Chapter XII.

The moduli space of stable curves

1. Introduction.

In this chapter we shall construct the moduli space $\overline{M}_{g,n}$ of stable n-pointed curves of genus g and look at its structure from various points of view. As a set, this space consists of all isomorphism classes of stable n-pointed curves of genus g. First we will put on $\overline{M}_{g,n}$ a structure of analytic space, and then we will see that this analytic space has a natural structure of an algebraic space. Only in Chapter XIV we will prove that $\overline{M}_{g,n}$ is, indeed, a projective variety. Finally we shall show that $\overline{M}_{g,n}$ is just a coarse reflection of a more fundamental object, the moduli stack $\overline{\mathcal{M}}_{g,n}$ of stable n-pointed curves of genus g.

Throughout the chapter we will make constant use of the construction of algebraic Kuranishi families, carried out in Chapter XI, which we now recall. Let $(C; p_1, \dots, p_n)$ be a stable, n-pointed curve of genus g. Then there exists an *algebraic* deformation

$$\begin{array}{c} \mathcal{C} \\ \big\downarrow{\scriptstyle \pi} \\ (X, x_0) \end{array} \qquad \sigma_i : X \to \mathcal{C}\,, \;\; i = 1, \dots, n\,, \qquad C = \pi^{-1}(x_0) \tag{1.1}$$

of $(C; p_1, \dots, p_n)$ having the following properties. Denote by C_y the fiber of π over y and let G_y be the automorphism group of $(C_y; \sigma_1(y), \dots, \sigma_n(y))$. Then

a) X is affine;
b) the family is Kuranishi at every point of X;
c) the action of the group G_{x_0} on the central fiber extends to compatible actions on $\mathcal{C}$ and X;
d) for every $y \in X$, the automorphism group G_y is equal to the stabilizer of y in G_{x_0}. In particular, G_y is a subgroup of G_{x_0};
e) for every $y \in X$, there is a G_y-invariant neighborhood U of y in X, for the analytic topology, such that any isomorphism (of n-pointed curves) between fibers over U is induced by an element of G_y.

A family with the above properties is called a standard algebraic Kuranishi family, while its restriction to the analytic neighborhood U is simply called a standard Kuranishi family.

E. Arbarello et al., *Geometry of Algebraic Curves*,
Grundlehren der mathematischen Wissenschaften 268,
DOI 10.1007/978-3-540-69392-5_4,

In Section 2 we begin by putting on $\overline{M}_{g,n}$ a natural structure of normal analytic space of dimension $3g-3+n$. This is done by patching together quotients of bases of standard Kuranishi families modulo the action of the automorphism groups of the central fibers. This patching procedure is based on the universal property of Kuranishi families. The local analytic neighborhoods in $\overline{M}_{g,n}$ are therefore of the type B/G, where B is a bounded simply connected domain in $\mathbb{C}^{3g-3+n}$, and G is a finite group acting linearly on it. Built in the definition of the analytic structure of $\overline{M}_{g,n}$ is its versal property: for every analytic family $\mathcal{Y} \to T$ of n-pointed, stable curves of genus g, the moduli map $t \mapsto [X_t]$ is an analytic morphism from T to $\overline{M}_{g,n}$. Using the fact that there exist stable curves with nontrivial automorphism group, we then proceed to show that there cannot exist a universal family of curves over $\overline{M}_{g,n}$. There is a surrogate which, in several practical applications, is almost as good as the nonexistent universal family. It consists in a family of stable n-pointed genus g curves

$$\eta : \mathcal{X} \to Z\,,$$

parameterized by a normal scheme Z, whose moduli map

$$m : Z \to \overline{M}_{g,n} \tag{1.2}$$

is finite and surjective. A first application of the existence of this family is the following. Using the valuative criterion for properness together with stable reduction, we prove that Z, and therefore $\overline{M}_{g,n}$, is compact. Another important application will come in Chapter XIV. There, in order to prove that the analytic space $\overline{M}_{g,n}$ is a projective variety, we will use the scheme Z as an intermediary. Indeed, using Seshadri's criterion and a small amount of geometric invariant theory, we will show that the line bundle $\eta_*(c_1(\omega_\eta)^2)$ is the first Chern class of an ample line bundle, proving that Z, and therefore $\overline{M}_{g,n}$, is projective.

In order to construct the family (1.2) and in particular to introduce the scheme Z, we need to make a digression on algebraic spaces. We do this in Section 3. Indeed, implicit in the algebraic nature of the Kuranishi families is the fact that $\overline{M}_{g,n}$ has a natural structure of algebraic space. To see this, recall that, as a set, $\overline{M}_{g,n}$ is the quotient of the Hilbert scheme $\mathrm{H}_{\nu,g,n}$ modulo the action of a projective group G and that the base of an algebraic Kuranishi family was constructed by taking slices in $\mathrm{H}_{\nu,g,n}$ that are transversal to the orbits of G. By compactness one can cover $\mathrm{H}_{\nu,g,n}$ with the images of finitely many sets of the type $G \times X_i$, $i = 1, \dots, N$, where X_i is the base of an algebraic Kuranishi family $\pi_i : \mathcal{C}_i \to X_i$. Let G_i be the automorphism group of the central fiber of π_i. The properties of algebraic Kuranishi families imply that the natural map $X_i/G_i \to \overline{M}_{g,n}$ is étale. Set

$$Y_i = X_i/G_i\,, \quad i = 1, \dots, N, \qquad X = \coprod_{i=1}^{N} X_i, \qquad Y = \coprod_{i=1}^{N} Y_i. \tag{1.3}$$

Then the étale map

$$\varphi : Y \longrightarrow \overline{M}_{g,n} \tag{1.4}$$

is surjective. Now, by definition, a separated algebraic space is an étale morphism $\psi : S \to M$ from an affine scheme to an analytic space such that $S \times_M S$ is a closed subscheme of $S \times S$. The scheme S should be considered as a sort of algebraic atlas for M, while the subscheme $S \times_M S$ should be regarded as the set of compatibility conditions for this atlas. In order to prove that $\overline{M}_{g,n}$ is a separated algebraic space, one then needs to show that $R = Y \times_{\overline{M}_{g,n}} Y$ is Zariski-closed in $Y \times Y$. This turns out to be an immediate consequence of the properness of the natural projection $q : \mathbf{I} \longrightarrow X \times X$, where

$$\mathbf{I} = \{(x, x', g) : x, x' \in X,\ \ g \in G,\ \ x' = gx\},$$

which is a consequence of Theorem (5.1) of Chapter X.

The construction of moduli spaces of curves as algebraic spaces shows implicitly that moduli spaces like $\overline{M}_{g,n}$ are also orbifolds. Orbifolds are the differential-geometric counterparts of stacks. Essentially, an orbifold is an analogue of a differentiable variety in which local charts are not open immersions, but rather quotients of open subsets of $\mathbb{R}^n$ by the actions of finite groups. In Section 4 we give the basic definitions and examples of the theory of orbifolds. Furthermore, we show how de Rham theory naturally extends to orbifolds. This is important in view of the fact that, when developing the intersection theory of $\overline{M}_{g,n}$, it will be useful to express intersection numbers as integrals of top degree differential forms over $\overline{M}_{g,n}$.

Section 5 contains a utilitarian introduction to stacks, closely motivated by the case of moduli of curves.[2] In studying Kuranishi families or moduli spaces of curves, we constantly have to deal with the automorphism group of a curve. It is the presence of curves with a nontrivial automorphism group that prevents the moduli space from being smooth and a universal family from existing. Using stacks is a way of effectively keeping track of these automorphism groups. When thinking of moduli spaces as stacks, the automorphism groups become an essential part of the definition.

A stack is, first of all, a groupoid. A groupoid is a pair $\mathcal{M} = (\mathcal{C}, p)$, where $\mathcal{C}$ is a category, and

$$p : \mathcal{C} \to Sch/S$$

[2]Warning: in Sections 5 through 8 we deviate from our general convention that "scheme" stands for "scheme of finite type over $\mathbb{C}$," and allow general schemes.

is a functor. The "fibers" of p are supposed to be categories in which all morphisms are isomorphisms. The fiber over a scheme T is denoted by $\mathcal{M}(T)$. To better understand the properties a stack is asked to satisfy, one should keep in mind the example of the moduli stack $\overline{\mathcal{M}}_{g,n}$ of stable, n-pointed, genus g curves. In this case we take as $\mathcal{C}$ the category in which the objects are the families

$$\begin{array}{c} \mathcal{X} \\ \downarrow \xi \\ T \end{array}$$

of stable n-pointed curves of genus g and in which a morphism

$$\varphi : \xi' \to \xi$$

between a family $\xi' : \mathcal{X}' \to T'$ and a family $\xi : \mathcal{X} \to T$ is a commutative diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "\xi'"'] & \mathcal{X} \arrow[d, "\xi"] \\ T' \arrow[r, "f"] & T \end{tikzcd}$$

inducing an isomorphism $\mathcal{X}' \cong T' \times_T \mathcal{X}$. The functor p assigns to a family $\xi : \mathcal{X} \to T$ its parameter space T:

$$p(\xi) = T\,.$$

In the case of $\overline{\mathcal{M}}_{g,n}$, given a scheme T, the category $\overline{\mathcal{M}}_{g,n}(T)$ is simply the category of families of stable n-pointed curves of genus g parameterized by T, in which a morphism

$$\varphi : \xi' \to \xi$$

between a family $\xi' : \mathcal{X}' \to T$ and a family $\xi : \mathcal{X} \to T$ is an isomorphism of schemes over T from $\mathcal{X}$ to $\mathcal{X}'$. This is how automorphisms of curves are encoded in the stack definition of moduli spaces. Any scheme M can be considered as a groupoid $M = (\mathcal{C}_M, p_M)$. Here, the objects of $\mathcal{C}_M$ are pairs (T, f) where $f : T \to M$ is a morphism of schemes. The morphisms $\varphi : (T, f) \to (T', f')$ are the morphisms $h : T \to T'$ with $f'h = f$. Finally, $p_M((T, f)) = T$. A groupoid $(\mathcal{C}, p)$ is (represented by) a scheme if, for some scheme M, there exists an isomorphism of groupoids

$$(\mathcal{C}_M, p_M) \cong (\mathcal{C}, p).$$

It follows from the definitions that, if a universal family $\mathcal{X} \to \overline{M}_{g,n}$ existed, then the groupoid $\overline{\mathcal{M}}_{g,n}$ would be represented by the scheme $\overline{M}_{g,n}$.

One of the advantages of the category of groupoids is that in this category quotients always exist. For example, if a group scheme G acts on a scheme X, then one can form a quotient stack $[X/G]$. As a first result, we prove that

$$\overline{\mathcal{M}}_{g,n} = [\mathrm{H}_{\nu,g,n}/PGL(N)]\,, \qquad N = (2\nu-1)(g-1)+\nu n\,.$$

In Section 6 we come to the second ingredient in the definition of a stack. This involves descent theory. Suppose that we are given a groupoid $\mathcal{M} = (\mathcal{C}, p)$, an étale surjective morphism of schemes $U \to T$, and an object ξ in $\mathcal{M}(U)$. The question is: when does ξ descend to T? In other words, when does there exists $\eta \in \mathcal{M}(T)$ with $f^*(\eta) \simeq \xi$? When $\mathcal{M} = \overline{\mathcal{M}}_{g,n}$, what we are given is a family of curves $\xi : X \to U$, and we look for conditions insuring the existence of a family $\eta : Y \to T$ with $f^*(\eta) \simeq \xi$. To understand these conditions, we consider the analogy between ordinary topology and étale topology. Instead of the étale map $U \to T$, we consider an open cover $\mathcal{U} = \{U_i\}$ of T. The collection of pairwise intersections $\{U_i \cap U_j\}$ is the topological counterpart of the fiber product $U \times_T U$, while the collection of triple intersections $\{U_i \cap U_j \cap U_k\}$ is the counterpart of the triple fiber product $U \times_T U \times_T U$. The datum of an object ξ_i on each U_i corresponds to the datum of an object ξ on U. An isomorphism φ_{ij} from $\xi_i | U_i \cap U_j$ to $\xi_i | U_i \cap U_j$ is translated into an isomorphism $\varphi : p_1^*\xi \to p_2^*\xi$, where p_1 and p_2 are the natural projections from $U \times_T U$ to U. The compatibility condition $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ on $\{U_i \cap U_j \cap U_k\}$ is translated into an appropriate "cocycle" condition for the isomorphism φ on $U \times_T U \times_T U$. If this cocycle condition is satisfied, (ξ, φ) are said to be a descent datum. However, this datum is not necessarily effective, meaning that the compatibility conditions are not always sufficient to make the object ξ "descend" from U to T. The first condition for a a groupoid to be a stack is that every étale descent datum is effective. To check this condition for the groupoid $\overline{\mathcal{M}}_{g,n}$, one has to use Grothendieck's descent theory for quasi-coherent sheaves, which we review in this same section.

In Section 7 we come to the third ingredient in the definition of a stack. This condition is almost automatically satisfied by the groupoid $\overline{\mathcal{M}}_{g,n}$, and it basically requires that a natural functor that can be concocted in terms of the *Isom* functor should, in fact, be a sheaf. Leaving the category of schemes to enter the category of stacks presents several advantages. Here are a few. First, as we observed, in the category of stacks one can take quotients. Secondly, looking at the case of curves, it makes sense to talk about a universal family of curves $\mathcal{C} \to \overline{\mathcal{M}}_{g,n}$ *over the stack* $\overline{\mathcal{M}}_{g,n}$. Moreover, as a stack, $\overline{\mathcal{M}}_{g,n}$ is smooth. In other words, in the category of stacks, modding out by finite groups destroys neither smoothness nor the property for a morphism to be étale. As another example, let us go back to (1.3) and (1.4). Look at the smooth variety X which is the disjoint union of bases $X_1, \dots, X_N$ of Kuranishi families

$\mathcal{C}_i \to X_i$ and consider the surjective moduli map $m : X \to \overline{M}_{g,n}$. As a map of schemes, m is not étale. To make it étale, one has to divide each X_i by the automorphism group of the central fiber of π_i. One of the advantages of replacing $\overline{M}_{g,n}$ with the stack $\overline{\mathcal{M}}_{g,n}$ is that, as a morphism of stacks, $m : X \to \overline{\mathcal{M}}_{g,n}$ is étale.

There is a class of stacks which is particularly manageable and which includes the moduli stacks $\overline{\mathcal{M}}_{g,n}$. This is the class of Deligne–Mumford stacks. We discuss these in Section 8. The first property that characterizes a Deligne–Mumford stack $\mathcal{M}$ is the existence of an étale surjective morphism $m : X \to \mathcal{M}$, where X is a scheme. The other requirement is that the diagonal morphism $\Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ should be representable. This last condition, for the stack $\overline{\mathcal{M}}_{g,n}$, translates into the following property which is the content of Proposition (3.10). Consider the family $\xi : \mathcal{C} \to X$ and the two projections $p_1, p_2 : X \times X \to X$. Then the natural projection

$$\mathbf{Isom}_{X\times X}(p_1^*\xi, p_2^*\xi) \longrightarrow X \tag{1.5}$$

is étale and surjective.

In Section 9, after digressing on Zariski's main theorem in the context of algebraic spaces, we state the basic result that, given a reduced, separated algebraic space X, there exists a scheme Z which is a finite Galois cover of X. A variant of the proof then yields the family (1.2).

The final section is devoted to the description of various natural morphisms between moduli stacks of curves. Building on the work done in Section 6 of Chapter X, we construct the universal curve

$$\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}\,,$$

the basic projection morphisms

$$\overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}\,,$$

and the basic clutching maps

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}\,,$$

where we adopt the notation introduced at the beginning of Section 7 of Chapter X. We also show that $\overline{\mathcal{C}}_{g,n}$ is naturally isomorphic to $\overline{\mathcal{M}}_{g,n+1}$.

We end this introduction by recalling some classical facts about elliptic curves that may be helpful to keep in mind in what follows.

Let $\mathbb{H}$ denote the upper half-plane. For $\tau \in \mathbb{H}$, we denote by E_τ the elliptic curve $\mathbb{C}/\Lambda_\tau$, where the lattice $\Lambda_\tau \cong \mathbb{Z}^2$ generated by 1 and τ acts by

$$(m_1, m_2) \cdot \tau = z + m_1 + m_2\tau.$$

It is well known that E_τ is embedded in $\mathbb{P}^2$ as a smooth cubic with affine equation

$$y^2 = 4x^3 + g_2(\tau)x + g_3(\tau)$$

by the map

$$\begin{cases} x = \mathcal{P}(z,\tau), \\ y = \mathcal{P}'(z,\tau), \end{cases}$$

where $\mathcal{P}(z,\tau)$ is the Weierstrass $\mathcal{P}$-function. This construction yields a family of smooth cubics

$$\alpha : \mathcal{C} \to \mathbb{H}. \tag{1.6}$$

This family, which is highly transcendental, has two remarkable properties. First of all, every elliptic curve appears in it, up to isomorphism, and, secondly, the family is everywhere Kuranishi. From g_2 and g_3 one constructs the j-function

$$j(\tau) = \frac{1728 g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2}.$$

Letting $\Gamma = SL_2(\mathbb{Z})$ denote the modular group acting as usual on $\mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d},$$

it is well known that $j(\tau)$ is Γ-invariant. Moreover, setting

$$\widehat{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\},$$

one has a natural extension of the action of Γ to $\widehat{\mathbb{H}}$. The points of $\mathbb{Q} \cup \{\infty\}$ are called cusps and are permuted transitively by Γ. There are bijections

$$\begin{array}{ccc} j : \Gamma\backslash\widehat{\mathbb{H}} & \xrightarrow{\cong} & \mathbb{P}^1 \\ \cup & & \cup \\ \Gamma\backslash\mathbb{H} & \xrightarrow{\cong} & \mathbb{P}^1 \setminus \{\infty\} \end{array} \tag{1.7}$$

Anticipating notation to be used below, we set

$$\begin{cases} Y(1) = \Gamma\backslash\mathbb{H}, \\ \overline{Y(1)} = \Gamma\backslash\widehat{\mathbb{H}}. \end{cases}$$

One may take a slightly different approach to the family (1.6). The semi-direct product $\Gamma \ltimes \mathbb{Z}^2$ acts on $\mathbb{H} \times \mathbb{C}$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (m_1, m_2) \right) \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + m_1 + m_2\tau}{c\tau + d} \right).$$

Taking quotients only modulo the second factor $\mathbb{Z}^2$, we obtain a family over $\mathbb{H}$ which is just (1.6). Taking instead the full quotient, we get a variety $\mathbf{E}^*$ fibered over $Y(1)$:

$$\pi : \mathbf{E}^* \to Y(1).$$

As is well known, for τ and τ' in $\mathbb{H}$, E_τ is isomorphic to $E_{\tau'}$ if and only if $j(\tau) = j(\tau')$. It follows that we may identify $M_{1,1}$ with $Y(1)$. One's first guess might be that then $\mathbf{E}^* \to M_{1,1}$ is the universal elliptic curve. However, this is spectacularly not correct. The point is that one is in particular dividing by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma \, .$$

This automorphism acts trivially on $\mathbb{H}$ but gives the -1 involution on each E_τ, so that the fibers of π are in fact $\mathbb{P}^1$'s. In addition, two fibers of π are special in that they correspond to elliptic curves with automorphisms other than ± 1. These are precisely

$$\begin{array}{llll} E_{\tau_0}\,, & \tau_0 = e^{\pi\sqrt{-1}}\,, & j(\tau_0) = 1728\,, & |\operatorname{Aut}(E_0)| = 4; \\ E_{\tau_1}\,, & \tau_1 = e^{2\pi\sqrt{-1}/3}\,, & j(\tau_1) = 0\,, & |\operatorname{Aut}(E_1)| = 6. \end{array}$$

Although $\mathbf{E}^* \to M_{1,1}$ is not a universal family of elliptic curves, if we consider the stack $\mathcal{M}_{1,1}$ which, roughly speaking, refines $M_{1,1}$ by adding the data $\operatorname{Aut}(E_\tau)$, then, as will be explained below, there is a universal family $\mathcal{E} \to \mathcal{M}_{1,1}$ of elliptic curves. Thus, in this case enlarging our concept of variety to include stacks, one resolves the issue of having a universal elliptic curve. It is worth noticing that, as will be seen in the following chapters, one may do enumerative geometry in a stack context. Perhaps the first instance of this is due to Mumford [549], who showed that $\operatorname{Pic}(\mathcal{M}_{1,1}) \cong \mathbb{Z}/12\mathbb{Z}$, where the left-hand side is the Picard group of the stack $\mathcal{M}_{1,1}$.

