\in\mathcal{G}_{\mathbf{z}}} h_{\tau}$. There exists a constant C_0 which depends only on the shape-regularity constant $\kappa_{\mathcal{G}}$ such that*

$$h_{\mathbf{z}} \leq C_0 h_{\tau} \qquad \forall \tau \in \mathcal{G}_{\mathbf{z}}.$$

Definition 4.3.24 (Clément interpolation). The Clément interpolation operator $Q_{\mathcal{G}} : L^1(\Gamma) \to S_{\mathcal{G}}^{1,0}$ is given by

$$Q_{\mathcal{G}} f := \sum_{\mathbf{z}\in\mathcal{I}} \pi_{\mathbf{z}}(f)\, b_{\mathbf{z}}.$$

The proof of the stability and the approximation property of the Clément interpolation employs local pullbacks to two-dimensional polygonal parameter domains and then follows the classical convergence proof in the two-dimensional parameter plane. The next assumption is illustrated in Fig. 4.7.

Assumption 4.3.25. *(a) For any $\mathbf{z} \in \mathcal{I}$, there is a two-dimensional convex and polygonal parameter domain $\widetilde{\omega_{\mathbf{z}}} \subset \mathbb{R}^2$ along with a bi-Lipschitz continuous mapping $\chi_{\mathbf{z}} : \widetilde{\omega_{\mathbf{z}}} \to \omega_{\mathbf{z}}$ which satisfies: For any $\tau \in \mathcal{G}_{\mathbf{z}}$, the pullback $\tilde{\tau} :=$*

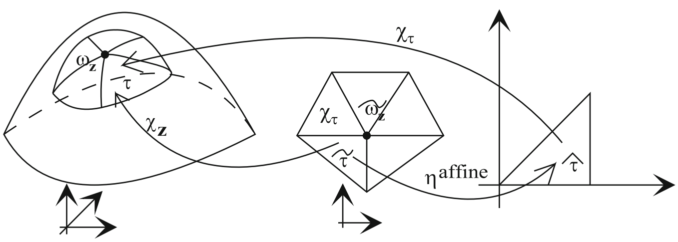


Fig. 4.7 Pullback of a surface patch to a two-dimensional parameter domain

$\chi_{\mathbf{z}}^{-1}(\tau)$ is a plane panel with straight edges. The pullback $\tilde{\tau}$ can be transformed to the reference element $\hat{\tau}$ by a regular, affine mapping which is denoted by η^{affine}.

(b) The reference mapping (see Definition 4.1.2) is denoted by $\chi_\tau : \hat{\tau} \to \tau$, where the reference element is always the unit triangle $\hat{\tau} = \widehat{S}_2$ because we only consider triangular panels. For curved panels, Assumption 4.3.18 holds so that

$$\chi_\tau = \chi \circ \chi_\tau^{\text{affine}}$$

where $\chi_\tau^{\text{affine}}$ is affine and $\chi : U \to V$ is independent of $\mathcal{G}$.

(c) For any $\tau \in \mathcal{G}$, the image $\chi_\tau^{\text{affine}}(\hat{\tau})$ is the plane triangle with straight edges which has the same vertices as τ, i.e., $\chi_\tau^{\text{affine}}$ is the componentwise affine interpolation of χ_τ.

Notation 4.3.26. *If τ, $\tilde{\tau}$, $\hat{\tau}$, χ_τ, $\chi_\tau^{\text{affine}}$, etc., appear in the same context their relationships are always as in Assumption 4.3.25.*

Let $g_{\mathbf{z}} \in L^\infty(\widetilde{\omega_{\mathbf{z}}})$ denote the surface element

$$g_{\mathbf{z}}(\mathbf{x}) := \sqrt{\det\left(\mathbf{J}_{\mathbf{z}}^{\intercal}(\mathbf{x})\,\mathbf{J}_{\mathbf{z}}(\mathbf{x})\right)} \quad \forall \mathbf{x} \in \widetilde{\omega_{\mathbf{z}}} \quad \text{a.e.},$$

where $\mathbf{J}_{\mathbf{z}}$ denotes the Jacobian of $\chi_{\mathbf{z}}$. Let the constants θ, Θ be defined by

$$\left\| g_{\mathbf{z}}^{-1} \right\|_{L^\infty(\widetilde{\omega_{\mathbf{z}}})} =: \theta \frac{|\widetilde{\omega_{\mathbf{z}}}|}{|\omega_{\mathbf{z}}|} \quad \text{and} \quad \| g_{\mathbf{z}} \|_{L^\infty(\tilde{\omega}_{\mathbf{z}})} =: \Theta \frac{|\omega_{\mathbf{z}}|}{|\widetilde{\omega_{\mathbf{z}}}|}. \tag{4.227}$$

Lemma 4.3.27. *Let Assumption 4.3.17 or Assumption 4.3.18 hold:*

(a) Then, θ, respectively Θ, in (4.227) can be bounded from below, respectively from above, by constants which depend only on the shape-regularity of the mesh, the ratios

$$c_1 := \max_{\mathbf{z} \in \mathcal{I}} \max_{\tau \in \mathcal{G}_{\mathbf{z}}} \left\{ \frac{|\omega_{\mathbf{z}}|}{|\tau|} \right\} \quad \textit{and} \quad C_1 := \max_{\mathbf{z} \in \mathcal{I}} \max_{\tau \in \mathcal{G}_{\mathbf{z}}} \left\{ \frac{|\tilde{\omega}_{\mathbf{z}}|}{|\tilde{\tau}|} \right\}, \tag{4.228}$$

and, for curved panels, on the global mapping χ (cf. Assumption 4.3.10).

(b) There exists a constant C_2 so that, for $i \in \{1, 2\}$ and any $\tilde{\mathbf{x}} \in \tilde{\tau} \subset \widetilde{\omega_{\mathbf{z}}}$, we have

$$\left\| \partial_i \left\{ \left(\chi_\tau - \chi_\tau^{\text{affine}} \right) \circ \eta^{\text{affine}}(\tilde{\mathbf{x}}) \right\} \right\| \leq C_2 \frac{\operatorname{diam} \tilde{\tau}}{2\,|\tilde{\tau}|} h_\tau^2. \tag{4.229}$$

Proof. Proof of part a. Let $\tau \in \mathcal{G}_{\mathbf{z}}$ and $\tilde{\tau} := \chi_{\mathbf{z}}^{-1}(\tau)$. The restriction $\chi_{\mathbf{z},\tau} := \chi_{\mathbf{z}}|_{\tilde{\tau}}$ can be written as

$$\chi_{\mathbf{z},\tau} = \chi_\tau \circ \eta^{\text{affine}},$$

where $\chi_\tau : \hat{\tau} \to \tau$ is the reference mapping as in Definition 4.1.2 and $\hat{\tau}$ is the unit triangle as in (4.13). Further, $\eta^{\text{affine}} : \tilde{\tau} \to \hat{\tau}$ is some affine map. For $\mathbf{x} \in \tau$, let

$\hat{\mathbf{x}} := \chi_\tau^{-1}(\mathbf{x})$ and $\tilde{\mathbf{x}} := \left(\eta^{\text{affine}}\right)^{-1}(\hat{\mathbf{x}})$. Then

$$\mathbf{J}_{\mathbf{z},\tau}(\tilde{\mathbf{x}}) = \mathbf{J}_\tau(\hat{\mathbf{x}})\,\mathbf{J}_{\text{affine}}(\tilde{\mathbf{x}}), \tag{4.230}$$

where $\mathbf{J}_{\mathbf{z},\tau}$, $\mathbf{J}_\tau$, $\mathbf{J}_{\text{affine}}$ are the Jacobi matrices of $\chi_{\mathbf{z},\tau}$, χ_τ, η^{affine}, and

$$g_{\mathbf{z}}(\tilde{\mathbf{x}}) = \sqrt{\det\left(\mathbf{J}_{\text{affine}}^{\mathsf{T}}(\tilde{\mathbf{x}})\,\mathbf{G}_\tau(\hat{\mathbf{x}})\,\mathbf{J}_{\text{affine}}(\tilde{\mathbf{x}})\right)} \quad \text{with} \quad \mathbf{G}_\tau(\hat{\mathbf{x}}) := \mathbf{J}_\tau^{\mathsf{T}}(\hat{\mathbf{x}})\,\mathbf{J}_\tau(\hat{\mathbf{x}}).$$

We introduce $\mathbf{G}_{\text{affine}} := \mathbf{J}_{\text{affine}}^{\mathsf{T}}\mathbf{J}_{\text{affine}}$ and employ the multiplication theorem for determinants to obtain

$$g_{\mathbf{z}}(\tilde{\mathbf{x}}) = \left|\det \mathbf{J}_{\text{affine}}\right| \sqrt{\det \mathbf{G}_\tau(\hat{\mathbf{x}})} = \frac{\sqrt{\det \mathbf{G}_\tau(\hat{\mathbf{x}})}}{2\,|\tilde{\tau}|}. \tag{4.231}$$

If τ is a plane triangle with straight edges then

$$\sqrt{\det \mathbf{G}_\tau(\hat{\mathbf{x}})} = 2\,|\tau|.$$

For curved panels, we have $\chi_\tau = \chi \circ \chi_\tau^{\text{affine}}$ (cf. Assumption 4.3.10) and obtain by arguing as in (4.231)

$$c_\chi 2\left|\tau^{\text{affine}}\right| \leq \sqrt{\det \mathbf{G}_\tau(\hat{\mathbf{x}})} \leq C_\chi 2\left|\tau^{\text{affine}}\right| \quad \text{with } \tau^{\text{affine}} := \chi_\tau^{\text{affine}}(\hat{\tau}),$$

where the constants $0 < c_\chi \leq C_\chi$ depend only on χ, i.e., are independent of the discretization parameters. From this we derive, by using the bi-Lipschitz continuity of χ and the shape-regularity of the surface mesh, the estimate

$$2cc_\chi\,|\tau| \leq 2c_\chi h_\tau^2 \leq \sqrt{\det \mathbf{G}_\tau(\hat{\mathbf{x}})} \leq 2C_\chi h_\tau^2 \leq 2CC_\chi\,|\tau|,$$

where c, C depend only on the shape-regularity constant $\kappa_{\mathcal{G}}$. Thus

$$\frac{cc_\chi}{c_1}\frac{|\omega_{\mathbf{z}}|}{|\widetilde{\omega_{\mathbf{z}}}|} \leq cc_\chi\frac{|\tau|}{|\tilde{\tau}|} \leq |g_{\mathbf{z}}(\tilde{\mathbf{x}})| \leq CC_\chi\frac{|\tau|}{|\tilde{\tau}|} \leq CC_\chi C_1\frac{|\omega_{\mathbf{z}}|}{|\widetilde{\omega_{\mathbf{z}}}|}.$$

Proof of part b. The statement is trivial for plane triangles with straight edges because the left-hand side in (4.229) is zero.

Let $\mathbf{z} \in \mathcal{I}$ and assume that $\tau \in \mathcal{G}_{\mathbf{z}}$ is a curved panel. For any $\tilde{\mathbf{x}} \in \tilde{\tau} \subset \widetilde{\omega_{\mathbf{z}}}$, we have

$$\left\|\partial_i\left\{\left(\chi_\tau - \chi_\tau^{\text{affine}}\right) \circ \eta^{\text{affine}}(\tilde{\mathbf{x}})\right\}\right\| \leq \sum_{j=1}^{2}\left\|\partial_j\left(\chi_\tau - \chi_\tau^{\text{affine}}\right)(\hat{\mathbf{x}})\right\|\left|\partial_i \eta_j^{\text{affine}}(\tilde{\mathbf{x}})\right|, \tag{4.232}$$

where $\hat{\mathbf{x}} = \eta^{\text{affine}}(\tilde{\mathbf{x}}) \in \hat{\tau}$.