A complementary approach to the issues raised above is the one of trying to *rigidify* the family of elliptic curves by adding additional data that kill the automorphisms groups of the E_s. This will be the central theme of Chapter XVI. Essentially, the additional data consists in considering the finite group of points of order N in each E_τ. For this, one sets

$$\Gamma(N) = \ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})).$$

Then the semi-direct product $\Gamma(N) \ltimes \mathbb{Z}^2$ acts on $\mathbb{H} \times \mathbb{C}$ as above. When $N \geq 3$, this action is free, so that the quotient $\mathbf{E}^*(N)$ is smooth. Moreover, the action of $\Gamma(N)$ on $\mathbb{H}$ extends to a free action on $\widehat{\mathbb{H}}$, giving rise to an open inclusion of smooth varieties $Y(N) = \Gamma(N)\backslash\mathbb{H} \subset \Gamma(N)\backslash\widehat{\mathbb{H}} =$

$\overline{Y(N)}$. The family of elliptic curves $\pi_N : \mathbf{E}^*(N) \to Y(N)$ can be completed to a family of nodal curves over $\overline{Y(N)}$, so that one gets a diagram

$$\begin{array}{ccc} \mathbf{E}^*(N) & \subset & \mathbf{E}(N) \\ {\scriptstyle \pi_N}\big\downarrow & & {\scriptstyle \overline{\pi}_N}\big\downarrow \\ Y(N) & \subset & \overline{Y(N)} \end{array}$$

Finally, setting $G_N = SL_2(\mathbb{Z}/N\mathbb{Z})$, we see that G_N acts naturally on the diagram

$$\begin{array}{ccc} Y(N) & \subset & \overline{Y(N)} \\ \big\downarrow & & \big\downarrow \\ M_{1,1} & \subset & \overline{M}_{1,1} \end{array}$$

and, consequently, $M_{1,1}$ and $\overline{M}_{1,1}$ are each represented as quotients of smooth varieties by finite groups.

2. Construction of moduli space as an analytic space.

Our goal in this section is to put a structure of analytic space on the set of isomorphism classes of stable P-pointed curves of genus g, where P is a finite set. The resulting space is called *the moduli space of stable P-pointed curves of genus g* and is denoted by $\overline{M}_{g,P}$. When $P = \{1, \dots, n\}$, one writes $\overline{M}_{g,n}$ for $\overline{M}_{g,P}$. The construction relies on the existence of standard Kuranishi families, as defined in Chapter XI, definition (6.8), proved in the same chapter. We shall need the following well-known elementary result, due to Henri Cartan [106].

Lemma (2.1). *Let G be a finite group acting on a complex manifold U. Then there is a unique structure of normal analytic space on the quotient U/G such that $U \to U/G$ is holomorphic.*

Without loss of generality, we may assume that the action of G is effective. Let u be a point of U, let p be its image in U/G, and denote by H the stabilizer of u. All sufficiently small open neighborhoods of p are of the form V/H, where V is a sufficiently small H-invariant open neighborhood of u, and conversely. Suppose that there exists a complex structure on U/G satisfying the requirements of the lemma. Then, if V and H are as above, a holomorphic function on V/H gives, by pullback, an H-invariant holomorphic function on V. Conversely, an H-invariant holomorphic function on V descends to a holomorphic function on V'/H, where V' is the open subset of V where the action of H is free, and hence to V/H, by Riemann's extension theorem. This proves the uniqueness.

In view of the uniqueness, to prove the existence, it suffices to put a structure of normal analytic space on all open sets of the form V/H, where H is the stabilizer of a point $u \in U$, and V is a sufficiently small H-invariant open neighborhood of u. So we may also assume that G

fixes a point $u \in U$ and that $V = U$. By Lemma (6.12) in Chapter XI, we may choose a system of coordinates centered at u in which G acts by linear transformations. We are thus reduced to the case where G acts linearly on $U = \mathbb{C}^n$. This is a special instance of the following more general result, whose full strength, in any case, we will later need.

LEMMA (2.2). *Let* $\operatorname{Spec}(A)$ *be a normal affine variety acted on algebraically by a finite group* Γ. *Then the ring* A^Γ *of* Γ*-invariant elements in* A *is an integrally closed finitely generated* $\mathbb{C}$*-algebra. Moreover, if* X *is the set of closed points of* $\operatorname{Spec}(A)$, *then the set of closed points of* $\operatorname{Spec}(A^\Gamma)$ *can be identified with* X/Γ.

Proof. Set $B = A^\Gamma$. Let $a_1, \dots, a_n$ be generators for A as a $\mathbb{C}$-algebra, so that $A = \mathbb{C}[a_1, \dots, a_n]$. Consider the polynomials

$$p_i(X) = \prod_{\gamma \in \Gamma} (X - \gamma a_i) \ , \qquad i = 1, \dots, n\,.$$

The coefficients of these polynomials are invariant under Γ, i.e., they belong to B. Let all these coefficients be $c_1, \dots, c_N$ and set $C = \mathbb{C}[c_1, \dots, c_N]$. Obviously, C is a subring of B, while A is integral over C since $p_i(a_i) = 0$. Moreover, A is a finitely generated C-module. To see this, given any polynomial $\alpha = p(a_1, \dots, a_n)$ with complex coefficients, one can use the integrality relations $p_i(a_i) = 0$ to recursively reduce α to a polynomial in $a_1, \dots, a_n$ with coefficients in C and degree in each variable bounded by the order of Γ minus one. Now, since C is noetherian and B is a C-submodule of the finitely generated C-module A, the C-module B is finitely generated as well; let $b_1, \dots, b_k$ be a set of generators of B over C. Then, clearly,

$$B = \mathbb{C}[c_1, \dots, c_N, b_1, \dots, b_k]\,.$$

We now show that B is integrally closed. Let K and L be the quotient fields of A and B. Suppose that $f \in L$ is integral over B. Since A is assumed to be integrally closed, f belongs to A. Since f is Γ-invariant, it belongs to B. It remains to show that the set of closed points of $\operatorname{Spec} B$ is X/Γ. We introduce the Reynolds operator

$$R : A \to B$$

defined by

$$R(a) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma a\,.$$

The Reynolds operator has the following elementary properties:

1) R is the identity on B;
2) if $a \in A$ and $b \in B$, then $R(ba) = bR(a)$.

An easy consequence of these properties is that, for any ideal $I \subset B$,

$$AI \cap B = I\,. \tag{2.3}$$

In fact, let $\sum a_i f_i$ be invariant under Γ, where $f_i \in I$, $a_i \in A$. Then, applying the Reynolds operator and property 2), we get

$$\sum a_i f_i = \sum R(a_i) f_i \in I\,.$$

It then follows that the map

$$\begin{aligned} X &\longrightarrow \max(B) \\ J &\mapsto J \cap B \end{aligned}$$

is surjective. In fact, given any maximal ideal I in B, by (2.3) we have that $AI \neq A$, so that, if J is any maximal ideal in A containing AI, then $J \cap B = I$. On the other hand, let $J \neq J'$ be elements of X. If $J' = \gamma J$ for some $\gamma \in \Gamma$, then clearly $J \cap B = J' \cap B$. Conversely, if J and J' belong to different orbits under Γ, we may pick an element f in A such that

$$\begin{aligned} &f \notin J \\ &f \in \Big(\bigcap_{\gamma J \neq J} \gamma J\Big) \cap \Big(\bigcap_{\gamma \in \Gamma} \gamma J'\Big)\,. \end{aligned}$$

But then

$$\begin{aligned} &Rf \in J' \cap B\,, \\ &Rf \notin J \cap B\,. \end{aligned}$$

In fact, the first summand of the right-hand side of the identity

$$R(f) = \frac{1}{|\Gamma|}\sum_{\gamma J \neq J} \gamma f + \frac{1}{|\Gamma|}\sum_{\gamma J = J} \gamma f$$

belongs to J by construction, while the second summand does not, as follows from the remark that the isotropy group Γ' of J acts trivially on $A/J = \mathbb{C}$, so that

$$\frac{1}{|\Gamma|}\sum_{\gamma J = J} \gamma f \equiv \frac{|\Gamma'|}{|\Gamma|} f \qquad \mod J\,.$$

Q.E.D.

Given any stable P-pointed genus g curve $(C; \{x_p\}_{p\in P})$, we shall write $[(C; \{x_p\}_{p\in P})]$ to indicate its isomorphism class. Consider a standard Kuranishi family

$$\mathcal{Y} \to (U, u_0) \qquad \tau_p : U \to \mathcal{Y}\,,\ p \in P\,, \qquad \varphi : C \xrightarrow{\simeq} \mathcal{Y}_{u_0}\,,$$

where of course φ is an isomorphism of P-pointed curves. We may suppose in addition that U is a bounded subset of $\mathbb{C}^{3g-3+n}$, $n = |P|$. Set $G = \mathrm{Aut}(C; \{x_p\}_{p\in P})$. There is a natural map of sets

$$\psi : U/G \hookrightarrow \overline{M}_{g,P}\,,$$

whose injectivity is a consequence of property iv) in the definition of standard Kuranishi family (cf. (6.8) in Chapter XI). By Lemma (2.1), U/G is normal. Since the point $[(C; \{x_p\}_{p\in P})] \in \overline{M}_{g,P}$ is entirely arbitrary, one can cover $\overline{M}_{g,P}$ by "charts" of this kind. This will put a structure of analytic space on $\overline{M}_{g,P}$ if one can show that the "changes of coordinates" are analytic. More precisely, suppose that

$$\eta : U'/G' \hookrightarrow \overline{M}_{g,P}$$

is another chart obtained with the above procedure and suppose that

$$A = \psi(U/G) \cap \eta(U'/G') \neq \emptyset\,.$$

First observe that the preimage of A under ψ or η is open. This follows from the fact that the families of curves we are dealing with are Kuranishi families at any point of their respective parameter spaces and from the universal property characterizing Kuranishi families. Now we have to prove that $\psi^{-1}\eta$ is analytic.

Clearly, it suffices to deal with the case where U' is "sufficiently small," in particular, where $\eta(U'/G')$ is contained in $\psi(U/G)$. Again, since our families are Kuranishi at every point, we get by universality a commutative diagram

$$\begin{tikzcd}
U' \arrow[rr, "\gamma"] \arrow[d, "\alpha"'] & & U \arrow[d, "\beta"] \\
U'/G' \arrow[rr, "\psi^{-1}\eta"] \arrow[dr, "\eta"'] & & U/G \arrow[dl, "\psi"] \\
& \overline{M}_{g,P} &
\end{tikzcd}$$

Since α is finite and holomorphic and β, γ are holomorphic, the map $\psi^{-1}\eta$ is holomorphic off the branch locus of α. Since U'/G' is normal and U/G can be realized as a bounded analytic subset of some $\mathbb{C}^N$, by Riemann's extension theorem $\psi^{-1}\eta$ is holomorphic everywhere.

This completes the construction of an analytic space structure on $\overline{M}_{g,P}$. What is already clear is that $\overline{M}_{g,P}$ is normal, since all the local patches U/G are. The results of Chapter X easily imply that $\overline{M}_{g,P}$ is separated and first countable, as we shall see in Section 3.

It follows from the construction that the analytic structure on $\overline{M}_{g,P}$ is natural in the following sense. Let

$$\psi : \mathcal{X} \to Z$$

be a family of stable P-pointed curves of genus g; then there is a morphism

$$m_\psi : Z \to \overline{M}_{g,P}$$

functorially attached to ψ such that, set-theoretically,

$$m_\psi(z) = \text{ isomorphism class of the } P\text{-pointed curve } \psi^{-1}(z)\,.$$

In addition, $\overline{M}_{g,P}$ dominates any variety having the above property. The map m_ψ is called the *moduli map of the family* ψ.

One denotes by $M_{g,P}$ the locus in $\overline{M}_{g,P}$ parameterizing smooth curves. It is an open subset of $\overline{M}_{g,P}$ since small deformations of smooth curves are smooth. Its complement, which parameterizes singular stable curves, is called the *boundary* of moduli space and is denoted by $\partial M_{g,P}$. Let x be a point of the boundary; it corresponds to a stable P-pointed genus g curve C with $\delta > 0$ nodes. Let U be the base of a (small) Kuranishi family for C. We know that, in suitable coordinates, the locus S in U parameterizing singular curves is the union of δ coordinate hyperplanes. This locus is obviously invariant under the action of $G = \mathrm{Aut}(C)$, so that, locally near x, the boundary $\partial M_{g,P}$ is just

$$S/G \subset U/G \subset \overline{M}_{g,P}\,. \tag{2.4}$$

As such, $\partial M_{g,P}$ is a closed codimension one analytic subvariety of $\overline{M}_{g,P}$. Of course, when $P = \{1, \dots, n\}$, we write $M_{g,n}$ for $M_{g,P}$ and $\partial M_{g,n}$ for $\partial M_{g,P}$.

We know from the explicit description of Kuranishi families that curves with two or more singular points occur in codimension two in $\overline{M}_{g,P}$. Thus, a general point of any component of $\partial M_{g,P}$ corresponds to a curve with a single node. On the other hand, we know (cf. Section 2 of Chapter X) that nodes come in different flavors. There are nonseparating nodes and separating ones; moreover, the latter are classified in different types, indexed by the different stable bipartitions of (g, P). It is obvious that for a node being separating or nonseparating, and its type as a separating node, are deformation invariants. Thus the locus Δ_{irr} in $\overline{M}_{g,P}$ parameterizing curves with at least one nonseparating node is a closed analytic subset of $\partial M_{g,P}$, and the same can be said of the locus $\Delta_{\mathcal{P}}$ parameterizing curves with at least one separating node of type $\mathcal{P}$, where $\mathcal{P}$ is a stable bipartition of (g, P). In Chapter XV, and again in Chapter XXI, we shall prove that $\overline{M}_{g,P}$ is always irreducible; an immediate consequence (see Section 10) is that Δ_{irr} and the $\Delta_{\mathcal{P}}$ are all irreducible. Thus,

$$\partial M_{g,P} = \Delta_{irr} \cup \left(\bigcup \Delta_{\mathcal{P}}\right),$$

where $\mathcal{P}$ runs through all stable bipartitions, is the decomposition of the boundary of $\overline{M}_{g,P}$ in irreducible components. In the sequel, following the conventions introduced in Section 2 of Chapter X, given a stable bipartition $\mathcal{P} = \{(a,A),(b,B)\}$ of (g,P), we shall normally write $\Delta_{a,A}$ or, equivalently, $\Delta_{b,B}$ to indicate $\Delta_{\mathcal{P}}$.

Let $(C; x_1, \dots, x_n)$ be a stable n-pointed curve of genus g, and let m be the corresponding point of $\overline{M}_{g,n}$. We know that a small neighborhood of m looks like U/G, where U is the base of a standard Kuranishi family for $(C; x_1, \dots, x_n)$, and G is the automorphism group of $(C; x_1, \dots, x_n)$. The point m can therefore be singular only if G is nontrivial. More precisely, m is a smooth point of $\overline{M}_{g,n}$ only in two cases. Either G acts trivially on U, or its fixed locus in U is a (smooth) divisor, in which case G is a cyclic group. The cases in which the first alternative occurs are implicitly described by Proposition(4.11) in Chapter XI: either G is trivial, or $g = 2$, $n = 0$ (resp., $g = 1$, $n = 1$), and the only nontrivial element of G is the hyperelliptic involution (resp., the symmetry about the marked point). To decide when the second alternative occurs, it is necessary to describe the divisor components of the locus Σ in $\overline{M}_{g,n}$ parameterizing curves with extra automorphisms; this locus is a closed analytic subspace of $\overline{M}_{g,n}$, as follows, for instance, from Lemma (6.11) in Chapter XI. A first result in this direction is the following.

Proposition (2.5). *Let Σ be the closed analytic subspace of $\overline{M}_{g,n}$ parameterizing curves with nontrivial automorphism group. Then*

i) *$\Sigma = \emptyset$ if and only if $g = 0$.*
ii) *$\Sigma = \overline{M}_{g,n}$ if and only if $g = 2$, $n = 0$ or $g = 1$, $n = 1$.*
iii) *In the remaining cases the divisor components of Σ are:*

a) *the closure in $\overline{M}_{1,2}$ of the locus parameterizing triples $(C; x_1, x_2)$ such that C is smooth and $2(x_1 - x_2)$ is linearly equivalent to zero;*
b) *the closure in $\overline{M}_{2,1}$ of the locus parameterizing pairs $(C; x)$ such that C is smooth and x is a Weierstrass point of C;*
c) *the closure in $\overline{M}_3$ of the locus parameterizing smooth hyperelliptic curves;*
d) *for any $g \geq 1$ and any n, the locus $\Delta_{1,\emptyset}$ in $\overline{M}_{g,n}$.*

Proof. If $(C; x_1, \dots, x_n)$ has a nontrivial automorphism group, it has an automorphism γ of prime order p. We have to classify the cases where γ propagates to all of $\overline{M}_{g,n}$ or along a codimension one subspace of $\overline{M}_{g,n}$. We first assume that C is smooth. Set $C' = C/\langle\gamma\rangle$. The quotient morphism $C \to C'$ is totally ramified at h points $x_1, \dots, x_h$. As the notation suggests, these include the marked points of C, since the latter are fixed for γ; thus $h \geq n$. Moreover, since C is stable, $h \geq 1$ if $g' = 1$ and $h \geq 3$ if $g' = 0$. We let $y_1, \dots, y_h$ be the images of $x_1, \dots, x_h$ in C'. Denoting by g' the genus of C', the Riemann–Hurwitz formula gives

$$2g - 2 = p(2g' - 2) + h(p - 1). \tag{2.6}$$

According to Lemma (6.11) in Chapter XI, the dimension of the subspace of moduli along which γ propagates is $\dim H^1(C, T_C(x_1 + \cdots + x_n))^\gamma$. The key to calculating this dimension is the observation that there is a natural isomorphism

$$H^1(C, T_C(-x_1 - \cdots - x_n))^\gamma \cong H^1(C', T_{C'}(-y_1 - \cdots - y_h)).$$

To show this, it is convenient to prove the dual statement, namely that the pullback of forms induces an isomorphism

$$H^0(C', \omega_{C'}^2(y_1 + \cdots + y_h)) \xrightarrow{\simeq} H^0(C, \omega_C^2(x_1 + \cdots + x_n))^\gamma. \tag{2.7}$$

We may choose local coordinates such that, near x_i, the morphism $C \to C'$ is of the form $z \mapsto z^p$. The automorphism γ acts on z by multiplication by a nontrivial pth root of unity ζ. The pullback of an element of $H^0(C', \omega_{C'}^2(y_1 + \cdots + y_h))$ is locally of the form

$$f(z^p)d(z^p)^2 = p^2 z^{2p-2} f(z^p) dz^2\,,$$

where f has at most a simple pole at the origin, and hence is actually a holomorphic quadratic differential on C. Conversely, let η be a γ-invariant element of $H^0(C, \omega_C^2(x_1 + \cdots + x_n))$ and write it locally as $a(z)dz^2$, where $a(z) = \sum a_i z^i$ has at most a simple pole at the origin. By γ-invariance we must have

$$a(z)dz^2 = a(\zeta z)d(\zeta z)^2 = \zeta^2 a(\zeta z)dz^2,$$

Thus a_i can be nonzero only when $i \equiv -2$ modulo p, and hence $a(z) = z^{p-2} b(z^p)$ for some holomorphic function b. It follows that

$$\eta = \frac{1}{p^2} z^{-p} b(z^p) d(z^p)^2$$

and therefore that η is the pullback of an element of $H^0(C', \omega_{C'}^2(y_1 + \cdots + y_h))$, proving (2.7). A consequence of (2.7) is that

$$d = \dim H^1(C, T_C(x_1 + \cdots + x_n))^\gamma = 3g' - 3 + h. \tag{2.8}$$

We now assume that $\dim \overline{M}_{g,n} - d$ does not exceed 1 and classify the cases where this occurs. Using (2.6), we find that

$$\dim \overline{M}_{g,n} - d = (p-1)(3g' - 3) + h\frac{3p - 5}{2} + n.$$

Thus, g' can only be equal to 1 or 0. If $g' = 1$, then $h \neq 0$ and $n = 0$, and the only possibility is that $p = h = 2$, so that $g = 2$. What this computation says, in effect, is that the locus of those genus 2 curves which are double coverings of an elliptic curve has dimension 2. Now suppose that $g' = 0$. It is straightforward to check that the only possibilities are:

(1) $p = 2,\ h = 4,\ n = 1$;
(2) $p = 2,\ h = 4,\ n = 2$;
(3) $p = 2,\ h = 6,\ n = 0$;
(4) $p = 2,\ h = 6,\ n = 1$;
(5) $p = 2,\ h = 8,\ n = 0$;
(6) $p = 3,\ h = 3,\ n = 1$.

The difference $\dim \overline{M}_{g,n} - d$ is zero only in the first and third cases, which correspond, respectively, to $g = n = 1$ and to $g = 2$, $n = 0$. This proves part ii) of the proposition. Cases (2), (4), and (5) correspond, respectively, to cases a), b), and c) in part iii) of the proposition. As for (6), it just says that there is a unique elliptic curve with an automorphism of order three (the quotient of $\mathbb{C}$ modulo the lattice generated by 1 and by a primitive third root ot unity).

We now turn to the divisor components of Σ which are entirely contained in the boundary $\partial M_{g,n}$. These may only be components of $\partial M_{g,n}$ itself. We know that a general member C of such a component has a single node, which is thus fixed for any automorphism of C. The automorphisms of C therefore induce automorphisms of its normalization N. However, by the analysis we carried out in the smooth case, it is immediate to check that, in general, N has no nontrivial automorphisms, with one exception: when N is the disjoint union of a 1-pointed genus 1 curve C_1 and an $(n+1)$-pointed genus $g-1$ curve C_2, it always has the nontrivial automorphism which restricts to the symmetry about the marked point on C_1 and to the identity on C_2. This takes care of case d) in part iii) of the proposition.

It remains to prove i). To do this, just observe that in genus $g \geq 1$ there are always curves with extra automorphisms for any value of n. We leave the construction of these curves, starting, for instance, from hyperelliptic ones, as an exercise for the reader.

Q.E.D.

It is in general very difficult to give a concrete description of $\overline{M}_{g,n}$, and this has been done only in a few low genus cases. Here are some of the simplest ones. Let us start with genus zero. In this case a stable curve has no nontrivial automorphisms, since an automorphism of $\mathbb{P}^1$ fixing three or more points is the identity. Thus, we expect a universal family to exist, and indeed we can easily construct one. Clearly, $\overline{M}_{0,3}$ is just a point, since any triple of distinct points on $\mathbb{P}^1$ is projectively equivalent, in a unique way, to the triple $(0, 1, \infty)$. We now turn to

$\overline{M}_{0,4}$. Consider the product $\mathbb{P}^1 \times \mathbb{P}^1$, let π' be the projection to the second factor, and denote by D'_1, D'_2, D'_3 the horizontal sections of π' corresponding to the points $0, 1, \infty$ on the first factor and by Δ the diagonal.

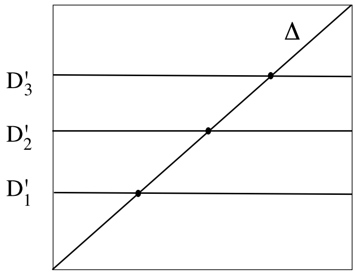

Figure 1.

Then blow up the three points where Δ meets D'_1, D'_2 or D'_3, denote by X the resulting surface, and by $\pi : X \to \mathbb{P}^1$ the composition of the contraction map $X \to \mathbb{P}^1 \times \mathbb{P}^1$ with π'. Clearly, $\pi : X \to \mathbb{P}^1$ has four distinguished nonintersecting sections D_1, D_2, D_3, and D_4, which are the proper transforms of D'_1, D'_2, D'_3, and Δ, respectively. This makes it into a family of 4-pointed genus zero curves. The fiber $\pi^{-1}(0)$ consists of two copies of $\mathbb{P}^1$ joined at one point, with the marked points labelled by 2 and 3 on one component, and those labelled by 1 and 4 on the other; in particular, it is a stable 4-pointed curve. The fibers $\pi^{-1}(1)$, $\pi^{-1}(\infty)$ can be similarly described. Then the moduli map $\mathbb{P}^1 \to \overline{M}_{0,4}$ attached to the family consisting of $\pi : X \to \mathbb{P}^1$ together with the sections D_1, D_2, D_3, D_4 is an isomorphism, and the family is a universal family.

The construction we have just carried out is the prototype of a general procedure which inductively constructs $\overline{M}_{0,n+1}$ out of the universal family on $\overline{M}_{0,n}$, as well as a universal family on it. We exemplify this for $\overline{M}_{0,5}$. Let $p_2 : X \times_{\overline{M}_{0,4}} X \to X$ be the projection to the second factor. Then $D_i \times_{\overline{M}_{0,4}} X$, $i = 1, \dots, 4$, are sections of p_2 and, together with it, constitute a family of stable 4-pointed curves of genus zero. We may apply to this family and to the diagonal Δ the general stabilization procedure described in Section 8 of Chapter X. The result is a family $Y \to X$ of stable 5-pointed curves of genus zero. Again, the moduli map $X \to \overline{M}_{0,5}$ is an isomorphism, and $Y \to X$ a universal family. This procedure shows that $\overline{M}_{0,n}$ is a smooth projective variety and that it carries a universal family which can be identified with the map $\pi : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ obtained by "forgetting" the $(n+1)$st point. If one takes this point of view, the ith section of π takes each n-pointed genus zero curve to the $(n+1)$-pointed curve obtained from it by attaching to the ith marked point a $\mathbb{P}^1$ with two marked points labelled with i and $n+1$, as illustrated in Fig. 2 below.

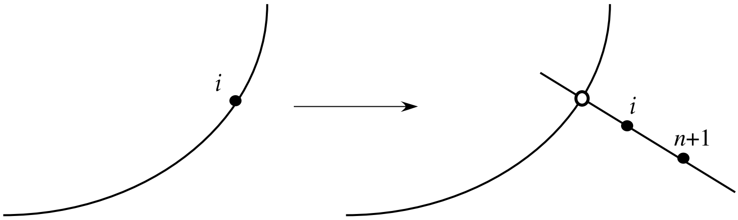

Figure 2.

We may apply what we have learned about $\overline{M}_{0,n}$ to construct other moduli spaces. The first of these is $\overline{M}_{1,1}$. Let $(E;x)$ be any smooth 1-pointed elliptic curve. Consider the group operation on E having x as origin, and let ι be the symmetry about x. Then the fixed points of the involution ι are precisely the four 2-torsion points of E, and the quotient E/ι is a $\mathbb{P}^1$. Conversely, given a stable 4-pointed curve of genus zero $(C;p_1,\dots,p_4)$, the double covering of C branched at $p_1,\dots,p_4$ is a genus 1 nodal curve E, which comes with four distinguished points $q_1,\dots,q_4$, the inverse images of the marked points of C. This gives us a stable 4-pointed elliptic curve. We forget about the labeling of q_1,q_2,q_3, and keep q_4 as a marked point on E. This directly gives us a 1-pointed elliptic curve when E is smooth. When E is not smooth, to get a stable 1-pointed elliptic curve, we also have to contract the component not containing q_4.

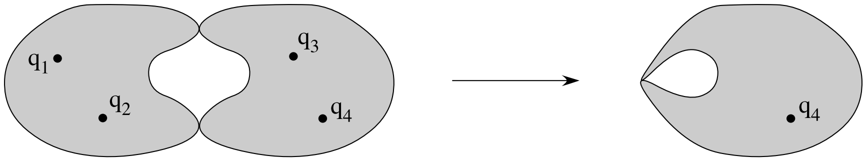

Figure 3.

By the previous discussion, we may get in this way all stable 1-pointed elliptic curves. This procedure defines a finite surjective morphism $\overline{M}_{0,4}\to\overline{M}_{1,1}$, and exhibits $\overline{M}_{1,1}$ as the quotient $\overline{M}_{0,4}/\mathfrak{S}_3$, where the symmetric group $\mathfrak{S}_3$ acts by changing the labeling of q_1, q_2, and q_3. An entirely similar construction shows that $\overline{M}_2$ can be identified with $\overline{M}_{0,6}/\mathfrak{S}_6$. In particular, $\overline{M}_{1,1}$ and $\overline{M}_2$ are both projective varieties; this is actually true for all moduli spaces $\overline{M}_{g,n}$, and Chapter XIV will be entirely devoted to proving it. As we explained above, neither $\overline{M}_{1,1}$ nor $\overline{M}_2$ carry a universal family.

Ideally, one would like to have, over $\overline{M}_{g,n}$, a universal family of stable curves, that is, one with the property that any family

$$\psi:\mathcal{X}\to Z$$

of stable curves is induced by the universal one via the map m_ψ. This, however, is impossible. In fact, let $(C; x_1, \dots, x_n)$ be a smooth n-pointed curve of genus g, and p the corresponding point in $\overline{M}_{g,n}$. If a universal family existed, it would locally be a pullback of a standard Kuranishi family

$$\mathcal{Y} \to (U, u_0)$$

for $(C; x_1, \dots, x_n)$, via a morphism

$$\alpha : V \to U\,,$$

where V is a neighborhood of p. Conversely, the Kuranishi family would be induced by the family on $\overline{M}_{g,n}$ via

$$\beta : U \to \overline{M}_{g,n}\,.$$

We claim that β has to be injective. In fact the composition $\alpha \circ \beta$ induces on (a neighborhood of u_0 in) U a deformation of $(C; x_1, \dots, x_n)$ which differs from the Kuranishi family we started with at most because the identification between the central fiber and $(C; x_1, \dots, x_n)$ has been changed by an automorphism of $(C; x_1, \dots, x_n)$. A deformation of this nature is induced via an automorphism of U. By uniqueness, $\alpha \circ \beta$ must coincide with this automorphism: thus β is injective. When $(C; x_1, \dots, x_n)$ has nontrivial automorphisms (or, in genus two, has nontrivial automorphisms other than the hyperelliptic involution), we reach a contradiction, since β maps any orbit of $\mathrm{Aut}(C; x_1, \dots, x_n)$ to one point of $\overline{M}_{g,n}$ and, by Proposition (4.11) in Chapter XI, $\mathrm{Aut}(C; x_1, \dots, x_n)$ acts nontrivially on U.

From the very construction of the analytic structure on $\overline{M}_{g,n}$ it follows that a universal family does exist on the open subset $\overline{M}^{\,0}_{g,n}$ of $\overline{M}_{g,n}$ whose points correspond to automorphism-free curves.

Something which is almost as good, in several practical applications, as a universal family on moduli, is the existence of such a family on a ramified covering of $\overline{M}_{g,n}$. In addition, the parameter space for this family may be taken to be a scheme rather than a mere analytic space. Formally, in Section 9, when we will have at our disposal some essential tools from the theory of stacks and algebraic spaces, we will prove the following result.

THEOREM (2.9). *There exists a family of stable n-pointed genus g curves $\eta : \mathcal{X} \to Z$, parameterized by a normal scheme Z, whose moduli map*

$$m : Z \to \overline{M}_{g,n}$$

is finite and surjective.

As we said, we will not prove this theorem in the current section. Here we shall only give one of its simplest consequences.

THEOREM (2.10). *$\overline{M}_{g,n}$ is compact.*

Proof. It suffices to show that the scheme Z in the statement of Theorem (2.9) is complete. For this, we use the valuative criterion of properness. Given an analytic map from $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ to Z,

$$f : \Delta^* \to Z,$$

which is meromorphic at the origin, we must show that f extends across the puncture, possibly after a base change on Δ of the form $z = \zeta^k$. Look at the pullback, via f, of the family $\mathcal{X} \to Z$. As explained in Section 4 of Chapter X, stable reduction implies that, after a base change, the induced family can be extended across the puncture to a family of stable curves over Δ. Therefore, if not yet f, at least the composition $m \circ f$ extends to a map from Δ to $\overline{M}_{g,n}$. By the finiteness of m this can be lifted, after another base change, to the required extension of f. Q.E.D.

As we mentioned, in Chapter XIV we shall prove that $\overline{M}_{g,n}$ is a projective variety. In our proof of this result, Theorem (2.9) will be essential. In fact, since $Z \to \overline{M}_{g,n}$ is finite and surjective, the projectivity of $\overline{M}_{g,n}$ follows from the one of Z. But on Z we have the great advantage of being able to work with the family $\eta : \mathcal{X} \to Z$.

3. Moduli spaces as algebraic spaces.

Our main goal in this section is to show that a small variant of the constructions carried out in the previous one puts a structure of algebraic space on $\overline{M}_{g,n}$. For this, we first need to digress on the theory of algebraic spaces. In this book we will use only foundational facts of this theory. Whenever, for a given property of algebraic spaces, a clear reference exists and is easily attainable, we will point the reader to it. When this is not the case, we will provide the necessary proofs.

Let Y be a set. We will say that a set R, together with a pair of maps

$$R \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Y$$

is an equivalence relation on Y if $(\alpha, \beta) : R \to Y \times Y$ is injective and $(\alpha, \beta)(R) \subset Y \times Y$ is an equivalence relation in the ordinary sense.

Now let B be a scheme, and Y a B-scheme. A *schematic equivalence relation*, or simply an *equivalence relation on Y*, is a B-scheme R together with a pair of morphisms

$$R \underset{t}{\overset{s}{\rightrightarrows}} Y$$

over B such that, for every B-scheme S,

$$\mathrm{Hom}_B(S,R) \underset{t_*}{\overset{s_*}{\rightrightarrows}} \mathrm{Hom}_B(S,Y)$$

is a set-theoretic equivalence relation. In particular, this implies that, for a schematic equivalence relation, $(s_*,t_*) : \mathrm{Hom}_B(S,R) \to \mathrm{Hom}_B(S,Y) \times \mathrm{Hom}_B(S,Y) = \mathrm{Hom}_B(S, Y \times_B Y)$ is injective for any S, i.e., that

$$(s,t) : R \to Y \times_B Y$$

is a *monomorphism.*

We will denote by $\eta : Y \times_B Y \to Y \times_B Y$ the involution interchanging the two factors, and by $\Delta : Y \to Y \times_B Y$ the diagonal morphism. Finally, we denote by $R\,{}_s\!\times_t R$ the fiber product induced by s and t. The following result follows from the definitions.

Proposition (3.1). *Let R and Y be B-schemes, and let $s,t : R \to Y$ be morphisms over B. Then $R \rightrightarrows Y$ is an equivalence relation if and only if $(s,t) : R \to Y \times_B Y$ is a monomorphism and there exist B-morphisms $u : Y \to R$, $i : R \to R$, and $m : R\,{}_s\!\times_t R \to Y$ such that the following diagrams commute:*

Reflexivity:

$$\begin{tikzcd} Y \arrow[r, "\Delta"] \arrow[dr, "u"'] & Y \times_B Y \\ & R \arrow[u, "(s,t)"'] \end{tikzcd}$$

Symmetry:

$$\begin{tikzcd} R \arrow[r, "(s,t)"] \arrow[d, "i"'] & Y \times_B Y \arrow[d, "\eta"] \\ R \arrow[r, "(s,t)"'] & Y \times_B Y \end{tikzcd}$$

Transitivity:

$$\begin{tikzcd} R\,{}_s\!\times_t R \arrow[r, "t \times s"] \arrow[dr, "m"'] & Y \times_B Y \\ & R \arrow[u, "(t,s)"'] \end{tikzcd}$$

In formulae:

$$(s,t)u = \Delta\,, \quad (s,t)i = \eta(s,t)\,, \quad (t,s)m = t \times s\,. \tag{3.2}$$

The points of R are sometimes called "arrows," and, given an arrow a, $s(a)$ stands for its "source" and $t(a)$ for its "target." One often refers to m as the "composition" of arrows and writes ab for $m(a,b)$.

We will say that a morphism of schemes $\pi : Y \to X$ is a *quotient* of an equivalence relation $s,t : R \rightrightarrows Y$ if it has the following properties:

1) $\pi s = \pi t$;
2) every morphism $f : Y \to Z$ such that $fs = ft$ is of the form $h\pi$ for a unique morphism $h : X \to Z$.

We will say that $\pi : Y \to X$ is an *effective quotient* of $R \rightrightarrows Y$ if, in addition,

3) the induced morphism $R \to Y \times_X Y$ is an isomorphism.

Sometimes, we shall write $X = Y/R$ to mean that X is an effective quotient of the equivalence relation. In the category of schemes, effective quotients seldom exist, so one has to enlarge the category to accommodate them. One candidate for this enlargement is the category of algebraic spaces.

The definition of algebraic space follows a simple philosophy: if you cannot beat them, join them. A practical, though slightly incorrect, way of defining an *algebraic space* is simply to say that it is an étale equivalence relation, in other words, an equivalence relation

$$R \underset{t}{\overset{s}{\rightrightarrows}} Y \tag{3.3}$$

where s and t are both étale morphisms.

However, defining an algebraic space in this way is like defining a manifold via an atlas. In this case, the atlas is Y, the scheme R corresponds to the disjoint union of the pairwise intersections of charts, and the two morphisms s and t dictate how the charts of the atlas are patched together along their mutual intersections.

Exactly as in the case of manifolds, one can free the definition of algebraic space from the specific choice of an atlas. We will do this in Section 9, after introducing algebraic stacks, by interpreting algebraic spaces as a particular class of stacks. At that point we will also explain what one means by morphism between two algebraic spaces. In this section we really do not need any of these notions.