Let $\widehat{T_p}\chi_\tau$ denote the p-th order Taylor expansion of χ_τ about the barycenter $\widehat{\mathbf{M}}$ of $\hat{\tau}$ and let $\chi_\tau^{\text{affine}} = \widehat{I_1}\chi_\tau$ be the affine interpolation at the vertices of $\hat{\tau}$. Then

$$\chi_\tau - \chi_\tau^{\text{affine}} = \left(\chi_\tau - \widehat{T_1}\chi_\tau\right) + \left(\widehat{T_1}\chi_\tau - \chi_\tau^{\text{affine}}\right) = \left(\chi_\tau - \widehat{T_1}\chi_\tau\right) + \widehat{I_1}\left(\widehat{T_1}\chi_\tau - \chi_\tau\right).$$

For $k = 0, 1$, this splitting leads to the estimate

$$\left\|\chi_\tau - \chi_\tau^{\text{affine}}\right\|_{C^k(\overline{\hat{\tau}})} \leq \left(1 + \left\|\widehat{I_1}\right\|_{C^k(\overline{\hat{\tau}}) \leftarrow C^k(\overline{\hat{\tau}})}\right) \left\|\chi_\tau - \widehat{T_1}\chi_\tau\right\|_{C^k(\overline{\hat{\tau}})}.$$

Standard error estimates for two-dimensional Taylor expansions result in

$$\left\|\chi_\tau - \widehat{T_1}\chi_\tau\right\|_{C^0(\overline{\hat{\tau}})} \leq \frac{1}{2} \max_{0 \leq j \leq 2} \left\|\partial_1^j \partial_2^{2-j}\chi_\tau\right\|_{C^0(\overline{\hat{\tau}})}.$$

Because $\partial_j \widehat{T_1}\chi_\tau = \widehat{T_0}\left(\partial_j\chi_\tau\right)$ we obtain

$$\left\|\partial_j\chi_\tau - \partial_j\widehat{T_1}\chi_\tau\right\|_{C^0(\overline{\hat{\tau}})} = \left\|\partial_j\chi_\tau - \widehat{T_0}\partial_j\chi_\tau\right\|_{C^0(\overline{\hat{\tau}})} \leq \max_{0 \leq i \leq 1} \left\|\partial_1^i \partial_2^{1-i}\partial_j\chi_\tau\right\|_{C^0(\overline{\hat{\tau}})}.$$

Thus

$$\left\|\chi_\tau - \chi_\tau^{\text{affine}}\right\|_{C^k(\overline{\hat{\tau}})} \leq \left(1 + \left\|\widehat{I_1}\right\|_{C^k(\overline{\hat{\tau}}) \leftarrow C^k(\overline{\hat{\tau}})}\right) \max_{0 \leq j \leq 2} \left\|\partial_1^j \partial_2^{2-j}\chi_\tau\right\|_{C^0(\overline{\hat{\tau}})}. \tag{4.233}$$

Next, we will estimate the first factor in (4.233). For any $w \in C^0\left(\overline{\hat{\tau}}\right)$, we have

$$\left\|\widehat{I_1}w\right\|_{C^0(\overline{\hat{\tau}})} = \max_{\hat{\mathbf{x}} \text{ is a vertex of } \hat{\tau}} |w(\hat{\mathbf{x}})| \leq \|w\|_{C^0(\overline{\hat{\tau}})}.$$

We denote the vertices of $\hat{\tau}$ by $\widehat{P}_1 = (0, 0)$, $\widehat{P}_2 = (1, 0)$, $\widehat{P}_3 = (1, 1)$ and the values of a continuous function w at $\widehat{P}_j$ by w_j, $1 \leq j \leq 3$. It is easy to see that

$$\left\|\partial_1\widehat{I_1}w\right\|_{C^0(\overline{\hat{\tau}})} = |w_2 - w_1| \leq \frac{|w_2 - w_1|}{\left\|\widehat{P}_2 - \widehat{P}_1\right\|} \leq \sup_{\hat{\mathbf{x}}, \hat{\mathbf{y}} \in \hat{\tau}} \frac{|w(\hat{\mathbf{x}}) - w(\hat{\mathbf{y}})|}{\|\hat{\mathbf{x}} - \hat{\mathbf{y}}\|} \leq \|w\|_{C^1(\overline{\hat{\tau}})}$$

and, similarly, we obtain the stability of the derivative ∂_2. Hence we have proved that the first factor in (4.233) is bounded from above by 2.

To estimate the second derivative of χ_τ in (4.233) we write the mapping $\chi_\tau^{\text{affine}}$ in the form

$$\chi_\tau^{\text{affine}}(\hat{\mathbf{x}}) = \mathbf{B}_\tau\hat{\mathbf{x}} + \mathbf{b}_\tau$$

with the (constant) Jacobi matrix $\mathbf{B}_\tau \in \mathbb{R}^{3\times 2}$ and $\mathbf{b}_\tau \in \mathbb{R}^3$. The columns of $\mathbf{B}_\tau$ are denoted by $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^3$. As in the proof of Lemma 4.3.6, we use

$$\partial^{\mu}\left(\chi\circ\chi_{\tau}^{\text{affine}}\right)=\sum_{\substack{\beta\in\mathbb{N}_0^3\\|\beta|=\mu_1}}\sum_{\substack{\nu\in\mathbb{N}_0^3\\|\nu|=\mu_2}}\frac{\mu!}{\beta!\nu!}\mathbf{a}_1^{\beta}\mathbf{a}_2^{\nu}\left(\partial^{\beta+\nu}\chi\right)\circ\chi_{\tau}^{\text{affine}}.$$

Next, we employ $\left|(\mathbf{B}_\tau)_{i,j}\right|\leq h_\tau$ and obtain for any $\mu\in\mathbb{N}_0^2$ with $|\mu|=2$

$$\sup_{\hat{\mathbf{x}}\in\hat{\tau}}\left|\partial^{\mu}\left(\chi\circ\chi_{\tau}^{\text{affine}}\right)(\hat{\mathbf{x}})\right|\leq C_3h_\tau^2,$$

where C_3 depends only on the derivatives of χ which, by Assumption 4.3.10, are independent of $\mathcal{G}$. Thus we have proved that

$$\left\|\chi_\tau-\chi_\tau^{\text{affine}}\right\|_{C^k(\hat{\tau})}\leq 2C_3h_\tau^2 \tag{4.234}$$

and it remains to estimate the last factor in (4.232). Because η^{affine} is affine, it is straightforward to show that $\mathbf{J}_{\text{affine}}^{-1}\in\mathbb{R}^{2\times 2}$ [cf. (4.230)] has column vectors $\mathbf{B}-\mathbf{A}$ and $\mathbf{C}-\mathbf{B}$, where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ denote the vertices of $\tilde{\tau}$. Hence

$$\mathbf{J}_{\text{affine}}=\frac{1}{2\,|\tilde{\tau}|}\begin{bmatrix}(\mathbf{C}-\mathbf{B})_2 & -(\mathbf{C}-\mathbf{B})_1\\ -(\mathbf{B}-\mathbf{A})_2 & (\mathbf{B}-\mathbf{A})_1\end{bmatrix}.$$

Consequently

$$\left|\partial_i\eta_j^{\text{affine}}\right|\leq\frac{\operatorname{diam}\tilde{\tau}}{2\,|\tilde{\tau}|}. \tag{4.235}$$

□

As a measure for the distortion of the local patches $\omega_{\mathbf{z}}$ by the pullback, we introduce the constant C_{d} by

$$C_{\mathrm{d}}:=\max_{\mathbf{z}\in\mathcal{I}}\left\{|\widetilde{\omega_{\mathbf{z}}}|^{-1/2}\operatorname{diam}\widetilde{\omega_{\mathbf{z}}}\right\}. \tag{4.236}$$

Theorem 4.3.28. *Let Assumption 4.3.25 be satisfied.*

There exist two constants c_1, c_2 depending only on the shape-regularity constant $\kappa_{\mathcal{G}}$ [cf. (4.17)], the constants C_{d} and C_1 [as in (4.228)], and, for curved panels, on the global chart χ so that

$$\|v-Q_{\mathcal{G}}v\|_{L^2(\tau)}\leq c_1h_\tau\|v\|_{H^1(\omega_\tau)}\quad\text{and}\quad\|Q_{\mathcal{G}}v\|_{H^1(\tau)}\leq\tilde{c}_1\|v\|_{H^1(\omega_\tau)} \tag{4.237a}$$

for all $v\in H^1(\Gamma)$ and all triangles $\tau\in\mathcal{G}$. Also,

$$\|v-Q_{\mathcal{G}}v\|_{H^{\sigma}(\Gamma)}\leq c_2h_{\mathcal{G}}^{s-\sigma}\|v\|_{H^s(\Gamma)}\quad\text{and}\quad\|Q_{\mathcal{G}}\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)}\leq\tilde{c}_2 \tag{4.237b}$$

for any $0\leq\sigma\leq s\leq 1$ and $v\in H^s(\Gamma)$.

Proof. We present the proof in eight steps (a)–(h).

(a) For $\mathbf{z} \in \mathcal{I}$, let $\chi_{\mathbf{z}} : \widetilde{\omega_{\mathbf{z}}} \to \omega_{\mathbf{z}}$ be the mapping as in Assumption 4.3.25. For $\varphi \in H^1(\omega_{\mathbf{z}})$, the pullback to $\widetilde{\omega_{\mathbf{z}}}$ is denoted by $\tilde{\varphi} := \varphi \circ \chi_{\mathbf{z}}$. The Lipschitz continuity of $\chi_{\mathbf{z}}$ implies that $\tilde{\varphi} \in H^1(\widetilde{\omega_{\mathbf{z}}})$.

We consider $\pi_{\mathbf{z}}(\varphi) \in \mathbb{C}$ as a constant function and obtain

$$\|\varphi - \pi_{\mathbf{z}}(\varphi)\|^2_{L^2(\omega_{\mathbf{z}})} = \int_{\tilde{\omega}_{\mathbf{z}}} g_{\mathbf{z}}(\tilde{\mathbf{x}}) \left|\tilde{\varphi}(\tilde{\mathbf{x}}) - \pi_{\mathbf{z}}(\varphi)\right|^2 d\tilde{\mathbf{x}}. \tag{4.238}$$

Case 1: First, we consider the case of flat panels with straight edges. Note that, for any $t \in \mathcal{G}_{\mathbf{z}}$ and $\tilde{t} := \chi_{\mathbf{z}}^{-1}(t)$, we have $g_{\mathbf{z}}|_{\tilde{t}} = |t| / \left|\widetilde{T}\right|$.