A typical example of an algebraic space is provided by the case of a finite group G acting on a scheme Y. In this case we set $R = G \times Y$, and we let $s : R \to Y$ and $t : R \to Y$ be, respectively, the projection and the action. This algebraic space is simply denoted by Y/G.

In practice, it does no harm to assume that the scheme Y in (3.3) is affine. An algebraic space $R \rightrightarrows Y$ is said to be *separated* if the map $(s,t) : R \to Y \times Y$ is a closed immersion. Since, over the complex numbers, an étale map is a local isomorphism for the underlying analytic structures, it is evident that, given a separated algebraic space $R \rightrightarrows Y$, an effective quotient M always exists in the analytic category. We say that M is the *underlying analytic space of the algebraic space* $R \rightrightarrows Y$, and we denote by $\pi : Y \to M$ the étale quotient map. Sometimes, we write $M = (Y/R)_{an}$.

We digress a moment to remark that the analytic space underlying a separated algebraic space is Hausdorff. For this, let $\{m_n\}$ be a sequence in M converging to points α and β. Write $\alpha = \pi(a)$, $\beta = \pi(b)$. Since π is étale, one can lift $\{m_n\}$ to sequences $\{x_n\}$, $\{y_n\}$ in Y such that $\{x_n\}$ converges to a and $\{y_n\}$ to b. By construction (x_n, y_n) belongs to $Y \times_M Y$. Since this is closed in $Y \times Y$, the point (a, b) belongs to $Y \times_M Y$, too. Thus $\alpha = \beta$.

Having said that, an alternative way of giving a *separated algebraic space over* $\mathbb{C}$ is to say that any one such space is defined by the datum of an étale, surjective, analytic map $\pi : Y \to M$, where M is an analytic variety, Y is an affine scheme, and $Y \times_M Y$ a closed subscheme of $Y \times Y$. This is what we shall mean when we will say that $\pi : Y \to M$ is a separated algebraic space. In particular, M will be a scheme exactly when the equivalence relation $Y \times_M Y \rightrightarrows Y$ is effective. Finally, we will say that the algebraic space $\pi : Y \to M$ is reduced, respectively irreducible, normal, complete if the underlying analytic space M is. In the next section we will use this simple way of viewing an algebraic space to see that the moduli space on n-pointed stable curves of given genus is indeed an algebraic space.

An elementary result about algebraic spaces that we will often use is the following.

THEOREM (3.4). *Let $X = (R \rightrightarrows Y)$ be an algebraic space. Then there exists an affine open dense subset $V \subset Y$ such that the induced relation $R_V \rightrightarrows V$ has an effective quotient. Hence, if M is the underlying analytic space of $R \rightrightarrows Y$, the scheme V/R_V is (isomorphic to) a dense open subset in M.*

A proof of this theorem can be found in [428], Prop. 5.19, p. 89, or [38], Prop. 4.5, p. 107.

Let X be an irreducible algebraic space. With the notation of Theorem (3.4), one defines *the field $K(X)$ of rational functions on X* by setting

$$K(X) = K(V/R_V)\,.$$

We will now prove the following result.

PROPOSITION (3.5). *$\overline{M}_{g,n}$ is a separated, normal, complete algebraic space.*

Proof. Consider a standard algebraic Kuranishi family

$$\xi : \mathcal{C} \longrightarrow (X_0, x_0)$$

as defined in Chapter XI, (6.7). Consider the natural map

$$\psi : X_0/G_{x_0} \longrightarrow \overline{M}_{g,n},$$

where G_{x_0} is the automorphism group of the central fiber of the family. By properties d) and e) in the definition of standard algebraic Kuranishi family, the map ψ is an étale morphism from the *affine scheme* X_0/G_{x_0} to the analytic space $\overline{M}_{g,n}$. The idea is then to take as Y a *finite* union of schemes of the form X_0/G_{x_0}. The first thing to show is that a finite number of these suffices to cover $\overline{M}_{g,n}$. We may argue as follows. Consider the natural map

$$m : \mathrm{H}_{\nu,g,n} \longrightarrow \overline{M}_{g,n}$$

and notice that it is obviously surjective. For each point $x \in \mathrm{H}_{\nu,g,n}$, let X_x be the parameter space for the (standard algebraic) Kuranishi family at x constructed in Proposition (6.5) of Chapter XI, and let G_x be the isotropy group of x. Recall that $\mathrm{H}_{\nu,g,n}$ is acted on algebraically by the projective group G and that X_x is a locally closed algebraic subvariety of $\mathrm{H}_{\nu,g,n}$ which is transverse to the orbits of G. It follows that $G \cdot X_x$ contains a Zariski-open subset of $\mathrm{H}_{\nu,g,n}$. By compactness we can cover $\mathrm{H}_{\nu,g,n}$ with finitely many sets of the type $G \times X_i$, $i = 1, \dots, N$. Set

$$Y_i = X_i/G_i\,, \quad i = 1, \dots, N, \qquad Y = Y = \coprod_{i=1}^{N} Y_i. \tag{3.6}$$

Then the étale map

$$\varphi : Y \longrightarrow \overline{M}_{g,n} \tag{3.7}$$

is surjective. Denote by φ_i the restriction of φ to Y_i. We now wish to show that

$$R = Y \times_{\overline{M}_{g,n}} Y$$

is Zariski-closed in $Y \times Y$. Set

$$X = \coprod_{i=1}^{N} X_i \tag{3.8}$$

and consider on X the family of curves $\xi : \mathcal{C} \to X$ induced by the universal family on $\mathrm{H}_{\nu,g,n}$. Denote by p_1 and p_2 the two projections from $X \times X$ to X. Look at the scheme

$$\mathbf{I} = \mathbf{Isom}_{X\times X}(p_1^*(\xi), p_2^*(\xi))\,. \tag{3.9}$$

Set-theoretically, $\mathbf{I}$ is nothing but the incidence correspondence in $X \times X \times G$ defined by

$$\mathbf{I} = \{(x, x', g) \mid x' = gx\},$$

where the X_i are viewed as embedded in the Hilbert scheme $\mathrm{H}_{\nu,g,n}$ on which the projective group G acts. By Theorem (5.1) of Chapter X, the natural projection

$$q : \mathbf{I} \longrightarrow X \times X$$

is finite. Thus, the composition η of this map with finite morphism

$$X \times X \longrightarrow Y \times Y$$

is also finite, so that $\eta(\mathbf{I})$ is a closed subscheme of $Y \times Y$. On the other hand, $\eta(\mathbf{I})$ is clearly equal to R.

This shows that $\overline{M}_{g,n}$ is a separated normal algebraic space. In particular, as we already observed, this means that the analytic space $\overline{M}_{g,n}$ is Hausdorff. The completeness of $\overline{M}_{g,n}$ is Theorem (2.10), which however depends on Theorem (2.9), to be proved later. Q.E.D.

In the sections of this chapter dealing with algebraic stacks, the scheme $\mathbf{I}$ in (3.9) will play a fundamental role. This is a good occasion to prove the following result.

Proposition (3.10). *The natural projection $q_1 : \mathbf{I} \to X$ is étale and surjective.*

Proof. Recall, from point e) of Definition (6.7) in Chapter XI, that every point y in X possesses a G_y-invariant neighborhood U such that $\{\gamma \in G \mid \gamma U \cap U \neq \emptyset\} \subset G_y = \mathrm{Aut}(\mathcal{C}_y)$. Let $\alpha : \mathcal{C}_U \to U$ be the restriction to U of the family ξ over X. The following simple lemma gives a local description of the two maps q and q_1.

Lemma (3.11). *Consider the Kuranishi family $\alpha : \mathcal{C}_U \to U$. Let p_1 and p_2 be the two projections from $U \times U$ to U. Consider the natural diagram*

$$\begin{array}{ccc} \mathbf{Isom}_{U\times U}(p_1^*\alpha, p_2^*\alpha) & \xrightarrow{\ q_1\ } & U \\ {\scriptstyle q}\big\downarrow & & \\ U \times U & & \end{array}$$

Let $C = C_{u_0}$ be the central fiber of α. Let $H = \mathrm{Aut}(C)$. Then, there is an isomorphism $\chi : H \times U \to \mathbf{Isom}_{U\times U}(p_1^*\alpha, p_2^*\alpha)$ such that $q_1\chi(g,u) = u$ and $q\chi(g,u) = (gu, u)$. In particular, q_1 is étale and surjective.

Proof. Set $\mathbf{I} = \mathbf{Isom}_{U\times U}(p_1^*\alpha, p_2^*\alpha)$. Define $\chi : H \times U \to \mathbf{I}$ by setting

$$\chi(g, u) = \{g^{-1} : C_{gu} \to C_u\}.$$

Since every isomorphism between two fibers of α is uniquely induced by an element of H, the morphism χ is, set-theoretically, a bijection. Set $k = |H|$. We then have a decomposition of $\mathbf{I}$ into irreducible components

$$\mathbf{I} = I_1 \cup \cdots \cup I_k \,.$$

We also have induced bijective morphisms $\chi : U \to I_i$ having the property that $q_1 \chi = \mathrm{id}_U$. But then χ is unramified. Thus, I_i must be smooth, and χ is an isomorphism.

Q.E.D.

Let us finish the proof of Proposition (3.10). Consider a point in $\mathbf{Isom}_{X \times X}(p_1^* \xi, p_2^* \xi)$ corresponding to an isomorphism $\varphi : \mathcal{C}_x \to \mathcal{C}_y$, where $(x, y) \in X \times X$. As an analytic neighborhood for φ, we take $\mathbf{Isom}_{U \times W}(p_1^*(\xi_{|U}), p_2^*(\xi_{|W}))$, where U (resp. W) is a neighborhood of x (resp. y), as described in point e) of Definition (6.7). It is then sufficient to show that

$$\mathbf{Isom}_{U \times W}(p_1^*(\xi_{|U}), p_2^*(\xi_{|W})) \longrightarrow W \tag{3.12}$$

is étale and surjective. Since $\xi_{|U} : \mathcal{C}_{|U} \to U$ and $\xi_{|W} : \mathcal{C}_{|W} \to W$ are Kuranishi families with isomorphic central fibers, we may assume that there is an isomorphism $\gamma : U \to W$ such that $\gamma^*(\xi_{|W}) = \xi_{|U}$. But then we are reduced to proving that

$$\mathbf{Isom}_{U \times U}(p_1^*(\xi_{|U}), p_2^*(\xi_{|U})) \longrightarrow U \tag{3.13}$$

is étale and surjective, which is the content of Lemma (3.11). Q.E.D.

In all of what we did so far, the presence of curves with nontrivial automorphism groups always appears as an impediment obstructing the existence of good quotients, of universal families, and so on. In the next sections, we will enter the realms of orbifolds and algebraic stacks, where the automorphism groups will no longer appear as a nuisance but rather as a physiological aspect of the structure.

4. The moduli space of curves as an orbifold.

The path we followed in putting an analytic structure on $\overline{M}_{g,n}$, by patching together quotients modulo finite groups of bases of Kuranishi families, suggests that $\overline{M}_{g,n}$ is just a shadow of a richer geometric structure. One way of formalizing what this consists of is by using the notion of orbifold, which we now introduce.

Let M be a Hausdorff topological space. A *V-cover for M* is a set $\mathcal{U}$ of connected open subsets of M such that

1) $M = \bigcup_{U \in \mathcal{U}} U$.

2) Given U and U' in $\mathcal{U}$ and $x \in U \cap U'$, there exists $W \in \mathcal{U}$ with $x \in W \subset U \cap U'$.
3) Each U comes equipped with an *orbifold local chart for* U, i.e., a triple (B, G, m), where B is a ball in $\mathbb{R}^n$, G is a finite group acting smoothly on B, and $m : B \to U$ is a continuous map inducing a homeomorphism between B/G and U.

In contrast with the usual definition of an orbifold local chart (see [612,614] or [514]), we are not asking that the action of G on B be effective. The example to keep in mind is the one of a Kuranishi family for a genus 2 curve C. If $\xi : \mathcal{C} \to B$ is a Kuranishi family for C, the group $\mathrm{Aut}(C)$ acts equivariantly on B and $\mathcal{C}$. But, while the hyperelliptic involution $\iota \in \mathrm{Aut}(C)$ acts nontrivially on $\mathcal{C}$, it instead acts trivially on B. In taking the quotient $B/G \subset M_2$, we want to retain a memory of this trivial action. This is the orbifold analogue of a feature which is an essential part of the stack definition of moduli spaces, where the relevant action of $\mathrm{Aut}(C)$ is the one on the family ξ and not on the base B.

We could continue in this vein to give a definition of an orbifold atlas on a space M by giving compatibility conditions between the various charts, but the notation quickly gets out of hand, and a more indirect approach is advisable.

A *Lie groupoid* $\mathbb{X}$ consists of the datum of two smooth manifolds X_0 and X_1 and five smooth structure maps

$$
\begin{aligned}
&s : X_1 \to X_0\,,\\
&t : X_1 \to X_0\,,\\
&m : X_1\ {}_s{\times}_t\ X_1 \to X_1\,,\\
&u : X_0 \to X_1\,,\\
&i : X_1 \to X_1\,,
\end{aligned}
\tag{4.1}
$$

satisfying formal properties that will be detailed below. Points of X_1 are called *arrows*, the map s is called the *source*, the map t the *target*, the map m the *composition*, the map u the *unit*, and the map i the *inverse*. Given arrows f and g with $t(f) = s(g)$, the composition $m(f, g)$ is also written fg. One also writes $i(g) = g^{-1}$. The following conditions must be satisfied:

i) $su(x) = x = tu(x)$, $\forall x \in X_0$.
ii) $ti(g) = s(g)$, $s(gh) = s(h)$, $t(gh) = t(g)$, whenever $t(h) = s(g)$.
iii) If $s(g) = x$, $t(g) = y$, then $gu(x) = g = u(y)g$, $g^{-1}g = u(x)$, $gg^{-1} = u(y)$.
iv) The composition m is associative.

Of course, all these properties can be expressed as the commutativity of a certain number of diagrams, including the ones in the statement of

Proposition (3.1). With the notation of that proposition, the equalities expressing these commutativity relations are the following:

$$
\begin{aligned}
&(s,t)u = \Delta\,, \quad (s,t)i = \eta(s,t)\,, \quad (t,s)m = (t\times s)\,,\\
&m\circ(ut, \mathrm{id}_{X_1}) = \mathrm{id}_{X_1}\,, \quad m\circ(\mathrm{id}_{X_1}, us) = \mathrm{id}_{X_1}\,,\\
&m\circ(\mathrm{id}_{X_1}, m) = m\circ(m, \mathrm{id}_{X_1})\,,
\end{aligned}
\tag{4.2}
$$

where in the first row we rewrote equalities (3.2). Also notice that, for the last equality to make sense, we are using the identification

$$
\begin{aligned}
(X_1 {}_s\!\times_t X_1) {}_{sm}\!\times_t X_1 &= (X_1 {}_s\!\times_t X_1) {}_{p_2}\!\times_{p_1} (X_1 {}_s\!\times_t X_1)\\
&= X_1 {}_s\!\times_{tm} (X_1 {}_s\!\times_t X_1),
\end{aligned}
\tag{4.3}
$$

where p_1 and p_2 are the two projections from $X_1 {}_s\!\times_t X_1$ to X_1.

A *proper étale Lie groupoid*, or simply an *orbifold groupoid*, is a Lie groupoid such that the two maps s and t are local diffeomorphisms while the map

$$(s,t) : X_1 \to X_0 \times X_0$$

is proper.

It is worth observing that the notion of Lie groupoid is a generalization of the one of equivalence relation. In fact, if we add to the axioms for an Lie groupoid the requirement that the map $(s,t) : X_1 \to X_0 \times X_0$ be a monomorphism, we obtain just the notion of equivalence relation, as the last two rows of (4.2) are then consequences of the first one.

In a sense, conditions i), ii), and iii) allow one to view X_1 as a generalization of a group acting on X_0, where each arrow acts by sending its source point to its target point.

If x is a point of X_0, one can easily verify that the set

$$G_x = \{g \in X_1 \mid s(g) = t(g) = x\} \tag{4.4}$$

is a finite group, which is called the *isotropy group at* x. The set $ts^{-1}(x)$ is called the *orbit* of x, and the orbit space $|\mathbb{X}|$ of $\mathbb{X}$ is the quotient

$$|\mathbb{X}| = X_0/\sim$$

where $x \sim y$ if and only if x and y belong to the same orbit. For $x \in X_0$, we will denote by $\overline{x}$ its class in $|\mathbb{X}|$.

A *morphism* $\varphi : \mathbb{X} \to \mathbb{Y}$ *of orbifold groupoids* consists of two smooth maps $\varphi_0 : X_0 \to Y_0$ and $\varphi_1 : X_1 \to Y_1$ commuting with all the structure maps defining the two orbifold groupoids.

Of course, one could also give the notion of *topological orbifold groupoid* by requiring that X_0 and X_1 are merely topological manifolds and by relaxing the C^∞ condition on the structure maps. At the other extreme, one could introduce the notion of *complex orbifold groupoid* by insisting that X_0 and X_1 be complex manifolds and that the structure maps be analytic.

An *orbifold structure* on a paracompact Hausdorff space M consists of an orbifold groupoid $\mathbb{X}$ and a homeomorphism $f : |\mathbb{X}| \to M$. An orbifold structure should be thought of as the analogue of a specific atlas on a manifold. As in the case of manifolds, it is seldom the case that different atlases can be compared directly, so one passes to a common refinement, and, in this way, one gets an atlas-free definition of a manifold. One can imitate this refinement procedure in the world of orbifold structures and thereby get to the notion of orbifold (freed from the choice of a specific atlas). For this and related matters, we refer to Chapter 1 in [3]. In the present book orbifolds will normally appear equipped with a specific orbifold structure. In Exercises A-1 and A-2 the reader will find the definition of the *orbifold quotient* $[M/G]$ *of a manifold* M *acted on by a finite group* G and will understand that, given an orbifold structure on a space M, every point in M has a neighborhood with an orbifold structure of the form $[B/G_x]$, where $x \in X_0$, and B is a chart around x, reconciling the notion of V-cover with the one of Lie groupoid.

Let us now equip the moduli space $\overline{M}_{g,n}$ with an orbifold structure. Let X be as in (3.8), so that $X = \coprod_i^N X_i$, each X_i is the (smooth) basis of a Kuranishi family, and the moduli map $m : X \to \overline{M}_{g,n}$ is surjective. Let $\mathcal{C} \to X$ be the total family over X. We define an orbifold groupoid $\overline{\mathbb{M}}_{g,n}$ in the following way. Consider the two projections p_1 and p_2 from $X \times X$ to X and set

$$\mathbf{I} = \mathbf{Isom}_{X\times X}(p_1^*\mathcal{C}, p_2^*\mathcal{C}) .$$

Since $X \times X$ is smooth, it follows from Theorem (5.1) in Chapter X that $\mathbf{I}$ is smooth as well. We then let

$$(\overline{\mathbb{M}}_{g,n})_0 = X , \quad (\overline{\mathbb{M}}_{g,n})_1 = \mathbf{I},$$

and we define s and t to be the natural projections from $\mathbf{I}$ onto the first and second factors of $X \times X$, respectively. The composition rule, the unit, and the inverse are the obvious ones. It is then an exercise to verify that there is a homeomorphism $\overline{m} : |\overline{\mathbb{M}}_{g,n}| \to \overline{M}_{g,n}$, giving $\overline{M}_{g,n}$ an orbifold structure.

As usual, enlarging a category (e.g., passing from manifolds to orbifolds) has the advantage of accommodating, inside the new category, operations that were not allowed in the old one, such as taking quotients. Let us then consider a *finite* group G acting on an orbifold $\mathbb{X}$. By this we mean that there are actions of G on X_0 and X_1 that are compatible with all structure morphisms. We then define an orbifold groupoid $[\mathbb{X}/G] = (Y_0, Y_1)$ in the following way:

$$Y_0 = X_0\,, \quad Y_1 = G \times X_1, \tag{4.5}$$

$$\begin{array}{ll} s_G : Y_1 \to Y_0 & \qquad t_G : Y_1 \to Y_0 \\ (\sigma, \varphi) \mapsto s(\varphi) & \qquad (\sigma, \varphi) \mapsto t(\sigma\varphi) \end{array} \tag{4.6}$$

The composition in Y_1 is given by

$$m_G((\sigma, \varphi), (\tau, \psi)) = (\sigma\tau, \tau^{-1} m(\varphi, \psi)). \tag{4.7}$$

The unit and the inverse are the obvious ones. It is then easy to verify that, under these rules, $[\mathbb{X}/G]$ is an orbifold groupoid. In Section 10 we will see an example of this situation when looking at the boundary of $\overline{M}_{g,n}$.

Next, let us say two words about the cohomology of orbifolds. We start with the de Rham complex. Let $\mathbb{X}$ be an orbifold groupoid. Set

$$\mathcal{A}^p(\mathbb{X}) = \{\varphi \in \mathcal{A}^p(X_0) \mid s^*\varphi = t^*\varphi\}.$$

For obvious reasons, the elements in $\mathcal{A}^p(\mathbb{X})$ are called *invariant forms.* Indeed, as we know, every point $\overline{x} \in |\mathbb{X}|$ has a neighborhood of the form B/G_x, where B is a local chart around x. Then the restrictions to B of the forms in $\mathcal{A}^p(\mathbb{X})$ are just the G_x-invariant forms on B. The differential $d : \mathcal{A}^p(\mathbb{X}) \to \mathcal{A}^{p+1}(\mathbb{X})$ is defined in the usual way, and the resulting cohomology groups are denoted with the symbol $H^p_{dR}(\mathbb{X})$. Satake proved that there is an isomorphism

$$H^*_{dR}(\mathbb{X}) \cong H^*(|\mathbb{X}|, \mathbb{R}), \tag{4.8}$$

where the right-hand-side denotes singular cohomology. When $\mathbb{X}$ is a complex orbifold groupoid, one can define in a similar fashion the vector space $\mathcal{A}^{p,q}(\mathbb{X})$ of invariant (p,q)-forms and the vector space $\Omega^p(\mathbb{X})$ of holomorphic p-forms.

Integration of forms requires some care. First of all, given B and G_x, as above, and a top-degree invariant form φ on $\mathbb{X}$, one defines

$$\int_{B/G_x} \varphi = \frac{1}{|G_x|} \int_B \varphi.$$

Now fix a locally finite cover of $\mathcal{U} = \{U_\alpha\}$ of $|\mathbb{X}|$, where U_α is of the form B_α/G_α, and let $\{\rho_\alpha\}$ be a partition of unit for $\bigcup_\alpha B_\alpha$ Then define

$$\int_{\mathbb{X}} \varphi = \sum_\alpha \int_{U_\alpha} \rho_\alpha \varphi .$$

One can then prove that, when $|\mathbb{X}|$ is compact, Poincaré duality holds in the sense that the pairing

$$\begin{gathered} H^p(\mathbb{X}) \times H^{n-p}(\mathbb{X}) \longrightarrow \mathbb{R} \\ (\varphi, \psi) \mapsto \int_{\mathbb{X}} \varphi \wedge \psi \end{gathered}$$

is nondegenerate.

We end this section by introducing the notion of divisor with normal crossings in a complex orbifold $\mathbb{X}$, presented as an orbifold groupoid (X_0, X_1). By definition, a divisor with normal crossings in $\mathbb{X}$ is just a divisor with normal crossings in X_0 which is X_1-invariant. The boundary $\partial M_{g,n}$ is a normal crossings divisor in $\overline{M}_{g,n}$, by the local description (2.4).

5. The moduli space of curves as a stack, I.

It is often useful to regard the moduli spaces $M_{g,n}$ and $\overline{M}_{g,n}$ as Deligne–Mumford stacks. The idea of stack is modeled on moduli spaces and on quotient spaces. In this brief introduction to stacks, we will continuously go back and forth between the abstract categorical concepts and their geometrical origins. In our treatement we will closely follow [167], [671], [190], and [94]. As already announced in the introduction to this chapter, in this section and in the following three, we deviate from our general convention that "scheme" stands for "scheme of finite type over $\mathbb{C}$" and allow general schemes.

As is generally done, we will introduce Deligne–Mumford stacks in three stages. First, we will introduce categories fibered in groupoids, then stacks, and, finally, algebraic stacks and Deligne–Mumford stacks.

Let S be a scheme and consider the category Sch/S of schemes over S. In what follows we will mostly consider the case $S = \operatorname{Spec}\mathbb{C}$.

A *category fibered in groupoids over* Sch/S or, more simply, a *groupoid over* S, is a pair $\mathcal{M} = (\mathcal{C}_\mathcal{M}, p_\mathcal{M})$, where $\mathcal{C}_\mathcal{M}$ is a category, and

$$p_\mathcal{M} : \mathcal{C}_\mathcal{M} \to Sch/S$$

is a functor satisfying the following two conditions:

A) Let $f : T' \to T$ be a morphism in Sch/S, and let η be an object in $\mathcal{C}_{\mathcal{M}}$ such that $p_{\mathcal{M}}(\eta) = T$. Then there exist an object ξ in $\mathcal{C}_{\mathcal{M}}$ and a morphism $\varphi : \xi \to \eta$ in $\mathcal{C}_{\mathcal{M}}$ with $p_{\mathcal{M}}(\varphi) = f$.

B) Every morphism $\varphi : \xi \to \eta$ in $\mathcal{C}_{\mathcal{M}}$ is *cartesian* in the following sense. Given any other arrow $\varphi' : \xi' \to \eta$ and a morphism $h : p_{\mathcal{M}}(\xi) \to p_{\mathcal{M}}(\xi')$ such that $p_{\mathcal{M}}(\varphi')h = p_{\mathcal{M}}(\varphi)$, there exists a unique morphism $\psi : \xi \to \xi'$ such that $p_{\mathcal{M}}(\psi) = h$ and $\varphi'\psi = \varphi$.

By abuse of language, we will refer to the objects of $\mathcal{C}_{\mathcal{M}}$ as to the objects of the groupoid $\mathcal{M}$, and given objects ξ and ξ' in $\mathcal{C}_{\mathcal{M}}$, we will write $\mathrm{Hom}_{\mathcal{M}}(\xi, \xi')$, instead of $\mathrm{Hom}_{\mathcal{C}_{\mathcal{M}}}(\xi, \xi')$.

A *morphism* $\alpha : \mathcal{M} \longrightarrow \mathcal{M}'$ *of groupoids over* Sch/S is a functor (also denoted by) $\alpha : \mathcal{C}_{\mathcal{M}} \to \mathcal{C}_{\mathcal{M}'}$ such that $p_{\mathcal{M}'} = \alpha p_{\mathcal{M}}$. The morphisms between $\mathcal{M}$ and $\mathcal{M}'$ form themselves a category $\mathrm{Hom}(\mathcal{M}, \mathcal{M}') = \mathrm{Hom}_{Sch/S}(\mathcal{C}_{\mathcal{M}}, \mathcal{C}_{\mathcal{M}'})$, whose arrows are the natural transformations between functors. Technically, one says that groupoids over Sch/S constitute a 2-category. We will not elaborate further on this notion, except to notice that some care must be exercised when dealing with commutativity question having to do with morphisms of groupoids, or, more generally, morphisms in a 2-category: one says that a diagram of groupoids and morphisms of groupoids over Sch/S

$$\begin{tikzcd} \mathcal{F} \arrow[r, "a"] \arrow[dr, "c"'] & \mathcal{G} \arrow[d, "b"] \\ & \mathcal{E} \end{tikzcd}$$

is *commutative* if there is given an isomorphism of functors between c and ba. This is in keeping with the fact that the good notion of "being essentially the same" for categories is the one of equivalence. When a morphism of groupoids is an equivalence of categories, we shall sometimes improperly say that it is an *isomorphism of groupoids.*

As we already mentioned, in this book, we will mostly be concerned with groupoids over $Sch/\mathbb{C}$. We simply call these *groupoids.* The definition of groupoid looks modeled after the following example. Take as $\mathcal{C}$ the category in which the objects are the families

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\xi"] \\ T \end{tikzcd}$$

of smooth (resp. stable, n-pointed) curves of genus g and in which a morphism

$$\varphi : \xi' \to \xi$$

between a family $\xi' : \mathcal{X}' \to T'$ and a family $\xi : \mathcal{X} \to T$ is a commutative diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "\xi'"'] & \mathcal{X} \arrow[d, "\xi"] \\ T' \arrow[r, "f"] & T \end{tikzcd}$$

inducing an isomorphism $\mathcal{X}' \cong T' \times_T \mathcal{X}$. The functor p assigns to a family $\xi : \mathcal{X} \to T$ its parameter space T:

$$p(\xi) = T\,.$$

With regard to morphisms, using the above notation, we set

$$p(\varphi) = f.$$

It is an easy exercise for the reader to prove that properties A) and B) are satisfied for the pair $(\mathcal{C}, p)$.

The groupoid of smooth, n-pointed, genus g curves is denoted with the symbol $\mathcal{M}_{g,n}$. The one of stable, n-pointed, genus g curves is denoted with the symbol $\overline{\mathcal{M}}_{g,n}$.

As suggested by the preceding examples, whenever a groupoid $\mathcal{M} = (\mathcal{C}, p)$ is given, it could help to think of an object $\xi \in \mathcal{C}$ as a "family over $p(\xi)$." The term groupoid has the following origin. Given a groupoid $\mathcal{M} = (\mathcal{C}, p)$, denote by $\mathcal{M}(T)$ the category whose objects are objects $\xi \in \mathcal{C}$ with $p(\xi) = T$ (i.e., the "families" over T) and whose morphisms are morphisms φ in $\mathcal{C}$ with $p(\varphi) = id$. Axiom B) tells us that a morphism φ in $\mathcal{C}$ is an isomorphism if, and only if, $p(\varphi)$ is. It follows that $\mathcal{M}(T)$ is a groupoid in the (more usual) sense that all morphisms in $\mathcal{M}(T)$ are isomorphisms. So one can view the functor $p : \mathcal{C} \to Sch/S$ as a "fibration" having groupoids as fibers:

$$\mathcal{M}(T) = p^{-1}(T).$$

The category $\mathcal{M}(T)$ is also called *the category of sections of* $\mathcal{M}$ *over* T.

Notice that, by axiom B), the object ξ in axiom A) is unique up to a unique isomorphism. We shall refer to this object, or rather to $\xi \to \eta$, as *a pullback of* η *to* T'. It is tempting to write $f^*(\eta)$ for ξ. The trouble is that pullbacks are generally not unique, and there is no way of singling out one which is "nicer" than the others. However, it is always possible to choose, for each object η in $\mathcal{C}$ and each arrow $T' \to T = p(\eta)$, a specific pullback $\xi \to \eta$. Technically, such a choice is called a *cleavage*. Once a cleavage has been chosen, it makes sense to write $f^*(\eta)$ for ξ, but one has to remember that the arrow $f^*(\eta) \to \eta$ is an essential part of the notion of pullback.

It is important to decide when two groupoids $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic. As we already mentioned, for this to happen, there must exist an equivalence of categories $F : \mathcal{C}_\mathcal{M} \to \mathcal{C}_{\mathcal{M}'}$ such that $p_\mathcal{M} = p_{\mathcal{M}'}F$. It is well known, and easy to prove, that for a functor F to be an equivalence, it is necessary and sufficient that:

i) F is *fully faithful,* meaning that, for every pair of objects ξ and ξ' in $\mathcal{M}$, the induced map

$$\mathrm{Hom}_\mathcal{M}(\xi', \xi) \longrightarrow \mathrm{Hom}_{\mathcal{M}'}(F(\xi'), F(\xi))$$

is bijective, and

ii) F is *essentially surjective,* meaning that every object η in $\mathcal{M}'$ is isomorphic to $F(\xi)$ for some object ξ in $\mathcal{M}$.

For groupoids, the following lemma holds.

Lemma (5.1). *A morphism $F : \mathcal{M} \to \mathcal{M}'$ of groupoids over Sch/S is an isomorphism if, and only if, for every T in Sch/S, the induced functor on fibers $F_T : \mathcal{M}(T) \to \mathcal{M}'(T)$ is an equivalence of categories.*

Proof. The only two nontrivial assertions hidden in this lemma are the following.

a) If F_T is fully faithful for every T, then F is fully faithful.
b) If F is essentially surjective, so is F_T for every T.

To prove a), it pays to use the following notation. Given a morphism $f : T' \to T$ and objects ξ' and ξ in $\mathcal{M}(T')$ and $\mathcal{M}(T)$, respectively, we set

$$\mathrm{Hom}^f_\mathcal{M}(\xi', \xi) = \{\varphi \in \mathrm{Hom}_\mathcal{M}(\xi', \xi) \mid p_\mathcal{M}(\varphi) = f\}.$$

Write $p = p_\mathcal{M}$ and $p' = p_{\mathcal{M}'}$. Since $p = p'F$, to prove a), it suffices to prove that F induces a bijection

$$\mathrm{Hom}^f_\mathcal{M}(\xi', \xi) \longrightarrow \mathrm{Hom}^f_{\mathcal{M}'}(F(\xi'), F(\xi)). \tag{5.2}$$

Consider the morphism $\varphi : f^*(\xi) \to \xi$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{M}(T)}(\xi', f^*(\xi)) \arrow[r, "F_T"] \arrow[d, "\varphi\circ"'] & \mathrm{Hom}_{\mathcal{M}'(T)}(F(\xi'), F(f^*(\xi))) \arrow[d, "F(\varphi)\circ"] \\
\mathrm{Hom}^f_\mathcal{M}(\xi', \xi) \arrow[r, "F"] & \mathrm{Hom}^f_{\mathcal{M}'}(F(\xi'), F(\xi))
\end{tikzcd}$$

By the definition of groupoid, the two vertical arrows are bijective, so that the bijectivity of F_T implies the bijectivity of (5.2). As far as b) is concerned, let η be an object in $\mathcal{M}'(T)$. By hypothesis, there are an object ξ' in $\mathcal{M}$ and an isomorphism $\varphi : \eta \to F(\xi')$. Set $f = p'(\varphi) : T \to T'$ and let $\xi = f^*(\xi')$. We have an isomorphism $\psi : \xi = f^*(\xi') \to \xi'$ for which $p'(F(\psi)) = p'(\varphi) = f$. By property B), there exists an isomorphism $\sigma : F(\xi) \to \eta$ with $p'(\sigma) = id_T$. But then ξ is an object of $\mathcal{M}(T)$, and σ is an isomorphism in $\mathcal{M}'(T)$ between η and $F_T(\xi)$. Q.E.D.

It is important to notice that any scheme X and any contravariant functor $F : Sch \to Sets$ can be considered as groupoids.

Let us start with the case of schemes. We will consider a scheme X as a groupoid $X = (\mathcal{C}_X, p_X)$, where the objects of $\mathcal{C}_X$ are pairs (T, f) with $f : T \to X$ a morphism of schemes. The morphisms $\varphi : (T, f) \to (T', f')$ are the morphisms $h : T \to T'$ with $f'h = f$. Finally, the projection p_X is defined by $p_X(T, f) = T$. A groupoid $\mathcal{M}$ is *(represented by) a scheme* X if there exists an *isomorphism* of groupoids $\alpha : X \xrightarrow{\sim} \mathcal{M}$. This condition is equivalent to the existence of an object ξ_X in $\mathcal{M}(X)$ having the following universal property: for every object ξ in $\mathcal{M}$, there exists a unique morphism $f : \xi \to \xi_X$. Of course, given the equivalence α, we have $\xi_X = \alpha(X, \mathrm{id}_X)$. If, in the examples above, we limit ourselves to families of smooth (stable) *automorphism-free* curves, then the corresponding groupoids are indeed (represented by) smooth schemes. But, as we know, this is not the case for the groupoid of smooth (resp. stable) n-pointed curves. The lack of a universal family over $\overline{M}_{g,n}$ can be rephrased by saying that, although any family of n-pointed, genus g stable curves $\xi : \mathcal{C} \to S$ induces a moduli map $m_\xi : S \to \overline{M}_{g,n}$, not every map $S \to \overline{M}_{g,n}$ induces a family over S. Let us see how, inherent in the concept of groupoid, is the cure for this asymmetry.

Let $\mathcal{M}$ be a groupoid over Sch. Given an object ξ in $\mathcal{M}(S)$, we think of S as a groupoid, and we define an induced morphism of groupoids

$$m_\xi : S \to \mathcal{M}\,,$$

by associating to every object in $S(T)$, i.e., to every arrow $f : T \to S$, a pullback $f^*(\xi)$ in $\mathcal{M}(T)$, and proceeding similarly for morphisms. By consonance, one might call m_ξ a moduli map of ξ; of course, m_ξ is not unique but depends on the choice of pullbacks. But now, conversely, given a morphism $\mu : S \to \mathcal{M}$, one gets an object ξ in $\mathcal{M}(S)$ by setting $\xi = \mu(\mathrm{id}_S)$. As we shall presently see, this sets up an equivalence of categories between $\mathcal{M}(S)$ and $\mathrm{Hom}(S, \mathcal{M})$, and the symmetry is reestablished.