Let $\tau \in \mathcal{G}_{\mathbf{z}}$. Then for any $\tilde{\mathbf{x}} \in \tilde{\tau} = \chi_{\mathbf{z}}^{-1}(\tau)$

$$\begin{aligned}
\tilde{\varphi}(\tilde{\mathbf{x}}) - \pi_{\mathbf{z}}\varphi &= \tilde{\varphi}(\tilde{\mathbf{x}}) - \frac{1}{|\omega_{\mathbf{z}}|}\int_{\omega_{\mathbf{z}}} \varphi = \tilde{\varphi}(\tilde{\mathbf{x}}) - \frac{1}{|\omega_{\mathbf{z}}|}\int_{\tilde{\omega}_{\mathbf{z}}} g_{\mathbf{z}}\tilde{\varphi} \\
&= \tilde{\varphi}(\tilde{\mathbf{x}}) - \frac{1}{|\omega_{\mathbf{z}}|}\sum_{t\in\mathcal{G}_{\mathbf{z}}}\int_{\tilde{t}} g_{\mathbf{z}}\tilde{\varphi} = \tilde{\varphi}(\tilde{\mathbf{x}}) - \frac{1}{|\omega_{\mathbf{z}}|}\sum_{t\in\mathcal{G}_{\mathbf{z}}}\frac{|t|}{\left|\widetilde{T}\right|}\int_{\tilde{t}}\tilde{\varphi} \qquad (4.239)\\
&= \sum_{t\in\mathcal{G}_{\mathbf{z}}}\frac{|t|}{|\omega_{\mathbf{z}}|}\left(\tilde{\varphi}(\tilde{\mathbf{x}}) - \pi_{\tilde{t}}\tilde{\varphi}\right)
\end{aligned}$$

with $\pi_{\tilde{t}}\tilde{\varphi} := \frac{1}{|\tilde{t}|}\int_{\tilde{t}}\tilde{\varphi}$. Applying the L^2-norm to both sides yields

$$\|\tilde{\varphi} - \pi_{\mathbf{z}}\varphi\|_{L^2(\tilde{\tau})} \leq \sum_{t\in\mathcal{G}_{\mathbf{z}}}\frac{|t|}{|\omega_{\mathbf{z}}|}\|\tilde{\varphi} - \pi_{\tilde{t}}\tilde{\varphi}\|_{L^2(\tilde{\tau})}. \tag{4.240}$$

Because $\widetilde{T} \subset \widetilde{\omega_{\mathbf{z}}}$ are both convex we may apply Corollary 2.5.12 to obtain

$$\begin{aligned}
\|\tilde{\varphi} - \pi_{\tilde{t}}\tilde{\varphi}\|_{L^2(\tilde{\tau})} &\leq \|\tilde{\varphi} - \pi_{\tilde{t}}\tilde{\varphi}\|_{L^2(\widetilde{\omega_{\mathbf{z}}})} \leq \left(1 + \sqrt{\frac{|\widetilde{\omega_{\mathbf{z}}}|}{|\tilde{t}|}}\right)\frac{\operatorname{diam}\widetilde{\omega_{\mathbf{z}}}}{\pi}|\tilde{\varphi}|_{H^1(\tilde{\omega}_{\mathbf{z}})} \\
&\leq \left(1 + \sqrt{C_1}\right)\frac{\operatorname{diam}\widetilde{\omega_{\mathbf{z}}}}{\pi}|\tilde{\varphi}|_{H^1(\tilde{\omega}_{\mathbf{z}})}, \qquad (4.241)
\end{aligned}$$

where C_1 is as in (4.228). Inserting this into (4.240) yields

$$\|\tilde{\varphi} - \pi_{\mathbf{z}}\varphi\|_{L^2(\tilde{\tau})} \leq \left(1 + \sqrt{C_1}\right)\frac{\operatorname{diam}\widetilde{\omega_{\mathbf{z}}}}{\pi}|\tilde{\varphi}|_{H^1(\tilde{\omega}_{\mathbf{z}})}.$$

We sum over all $\tilde{\tau} \subset \widetilde{\omega_{\mathbf{z}}}$ and apply a Cauchy–Schwarz inequality to derive the estimate

$$\|\tilde{\varphi} - \pi_{\mathbf{z}}\varphi\|_{L^2(\tilde{\omega}_{\mathbf{z}})} \leq \sqrt{\operatorname{card}\mathcal{G}_{\mathbf{z}}}\left(1 + \sqrt{C_1}\right)\frac{\operatorname{diam}\widetilde{\omega_{\mathbf{z}}}}{\pi}|\tilde{\varphi}|_{H^1(\tilde{\omega}_{\mathbf{z}})},$$

where the number of panels (card $\mathcal{G}_{\mathbf{z}}$) is bounded by a constant which depends only on the shape-regularity of the surface mesh.

The combination with (4.238) leads to

$$\begin{aligned}
\|\varphi - \pi_{\mathbf{z}}(\varphi)\|_{L^2(\omega_{\mathbf{z}})} &\leq \sqrt{\|g_{\mathbf{z}}\|_{L^\infty(\tilde{\omega}_{\mathbf{z}})}}\, \|\tilde{\varphi} - \pi_{\mathbf{z}}\varphi\|_{L^2(\tilde{\omega}_{\mathbf{z}})} \\
&\overset{(4.227)}{\leq} C_4 \sqrt{|\omega_{\mathbf{z}}|} \frac{\operatorname{diam} \widetilde{\omega_{\mathbf{z}}}}{\sqrt{|\widetilde{\omega_{\mathbf{z}}}|}} |\tilde{\varphi}|_{H^1(\tilde{\omega}_{\mathbf{z}})} \\
&\leq C_4 C_{\mathrm{d}} h_{\mathbf{z}} |\tilde{\varphi}|_{H^1(\tilde{\omega}_{\mathbf{z}})}
\end{aligned}$$

with $C_4 := \sqrt{\Theta \operatorname{card} \mathcal{G}_{\mathbf{z}}}\left(1 + \sqrt{C_1}\right)/\pi$. From Lemma 4.3.6 resp. Lemma 4.3.13 we obtain

$$|\tilde{\varphi}|^2_{H^1(\tilde{\omega}_{\mathbf{z}})} = \sum_{\tilde{\tau} \subset \widetilde{\omega_{\mathbf{z}}}} |\tilde{\varphi}|^2_{H^1(\tilde{\tau})} \leq C_5 \sum_{\tau \in \mathcal{G}_{\mathbf{z}}} |\varphi|^2_{H^1(\tau)} \tag{4.242}$$

and, finally, for any $\tau \subset \omega_{\mathbf{z}}$

$$\|\varphi - \pi_{\mathbf{z}}(\varphi)\|_{L^2(\omega_{\mathbf{z}})} \leq \widetilde{C}_6 h_{\mathbf{z}} |\varphi|_{H^1(\omega_{\mathbf{z}})} \leq C_0 \widetilde{C}_6 h_\tau |\varphi|_{H^1(\omega_{\mathbf{z}})} \quad \text{with} \quad \widetilde{C}_6 = C_{\mathrm{d}} C_4 \sqrt{C_5}. \tag{4.243}$$

Case 2: Next, we consider the general case of curved panels. As in (4.239) we derive

$$\begin{aligned}
\tilde{\varphi} - \pi_{\mathbf{z}}\varphi &= \sum_{t \in \mathcal{G}_{\mathbf{z}}} \frac{|t|}{|\omega_{\mathbf{z}}|} \left(\tilde{\varphi} - \frac{1}{|\widetilde{T}|} \int_{\tilde{t}} \frac{|\widetilde{T}|}{|t|} g_{\mathbf{z}} \tilde{\varphi} \right) \\
&= \sum_{t \in \mathcal{G}_{\mathbf{z}}} \frac{|t|}{|\omega_{\mathbf{z}}|} \left\{ (\tilde{\varphi} - \pi_{\tilde{t}}\tilde{\varphi}) + \frac{1}{|\widetilde{T}|} \int_{\tilde{t}} d_t \tilde{\varphi} \right\}
\end{aligned} \tag{4.244}$$

with $d_t := 1 - \frac{|\widetilde{T}|}{|t|} g_{\mathbf{z}}|_{\tilde{t}}$. The first difference in (4.244) can be estimated as in the case of flat panels while, for the second one, we will derive an estimate of d_t. We use the notation as in Assumption 4.3.25 and employ the splitting

$$d_t = \left(1 - \frac{\left|t^{\text{affine}}\right|}{|t|}\right) + \frac{1}{2|t|}\left(2\left|t^{\text{affine}}\right| - 2\left|\widetilde{T}\right|\left(g_{\mathbf{z}}|_{\tilde{t}}\right)\right), \tag{4.245}$$

where $t^{\text{affine}} = \chi_t^{\text{affine}}(\hat{\tau})$ is the plane triangle with straight edges which interpolates τ at its vertices.

We start by estimating the second term in (4.245). We employ the representation (4.231) for Gram's determinant to obtain

$$2\left|\widetilde{T}\right|\left(g_{\mathbf{z}}|_{\tilde{t}}\right) = g_t,$$

where g_t is Gram's determinant of the reference map $\chi_t : \hat{\tau} \to t$, i.e.

$$g_t = \|\partial_1 \chi_t \times \partial_2 \chi_t\| .$$

The area $2\left|t^{\text{affine}}\right|$ can be expressed by

$$2\left|t^{\text{affine}}\right| = \left\|\partial_1 \chi_t^{\text{affine}} \times \partial_2 \chi_t^{\text{affine}}\right\| =: g_t^{\text{affine}} \tag{4.246}$$

Hence

$$\left|2\left|t^{\text{affine}}\right| - 2\left|\widetilde{T}\right|(g_{\mathbf{z}}|_{\tilde{t}})\right| = \left|g_t^{\text{affine}} - g_t\right| = \left|\left\|\partial_1 \chi_t^{\text{affine}} \times \partial_2 \chi_t^{\text{affine}}\right\| - \|\partial_1 \chi_t \times \partial_2 \chi_t\|\right| \tag{4.247}$$

$$\begin{aligned} &\le \left\|\partial_1 \chi_t^{\text{affine}} \times \partial_2 \chi_t^{\text{affine}} - (\partial_1 \chi_t \times \partial_2 \chi_t)\right\| \\ &\le \left\|\partial_1\left(\chi_t - \chi_t^{\text{affine}}\right) \times \partial_2 \chi_t\right\| \\ &\quad + \left\|\partial_1 \chi_t^{\text{affine}} \times \partial_2\left(\chi_t - \chi_t^{\text{affine}}\right)\right\| . \end{aligned}$$

We employ (4.234) to obtain

$$\left|2\left|t^{\text{affine}}\right| - 2\left|\widetilde{T}\right|(g_{\mathbf{z}}|_{\tilde{t}})\right| \le 2C_3 h_t^2 \left(\|\partial_2 \chi_t\|_{\mathbf{L}^\infty\left(\widehat{\tau}\right)} + \left\|\partial_1 \chi_t^{\text{affine}}\right\|_{\mathbf{L}^\infty(\hat{\tau})}\right).$$

The estimate $\left\|\partial_1 \chi_t^{\text{affine}}\right\|_{\mathbf{L}^\infty(\hat{\tau})} \le h_t$ is obvious because t^{affine} interpolates t in its vertices. For the other term, we use

$$\|\partial_2 \chi_t\|_{\mathbf{L}^\infty(\hat{\tau})} = \left\|\sum_{j=1}^{3}\left(\partial_j \chi \circ \chi_t^{\text{affine}}\right) \partial_2\left(\chi_t^{\text{affine}}\right)_j\right\|_{\mathbf{L}^\infty(\hat{\tau})} \le C h_t,$$

where C depends only on the global chart χ but not on the discretization parameters. In summary we have proved that

$$\frac{\left|\left|t^{\text{affine}}\right| - \left|\widetilde{T}\right|(g_{\mathbf{z}}|_{\tilde{t}})\right|}{|t|} \le \frac{C_3 C h_t}{c},$$

where c depends only on the shape-regularity of the mesh and the global chart χ. The first term of the sum in (4.245) can be estimated by using (4.247)

$$\begin{aligned} \left|1 - \frac{\left|t^{\text{affine}}\right|}{|t|}\right| &= \left|\frac{|t| - \left|t^{\text{affine}}\right|}{|t|}\right| \le |t|^{-1} \int_{\hat{\tau}} \left|g_t - g_t^{\text{affine}}\right| d\mathbf{x} \\ &\le \frac{C_3 C h_t^3}{|t|} \le \frac{C_3 C h_t}{c}. \end{aligned}$$

This finishes the estimate of d_t

$$|d_t| \leq 2\frac{C_3 C}{c} h_t.$$

Inserting this into (4.244) and proceeding as in the case of flat panels yields

$$\begin{aligned}\|\tilde{\varphi} - \pi_{\mathbf{z}}\varphi\|_{L^2(\tilde{\tau})} &\leq \sum_{t\in\mathcal{G}_{\mathbf{z}}} \frac{|t|}{|\omega_{\mathbf{z}}|} \|\tilde{\varphi} - \pi_{\tilde{t}}\tilde{\varphi}\|_{L^2(\tilde{\tau})} + \sum_{t\in\mathcal{G}_{\mathbf{z}}} \frac{|t|}{|\omega_{\mathbf{z}}|} 2\frac{C_3 C}{c} h_t \sqrt{\frac{|\tilde{\tau}|}{|\widetilde{T}|}} \|\tilde{\varphi}\|_{L^2(\tilde{T})} \\ &\overset{(4.241)}{\leq} \left(1 + \sqrt{C_1}\right) \frac{\operatorname{diam}\widetilde{\omega_{\mathbf{z}}}}{\pi} |\tilde{\varphi}|_{H^1(\tilde{\omega}_{\mathbf{z}})} + 2\frac{C_3 C\sqrt{C_1}}{c} h_{\mathbf{z}} \|\tilde{\varphi}\|_{L^2(\tilde{\omega}_{\mathbf{z}})}.\end{aligned}$$

We sum over all $\tilde{\tau} \subset \widetilde{\omega_{\mathbf{z}}}$ and apply a Cauchy–Schwarz inequality to derive the estimate

$$\begin{aligned}\|\tilde{\varphi} - \pi_{\mathbf{z}}\varphi\|_{L^2(\tilde{\omega}_{\mathbf{z}})} \leq \sqrt{\operatorname{card}\mathcal{G}_{\mathbf{z}}} \Big\{ &\left(1 + \sqrt{C_1}\right) \frac{\operatorname{diam}\widetilde{\omega_{\mathbf{z}}}}{\pi} |\tilde{\varphi}|_{H^1(\tilde{\omega}_{\mathbf{z}})} \\ &+ 2\frac{C_3 C\sqrt{C_1}}{c} h_{\mathbf{z}} \|\tilde{\varphi}\|_{L^2(\tilde{\omega}_{\mathbf{z}})} \Big\}.\end{aligned}$$

From Lemma 4.3.6 resp. Lemma 4.3.13 we obtain the scaling relations

$$|\tilde{\varphi}|^2_{H^1(\tilde{\omega}_{\mathbf{z}})} \leq C_5 |\varphi|^2_{H^1(\omega_{\mathbf{z}})} \text{ and } \tilde{c}_5 \frac{|\widetilde{\omega_z}|}{|\omega_{\mathbf{z}}|} \|\varphi\|^2_{L^2(\omega_{\mathbf{z}})} \leq \|\tilde{\varphi}\|^2_{L^2(\tilde{\omega}_{\mathbf{z}})} \leq \widetilde{C}_5 \frac{|\widetilde{\omega_z}|}{|\omega_{\mathbf{z}}|} \|\varphi\|^2_{L^2(\omega_{\mathbf{z}})}$$

and, finally, for any $\tau \subset \omega_{\mathbf{z}}$

$$\|\varphi - \pi_{\mathbf{z}}\varphi\|_{L^2(\omega_{\mathbf{z}})} \leq \widehat{C}_6 h_{\mathbf{z}} \|\varphi\|_{H^1(\omega_{\mathbf{z}})} \leq C_0 \widehat{C}_6 h_\tau \|\varphi\|_{H^1(\omega_{\mathbf{z}})}, \tag{4.248}$$

where $\widehat{C}_6$ depends on C_1, C_{d}, C_5, $\tilde{c}_5$, $\widetilde{C}_5$, and $\operatorname{card}\mathcal{G}_{\mathbf{z}}$. Let $C_6 := \max\left\{\widetilde{C}_6, \widehat{C}_6\right\}$ [cf. (4.243)].

(b) Let $\tau \in \mathcal{G}$. The set of vertices of τ is denoted by $\mathcal{I}_\tau$. Then

$$\sum_{\mathbf{x}\in\mathcal{I}_\tau} b_{\mathbf{x}} = 1 \qquad \text{on } \tau.$$

By using Step a, we derive

$$\begin{aligned}\|\varphi - Q_{\mathcal{G}}\varphi\|_{L^2(\tau)} &= \left\| \sum_{\mathbf{z}\in\mathcal{I}_\tau} b_{\mathbf{z}} \left(\varphi - \pi_{\mathbf{z}}(\varphi)\right) \right\|_{L^2(\tau)} \leq \sum_{\mathbf{z}\in\mathcal{I}_\tau} \|b_{\mathbf{z}} \left(\varphi - \pi_{\mathbf{z}}(\varphi)\right)\|_{L^2(\tau)} \\ &\leq \sum_{\mathbf{z}\in\mathcal{I}_\tau} \|\varphi - \pi_{\mathbf{z}}(\varphi)\|_{L^2(\tau)} \leq \sum_{\mathbf{z}\in\mathcal{I}_\tau} \|\varphi - \pi_{\mathbf{z}}(\varphi)\|_{L^2(\omega_{\mathbf{z}})}\end{aligned}$$

$$
\begin{aligned}
&\leq C_0 C_6 h_\tau \sum_{\mathbf{z}\in\mathcal{I}_\tau} \|\varphi\|_{H^1(\omega_\mathbf{z})} \\
&\leq \sqrt{3} C_0 C_6 h_\tau \sqrt{\sum_{\mathbf{z}\in\mathcal{I}_\tau} \|\varphi\|^2_{H^1(\omega_\mathbf{z})}} \qquad (4.249)\\
&\leq \sqrt{3} C_0 C_6 h_\tau \sqrt{\sum_{t\in\omega_\tau} \sum_{\mathbf{z}\in\mathcal{I}_\tau : t\subset\omega_\mathbf{z}} \|\varphi\|^2_{H^1(t)}} \\
&\leq C_7 h_\tau \|\varphi\|_{H^1(\omega_\tau)} \qquad (4.250)
\end{aligned}
$$

with $C_7 := 3C_0C_6$.

(c) By summing over all panels we obtain

$$
\begin{aligned}
\|\varphi - Q_\mathcal{G}\varphi\|^2_{L^2(\Gamma)} &= \sum_{\tau\in\mathcal{G}} \|\varphi - Q_\mathcal{G}\varphi\|^2_{L^2(\tau)} \leq C_7^2 h_\mathcal{G}^2 \sum_{\tau\in\mathcal{G}} \|\varphi\|^2_{H^1(\omega_\tau)} \\
&= C_7^2 h_\mathcal{G}^2 \sum_{t\in\mathcal{G}} \sum_{\tau\in\mathcal{G}:t\subset\omega_\tau} \|\varphi\|^2_{H^1(t)} \leq C_8^2 h_\mathcal{G}^2 \|\varphi\|^2_{H^1(\Gamma)},
\end{aligned}
$$

where $C_8 = C_7 C_\sharp^{1/2}$ and

$$
C_\sharp := \max_{t\in\mathcal{G}} \operatorname{card}\{\tau \in \mathcal{G} : t \subset \omega_\tau\}
$$

depends only on the shape-regularity constant.

(d) For the $L^2(\Gamma)$-stability we repeat the first steps of (4.249) to obtain

$$
\|Q_\mathcal{G}\varphi\|_{L^2(\tau)} \leq \sum_{\mathbf{z}\in\mathcal{I}_\tau} \|\pi_\mathbf{z}(\varphi)\|_{L^2(\omega_\mathbf{z})}.
$$

The Cauchy-Schwarz inequality yields

$$
|\pi_\mathbf{z}(\varphi)| \leq |\omega_\mathbf{z}|^{-1/2} \|\varphi\|_{L^2(\omega_\mathbf{z})}
$$

and as in (4.250) we derive

$$
\|Q_\mathcal{G}\varphi\|_{L^2(\tau)} \leq \sum_{\mathbf{z}\in\mathcal{I}_\tau} \|\varphi\|_{L^2(\omega_\mathbf{z})} \leq \sqrt{3}\, \|\varphi\|_{L^2(\omega_\tau)}. \qquad (4.251)
$$

A summation as in Step c results in the $L^2(\Gamma)$-stability of the Clément interpolation operator

$$
\|Q_\mathcal{G}\varphi\|_{L^2(\Gamma)} \leq \sqrt{3C_\sharp}\, \|\varphi\|_{L^2(\Gamma)}. \qquad (4.252)
$$

(e) From Step c and Step d we conclude that

$$\|\varphi - Q_{\mathcal{G}}\varphi\|_{L^2(\Gamma)} \leq C_9 \|\varphi\|_{L^2(\Gamma)} \quad \text{and} \quad \|\varphi - Q_{\mathcal{G}}\varphi\|_{L^2(\Gamma)} \leq C_8 h_{\mathcal{G}} \|\varphi\|_{H^1(\Gamma)}$$

hold with $C_9 := 1 + \sqrt{3C_\sharp}$. Hence the approximation result for the intermediate Sobolev spaces $H^s(\Gamma)$, $s \in]0,1[$, follows by interpolation as in the proof of Theorem 4.1.33.

(f) For the local H^1-stability we proceed as in Step d, respectively as in (4.249). Recall the definition of the surface gradient as in (4.200) and (4.201) to derive

$$|Q_{\mathcal{G}}\varphi|_{H^1(\tau)} = \left\| \sum_{\mathbf{z}\in\mathcal{I}_\tau} \pi_{\mathbf{z}}(\varphi) \nabla_S b_{\mathbf{z}} \right\|_{L^2(\tau)} = \left\| \sum_{\mathbf{z}\in\mathcal{I}_\tau} \left(\pi_{\mathbf{z}}(\varphi) - \pi_{\mathbf{z}_0}(\varphi)\right) \nabla_S b_{\mathbf{z}} \right\|_{L^2(\tau)} \tag{4.253}$$

for any fixed $\mathbf{z}_0 \in \mathcal{I}_\tau$. Let $\bar{\varphi}_\tau := \frac{1}{|\omega_\tau|}\int_{\omega_\tau}\varphi$. Then $\pi_{\mathbf{z}}(\bar{\varphi}_\tau) = \pi_{\mathbf{z}_0}(\bar{\varphi}_\tau) = \bar{\varphi}_\tau$ and

$$\begin{aligned}
\left|\pi_{\mathbf{z}}(\varphi) - \pi_{\mathbf{z}_0}(\varphi)\right| &\leq |\pi_{\mathbf{z}}(\varphi) - \pi_{\mathbf{z}}(\bar{\varphi}_\tau)| + \left|\pi_{\mathbf{z}_0}(\bar{\varphi}_\tau) - \pi_{\mathbf{z}_0}(\varphi)\right| \\
&\leq \left| \frac{1}{|\omega_{\mathbf{z}}|} \int_{\omega_{\mathbf{z}}} (\varphi - \bar{\varphi}_\tau) \right| + \left| \frac{1}{|\omega_{\mathbf{z}_0}|} \int_{\omega_{\mathbf{z}_0}} (\bar{\varphi}_\tau - \varphi) \right| \\
&\leq \frac{\|\varphi - \bar{\varphi}_\tau\|_{L^2(\omega_{\mathbf{z}})}}{|\omega_{\mathbf{z}}|^{1/2}} + \frac{\|\varphi - \bar{\varphi}_\tau\|_{L^2(\omega_{\mathbf{z}_0})}}{|\omega_{\mathbf{z}_0}|^{1/2}}.
\end{aligned}$$

In a similar fashion to (4.243) and (4.248) one derives for $D \in \{\omega_{\mathbf{z}}, \omega_{\mathbf{z}_0}\}$

$$\|\varphi - \bar{\varphi}_\tau\|_{L^2(D)} \leq \|\varphi - \bar{\varphi}_\tau\|_{L^2(\omega_\tau)} \leq \widetilde{C}_7 (\operatorname{diam}\omega_\tau) \|\varphi\|_{H^1(\omega_\tau)}.$$

Hence

$$\left|\pi_{\mathbf{z}}(\varphi) - \pi_{\mathbf{z}_0}(\varphi)\right| \leq C_{10} \|\varphi\|_{H^1(\omega_\tau)}, \tag{4.254}$$

where C_{10} depends only on the shape-regularity constant and the global parametrization χ.

In Theorem 4.4.2 (with $\ell = 1$ and $m = 0$), we will prove the *inverse inequality* and, thus, obtain the estimate

$$\|\nabla_S b_{\mathbf{z}}\|_{L^2(\tau)} \leq C h_\tau^{-1} \|b_{\mathbf{z}}\|_{L^2(\tau)} \leq C h_\tau^{-1} |\tau|^{1/2} \leq C_{11}, \tag{4.255}$$

where C_{11} depends only on the shape-regularity of the mesh and the global parametrization χ.