Actually, we shall prove something slightly more general. Let $\mathcal{M}$ be a groupoid over Sch. Consider the category $\widetilde{\mathcal{C}}$ whose objects are the morphisms of groupoids $S \to \mathcal{M}$, where S is a scheme, and whose arrows are commutative triangles. More precisely, an arrow from $\beta : T \to \mathcal{M}$ to $\alpha : S \to \mathcal{M}$ is a pair consisting of a morphism $f : T \to S$ and an isomorphism of functors between β and αf. As the reader will easily check, the functor $\widetilde{p} : \widetilde{\mathcal{C}} \to Sch$ which attaches S to $\alpha : S \to \mathcal{M}$ makes $\widetilde{\mathcal{M}} = (\widetilde{\mathcal{C}}, \widetilde{p})$ into a category fibered in groupoids over Sch. By definition we have $\widetilde{\mathcal{M}}(S) = \mathrm{Hom}(S, \mathcal{M})$. Moreover, $\widetilde{\mathcal{M}}$ is endowed with a canonical cleavage: given a morphism $f : T \to S$, one can take as pullback of an object $\alpha : S \to \mathcal{M}$ simply the composition $\alpha f : T \to \mathcal{M}$ mapping to α via the pair consisting of f and the identity isomorphism of functors. In fact, this cleavage is a *splitting*, meaning that it contains all the identities and is closed under composition. There is an obvious functor $F : \widetilde{\mathcal{C}} \to \mathcal{C}$. As above, we associate to $\alpha : S \to \mathcal{M}$ the object $F(\alpha) = \alpha(\mathrm{id}_S)$ in $\mathcal{M}(S)$. Similarly, given an arrow φ in $\widetilde{\mathcal{C}}$, consisting of a morphism of schemes $f : T \to S$ and of an isomorphism of functors $\beta \cong \alpha f$, we first get an isomorphism $\beta(\mathrm{id}_T) \cong \alpha f(\mathrm{id}_T) = \alpha(f)$. On the other hand, since α is a functor, it gives an arrow $\alpha(f) \to \alpha(\mathrm{id}_S)$. Composing the two, we get an arrow $F(\varphi) : F(\beta) = \beta(\mathrm{id}_T) \to \alpha(\mathrm{id}_S) = F(\alpha)$. We leave it to the reader to check that what we have defined is indeed a functor. It is clear that $\widetilde{p} = pF$, and hence that F can be viewed as a morphism $\widetilde{\mathcal{M}} \to \mathcal{M}$. The next result is a special instance of the *2-categorical Yoneda lemma.*

LEMMA (5.3). *The morphism $F : \widetilde{\mathcal{M}} \to \mathcal{M}$ is an equivalence of categories fibered in groupoids over Sch.*

Here is a sketch of the proof. First, we define a functor $G : \mathcal{C} \to \widetilde{\mathcal{C}}$. Suppose that ξ is an object in $\mathcal{M}(S)$ and $f : T \to S$ is a morphism of schemes. As above, we set $m_\xi(f) = f^*\xi$. If $\varphi : U \to T$ is a morphism of schemes, and $h = f\varphi$, then by cartesianness there is a unique arrow $m_\xi(\varphi) : m_\xi(h) \to m_\xi(f)$ lying above φ and making the diagram

$$\begin{tikzcd} m_\xi(h) \arrow[d, "m_\xi(\varphi)"'] \arrow[dr] & \\ m_\xi(f) \arrow[r] & \xi \end{tikzcd}$$

commute. Cartesianness also immediately shows that m_ξ is a functor $S \to \mathcal{M}$. We set $G(\xi) = m_\xi$. Now let $\alpha : \eta \to \xi$ be an arrow in $\mathcal{C}$ lying above a morphism of schemes $a : T \to S$. Let $b : U \to T$ be a morphism of schemes and set $c = ab$. Then $m_\eta(b)$ and $m_\xi(c)$ are both pullbacks of ξ to U and hence are canonically isomorphic. As $U \to T$ varies, these isomorphisms give an isomorphism of functors between m_η and $m_\xi a$, and hence an arrow $G(\alpha) : G(\eta) \to G(\xi)$. We leave to the reader the easy task of checking that G is a functor.

We claim that FG is isomorphic to the identity functor on $\mathcal{M}$, and GF to the identity on $\widetilde{\mathcal{M}}$. First of all, when ξ is an object in $\mathcal{M}(S)$, $FG(\xi)$ is just $\mathrm{id}_S^*(\xi)$, which is canonically isomorphic to ξ; in fact, in the definition of G, we can arrange things so that $\mathrm{id}_S^*(\xi)$ is just ξ, and $\mathrm{id}_S^*(\xi) \to \xi$ the identity. If we do this, FG turns out to be the identity functor.

Now we turn to GF. Let $\alpha : S \to \mathcal{M}$ be a morphism where S is a scheme. In other words, α is a base-preserving functor from the category of morphisms of schemes $T \to S$ to $\mathcal{C}$. This simply means that, for each morphism $a : T \to S$, we are given an object $\alpha(a)$ in $\mathcal{M}(T)$ and that, for any morphism $b : U \to T$, $\alpha(ab)$ is a pullback of $\alpha(a)$ via b. By the essential uniqueness of pullbacks, the objects $\alpha(a)$ are determined, up to a unique isomorphism, by $\alpha(\mathrm{id}_S) = F(\alpha)$. By the definition of G, this sets up a canonical isomorphism between α and $GF(\alpha)$. It is not difficult, but tedious, to check that this defines an isomorphism of functors between GF and the identity, ending the proof of (5.3).

As we have announced, a contravariant functor $F : Sch \to Sets$ can also be considered as a groupoid. For this groupoid, also denoted by F, the objects of $\mathcal{C}_F$ are pairs (T, ξ), where T is a scheme and $\xi \in F(T)$. A morphism $(T, \xi) \to (T', \xi')$ is a morphism $f : T \to T'$ such that $F(f)(\xi') = \xi$. The symbol $F(T)$ unambiguosly denotes both the set $F(T)$ and the fiber over T of F considered as a groupoid. When we choose as F the functor of points $h_X = \mathrm{Hom}(-, X)$ of a scheme X, the groupoids associated to h_X and X coincide. A contravariant functor $F : Sch \to Sets$ is said to be *representable* if there exists a scheme X and a groupoid isomorphism between F an h_X.

We now make an important remark regarding $\overline{\mathcal{M}}_{g,n}$ and $\mathcal{M}_{g,n}$. In Section 3 we constructed the spaces $M_{g,n}$ and $\overline{M}_{g,n}$ as algebraic spaces. We will see in Chapter XIV that they are actually schemes, and as such we will treat them now. Let us concentrate our attention on stable curves, the case of smooth curves being completely similar. Consider the scheme $\overline{M}_{g,n}$ and the groupoid $\overline{\mathcal{M}}_{g,n}$. A third object is linked to n-pointed, stable curves of genus g, namely the contravariant functor

$$\overline{F}_{g,n} : Sch/\mathbb{C} \longrightarrow Sets$$

defined as follows. For every scheme T, we set

$$\overline{F}_{g,n}(T) = \left\{ \begin{array}{c} \text{Families of } n\text{-pointed genus } g \\ \text{stable curves parameterized by } T \end{array} \right\} \Big/ \text{isomorphisms.}$$

An element of $\overline{F}_{g,n}(T)$ is denoted by $[\xi : \mathcal{X} \to T]$, and for every morphism $f : T \to T'$ and every element $[\xi' : \mathcal{X}' \to T']$ in $\overline{F}_{g,n}(T')$, we set $F(f) = [\mathcal{X}' \times_{T'} T \to T]$. We now have three groupoids and two obvious morphisms:

$$\overline{\mathcal{M}}_{g,n} \xrightarrow{\alpha} \overline{F}_{g,n} \xrightarrow{\beta} \overline{M}_{g,n} \,. \tag{5.4}$$

In the rightmost groupoid we are looking at moduli as a scheme (or as an algebraic space). In the central one we look at moduli as a functor. In the first one we consider moduli as a bona fide groupoid. In a sense, we can regard α and β as forgetful functors. The functor α associates to the object $\xi : \mathcal{X} \to T$ of $\overline{\mathcal{M}}_{g,n}$ the object $[\xi]$ of $\overline{F}_{g,n}$. The functor β associates to the object $[\xi]$ the object (T, f) of $\overline{M}_{g,n}$, where $f : T \to \overline{M}_{g,n}$ is the morphism induced by the family ξ. Neither α nor β is an isomorphism of groupoids. The fact that β is not is a restatement of the fact that $\overline{F}_{g,n}$ is not representable or, which is the same, that there is no universal family of curves over $\overline{M}_{g,n}$ or, yet in other words, that there is no object in $\mathcal{C}_{\overline{F}_{g,n}}$ mapping, via β, to $(\overline{M}_{g,n}, \mathrm{id}_{\overline{M}_{g,n}})$. Clearly, α also fails to be an isomorphism of groupoids. Indeed, given an object $\xi : \mathcal{X} \to T$ in $\overline{\mathcal{M}}_{g,n}(T)$, we have

$$\mathrm{Hom}_{\overline{\mathcal{M}}_{g,n}(T)}(\xi, \xi) = \{\text{isomorphisms} \quad \varphi : \mathcal{X} \to \mathcal{X} \mid \xi = \varphi\xi\},$$

while

$$\mathrm{Hom}_{\overline{F}_{g,n}(T)}([\xi], [\xi]) = \{\mathrm{id}_{\mathcal{X}}\}.$$

In particular, when $T = \{pt\}$ is a single point, and $\mathcal{X} = X$ a stable curve,

$$\mathbf{Isom}_{\overline{\mathcal{M}}_{g,n}(pt)}(X, X) = \mathrm{Aut}(X)\,, \qquad \mathbf{Isom}_{\overline{F}_{g,n}(\{pt\})}([X], [X]) = \{\mathrm{id}_X\}.$$

The presence of stable curves with nontrivial automorphism group, which is the cause for the nonrepresentability of the moduli functor $\overline{F}_{g,n}$, is actually the distinctive feature of the geometric fibers of the moduli groupoid $\overline{\mathcal{M}}_{g,n}$.

A very important example of groupoid is the following. Suppose that a group scheme G acts on a scheme X. Then one can form the *quotient groupoid*

$$[X/G] = (\mathcal{P}_{G,X}, p)\,, \tag{5.5}$$

where $\mathcal{P}_{G,X}$ is the category whose objects are pairs (π, σ_π), where $\pi : E \to T$ is a principal G-bundle, and $\sigma_\pi : E \to X$ is a G-equivariant map. A morphism between (π, σ_π) and (π', σ'_π) is a pair of commutative diagrams

$$\begin{tikzcd} E' \arrow[r, "\varphi"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ T' \arrow[r, "f"] & T \end{tikzcd} \qquad \begin{tikzcd} E' \arrow[r, "\varphi"] \arrow[dr, "\sigma_{\pi'}"'] & E \arrow[d, "\sigma_\pi"] \\ & X \end{tikzcd}$$

the first one of which is cartesian. Finally, the projection

$$p : \mathcal{P}_{G,X} \longrightarrow Sch$$

is given by

$$(\pi\,, \sigma_\pi) \quad \mapsto \quad T\,,$$

so that the fiber $\mathcal{P}_{G,X}(T)$ is the category of principal G-bundles over T, equipped with a G-equivariant map from their total space to X. Notice that only when G acts freely on X and the quotient X/G exists as a scheme, the groupoid $[X/G]$ is represented by X/G. Indeed, in this case, X is a principal G-bundle over X/G and every principal G-bundle $\pi : E \to T$, equipped with a G-equivariant morphism $\sigma_\pi : E \to X$, is isomorphic to the pullback bundle $X \times_f T$, via a unique map $f : T \to X/G$:

$$\begin{tikzcd} X \times_f T \cong E \arrow[r, "\sigma_\pi"] \arrow[d, "\pi"'] & X \arrow[d] \\ T \arrow[r, "f"] & X/G \end{tikzcd}$$

On the other hand, if the action of G is not free, X/G may well exist as a scheme without the groupoid $[X/G]$ being representable. The case is the one where $X = \{pt\}$ is a single point. Then, for obvious reasons, one sets

$$[\{pt\}/G] = BG\,.$$

The geometric example we have in mind is of course the Hilbert scheme $\mathrm{H}_{\nu,g,n}$ of ν-log-canonically embedded n-pointed stable curves of genus g (where $\nu \geq 3$). As we saw in Section 5 of Chapter XI, $\mathrm{H}_{\nu,g,n}$ is acted on by $PGL(N)$, where $N = (2\nu - 1)(g - 1) + \nu n$.

Theorem (5.6). *The moduli groupoid $\overline{\mathcal{M}}_{g,n}$ is isomorphic to the quotient groupoid $[\mathrm{H}_{\nu,g,n}\,/PGL(N)]$.*

Proof. Let us define a morphism

$$\Phi : \overline{\mathcal{M}}_{g,n} \longrightarrow [\mathrm{H}_{\nu,g,n}\,/PGL(N)]\,.$$

Given an object in $\mathcal{C}$, that is, a family $\xi : \mathcal{X} \to T$ of stable n-pointed genus g curves, $\Phi(\xi)$ must consist of a G-bundle $\pi : E \to T$ and a G-equivariant map $\sigma_\pi : E \to \mathrm{H}_{\nu,g,n}$. As far as the bundle is concerned, we let $\pi : E \to T$ be the principal G-bundle associated to the projective bundle $\mathbb{P}_\xi = \mathbb{P}(\xi_*(\omega_\xi^\nu(\nu D))) \to T$, where D is the divisor of the canonical sections of ξ. Consider the canonically trivialized G-bundle

$$\pi^*\mathbb{P}_\xi \to E$$

and the pulled-back family

$$\eta : \mathcal{Z} = \mathcal{X} \times_\pi E \to E\,.$$

There is a canonical isomorphism

$$\mathbb{P}_\eta \cong \pi^*\mathbb{P}_\xi.$$

The canonical trivialization of $\mathbb{P}_\eta$ exhibits $\mathcal{Z} \to E$ as a family of ν-log-canonically embedded curves and therefore gives a G-equivariant morphism

$$\sigma_\pi : E \longrightarrow \mathrm{H}_{\nu,g,n} \ .$$

The definition of Φ on objects is now completed. The definition of Φ on morphisms and the proof that Φ is indeed a morphism of groupoids are straightforward and are left to the reader. Let us show that Φ is an isomorphism. We use Lemma (5.1). We must show that, for every scheme T, the functor Φ_T is fully faithful and essentially surjective. For the first point, we must prove that Φ_T induces a bijection

$$\mathrm{Hom}_{\mathcal{C}(T)}(\xi, \xi) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{P}(T)}(\Phi(\xi), \Phi(\xi)).$$

Equivalently, we must show that if $\xi : \mathcal{X} \to T$ is a family of stable, n-pointed curves of genus g, then the automorphisms of this family and the automorphisms of the projective bundle $P_\xi \to T$ determine each other. Looking at the fiberwise ν-canonical embedding

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathbb{P}^*_\xi \arrow[dl] \\ T & \end{tikzcd}$$

it is clear that the only thing to prove is that any automorphism γ of the family $\xi : \mathcal{X} \to T$ is induced by one of the bundles $\mathbb{P}_\xi$. This is certainly true locally, where the projective bundle can be trivialized. But then it is also true globally because, on each fiber $\mathcal{X}_t$, the automorphism γ_t is uniquely induced, via φ_t, by a projective automorphism of $(\mathbb{P}_\xi)_t$. We now address the essential surjectivity of Φ_T. Let then (π, σ_π) be an object of $\mathcal{P}$, so that $\pi : E \to T$ is a principal G-bundle, and $\sigma_\pi : E \to \mathrm{H}_{\nu,g,n}$ is a G-equivariant map. Look at the universal family $\mathcal{Y} \to \mathrm{H}_{\nu,g,n}$ and form the cartesian diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d, "\eta"'] & \mathcal{Y} \arrow[d] \\ E \arrow[r, "\sigma_\pi"] & \mathrm{H}_{\nu,g,n} \end{tikzcd}$$

The group G acts equivariantly and *freely* on E and $\mathcal{Z}$. We can then form the quotient family $\xi : \mathcal{Z}/G = \mathcal{X} \to T = E/G$. It is now an exercise to prove that, indeed, $\Phi_T(\xi)$ is isomorphic to (π, σ_π). Q.E.D

6. The classical theory of descent for quasi-coherent sheaves.

In this section we recall the simplest instance of Grothendieck's descent theory, namely faithfully flat descent for quasicoherent sheaves. To explain what this means, consider a morphism of schemes $X \to Y$, and a quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$. Via the two projections p_1 and p_2 of $X \times_Y X$ to the two factors, $\mathcal{F}$ pulls back to $\mathcal{F}_1 = p_1^*\mathcal{F}$ and $\mathcal{F}_2 = p_2^*\mathcal{F}$. Write p_{12}, p_{13}, and p_{23} to indicate the projections of $X \times_Y X \times_Y X$ to $X \times_Y X$ obtained by omitting the third, second, and first components, respectively, and q_1, q_2, q_3 to indicate the three projections of $X \times_Y X \times_Y X$ to X. We have the usual simplicial diagram

$$\begin{tikzcd} X \times_Y X \times_Y X \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X \times_Y X \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r] & Y \end{tikzcd}$$

Notice that $p_1p_{12} = q_1 = p_1p_{13}$, $p_2p_{12} = q_2 = p_1p_{23}$, and $p_2p_{13} = q_3 = p_2p_{23}$. By *descent data* for $\mathcal{F}$ relative to $X \to Y$ we mean an isomorphism $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ such that the following "benzene" diagram commutes:

$$\begin{tikzcd} & p_{12}^*\mathcal{F}_2 \arrow[r, equal] & p_{23}^*\mathcal{F}_1 \arrow[dr, "p_{23}^*\varphi"] & \\ p_{12}^*\mathcal{F}_1 \arrow[ur, "p_{12}^*\varphi"] \arrow[dr, equal] & & & p_{23}^*\mathcal{F}_2 \\ & p_{13}^*\mathcal{F}_1 \arrow[r, "p_{13}^*\varphi"] & p_{13}^*\mathcal{F}_2 \arrow[ur, equal] & \end{tikzcd} \tag{6.1}$$

We will refer to this condition as the *cocycle condition.* Very roughly speaking, the existence of φ tells us that $\mathcal{F}$ "looks the same" at points belonging to the same fiber of $X \to Y$, and the cocycle condition guarantees that the ensuing identifications are consistent.

When $\mathcal{F}$ is the pullback of a quasicoherent $\mathcal{O}_Y$-module, there is a canonical isomorphism between $\mathcal{F}_1$ and $\mathcal{F}_2$ which provides $\mathcal{F}$ with canonical descent data. The problem of descent is to decide whether this process can be inverted, that is, whether a quasicoherent $\mathcal{O}_X$-module with descent data comes from an $\mathcal{O}_Y$-module. One may ask a similar question for morphisms. There is an obvious notion of morphism of quasicoherent $\mathcal{O}_X$-modules with descent data, and one may wonder whether morphisms between modules with descent data which arise by pullback from $\mathcal{O}_Y$-modules $\mathcal{F}$ and $\mathcal{G}$ do descend to morphisms between $\mathcal{F}$ and $\mathcal{G}$. The answer to both questions is yes if one assumes that $X \to Y$ be *faithfully flat*, that is, flat and surjective. Actually, one also has to assume that $X \to Y$ is quasi-compact. This is automatic when one deals only with schemes of finite type over a field, as we do in the rest of the book.

THEOREM (6.2). *Let $\pi : X \to Y$ be a faithfully flat and quasi-compact morphism of schemes. Then the pullback functor*

$$\{\textit{quasicoherent } \mathcal{O}_Y\textit{-modules}\} \to \left\{ \begin{matrix} \textit{quasicoherent } \mathcal{O}_X\textit{-modules with} \\ \textit{descent data relative to } \pi \end{matrix} \right\}$$

is an equivalence of categories.

We postpone the proof for a moment, except for the following simple observation.

REMARK (6.3). There is one case in which the conclusion of Theorem (6.2) holds true, even without faithful flatness hypothesis, and this is the one when π admits a section, that is, a right inverse. Call this σ and consider the two morphisms $\tau_1 = (\mathrm{id}, \sigma)$ and $\tau_2 = (\sigma, \mathrm{id})$ from X to $X \times_Y X$:

$$Y \xrightarrow{\sigma} X \underset{\tau_2}{\overset{\tau_1}{\rightrightarrows}} X \times_Y X$$

Clearly,

$$\pi\sigma = \mathrm{id}_Y\,, \quad p_1\tau_1 = p_2\tau_2 = \mathrm{id}_X\,, \quad p_1\tau_2 = p_2\tau_1 = \sigma\pi\,.$$

In particular, if a quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ is of the form $\pi^*\mathcal{H}$ for some $\mathcal{H}$, then $\mathcal{H} = \sigma^*\mathcal{F}$. Similarly, any morphism $\mathcal{H} \to \mathcal{K}$ of quasicoherent $\mathcal{O}_Y$-modules can be recovered from $\pi^*\mathcal{H} \to \pi^*\mathcal{K}$ as $\sigma^*\pi^*\mathcal{H} \to \sigma^*\pi^*\mathcal{K}$. Thus, $\mathrm{Hom}(\mathcal{H}, \mathcal{K}) \to \mathrm{Hom}(\pi^*\mathcal{H}, \pi^*\mathcal{K})$ is injective.