By inserting (4.254) and (4.255) into (4.253) we derive

$$|Q_{\mathcal{G}}\varphi|_{H^1(\tau)} \leq 3C_{10}C_{11} \|\varphi\|_{H^1(\omega_\tau)}.$$

The combination with (4.251) leads to the local stability with respect to the $\|\cdot\|_{H^1(\tau)}$-norm and a summation over all panels as in Step c results in the global H^1-stability

$$\|Q_{\mathcal{G}}\varphi\|_{H^1(\Gamma)} \leq C_{12}\,\|\varphi\|_{H^1(\Gamma)}\,.$$

(g) Applying Proposition 2.1.62 with $X_0 = Y_0 = L^2(\Gamma)$ and $X_1 = Y_1 = H^1(\Gamma)$ we obtain by interpolation of (4.252)

$$\|Q_{\mathcal{G}}\|_{H^s(\Gamma)\leftarrow H^s(\Gamma)} \leq \|Q_{\mathcal{G}}\|^{1-s}_{L^2(\Gamma)\leftarrow L^2(\Gamma)}\,\|Q_{\mathcal{G}}\|^{s}_{H^1(\Gamma)\leftarrow H^1(\Gamma)} \leq C_{13}$$

with $C_{13} := \left(3C_\sharp\right)^{\frac{1-s}{2}}\, c_{12}^s$.

(h) Part e and g imply that

$$\|\varphi - Q_{\mathcal{G}}\varphi\|_{L^2(\Gamma)} \leq C_8 h_{\mathcal{G}}^s \|\varphi\|_{H^s(\Gamma)} \text{ and } \|\varphi - Q_{\mathcal{G}}\varphi\|_{H^s(\Gamma)} \leq (1 + C_{13})\|\varphi\|_{H^s(\Gamma)}\,.$$

We apply Proposition 2.1.62 with $T = I - Q_{\mathcal{G}}$, $Y_0 = Y_1 = H^s(\Gamma)$, $X_0 = L^2(\Gamma)$, $X_1 = H^s(\Gamma)$, and $\theta = \sigma/s \in [0,1]$ to interpolate these two inequalities. The result is

$$\begin{aligned}\|T\|_{H^\sigma(\Gamma)\leftarrow H^s(\Gamma)} &\leq \|T\|^{1-\theta}_{L^2(\Gamma)\leftarrow H^s(\Gamma)}\,\|T\|^{\theta}_{H^s(\Gamma)\leftarrow H^s(\Gamma)} \leq \left(C_8 h_{\mathcal{G}}^s\right)^{1-\theta}(1 + C_{13})^\theta \\ &= C_{14} h_{\mathcal{G}}^{s-\sigma}\end{aligned}$$

with $C_{14} := C_8^{1-\sigma/s}\,(1 + C_{13})^{\sigma/s}$ and this implies the first estimate in (4.237b). $\square$

In Sect. 9 we will need an estimate of the surface metric on $\omega_{\mathbf{z}}$ compared with the two-dimensional Euclidean metric on $\widetilde{\omega}_{\mathbf{z}}$. Since $\omega_{\mathbf{z}}$ may consist of several panels, the local Assumptions 4.3.17 and 4.3.18 have to be supplemented by the following, more global Assumption 4.3.29 which states that Γ has to satisfy a cone-type condition and that the minimal angle of the surface mesh has to be bounded below by a positive constant (see Fig. 4.8).

Assumption 4.3.29. *1. For all $\tau \in \mathcal{G}$, $\mathbf{x} \in \Gamma\backslash\tau$ and $\mathbf{y} \in \tau$, there exist $c > 0$ and an $\mathbf{x}_0 \in \overline{\tau}$ such that*

$$\|\mathbf{x} - \mathbf{x}_0\| = \operatorname{dist}(\mathbf{x}, \tau) \quad \textit{and} \quad \|\mathbf{x} - \mathbf{y}\|^2 \geq c\left(\|\mathbf{x} - \mathbf{x}_0\|^2 + \|\mathbf{x}_0 - \mathbf{y}\|^2\right).$$

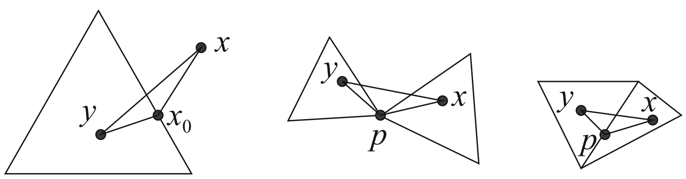


Fig. 4.8 Illustration of the cone and the angle condition for the surface mesh

2. *For all* $\tau, t \in \mathcal{G}$ *whose intersection consists of at most one point, there exists a point* $\mathbf{p}$ *of* t *such that*

$$\|\mathbf{x}-\mathbf{y}\| \geq c\left(\|\mathbf{x}-\mathbf{p}\|+\|\mathbf{p}-\mathbf{y}\|\right) \qquad \forall \mathbf{x} \in \tau, \forall \mathbf{y} \in t.$$

3. *For all* $\tau, t \in \mathcal{G}$ *with exactly one common edge* $\overline{\tau} \cap \overline{t} = E$ *and for all* $\mathbf{x} \in \tau$, $\mathbf{y} \in t$ *there exists a point* $\mathbf{p} \in E$ *such that*

$$\|\mathbf{y}-\mathbf{x}\| \geq c\left(\|\mathbf{y}-\mathbf{p}\|+\|\mathbf{p}-\mathbf{x}\|\right).$$

Lemma 4.3.30. *Let Assumption 4.3.29 be satisfied and let Assumption 4.3.17 or Assumption 4.3.18 hold. Then*

$$c\,\|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}\| \leq \frac{\operatorname{diam}\widetilde{\omega_{\mathbf{z}}}}{h_{\mathbf{z}}}\,\|\chi_{\mathbf{z}}(\tilde{\mathbf{x}})-\chi_{\mathbf{z}}(\tilde{\mathbf{y}})\| \leq C\,\|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}\| \quad \forall \tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \widetilde{\omega_{\mathbf{z}}},$$

where C *depends only on the global chart* χ *but is independent of the surface mesh.*

Proof. (a) Let $\tau \in \mathcal{G}_{\mathbf{z}}$ be a surface triangle with vertices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$. First, we will prove the statement for $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \tilde{\tau} = \chi_{\mathbf{z}}^{-1}(\tau)$.

Let $\tau^{\text{affine}} := \chi_\tau^{\text{affine}}(\hat{\tau})$ be the plane triangle with straight edges which interpolates τ in its vertices. Note that $\chi\left(\tau^{\text{affine}}\right) = \tau$. Hence

$$\chi(\mathbf{x})-\chi(\mathbf{y}) = \mathbf{J}_\chi(\mathbf{w})(\mathbf{x}-\mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in \tau^{\text{affine}}$$

where $\mathbf{J}_\chi \in \mathbb{R}^{3\times 3}$ is the Jacobi matrix of the global chart χ and $\mathbf{w}$ is some point in $\overline{\mathbf{xy}}$. Note that the largest and the smallest eigenvalues $\lambda_{\max}$ and $\lambda_{\min}$ of the positive definite Gram matrix $\mathbf{G}_\chi$ depend only on the global chart χ and are, in particular, independent of the discretization parameters. Thus

$$\sqrt{\lambda_{\min}}\,\|\mathbf{x}-\mathbf{y}\| \leq \|\chi(\mathbf{x})-\chi(\mathbf{y})\| \leq \sqrt{\lambda_{\max}}\,\|\mathbf{x}-\mathbf{y}\| \qquad \forall \mathbf{x}, \mathbf{y} \in \tau^{\text{affine}}.$$

Let $\mathbf{G}_\tau^{\text{affine}} \in \mathbb{R}^{2\times 2}$ denote the (constant) Gram matrix of $\chi_\tau^{\text{affine}}$. From Lemma 4.3.5 we conclude that

$$\left\|\chi_\tau^{\text{affine}}(\hat{\mathbf{x}})-\chi_\tau^{\text{affine}}(\hat{\mathbf{y}})\right\| = \left\langle \mathbf{G}_\tau^{\text{affine}}(\hat{\mathbf{x}}-\hat{\mathbf{y}}), (\hat{\mathbf{x}}-\hat{\mathbf{y}})\right\rangle^{1/2} \leq \sqrt{2}h_\tau\,\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|$$

for all $\hat{\mathbf{x}}, \hat{\mathbf{y}} \in \hat{\tau}$. Because the matrix $\mathbf{G}_\tau^{\text{affine}}$ is symmetric and positive definite, its minimal eigenvalue $\lambda_{\min}^{\text{affine}}$ can be expressed by

$$\lambda_{\min}^{\text{affine}} = \left\|\left(\mathbf{G}_\tau^{\text{affine}}\right)^{-1}\right\|^{-1}.$$

We employ Lemma 4.3.5 to obtain

$$\left\|\chi_\tau^{\text{affine}}(\hat{\mathbf{x}})-\chi_\tau^{\text{affine}}(\hat{\mathbf{y}})\right\| \geq c h_\tau\,\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|$$

for all $\hat{\mathbf{x}}, \hat{\mathbf{y}} \in \hat{\tau}$, where C depends only on the shape-regularity of the mesh. Thus we have proved that

$$c\sqrt{\lambda_{\min}}h_\tau \|\hat{\mathbf{x}} - \hat{\mathbf{y}}\| \leq \|\chi_\tau(\hat{\mathbf{x}}) - \chi(\hat{\mathbf{y}})\| \leq \sqrt{2\lambda_{\max}}h_\tau \|\hat{\mathbf{x}} - \hat{\mathbf{y}}\|$$

for all $\hat{\mathbf{x}}, \hat{\mathbf{y}} \in \hat{\tau}$. Finally, we replace $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ by $\eta^{\text{affine}}(\tilde{\mathbf{x}})$ and $\eta^{\text{affine}}(\tilde{\mathbf{y}})$. From (4.235) we derive the estimate for the largest eigenvalue $\lambda_\eta^{\max}$ of the Gram matrix $\mathbf{G}_\eta$ of η^{affine}

$$\sqrt{\lambda_\eta^{\max}} \overset{(4.235)}{\leq} \sqrt{2}\frac{\operatorname{diam}\tilde{\tau}}{2|\tilde{\tau}|} \leq C \operatorname{diam}^{-1}\tilde{\tau},$$

where C depends only on the shape-regularity constant and the global chart χ.

For the smallest eigenvalue we use

$$\mathbf{G}_\eta^{-1} = \begin{bmatrix} \|\tilde{\mathbf{e}}_1\|^2 & \langle\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\rangle \\ \langle\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\rangle & \|\tilde{\mathbf{e}}_2\|^2 \end{bmatrix},$$

where $\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{C}}$ denote the vertices of $\tilde{\tau}$ and $\tilde{\mathbf{e}}_1 = \widetilde{\mathbf{B}} - \widetilde{\mathbf{A}}$, $\mathbf{e}_2 = \widetilde{\mathbf{C}} - \widetilde{\mathbf{B}}$. Thus

$$\left\|\mathbf{G}_\eta^{-1}\right\| \leq 2 \operatorname{diam}\tilde{\tau}$$

and the minimal eigenvalue $\lambda_\eta^{\min}$ satisfies

$$\sqrt{\lambda_\eta^{\min}} = \left\|\mathbf{G}_\eta^{-1}\right\|^{-1/2} \geq \frac{1}{\sqrt{2}\operatorname{diam}\tilde{\tau}}.$$

The combination of these estimates leads to

$$c\frac{h_\tau}{\operatorname{diam}\tilde{\tau}}\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\| \leq \|\chi_{\mathbf{z}}(\tilde{\mathbf{x}}) - \chi_{\mathbf{z}}(\tilde{\mathbf{y}})\| \leq C\frac{h_\tau}{\operatorname{diam}\tilde{\tau}}\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\| \qquad \forall \tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \tilde{\tau}. \tag{4.256}$$

(b) We assume that $\mathcal{G}_{\mathbf{z}}$ contains more than one panel and consider the case that $\mathbf{x}$ and $\mathbf{y}$ belong to different panels $\tau, t \in \mathcal{G}_{\mathbf{z}}$. Note that

$$c\frac{h_{\mathbf{z}}}{\operatorname{diam}\widetilde{\omega_{\mathbf{z}}}} \leq \frac{h_\tau}{\operatorname{diam}\tilde{\tau}} \leq C\frac{h_{\mathbf{z}}}{\operatorname{diam}\widetilde{\omega_{\mathbf{z}}}} \qquad \forall \tau \in \mathcal{G}_{\mathbf{z}},$$

where c and C depend only on the global chart χ and the shape-regularity constant. Assumption 4.3.29 implies that one of the following two cases is satisfied:

(i) The panels τ and t share exactly one common edge $\overline{\tau} \cap \overline{t} = E$. Then there exists a point $\mathbf{p} \in E$ such that

$$\|\mathbf{y} - \mathbf{x}\| \geq c\left(\|\mathbf{y} - \mathbf{p}\| + \|\mathbf{p} - \mathbf{x}\|\right).$$

The combination of (4.256) and a triangle inequality leads to

$$\|\mathbf{y}-\mathbf{x}\| \geq \tilde{c}\frac{h_{\mathbf{z}}}{\operatorname{diam}\widetilde{\omega_{\mathbf{z}}}}\left(\|\tilde{\mathbf{y}}-\tilde{\mathbf{p}}\|+\|\tilde{\mathbf{p}}-\tilde{\mathbf{x}}\|\right) \geq \tilde{c}\frac{h_{\mathbf{z}}}{\operatorname{diam}\widetilde{\omega_{\mathbf{z}}}}\|\tilde{\mathbf{y}}-\tilde{\mathbf{x}}\|$$

with $\tilde{\mathbf{p}} := \chi_{\mathbf{z}}^{-1}(\mathbf{p})$. For the upper estimate we use $\overline{\tilde{\mathbf{x}}\tilde{\mathbf{y}}} \subset \widetilde{\omega_{\mathbf{z}}}$ since $\widetilde{\omega_{\mathbf{z}}}$ is convex. Let $(\widetilde{\mathbf{p}}_i)_{i=0}^{q}$ be the minimal number of points lying on $\overline{\tilde{\mathbf{x}}\tilde{\mathbf{y}}}$ such that

$$\tilde{\mathbf{p}}_0 = \tilde{\mathbf{x}}, \quad \tilde{\mathbf{p}}_q = \mathbf{y}, \quad \text{and} \quad \forall 1 \leq i \leq q : \overline{\tilde{\mathbf{p}}_{i-1}\tilde{\mathbf{p}}_i} \text{ is contained in some } \tilde{\tau} \subset \widetilde{\omega_{\mathbf{z}}}.$$

Let $\mathbf{p}_i = \chi_{\mathbf{z}}(\tilde{\mathbf{p}}_i)$, $1 \leq i < q$. Then the upper estimate follows from

$$\|\mathbf{y}-\mathbf{x}\| \leq \sum_{i=1}^{q}\|\mathbf{p}_i-\mathbf{p}_{i-1}\| \leq C\frac{h_{\mathbf{z}}}{\operatorname{diam}\widetilde{\omega_{\mathbf{z}}}}\sum_{i=1}^{q}\|\tilde{\mathbf{p}}_i-\tilde{\mathbf{p}}_{i-1}\| = C\frac{h_{\mathbf{z}}}{\operatorname{diam}\widetilde{\omega_{\mathbf{z}}}}\|\tilde{\mathbf{y}}-\tilde{\mathbf{x}}\|.$$

(ii) τ and t share exactly one common point $\{\mathbf{z}\} = \overline{\tau} \cap \overline{t}$. Then

$$\|\mathbf{x}-\mathbf{y}\| \geq c\left(\|\mathbf{x}-\mathbf{z}\|+\|\mathbf{z}-\mathbf{y}\|\right)$$

and the rest of the proof is just a repetition of the arguments as in Case i. □

Lemma 4.3.31. *Let Assumption 4.3.17 or Assumption 4.3.18 be satisfied. For* $\tau \in \mathcal{G}_{\mathbf{z}}$, *let* $\chi_{\mathbf{z},\tau} := \chi_{\mathbf{z}}|_{\tilde{\tau}}$, *where* $\tilde{\tau} := \chi_{\mathbf{z}}^{-1}(\tau)$. *Then, for any* $\mu \in \mathbb{N}_0^2$ *with* $k := |\mu|$,

$$\|\partial^{\mu}\chi_{\mathbf{z},\tau}\|_{\mathbf{L}^{\infty}(\tilde{\tau})} \leq C\left(\frac{h_{\mathbf{z}}}{\operatorname{diam}\widetilde{\omega_{\mathbf{z}}}}\right)^{k},$$

where C *depends only on* k, C_1 *as in (4.228),* C_{d} *as in (4.236), and the global chart* χ.

Proof. Recall that $\chi_{\mathbf{z},\tau} = \chi \circ \kappa^{\text{affine}}$, where $\kappa^{\text{affine}} = \chi_{\tau}^{\text{affine}} \circ \eta^{\text{affine}}$ is affine. As in the proof of Lemma 4.3.6, we use

$$\partial^{\mu}\left(\chi \circ \kappa^{\text{affine}}\right) = \sum_{\substack{\beta \in \mathbb{N}_0^2 \\ |\beta| = \mu_1}}\sum_{\substack{\nu \in \mathbb{N}_0^2 \\ |\nu| = \mu_2}}\frac{\mu!}{\beta!\nu!}\mathbf{a}_1^{\beta}\mathbf{a}_2^{\nu}\left(\partial^{\beta+\nu}\chi\right) \circ \kappa^{\text{affine}},$$

where $\mathbf{a}_1, \mathbf{a}_2$ are the column vectors of the Jacobi matrix of κ^{affine}, that is,

$$(\mathbf{a}_i)_j = \partial_j \kappa_i^{\text{affine}} = \sum_{k=1}^{2}\partial_k \chi_{\tau,i}^{\text{affine}}\partial_j \eta_j^{\text{affine}}.$$

We have $\left|\partial_k \chi_{\tau,i}^{\text{affine}}\right| \leq h_\tau$ and from (4.235) we conclude that $\left|\partial_i \eta_j^{\text{affine}}\right| \leq \operatorname{diam} \tilde{\tau} / (2 |\tilde{\tau}|)$. This leads to $\left|(\mathbf{a}_i)_j\right| \leq h_\tau \operatorname{diam} \tilde{\tau} / |\tilde{\tau}|$. Thus

$$\left|\partial^\mu \left(\chi \circ \kappa^{\text{affine}}\right)\right| \leq C \left(\frac{h_\tau \operatorname{diam} \tilde{\tau}}{|\tilde{\tau}|}\right)^k \leq C \left(C_1 C_{\mathrm{d}} \frac{h_{\mathbf{z}}}{\operatorname{diam} \widetilde{\omega_{\mathbf{z}}}}\right)^k,$$

where C depends only on k and the global chart χ. □

4.4 Inverse Estimates

The spaces $H^s(\Gamma)$ form a scale:

$$H^s(\Gamma) \subseteq H^t(\Gamma), \quad \text{for } t \leq s \tag{4.257}$$

with a continuous embedding: there exists some $C(s,t) > 0$ such that

$$\|u\|_{H^t(\Gamma)} \leq C(s,t) \|u\|_{H^s(\Gamma)}, \qquad \forall u \in H^s(\Gamma). \tag{4.258}$$

Note that the range of s and t may be bounded by the smoothness of the surface (see Sect. 2.4). In general, the inverse of this inequality is false.

Exercise 4.4.1. *Find a sequence of functions $(u_n)_{n\in\mathbb{N}} \in C^\infty([0,1])$ which contradicts the inverse of (4.258) for $s = 0$ and $t = 1$, i.e., which satisfies*

$$\lim_{n\to\infty} \|u_n\|_{H^1([0,1])} / \|u_n\|_{L^2([0,1])} = \infty.$$

However, for boundary element functions there is a valid inverse of (4.258), a so-called *inverse inequality*, where the constant C depends on the dimension of the boundary element space. In the following we will assume that the maximal mesh width h is bounded above by a global constant h_0. For example, we can choose $h_0 = \operatorname{diam} \Gamma$ or otherwise $h_0 = 1$ for sufficiently fine surface meshes. Recall the definition of $\mathbb{P}_k^\tau$ as in (4.67).

Theorem 4.4.2. *Let either Assumption 4.3.17 or Assumption 4.3.18 hold. We have for $0 \leq m \leq \ell$, all $\tau \in \mathcal{G}$ and all $v \in \mathbb{P}_k^\tau$:*

$$\|v\|_{H^\ell(\tau)} \leq C h_\tau^{m-\ell} \|v\|_{H^m(\tau)}.$$

The constant C depends only on h_0, ℓ, k and, for a polyhedral surface, on the shape-regularity of the mesh $\mathcal{G}$ via the constant $\kappa_{\mathcal{G}}$ from Definition 4.1.12. In the case of a curved surface it also depends on the derivatives of the global transformations χ, χ^{-1} up to the order k.

Proof. Owing to the h-independent equivalence of the norms $\|v\|_{H^\ell(\tau)}$ and $\|\tilde{v}\|_{H^\ell(\tau^{\text{affine}})}$ from Corollary 4.3.12, it suffices to consider the case of a plane polyhedral surface.

Case 1: $m = 0$. Since $\mathbb{P}_k^\tau$ is finite-dimensional, all norms on $\mathbb{P}_k^\tau$ are equivalent: There exists a positive constant C_ℓ such that for $0 \leq j \leq \ell$

$$\|\hat{v}\|_{H^j(\hat{\tau})} \leq C_\ell \|\hat{v}\|_{L^2(\hat{\tau})} \qquad \forall \hat{v} \in \mathbb{P}_k^{\hat{\tau}}.$$

With Lemma 4.3.6 or Lemma 4.3.13 it follows for all $v \in \mathbb{P}_k^\tau$ that

$$|v|_{H^j(\tau)} \leq C_1 h_\tau^{1-j} |\hat{v}|_{H^j(\hat{\tau})} \leq C_\ell C_1 h_\tau^{1-j} \|\hat{v}\|_{L^2(\hat{\tau})} \leq C_\ell C_1 C_2 h_\tau^{-j} \|v\|_{L^2(\tau)}.$$

For the $\|\cdot\|_{H^\ell}$-norm, by summing the squares of the seminorm we obtain

$$\|v\|_{H^\ell(\tau)} \leq C h_\tau^{-\ell} \|v\|_{L^2(\tau)}, \tag{4.259}$$

where C depends on ℓ, k and the upper bound of the mesh width h_0.

Case 2: $0 < m \leq \ell$. For $\ell - m \leq n \leq \ell$ and $|\alpha| = n$ we write $\partial^\alpha v = \partial^\beta \partial^{\alpha-\beta}$ with $|\beta| = \ell - m$ and $\beta \leq \alpha$ componentwise. Then with Case 1 we have

$$\|\partial^\alpha v\|_{L^2(\tau)} \leq \left|\partial^{\alpha-\beta} v\right|_{H^{\ell-m}(\tau)} \leq C h_\tau^{m-\ell} \left\|\partial^{\alpha-\beta} v\right\|_{L^2(\tau)} \leq C h_\tau^{m-\ell} |v|_{H^{n-\ell+m}(\tau)}.$$

Since $|\alpha| = n$ was arbitrary, this result and $n - \ell + m \leq m$ together yield

$$|v|_{H^n(\tau)} \leq C h_\tau^{m-\ell} |v|_{H^{n-\ell+m}(\tau)} \leq C h_\tau^{m-\ell} \|v\|_{H^m(\tau)} \tag{4.260}$$

for an arbitrary $\ell - m \leq n \leq \ell$. (Note that the constant C in (4.260) depends on n, m, and ℓ. However, n and m are from the finite set $\{0, 1, \ldots, \ell\}$ and – by taking the maximum over n and m – results in a constant C which does not depend on n and m but on ℓ instead.) Inequality (4.259) for $\ell \leftarrow \ell - m$ as well as Estimate (4.260) finally yield the assertion

$$\begin{aligned}
\|v\|_{H^\ell(\tau)}^2 &= \|v\|_{H^{\ell-m}(\tau)}^2 + \sum_{n=\ell-m+1}^{m} |v|_{H^n(\tau)}^2 \\
&\leq C \left\{ h_\tau^{2(m-\ell)} \|v\|_{L^2(\tau)}^2 + \sum_{n=\ell-m+1}^{m} h_\tau^{2(m-\ell)} \|v\|_{H^m(\tau)}^2 \right\} \\
&\leq C h_\tau^{2(m-\ell)} \|v\|_{H^m(\tau)}^2.
\end{aligned}$$

□

The global version of Theorem 4.4.2 requires the quasi-uniformity of the surface mesh $\mathcal{G}$.