Now suppose that $\mathcal{F}$ is a quasicoherent $\mathcal{O}_X$-module with descent data $p_1^*\mathcal{F} \to p_2^*\mathcal{F}$ and pull back these via τ_1. What we get is an isomorphism

$$\mathcal{F} = \tau_1^* p_1^* \mathcal{F} \to \tau_1^* p_2^* \mathcal{F} = \pi^*\sigma^*\mathcal{F}\,. \tag{6.4}$$

Thus, $\mathcal{F}$ is the pullback via π of a quasicoherent sheaf on Y, namely $\sigma^*\mathcal{F}$. The isomorphism between $\mathcal{F}$ and $\pi^*\sigma^*\mathcal{F}$ holds also if we take descent data into account. This is a direct consequence of the cocycle relation. Define a morphism $\tau_{12} : X \times_Y X \to X \times_Y X \times_Y X$ by setting $\tau_{12} = (p_1, p_2, \sigma\pi p_1) = (p_1, p_2, \sigma\pi p_2)$. Then one immediately checks that

$$p_{12}\tau_{12} = \mathrm{id}\,, \quad p_{13}\tau_{12} = \tau_1 p_1\,, \quad p_{23}\tau_{12} = \tau_1 p_2\,.$$

Since (6.1) commutes, pulling it back by means of τ_{12}, we get another commutative diagram, which, taking into account the above identities and recalling that $p_2\tau_1 = \sigma\pi$, reduces to

$$\begin{array}{ccccccc}
p_1^*\mathcal{F} & & \longrightarrow & & & & p_2^*\mathcal{F} \\
\downarrow & & & & & & \downarrow \\
p_1^*\tau_1^*p_2^*\mathcal{F} & = & p_1^*\pi^*\sigma^*\mathcal{F} & = & p_2^*\pi^*\sigma^*\mathcal{F} & = & p_2^*\tau_1^*p_2^*\mathcal{F}
\end{array}$$

This shows that the descent data for $\mathcal{F}$ and those for $\pi^*\sigma^*\mathcal{F}$ correspond to each other via the isomorphism $\mathcal{F} \to \pi^*\sigma^*\mathcal{F}$ given by (6.4).

Finally, let $\mathcal{F}$ and $\mathcal{G}$ be quasicoherent $\mathcal{O}_X$-modules with descent data and suppose that α is a morphism between them. In other words, $\alpha : \mathcal{F} \to \mathcal{G}$ is such that

$$\begin{array}{ccc}
p_1^*\mathcal{F} & \xrightarrow{p_1^*\alpha} & p_1^*\mathcal{G} \\
\downarrow & & \downarrow \\
p_2^*\mathcal{F} & \xrightarrow{p_2^*\alpha} & p_2^*\mathcal{G}
\end{array}$$

commutes. Pulling back this diagram via τ_1 yields another commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\alpha"] \arrow[d] & \mathcal{G} \arrow[d] \\ \pi^*\sigma^*\mathcal{F} \arrow[r, "\pi^*\sigma^*\alpha"] & \pi^*\sigma^*\mathcal{G} \end{tikzcd}$$

This means that α is the pullback of the morphism of $\mathcal{O}_Y$-modules $\sigma^*\alpha : \sigma^*\mathcal{F} \to \sigma^*\mathcal{G}$.

We now prove (6.2). Let us recapitulate what must be shown. Recall that a diagram of mappings of sets

$$A \xrightarrow{a} B \underset{b_2}{\overset{b_1}{\rightrightarrows}} C$$

is said to be *exact* if a is injective and its image is the *equalizer* of the pair of mappings b_1, b_2, that is, the set of those element of B which map to the same element of C via b_1 and b_2. First of all, we must prove that, if $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent $\mathcal{O}_Y$-modules and we set $\mathcal{F}' = \pi^*\mathcal{F}$, $\mathcal{F}'' = p_1^*\mathcal{F}' = p_2^*\mathcal{F}'$, then the diagram

$$\text{(6.5)} \qquad \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F},\mathcal{G}) \xrightarrow{\pi^*} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}',\mathcal{G}') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{O}_{X\times_Y X}}(\mathcal{F}'',\mathcal{G}'')$$

is exact. Then we must show that any quasicoherent $\mathcal{O}_X$-module with descent data comes by pullback from a quasicoherent $\mathcal{O}_Y$-module; the latter is then automatically unique, up to a unique isomorphism, by the exactness of (6.5). By Remark (6.3), the conclusion of the theorem is valid if π has a section. The idea is to reduce to this case via the base change $X \to Y$. Once this has been done, however, we must push down from X to Y what we have obtained. Here is where the faithful flatness assumption comes into play. Recall in fact that a module M over a commutative ring A is said to be *faithfully flat* if the exactness of any sequence of homomorphisms of A-modules is equivalent to the exactness of the sequence obtained by tensoring it with M. It is an elementary result (see, for instance, [503]) that a commutative A-algebra B is faithfully flat if and only if it is flat and $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is onto, that is, if and only if f is a faithfully flat morphism of schemes. It is then at least plausible that questions having to do with exactness of sequences on Y could be decided by examining the corresponding questions on X. Before we explain this in detail, however, it is best to perform a couple of reductions. First of all, we claim that it suffices to prove (6.2) when Y is affine. In fact, the injectivity of π^* in (6.5), that is, uniqueness of descent for homomorphisms, is a local property on Y. On the other hand, if a homomorphism between $\mathcal{F}'$ and $\mathcal{G}'$ descends locally to a homomorphism between $\mathcal{F}$ and $\mathcal{G}$, the compatibility of the descended homomorphisms can be checked locally and follows from uniqueness if the theorem is known to hold when Y is affine. Likewise, if a quasicoherent sheaf $\mathcal{G}$ with descent data on X descends to a quasicoherent sheaf $\mathcal{F}_i$ on each open set Y_i of a cover of Y, then the $\mathcal{F}_i$ are isomorphic, via a unique isomorphism, on any affine contained in any one of the overlaps of the Y_i and hence patch together to yield a quasicoherent sheaf $\mathcal{F}$ which lifts to $\mathcal{G}$.

The second reduction is that we may suppose that X is also affine. In fact, if Y is affine, then X is the union of finitely many affines X_i. We write X' for the disjoint union of the X_i, which is then affine. Clearly, $X' \to Y$ is faithfully flat and factors through $X \to Y$. Equally clearly, if the theorem holds for $X' \to Y$, it holds a fortiori for $X \to Y$. In conclusion, in proving (6.2) we may assume that both X and Y are affine.

We therefore assume that $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} A'$, that π corresponds to a ring homomorphism $\alpha : A \to A'$, and that A' is faithfully flat over A. We set $A'' = A' \otimes_A A'$; moreover, for any A-module M, we set $M' = M \otimes_A A'$, $M'' = M \otimes_A A''$. There are two natural homomorphisms β_1 and β_2 from A' to A'', given by $b \mapsto b \otimes 1$ and $b \mapsto 1 \otimes b$, corresponding to the two projections $X \times_Y X \to X$. For any A'-module H, these give rise to two tensor products $H \otimes_{A'} A''$, which we denote by H_1 and H_2. In the language of rings and modules, descent data for the quasicoherent $\mathcal{O}_X$-module $\widetilde{H}$ correspond to a homomorphism $H_1 \to H_2$ of A''-modules, satisfying an obvious cocycle condition. We shall refer to H, equipped with such a homomorphism, as a *module with descent data*. An A'-module of the form M' comes equipped with natural descent data. For any pair M, N of A-modules, the homomorphisms β_1 and β_2 give two distinct homomorphisms from $\operatorname{Hom}_{A'}(M', N')$ to $\operatorname{Hom}_{A''}(M'', N'')$. Rephrased in terms of modules over rings, what we have to prove is then:

i) for any pair M, N of A-modules, the diagram

$$\operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_{A'}(M', N') \rightrightarrows \operatorname{Hom}_{A''}(M'', N'')$$

is exact;

ii) Any A'-module with descent data is isomorphic to one of the form M', for some A-module M.

The key to proving i) is the following special case of i) itself.

LEMMA (6.6). *For any A-module N, the diagram*

$$N \longrightarrow N' \rightrightarrows N'' \tag{6.7}$$

is exact.

The proof is based on the idea, mentioned earlier, of performing a faithfully flat base change $A \to B$ such that, setting $B' = B \otimes_A A'$, $B'' = B' \otimes_B B'$, the ring homomorphism $B' \to B''$ has a left inverse (i.e., such that $\operatorname{Spec} B'' \to \operatorname{Spec} B'$ has a section). One then deduces exactness from the existence of the left inverse, and finally exactness can be "pushed down" to A by faithful flatness. The required base change is provided to us for free; it suffices to take A' as B, since $\beta_i : A' \to A''$ has a left inverse, namely the multiplication homomorphism $A'' = A' \otimes_A A' \to A'$.

We now show that this idea can be made into an actual proof. We claim that the lemma is true if $\alpha : A \to A'$ has a left inverse. Actually, we have already proved this in Remark (6.3), but let us do it again here. Let ρ be a left inverse of α. We define homomorphisms $\sigma_i : A'' \to A'$, $i = 1, 2$, by $\sigma_1(b \otimes b') = \rho(b')b$, $\sigma_2(b \otimes b') = \rho(b)b'$. Then $\sigma_i\beta_i = \mathrm{id}$ and $\sigma_i\beta_j = \alpha\rho$ when $i \neq j$. Now tensor with N and call with the same names the resulting homomorphisms. We see that $N \to N'$ has a left inverse, and hence is injective. On the other hand, if $\beta_1(n') = \beta_2(n')$, then $n' = \sigma_1\beta_1(n') = \alpha\rho(n')$. This proves the claim.

Now observe that, if B is an A-algebra, the diagram obtained by tensoring (6.7) with B is just

$$N \otimes_A B \longrightarrow (N \otimes_A B) \otimes_B B' \rightrightarrows (N \otimes_A B) \otimes_B B''$$

as can be easily checked. Thus, when $B \to B'$ has a left inverse, this diagram is exact. If, in addition, B is a faithfully flat A-algebra, then the original diagram (6.7) is also exact. This proves the lemma.

Our next task is to deduce i) from the lemma. The latter says in particular that M is an A-submodule of M', which is an A'-submodule of M'', and similarly for N, N', and N''. Since N injects into N', the homomorphism $\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{A'}(M', N')$ is injective. Now let $\xi : M' \to N'$ be a homomorphism which belongs to the kernel of the pair of homomorphisms from $\operatorname{Hom}_{A'}(M', N')$ to $\operatorname{Hom}_{A''}(M'', N'')$. To show that ξ comes from $\operatorname{Hom}_A(M, N)$, it suffices to show that $\xi(m \otimes 1) \in N \otimes 1$ for any $m \in M$. But $\xi(m \otimes 1)$ is an element of the kernel of the pair of homomorphisms from N' to N'', so by the lemma it belongs to $N \otimes 1$. This proves i).

To prove ii), we may argue as follows. Let H be an A'-module with descent data $v : H_1 \to H_2$, and let $\gamma_i : H \to H_i$, $i = 1, 2$, be the natural homomorphisms. We must find an A-module M such that H is isomorphic, as a module with descent data, to M'. There is a natural candidate for M. Suppose in fact that an M exists. Then both H_1 and H_2 can be identified with M'', and the composition of the two identifications is just v. But then Lemma (6.6) says that M injects in H and gets identified with

$$N = \{h \in H : v\gamma_1(h) = \gamma_2(h)\}.$$

Thus, all that needs to be done is to prove that the homomorphism $N \otimes_A A' \to H$ is an isomorphism. We resort to the same trick used to prove i). Let B be a faithfully flat A-algebra such that $B \to B'$ has a left inverse (for instance, $B = A'$). Change ring by tensoring everything with B. Write $\overline{N}$ for $N \otimes_A B$, $\overline{H}$ for $H \otimes_A B$, $\overline{v}$ for $v \otimes \mathrm{id} : \overline{H}_1 \to \overline{H}_2$, and so on. Since B is A-flat, $\overline{N}$ is the set of all $h \in \overline{H}$ such that $\overline{v}\,\overline{\gamma}_1(h) = \overline{\gamma}_2(h)$. Since ii) is true for $B \to B'$, by Remark (6.3), we know that $\overline{N} \otimes_B B' \to \overline{H}$ is an isomorphism. On the other hand, this isomorphism can be obtained by tensoring with B the homomorphism $N \otimes_A A' \to H$, which is then also an isomorphism, by faithful flatness. This completes the proof of ii) and of (6.2).

The category $QCoh$ of quasi-coherent sheaves over schemes of finite type is an example of a fibered category (endowed with a cleavage). The theory of descent is best formalized in the framework of these categories. In this language, Grothendieck's theorem of descent for quasi-coherent sheaves can be stated by saying that *the fibered category $QCoh$ is a stack for the faithfully flat, quasi-compact topology.* We will not introduce general fibered categories in this book. As far as moduli spaces are concerned, it will suffice to restrict our attention to the particular case of categories fibered in groupoids ($QCoh$ is not one such), and for these categories, we will formally treat descent only in the étale topology. This is what we are doing in the next section.

7. The moduli space of curves as a stack, II.

Let $\mathcal{M} = (\mathcal{C}, p)$, with $p : \mathcal{C} \to Sch$, be a category fibered in groupoids or, briefly, a groupoid. Let T be a scheme, let ξ be an object of $\mathcal{M}(U)$, and let $f : U \to T$ be an étale surjective morphism. We consider the two projections p_1 and p_2 from $U \times_T U$ to the two factors. As before, we write p_{12}, p_{13}, and p_{23} to indicate the projections of $U \times_T U \times_T U$ to $U \times_T U$ obtained by omitting the third, second, and first component, respectively, and q_1, q_2, q_3 to indicate the three projections of $U \times_T U \times_T U$ to X, so that $p_1p_{12} = q_1 = p_1p_{13}$, $p_2p_{12} = q_2 = p_1p_{23}$, and $p_2p_{13} = q_3 = p_2p_{23}$. A *descent datum for ξ, relative to $f : U \to T$,* is an isomorphism $\varphi : p_1^*\xi \to p_2^*\xi$ such that the following diagram commutes:

(7.1)
$$\begin{tikzcd}[column sep=small]
 & p_{12}^*p_2^*\xi \arrow[r, equal] & p_{23}^*p_1^*\xi \arrow[dr, "p_{23}^*\varphi"] & \\
p_{12}^*p_1^*\xi \arrow[ur, "p_{12}^*\varphi"] \arrow[dr, equal] & & & p_{23}^*p_2^*\xi \arrow[dl, equal] \\
 & p_{13}^*p_1^*\xi \arrow[r, "p_{13}^*\varphi"] & p_{13}^*p_2^*\xi &
\end{tikzcd}$$

In other words, we must have

(7.2)
$$p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi : q_1^*\xi \longrightarrow q_3^*\xi\,.$$

A descent datum for ξ, relative to f, is said to be *effective* if there exist an object $\eta \in \mathcal{M}(T)$ and an isomorphism $\psi : f^*(\eta) \to \xi$ such that

$$\varphi = (p_2^*\psi) \circ (p_1^*\psi)^{-1} . \tag{7.3}$$

This last condition just says that ψ identifies the descent data for $f^*(\eta)$ with those of ξ. The intuitive meaning of these definitions is clear when we translate the language of the usual topology into the language the étale topology. An open cover $\mathcal{U} = \{U_i\}$ of T is translated into a surjective étale map $U \to T$, the collection of pairwise intersections $\{U_i \cap U_j\}$ is translated into the fiber product $U \times_T U$, while the collection of triple intersections $\{U_i \cap U_j \cap U_k\}$ is translated into the triple fiber product $U \times_T U \times_T U$. The datum of an object ξ_i on each U_i corresponds to the datum of an object ξ on U. An isomorphism φ_{ij} from $\xi_i|U_i \cap U_j$ to $\xi_i|U_i \cap U_j$ is translated into the descent datum $\varphi : p_1^*\xi \to p_2^*\xi$. The compatibility condition $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ on $\{U_i \cap U_j \cap U_k\}$ is translated into the cocycle condition (7.2).

We are now ready to define what we mean by a *stack in groupoids for the étale topology* or, as we will usually say for the sake of brevity, a *stack.* Such an object is a groupoid $\mathcal{M} = (\mathcal{C}, p)$ having the following two properties.

1) Every (étale) descent datum is effective.
2) Given a scheme S and objects ξ and η in $\mathcal{M}(S)$, the functor

$$Isom_S(\xi, \eta) : Sch/S \longrightarrow Sets$$

which associates to a morphism $f : T \to S$ the set of isomorphisms in $\mathcal{M}(T)$ between $f^*\xi$ and $f^*\eta$ is a sheaf in the étale topology.

We recall that a contravariant functor $F : Sch/S \to Sets$ is a sheaf in the étale topology if, for every étale surjective morphism $\pi : X \to Y$ of S-schemes, the diagram

$$F(Y) \xrightarrow{F(\pi)} F(X) \overset{F(p_1)}{\underset{F(p_2)}{\rightrightarrows}} F(X \times_Y X)$$

is exact. Notice that, when condition 2) is satisfied, (7.3) implies that η is unique up to a unique isomorphism.

The following fundamental theorem is due to Grothendieck.

Theorem (7.4). *Let S be a scheme. Let $F : Sch/S \to Sets$ be a contravariant, representable functor. Then F is a sheaf for the étale topology.*

Proof. Assume that F is represented by a scheme Z, so that $F = \mathrm{Hom}(-, Z)$. Given an étale surjective morphism $\pi : X \to Y$ of S-schemes, we must prove the exactness of the diagram

$$\mathrm{Hom}(Y,Z) \xrightarrow{\pi^*} \mathrm{Hom}(X,Z) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}(X \times_Y X, Z)\,\cdot$$

Following, almost word by word, the arguments we used to prove the exactness of (6.5), we are reduced to the case in which $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} A'$. In an analogous way, we may furthermore assume that $Z = \operatorname{Spec} B$. The above sequence may then be identified with

$$\mathrm{Hom}(B,A) \longrightarrow \mathrm{Hom}(B,A') \rightrightarrows \mathrm{Hom}(B, A' \otimes_A A')\,\cdot \tag{7.5}$$

Recall that étale surjective morphisms are faithfully flat, so that A' is a faithfully flat A-algebra. Therefore the exactness of (7.5) follows from the exactness of the sequence of rings

$$A \longrightarrow A' \rightrightarrows A' \otimes_A A'\,\cdot$$

Q.E.D.

A *morphism between two stacks* $\mathcal{M}$ *and* $\mathcal{M}'$ is just a morphism between the underlying groupoids. As we already mentioned, we may consider a scheme S as a groupoid, and it is easily seen that this groupoid is indeed a stack.

THEOREM (7.6). *The groupoids $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ are stacks (in groupoids in the étale topology).*

In view of Theorem (5.6), this is a consequence of the following general result.

THEOREM (7.7). *Given a group scheme G acting on a scheme X, the quotient groupoid $[X/G]$ is a stack.*

To prove (7.7), one needs to know more about descent than we have explained. Therefore we shall follow a different path. A sketch of proof of (7.7) can be found in [190], Proposition 2.1.