Theorem 4.4.3. *Let either Assumption 4.3.17 or Assumption 4.3.18 hold. Then we have for all $t, s \in \{0, 1\}$, $t \leq s$, the estimate*

$$\forall v \in S_{\mathcal{G}}^{p,0} : \|v\|_{H^s(\Gamma)} \leq C h^{t-s} \|v\|_{H^t(\Gamma)}. \tag{4.261}$$

The constant C depends only on h_0, p and, for a polyhedral surface, on the shape-regularity and quasi-uniformity of the mesh $\mathcal{G}$ via the constants $\kappa_{\mathcal{G}}$ and $q_{\mathcal{G}}$ from Definitions 4.1.12 and 4.1.13. In the case of a curved surface it also depends on the derivatives of the global transformations χ, χ^{-1} up to the order k.

Proof. From Theorem 4.4.2 we have

$$\begin{aligned}\|v\|_{H^s(\Gamma)}^2 &= \sum_{\tau \in \mathcal{G}} \|v\|_{H^s(\tau)}^2 \leq C \sum_{\tau \in \mathcal{G}} h_\tau^{2(t-s)} \|v\|_{H^t(\tau)}^2 \leq C \left(\min_{\tau \in \mathcal{G}} h_\tau\right)^{2(t-s)} \|v\|_{H^t(\Gamma)}^2 \\ &\leq \left(C q_{\mathcal{G}}^{2(s-t)}\right) h^{2(t-s)} \|v\|_{H^t(\Gamma)}^2 .\end{aligned}$$

□

Theorem 4.4.3 can be generalized in various ways. In the following we will cite results from [75].

Remark 4.4.4. *(a) Theorem 4.4.3 holds for all $t, s \in \mathbb{R}$ with $0 \leq t \leq s \leq 1$ or $-1 \leq t \leq 0 \wedge s = 0$ (see [75, Theorems 4.1, 4.6]).*
(b) Theorem 4.4.3 is valid for the space $S_{\mathcal{G}}^{p,-1}$ for all $t, s \in \mathbb{R}$ with $t = 0 \wedge 0 \leq s < 1/2$ or $-1 \leq t \leq 0 \wedge s = 0$ (see [75, Theorems 4.2, 4.6]).

We will also require estimates between different L^p-norms and discrete ℓ^p-norms for boundary element functions and, thus, we again start with a local result. Here we will always consider the situation where a Lagrange basis is chosen for $\mathbb{P}_k^{\hat{\tau}}$ on $\hat{\tau}$. $\Sigma_\kappa = \left\{\widehat{\mathbf{P}}_i : i \in \iota_k^{\hat{\tau}}\right\}$ denotes the set of nodal points on $\widehat{\tau}$. The Lagrange basis $\left(\widehat{N}_i\right)_{i \in \iota_k^{\hat{\tau}}}$ of $\mathbb{P}_k^{\hat{\tau}}$ satisfies

$$\widehat{N}_i \left(\widehat{\mathbf{P}}_j\right) = \delta_{i,j} \qquad \forall i, j \in \iota_k^{\hat{\tau}}.$$

A vector of coefficients $\mathbf{w} := (w_i)_{i \in \iota_k^{\hat{\tau}}}$ is put into relation with the associated polynomial $\hat{w} \in \mathbb{P}_k^{\hat{\tau}}$ on the reference element by means of

$$\hat{w} := \widehat{P}\mathbf{w} := \sum_{i \in \iota_k^{\hat{\tau}}} w_i \widehat{N}_i .$$

We define the "lifted" function

$$w := P_\tau \mathbf{w} := \sum_{i \in \iota_k^{\hat{\tau}}} w_i N_i \quad \text{with} \quad N_i = \widehat{N}_i \circ \chi_\tau^{-1}$$

analogously.

Theorem 4.4.5. *Let either Assumption 4.3.17 or Assumption 4.3.18 hold. For all* $\tau \in \mathcal{G}$ *and all* $\mathbf{w} := (w_i)_{i \in \iota_k^{\hat{\tau}}}$ *we have*

$$\tilde{c} h_\tau \|\mathbf{w}\|_{\ell^2} \le \|P_\tau \mathbf{w}\|_{L^2(\tau)} \le \widetilde{C} h_\tau \|\mathbf{w}\|_{\ell^2} .$$

The constants $\tilde{c}$ *and* $\widetilde{C}$ *depend on the parameters qualitatively in the same way as does* C *in Theorem 4.4.3.*

Proof. From Lemma 4.3.6 or Lemma 4.3.13 we have

$$c h_\tau \|\hat{w}\|_{L^2\left(\widehat{\tau}\right)} \le \|w\|_{L^2(\tau)} \le C h_\tau \|\hat{w}\|_{L^2\left(\widehat{\tau}\right)} \quad \text{with} \quad \hat{w} := w \circ \chi_\tau .$$

Since all norms are equivalent on $\mathbb{P}_k^{\hat{\tau}}$, we have

$$c_k \|\hat{w}\|_{H^{k+1}(\hat{\tau})} \le \|\hat{w}\|_{L^2(\hat{\tau})} \le C_k \|\hat{w}\|_{H^{k+1}\left(\widehat{\tau}\right)} .$$

The equivalence of the $H^{k+1}(\hat{\tau})$-norm and the $[\cdot]_{k+1}$-norm follows from Lemma 4.3.1. Since $\hat{w} \in \mathbb{P}_k^{\hat{\tau}}$,

$$[\hat{w}]_{k+1} = |\hat{w}|_{H^{k+1}(\hat{\tau})} + \sum_{\mathbf{z} \in \Sigma_k} |\hat{w}(\mathbf{z})| = \sum_{\mathbf{z} \in \Sigma_k} |\hat{w}(\mathbf{z})| = \sum_{i \in \iota_p^{\hat{\tau}}} |w_i| = \|\mathbf{w}\|_{\ell^1} . \quad (4.262)$$

Since $\sharp \Sigma_k$ is finite, there exist positive constants c, C depending only on the cardinality of Σ_k, i.e., on k, such that

$$c \|\mathbf{w}\|_{\ell^2} \le \|\mathbf{w}\|_{\ell^1} \le C \|\mathbf{w}\|_{\ell^2} .$$

Combining all these results, we have thus proved that

$$\tilde{c} h_\tau \|\mathbf{w}\|_{\ell^2} \le \|w\|_{L^2(\tau)} \le \widetilde{C} h_\tau \|\mathbf{w}\|_{\ell^2} .$$

□

Corollary 4.4.6. *Let the conditions from Theorem 4.4.5 be satisfied. Then*

$$\hat{c} h_\tau \|w\|_{L^\infty(\tau)} \le \|w\|_{L^2(\tau)} \le \widehat{C} h_\tau \|w\|_{L^\infty(\tau)}$$

for all $w \in \mathbb{P}_k^\tau$. The constants $\hat{c}, \widehat{C}$ qualitatively depend on the parameters in the same way as do $\tilde{c}, \widetilde{C}$ in Theorem 4.4.3.

Proof. If we combine Theorem 4.4.5 with the norm equivalence on finite-dimensional spaces for $\mathbf{w} = (w_i)_{i \in \iota_p^{\hat{\tau}}}$ and $w = P_\tau \mathbf{w}$ it follows that

$$\|w\|_{L^2(\tau)} \leq C h_\tau \|\mathbf{w}\|_{\ell^2} \leq \widehat{C} h_\tau \|\mathbf{w}\|_{\ell^\infty} \leq \widehat{C} h_\tau \|w\|_{L^\infty(\tau)}.$$

Conversely, with the notation from the proof of Theorem 4.4.5 we have

$$\begin{aligned}\|w\|_{L^\infty(\tau)} &= \|\hat{w}\|_{L^\infty(\hat{\tau})} \leq C \|\hat{w}\|_{H^{k+1}(\hat{\tau})} \leq C' [\hat{w}]_{k+1} \overset{(4.262)}{=} C' \|\mathbf{w}\|_{\ell^1} \\ &\leq C'' \|\mathbf{w}\|_{\ell^\infty} \leq C''' \|\mathbf{w}\|_{\ell^2}.\end{aligned}$$

Note that the constants in this estimate depend on the cardinality of Σ_k, i.e., on k. From Theorem 4.4.5 we thus have the lower bound. $\square$

The global version of Theorem 4.4.5 shows an equivalence between boundary element functions and the associated coefficient vector. Let $(b_i)_{i=1}^N$ be the Lagrange basis of the boundary element space S. We define the operator $P : \mathbb{R}^N \to S$ for $\mathbf{w} = (w_i)_{i=1}^N$ by

$$P\mathbf{w} = \sum_{i=1}^N w_i b_i.$$

Theorem 4.4.7. *Let Assumption 4.3.17 or Assumption 4.3.18 hold. Then for all $\mathbf{w} \in \mathbb{R}^N$*

$$\check{c} h \|\mathbf{w}\|_{\ell^2} \leq \|P\mathbf{w}\|_{L^2(\Gamma)} \leq \check{C} h \|\mathbf{w}\|_{\ell^2}.$$

The constants $\check{c}, \check{C}$ qualitatively depend on the parameters in the same way as $\tilde{c}, \widetilde{C}$ do in Theorem 4.4.5.

Proof. Let $\mathbf{w} \in \mathbb{R}^N$ be the coefficient vector of the boundary element function $w = P\mathbf{w}$. For $\tau \in \mathcal{G}$ we can associate a global index $\operatorname{ind}(m, \tau) \in \{1, 2, \ldots, N\}$ on τ with every local degree of freedom $m \in \iota_k^{\hat{\tau}}$. We set $\mathbf{w}_\tau := (\mathbf{w}_{\tau,m})_{m \in \iota_k^{\hat{\tau}}} := \left(\mathbf{w}_{\operatorname{ind}(m,\tau)}\right)_{m \in \iota_k^{\hat{\tau}}}$. With Theorem 4.4.5 we obtain

$$\|P\mathbf{w}\|_{L^2(\Gamma)}^2 = \sum_{\tau \in \mathcal{G}} \|P_\tau \mathbf{w}\|_{L^2(\tau)}^2 \leq C h^2 \sum_{\tau \in \mathcal{G}} \|\mathbf{w}_\tau\|_{\ell^2}^2.$$

The constant

$$M := \max_{i \in \{1,2,\ldots,N\}} \sharp \left\{(m, \tau) \in \iota_p^{\hat{\tau}} \times \mathcal{G} : i = \operatorname{ind}(m, \tau)\right\}$$

depends only on the polynomial degree k and on the shape-regularity of the surface mesh. It thus follows that

$$\|P\mathbf{w}\|^2_{L^2(\Gamma)} \leq CMh^2 \|\mathbf{w}\|^2_{\ell^2}.$$

The lower bound can be found in a similar way. □

Corollary 4.4.8. *Let either Assumption 4.3.17 or Assumption 4.3.18 hold and let $(b_i)_{i\in\mathcal{I}}$ denote the nodal basis for the boundary element space S. Then*

$$\|b_i\|_{L^\infty(\Gamma)} \leq \breve{C}_1. \tag{4.263}$$

The constant $\breve{C}_1$ depends only on the shape-regularity of the mesh and the polynomial degree of S.

If $S = S^{k,0}_{\mathcal{G}}$ for some $k \geq 1$ then

$$|b_i|_{W^{1,\infty}(\Gamma)} := \|\nabla_S b_i\|_{L^\infty(\Gamma)} \leq \breve{C}_2 h_\tau^{-1} \quad \text{for any } \tau \subset \operatorname{supp} b_i. \tag{4.264}$$

The full $W^{1,\infty}(\Gamma)$-norm is given by $\|\cdot\|_{W^{1,\infty}(\Gamma)} := \max\left\{\|\cdot\|_{L^\infty(\Gamma)}, |\cdot|_{W^{1,\infty}(\Gamma)}\right\}$ and hence

$$\|b_i\|_{W^{1,\infty}(\Gamma)} \leq \breve{C}_3 h_\tau^{-1} \quad \text{for any } \tau \subset \operatorname{supp} b_i. \tag{4.265}$$

Proof. Let $\mathbf{e}_i \in \mathbb{R}^{\mathcal{I}}$ denote the vector with $(\mathbf{e}_i)_i = 1$ and $(\mathbf{e}_i)_j = 0$ otherwise, i.e., $b_i = P\mathbf{e}_i$, Let $\tau \subset \operatorname{supp} b_i$. The combination of Corollary 4.4.6 and Theorem 4.4.5 leads to

$$\|b_i\|_{L^\infty(\tau)} \leq (\hat{c}h_\tau)^{-1} \|b_i\|_{L^2(\tau)} \leq \widetilde{C}/\hat{c}.$$

Because $b_i|_\tau = 0$ for all $\tau \in \mathcal{G}_i$ with $\tau \not\subset \operatorname{supp} b_i$ we have proved (4.263).

For the proof of the second estimate we observe that – as in the proof of Theorem 4.4.3 – it suffices to consider plane panels with straight edges. Hence $\nabla_S b_i$ is a polynomial on every panel τ so that

$$\begin{aligned}\hat{c}h_\tau \|\nabla_S b_i\|_{L^\infty(\tau)} &\overset{\text{Cor. 4.4.6}}{\leq} \|\nabla_S b_i\|_{L^2(\tau)} \overset{\text{Theo. 4.4.2}}{\leq} Ch_\tau^{-1} \|b_i\|_{L^2(\tau)} \\ &\overset{\text{Theo. 4.4.5}}{\leq} C\widetilde{C}\, \|\mathbf{e}_i\|_{\ell^2} = C\widetilde{C}\end{aligned}$$

from which the assertion follows. □

We can also analyze how far the constants in the norm equivalences depend on the mesh width h in the case of the ℓ^p and $L^p(\Gamma)$-norms with $1 \leq p \leq \infty$. Here we will only require the cases $p = 2$ and $p = \infty$ and refer to [75] for the more general case.

4.5 Condition of the System Matrices

One of the first applications of the inverse inequalities is the estimation of the condition of the system matrices of the integral operators.

Lemma 4.5.1. *Let Assumption 4.3.17 or Assumption 4.3.18 hold. Let* $\mathbf{K}$ *be the system matrix associated with the Galerkin discretization of the single layer operator* V *for the Laplace problem. Then we have*

$$\operatorname{cond}_2(\mathbf{K}) \leq C h^{-1}.$$

The constant C *depends only on the polynomial degree* p *and the shape-regularity and the quasi-uniformity of the surface mesh* $\mathcal{G}$*, more specifically on the constants* $\kappa_{\mathcal{G}}$ *and* $q_{\mathcal{G}}$ *from Definitions 4.1.12 and 4.1.13. In the case of curved surfaces it also depends on the derivatives of the global transformations* χ*,* χ^{-1} *up to the order* k*.*

Proof. Since $\mathbf{K}$ is symmetric and positive definite, we have

$$\operatorname{cond}_2(\mathbf{K}) = \frac{\lambda_{\max}(\mathbf{K})}{\lambda_{\min}(\mathbf{K})}.$$

In the following we will thus estimate the eigenvalues of $\mathbf{K}$. It follows from the continuity and the $H^{-1/2}$-ellipticity of the bilinear form $(V\cdot,\cdot)_0 : H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to \mathbb{K}$ that there exist two positive constants γ and C_c such that

$$\gamma \|u\|^2_{H^{-1/2}(\Gamma)} \leq (Vu, u)_0 \leq C_c \|u\|^2_{H^{-1/2}(\Gamma)} \qquad \forall u \in H^{-1/2}(\Gamma).$$

From this it follows with Theorem 4.4.7 that

$$\begin{aligned}\lambda_{\max}(\mathbf{K}) &= \max_{\mathbf{w}=(w_i)_i \in \mathbb{R}^N \setminus \{0\}} \frac{\langle \mathbf{Kw}, \mathbf{w}\rangle}{\|\mathbf{w}\|^2} \leq C h^2 \max_{w \in S\setminus\{0\}} \frac{(Vw, w)_0}{\|w\|^2_{L^2(\Gamma)}} \\ &\leq C h^2 C_c \max_{w \in S\setminus\{0\}} \frac{\|w\|^2_{H^{-1/2}(\Gamma)}}{\|w\|^2_{L^2(\Gamma)}} \leq C h^2 C_c.\end{aligned}$$

By Theorem 4.4.7 and Remark 4.4.4 we have for the smallest eigenvalue

$$\begin{aligned}\lambda_{\min}(\mathbf{K}) &= \min_{\mathbf{w}=(w_i)_i \in \mathbb{R}^N \setminus \{0\}} \frac{\langle \mathbf{Kw}, \mathbf{w}\rangle}{\|\mathbf{w}\|^2} \geq C h^2 \min_{w \in S\setminus\{0\}} \frac{(Vw, w)_0}{\|w\|^2_{L^2(\Gamma)}} \\ &\geq C h^2 \gamma \min_{w \in S\setminus\{0\}} \frac{\|w\|^2_{H^{-1/2}(\Gamma)}}{\|w\|^2_{L^2(\Gamma)}} \geq C' h^2 \gamma h.\end{aligned}$$

Thus

$$\lambda_{\max}(\mathbf{K}) / \lambda_{\min}(\mathbf{K}) \leq C h^{-1}$$

and the lemma follows. □

Exercise 4.5.2. *Show that the system matrix* $\mathbf{K}$ *associated with the hypersingular operator also satisfies the estimate*

$$\operatorname{cond}_2(\mathbf{K}) \leq C h^{-1}$$

under the conditions of Lemma 4.5.1.

Remark 4.5.3. *For the condition of the* mass matrix $\mathbf{M} := \left(\left(b_i, b_j \right)_{L^2(\Gamma)} \right)_{i,j=1}^N$ *we have*

$$\operatorname{cond}_2(\mathbf{M}) \leq C.$$

Proof. Since

$$\langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle = (P\mathbf{w}, P\mathbf{w})_{L^2(\Gamma)}$$

we can apply Theorem 4.4.7:

$$\check{c}^2 h^2 \leq \min_{\mathbf{w} \in \mathbb{R}^N \setminus \{0\}} \frac{\langle \mathbf{M}\mathbf{w}, \mathbf{w} \rangle}{\|\mathbf{w}\|^2} \leq \max_{\mathbf{w} \in \mathbb{R}^N \setminus \{0\}} \frac{\langle \mathbf{M}\mathbf{w}, \mathbf{w} \rangle}{\|\mathbf{w}\|^2} \leq \check{C}^2 h^2,$$

from which we have the estimate of the condition with $C = \check{C}^2 / \check{c}^2$. □

Estimating the condition of system matrices for equations of the second kind is more problematic, as the stability of the Galerkin discretization for these equations is in many cases still an open question. If we assume that the h-independent stability of the discrete operators is given, the condition of the system matrices for equations of the second kind can be determined in terms of an h-independent constant in the same way as before.

4.6 Bibliographical Remarks and Further Results

In the present chapter, we introduced spaces of piecewise polynomial functions on the boundary manifold Γ, and established approximation properties of these spaces, as the meshwidth h tends to zero, in several function spaces of Sobolev type on Γ. These boundary element spaces are, in a sense, Finite Element spaces on the boundary surface Γ. We also presented a general framework for the convergence analysis of Galerkin boundary element methods, in particular necessary and sufficient conditions for the quasi-optimality of the Galerkin solutions to hold.

For reasons of space, our presentation does not cover the most general cases. For example, the surface meshes upon which the boundary element spaces are built did not allow for local mesh refinement or, more importantly, for *anisotropic local refinements* for example in the vicinity of edges (see, e.g., [75, 87, 234]).

Most of our results do extend to so-called graded, anisotropic meshes (cf. [104, 107, 108]). In addition, besides mesh refinement, analogs of spectral methods or even a combination of mesh refinement and order increase, the so-called *hp-Version BEM*, is conceivable (cf. [222] and the references therein).

Further, for particular classes of boundary integral equations, special choices of subspaces may yield large gains in accuracy versus number of degrees of freedom. Let us mention, for example, the case of high frequency acoustic scattering. Here, the stability of the boundary integral operators depends, of course, on the problem's wave number but, in addition, also the solutions contain high-frequency components which are smooth, but highly oscillatory at large wave numbers, and therefore poorly captured by standard boundary element spaces, unless the fine scale of the unknown functions on the boundary is resolved by sufficient mesh refinement. This strategy may lead, however, to prohibitively large numbers of degrees of freedom. A better approach may be to augment the standard boundary element spaces by explicitly known, dominant asymptotic components of the unknown solution. In high frequency acoustics and electromagnetics, in particular for BIEs obtained from the direct method (where the unknowns are Cauchy data of the domain unknowns), strong results on the asymptotic structure of the solution are available from geometrical optics. These can be used to build boundary element spaces with no or a reduced preasymptotic convergence regime at high wave numbers. We refer e.g. to [5, 57, 153] for recent work on wave number independent Galerkin BEM for acoustics problems.

In this chapter, and throughout this book, we focused on *Galerkin* BEM. We do emphasize, however, that the alternative *collocation* BEM do constitute a powerful competition; for collocation BEM on polyhedra, however, the theory of stability and quasi-optimality is much less mature that in the Galerkin case. Still, since collocation methods do not require the numerical evaluation of double surface integrals, they offer a substantial gain in accuracy versus CPU time.

For this reason, in recent years substantial work has been devoted to collocation based BIEs for high frequency acoustic and electromagnetic scattering. We mention in particular the work of O. Bruno et al. (e.g. [34,35,161]) which is a collocation type boundary element method which combines incorporation of high frequency asymptotics with a degenerate coordinate transformation of the surface in the presence of edges or vertices and a Nyström type collocation procedure. The mathematical error analysis of this method is in progress.

The a priori asymptotic error bounds for Galerkin BEM developed in Sect. 4.2 show that Galerkin BEM exhibit superconvergence in negative Sobolev norms on Γ. This allows us, in particular, to deduce corresponding results for *postprocessed* Galerkin approximations which can be obtained as smooth functionals of the solution. Importantly, *the insertion of the Galerkin solution into the representation formula* is such a postprocessing operation. Therefore superconvergent pointwise approximations of the solution to the underlying boundary value problem at interior points of the domain result usually from Galerkin boundary element approximations. Note that our analysis in Sect. 4.2 reveals the crucial role of Galerkin orthogonality of the discretization in the derivation of superconvergence estimates in negative order norms (indeed, for other discretization schemes such as collocation or Nyström methods, such superconvergence results either do not hold or only with a much smaller gain in asymptotic convergence order). We finally note that the superconvergence error bounds for the solution at points $\mathbf{x}$ in the interior of both the domain Ω or its complement deteriorate as $\mathbf{x}$ approaches Γ. Nevertheless, this deterioration can be remedied and postprocessing procedures can be designed to recover superconvergent solution values and normal and tangential derivatives (required, for example, in shape optimization or uncertainty quantification) of arbitrary order from the Galerkin solution such that the superconvergence bounds are uniform in the distance of $\mathbf{x}$ to the boundary Γ. For the details, we refer to [213].

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