Proof. We only deal with $\overline{\mathcal{M}}_{g,n}$, the case of $\mathcal{M}_{g,n}$ being completely analogous. Property 2) in the definition of stack is easily checked. In fact, given families of stable curves $\xi : X \to S$ and $\eta : Y \to S$, that is, objects in $\overline{\mathcal{M}}_{g,n}(S)$, the functor $Isom_S(\xi, \eta)$ is represented by the scheme $\mathbf{Isom}_S(X, Y)$, introduced in Section 7 of Chapter IX. Thus, property 2) follows from Theorem (7.4). We turn to property 1). Let then

$$T \longrightarrow T'$$

be a surjective étale morphism, and let

$$\xi : X \longrightarrow T$$

be a family of stable curves endowed with descent data $\varphi : p_1^*(\xi) \to p_2^*(\xi)$,

$$\begin{tikzcd}[column sep=small] p_1^*(X) = T \times_{T'} X \arrow[rr, "\varphi"] \arrow[dr, "p_1^*(\xi)"'] & & X \times_{T'} T = p_2^*(X) \arrow[dl, "p_2^*(\xi)"] \\ & T \times_{T'} T & \end{tikzcd} \tag{7.8}$$

To check property 1), we must produce a family of stable curves $\eta : Y \to T'$ such that $\xi = \pi^*(\eta)$. The construction of the family η is a typical descent construction and, as is often the case, will be reduced to the theory of descent of quasi-coherent sheaves. Let us illustrate the two basic steps of this reduction.

We need some notation. Consider the family $\xi : X \to T$. Denote by L_ξ the line bundle $(\omega_\xi(D))^3$, where D is the divisor of marked points in the fibers of ξ, and consider the dual direct image bundle $E_\xi = \xi_*(L_\xi)^\vee$. The total space X of the family ξ can be viewed as embedded in $\mathbb{P}(E_\xi)$:

$$\begin{tikzcd} X \subset \mathbb{P}(E_\xi) \arrow[d, "\xi"] \\ T \end{tikzcd}$$

Step 1. From the descent data for ξ we deduce descent data for the vector bundle (or better, locally free sheaf) E_ξ. From the theory of descent on $QCoh$, we get a vector bundle E' over T' with $\pi^*(E') = E$.

Step 2. At this stage there is a diagram

$$\begin{tikzcd} \mathbb{P} = \mathbb{P}(E_\xi) \arrow[r, "q"] \arrow[d] & \mathbb{P}(E') = \mathbb{P}' \arrow[d] \\ T \arrow[r, "\pi"] & T' \end{tikzcd} \tag{7.9}$$

Look at the étale morphism $q : \mathbb{P} \to \mathbb{P}'$. The descent data for $\xi : X \to T$ relative to the étale cover $\pi : T \to T'$ determine descent data for the ideal sheaf $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}}$ with respect to the étale cover $q : \mathbb{P} \to \mathbb{P}'$. Using again the theory of descent for $QCoh$, we get the subscheme $Y \subset \mathbb{P}'$ and the family $\eta : Y \to T'$.

Before embarking on the actual proof of the two steps above, we need some preparation. Given a morphism $U \to U'$ and a cartesian diagram of families of stable curves

$$
\begin{tikzcd}
Z \arrow[r, "h"] \arrow[d, "\alpha"'] & Z' \arrow[d, "\beta"] \\
U \arrow[r, "f"] & U'
\end{tikzcd}
$$

we have canonical isomorphisms:

$$
\sigma_{h,f} : h^*(L_\beta) \xrightarrow{\sim} L_\alpha\,, \qquad \tau_{h,f} : f^*(E_\beta) \xrightarrow{\sim} E_\alpha\,. \tag{7.10}
$$

Given two composable cartesian squares of families of stable curves

$$
\begin{tikzcd}
Z \arrow[r, "h"] \arrow[d, "\alpha"'] & Z' \arrow[r, "k"] \arrow[d, "\beta"'] & Z'' \arrow[d, "\gamma"'] \\
U \arrow[r, "f"] & U' \arrow[r, "g"] & U''
\end{tikzcd}
$$

one can easily check the equalities

$$
\begin{aligned}
\sigma_{kh,gf} &= \sigma_{h,f} h^*(\sigma_{k,g}) : \ (kh)^*(L_\gamma) \xrightarrow{\sim} L_\alpha, \\
\tau_{kh,gf} &= \tau_{h,f} f^*(\tau_{k,g}) \ : \ (gf)^*(E_\gamma) \xrightarrow{\sim} E_\alpha.
\end{aligned} \tag{7.11}
$$

We return to the étale cover $\pi : T \to T'$ and to the descent datum $\varphi : p_1^*(X) \to p_2^*(X)$ for the family $\xi : X \to T$. Look at the diagram

$$
\begin{tikzcd}
X \arrow[d, "\xi"'] & p_1^*(X) \arrow[l, "p_1"'] \arrow[r, "\varphi"] \arrow[d, "p_1^*(\xi)"'] & p_2^*(X) \arrow[r, "p_2"] \arrow[d, "p_2^*(\xi)"'] & X \arrow[d, "\xi"'] \\
T & T \times_{T'} T \arrow[l, "p_1"'] \arrow[r, equal] & T \times_{T'} T \arrow[r, "p_2"] & T
\end{tikzcd}
$$

Using (7.11) and some patience, the reader will see that the isomorphism

$$
\varphi_\xi = \tau^{-1}_{p_2,p_2} \tau^{-1}_{\varphi,id} \tau_{p_1,p_1} : \ p_1^* E_\xi \longrightarrow p_2^* E_\xi
$$

satisfies the cocycle condition for the étale cover $\pi : T \to T'$, thus defining descent data for the coherent $\mathcal{O}_T$-module E_ξ. From the theory of descent on $QCoh$ we get a quasi-coherent $\mathcal{O}_{T'}$-module E' such that $E_\xi = \pi^*(E')$. Since π is étale, E' also is locally free. In particular $E = \pi^*(E')$. This concludes the first step of the proof.

Now we have the cartesian diagram (7.9). As we already mentioned, the descent data for $\xi : X \to T$, relative to the étale cover $\pi : T \to T'$, determine descent data for the ideal sheaf $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}}$ with respect to the étale cover $q : \mathbb{P} \to \mathbb{P}'$. Again by descent in $QCoh$, we get an $\mathcal{O}_{\mathbb{P}'}$-module $\mathcal{G}$ such that $q^*(\mathcal{G}) = \mathcal{I}_X$. As q is étale, and hence faithfully flat, it follows that $\mathcal{G}$ is a sheaf of ideals in $\mathcal{O}_{\mathbb{P}'}$. This sheaf of ideals defines a subscheme $Y \subset \mathbb{P}'$ such that $X \cong q^*(Y) = Y \times_{\mathbb{P}'} \mathbb{P}$. But then

$$X \cong Y \times_{\mathbb{P}'} \mathbb{P} = Y \times_{\mathbb{P}'} \mathbb{P}' \times_{T'} T \cong Y \times_{T'} T\,.$$

We then get a cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\xi"'] & Y \arrow[d, "\eta"] \\ T \arrow[r, "\pi"] & T' \end{tikzcd}$$

Since π is étale, $\eta : Y \to T'$ is a family of stable curves, as desired.
Q.E.D.

We end this section by saying a few words about *fiber products* of stacks. These are defined in the following way. Suppose that $\alpha : \mathcal{M} \to \mathcal{P}$ and $\beta : \mathcal{N} \to \mathcal{P}$ are morphisms of stacks. Then $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ is the groupoid whose objects are defined by

$$(\mathcal{M} \times_{\mathcal{P}} \mathcal{N})(T) = \{(\xi, \eta, \varphi) : (\xi, \eta) \in \mathcal{M}(T) \times \mathcal{N}(T),\ \varphi \in \mathrm{Isom}_T(\alpha(\xi), \beta(\eta))\}\,,$$

for every scheme T. A morphism in the category $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ between two objects (ξ, η, φ) and (ξ', η', φ') is a pair (ψ_1, ψ_2), where $\psi_1 : \xi \to \xi'$ is a morphism in $\mathcal{M}$ and $\psi_2 : \eta \to \eta'$ is a morphism in $\mathcal{N}$, with $p_{\mathcal{M}}(\psi_1) = p_{\mathcal{N}}(\psi_2)$ and $\varphi'\alpha(\psi_1) = \beta(\psi_1)\varphi$. It can be easily checked that such a groupoid is indeed a stack. As an exercise in the language of 2-categories, and more specifically in the definition of a commutative diagram in the category of stacks, the reader is encouraged to state the universal property satisfied by the fiber product of a stacks as just defined.

8. Deligne–Mumford stacks.

As we already mentioned, we may view a scheme S as a stack, by considering the stack associated to the functor of points of S. It is in this sense that we will talk about morphisms between schemes and stacks. As we already observed, a morphism f from a scheme S to a stack $\mathcal{M}$ is equivalent to the datum of an object ξ in $\mathcal{M}(S)$; indeed, $\xi = f(\mathrm{id}_S)$. When a stack is isomorphic to a scheme, we will say that it is *represented* by this scheme. We also observed that, when talking about representable groupoids, the word isomorphism is crucial. As an exercise in subtleties, and using Lemma (5.1), the reader should give a detailed proof of the fact that, given a groupoid $\mathcal{M}$ and a morphism $S \to \mathcal{M}$, the groupoid $\mathcal{M} \times_{\mathcal{M}} S$ is represented by S.

A morphism of stacks $f : \mathcal{M} \to \mathcal{N}$ is said to be *representable* if, for every scheme S and every morphism $S \to \mathcal{N}$, the fiber product $\mathcal{M} \times_{\mathcal{N}} S$ is a scheme. Given a stack $\mathcal{M}$, let us consider the diagonal morphism $\Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ (which is defined in the obvious way). The following lemma explains what the representability of Δ means.

Lemma (8.1). *$\Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is representable if and only if every morphism from a scheme S to $\mathcal{M}$ is.*

Proof. Let $f : S \to \mathcal{M}$ and $g : T \to \mathcal{M}$ be morphisms from schemes to $\mathcal{M}$. Look at $(f, g) : S \times T \to \mathcal{M} \times \mathcal{M}$. From the universal property of the fiber product we get an isomorphism

$$S \times_{\mathcal{M}} T \cong \mathcal{M} \times_{\mathcal{M} \times \mathcal{M}} (S \times T). \tag{8.2}$$

Now suppose that Δ is representable. The right-hand side of (8.2) is then a scheme. Since this is true for every f and g, this means that f (and g) is representable. Conversely, assume that every morphism $S \to \mathcal{M}$ is representable. Let $h : S \to \mathcal{M} \times \mathcal{M}$ be a morphism. We must show that $\mathcal{M} \times_{\mathcal{M} \times \mathcal{M}} S$ is a scheme. Write $h = (f, g) \circ \Delta_S$, where $(f, g) : S \times S \to \mathcal{M} \times \mathcal{M}$. Then

$$\mathcal{M} \times_{\mathcal{M} \times \mathcal{M}} S = (\mathcal{M} \times_{\mathcal{M} \times \mathcal{M}} (S \times S)) \times_{S \times S} S \cong (S \times_{\mathcal{M}} S) \times_{S \times S} S\,,$$

and the stack on the right-hand side is a scheme, since $S \to \mathcal{M}$ is representable. Q.E.D.

Let $\mathbf{P}$ be a property of morphisms of schemes which is stable under base change as, for example, being surjective, flat, faithfully flat, étale, unramified, quasi-compact, separated, or of finite type. Then, by definition, a *representable morphism* $f : \mathcal{M} \to \mathcal{N}$ *satisfies* $\mathbf{P}$ if, for every morphism $S \to \mathcal{M}$, where S is a scheme, the morphism of schemes $\mathcal{M} \times_{\mathcal{N}} S \to S$ satisfies $\mathbf{P}$.

A *Deligne–Mumford stack* is a stack $\mathcal{M}$ having the following two properties.

1) The diagonal $\Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is representable, quasi-compact, and separated.
2) There exist a scheme X and an étale surjective morphism $\alpha : X \to \mathcal{M}$.

The morphism α is also called *an atlas for* $\mathcal{M}$. As far as terminology is concerned, the reader should be aware of the fact that the original name given by Deligne and Mumford to what we now call a Deligne–Mumford stack is "algebraic stack." On the other hand, an *Artin stack* is a stack satifying 1) and 2) with the word "étale" substituted with "smooth." We now prove the following theorem.

Theorem (8.3). *$\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ are Deligne–Mumford stacks.*

Proof. We prove the theorem for $\overline{\mathcal{M}}_{g,n}$, the proof for $\mathcal{M}_{g,n}$ being completely analogous. Set $\mathcal{M} = \overline{\mathcal{M}}_{g,n}$. The representabilty of $\Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is straightforward. Let $h : S \to \mathcal{M} \times \mathcal{M}$ be a morphism. The datum of h is equivalent to the datum of two families of stable pointed

curves $\xi : X \to S$ and $\eta : Y \to S$ in $\mathcal{M}(S)$. From the representability results of Section 7 of Chapter IX we get:
(8.4)
$$\begin{aligned}(\mathcal{M} \times_{\mathcal{M}\times\mathcal{M}} S)\,(T) &= \{(f,\alpha) \mid f : T \to S\,,\ \ \alpha \in \mathrm{Isom}_T(f^*\xi, f^*\eta)\}\\ &= \{(f,\beta) \mid f : T \to S\,,\ \ \beta \in \mathrm{Hom}_S(T, \mathbf{Isom}_S(\xi,\eta))\}\\ &= \mathrm{Hom}(T, \mathbf{Isom}_S(\xi,\eta)).\end{aligned}$$

Therefore, $\mathcal{M} \times_{\mathcal{M}\times\mathcal{M}} S$ is represented by $\mathbf{Isom}_S(\xi,\eta)$, which is separated and quasi-compact. This proves property 1). Before proving the second property, let us observe that, given morphisms $f : S \to \mathcal{M}$ and $g : T \to \mathcal{M}$, where S and T are schemes, or equivalently given two families of stable curves, $\xi : X \to S$ in $\mathcal{M}(S)$ and $\eta : Y \to T$ in $\mathcal{M}(T)$, proceeding as in (8.4), we get

$$S \times_{\mathcal{M}} T = \mathbf{Isom}_{S\times T}(p_1^*\xi, p_2^*\eta), \tag{8.5}$$

where $p_1 : S \times T \to S$ and $p_2 : S \times T \to T$ are the two projections. We now proceed to prove property 2). Let us go back to the smooth variety X defined in (3.8). Recall that X is the disjoint union of a finite number of "slices", $X_1, \dots, X_N$, in the Hilbert scheme $\mathrm{H}_{\nu,g,n}$. Each one of these slices is a smooth affine $(3g-3+n)$-dimensional subvariety of $\mathrm{H}_{\nu,g,n}$ which is transversal to the orbits of $G = PGL(N)$ and satisfies all the properties listed in definition (6.7) of Chapter XI. The restriction to X of the universal family over $\mathrm{H}_{\nu,g,n}$ yields a family of stable curves $\xi : \mathcal{C} \to X$ and hence a morphism

$$\alpha : X \to \mathcal{M}\,. \tag{8.6}$$

We wish to prove that α is étale and surjective. For this, we must prove that, for every morphism f from a scheme S to $\mathcal{M}$, the induced morphism

$$X \times_{\mathcal{M}} S = \mathbf{Isom}_{X\times S}(p_1^*\xi, p_2^*\eta) \longrightarrow S \tag{8.7}$$

is étale and surjective. Let $\eta : \mathcal{X} \to S$ be the family corresponding to the morphism $f : S \to \mathcal{M}$. Since being étale is a local property and since, locally on S, the family η is the pullback of the family $\xi : \mathcal{C} \to X$, we are reduced to showing that the natural projections

$$X \times_{\mathcal{M}} X = \mathbf{Isom}_{X\times X}(p_1^*\xi, p_2^*\xi) \longrightarrow X \tag{8.8}$$

are étale and surjective, but this is exactly the content of Proposition (3.10).

Q.E.D.

Let us go back to the étale map

$$\varphi : Y \to \overline{M}_{g,n}$$

defined in (3.7). The scheme Y is the disjoint union of schemes $Y_1, \dots, Y_N$, and each of these is a quotient $Y_i = X_i/G_i$, where X_i is a Kuranishi family obtained as a slice of $\mathrm{H}_{\nu,g,n}$. Certainly, the composite map

$$\beta : X = \coprod X_i \to \overline{M}_{g,n}$$

is not étale. The scheme Y has the advantage of mapping surjectively and in an étale manner onto $\overline{M}_{g,n}$. The scheme X has the advantage of being the basis of family of stable curves, which in turn determines the moduli map β. These two advantages cannot be reconciled in the world of algebraic spaces or in the world of schemes. In proving that

$$\alpha : X \to \overline{\mathcal{M}}_{g,n}$$

is étale and surjective, we showed that this reconciliation is possible in the world of algebraic stacks.

The coarse moduli space of a stack

From now on all the stacks and schemes we consider will be over $\mathbb{C}$. We start with a definition and a general remark. A *geometric point of a stack* $\mathcal{M}$ is, by definition, a connected component of the groupoid $\mathcal{M}(\mathrm{Spec}(\mathbb{C}))$, that is, an isomorphism class of objects in $\mathcal{M}(\mathrm{Spec}(\mathbb{C}))$. Let S be a scheme, and $f : \mathcal{M} \to S$ a morphism. Then, from the definition of morphism it follows that if ξ and ξ' belong to the same connected component of $\mathcal{M}(\mathrm{Spec}(\mathbb{C}))$, then $f(\xi) = f(\xi')$.

A *coarse moduli space for a stack* $\mathcal{M}$ is a scheme M together with a morphism $m : \mathcal{M} \to M$ inducing a bijection on geometric points and such that every morphism from $\mathcal{M}$ to a scheme factors through M.

In the next chapter we will prove that the analytic space $\overline{M}_{g,n}$ is a projective variety. We are going to use this property to show that $\overline{M}_{g,n}$ is a coarse moduli space for $\overline{\mathcal{M}}_{g,n}$. The analogous statement regarding $M_{g,n}$ and $\mathcal{M}_{g,n}$ immediately follows from this. The morphism $m : \overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ is readily described. If T is a scheme, the functor

$$m_T : \overline{\mathcal{M}}_{g,n}(T) \to \overline{M}_{g,n}(T)$$

is defined as follows. Given a family $\xi : X \to T$, $m_T(\xi)$ is nothing but the moduli map $m_T(\xi) : T \to \overline{M}_{g,n}$. The statement about geometric points is an immediate consequence of the general remark we made at the beginning of this subsection. Finally, let S be a scheme, and $f : \overline{\mathcal{M}}_{g,n} \to S$ a morphism. Consider the Kuranishi étale cover (8.6)

$$\alpha : X \to \overline{\mathcal{M}}_{g,n}.$$

We get morphisms $m\alpha : X \to \overline{M}_{g,n}$ and $f\alpha : X \to S$ with the property that $f\alpha$ factors, set-theoretically, through $m\alpha$ via a map $h : \overline{M}_{g,n} \to S$. Since $m\alpha$ is finite, h is necessarily analytic. We now use the fact that $\overline{M}_{g,n}$ is projective, together with Proposition 15 of [626], to conclude that h is algebraic.

An orbifold-like definition of Deligne–Mumford stacks

Let $\mathcal{M}$ be a Deligne–Mumford stack, and let $X_0 \to \mathcal{M}$ be an étale surjective morphism, where X_0 is a scheme. The fiber product $X_1 = X_0 \times_{\mathcal{M}} X_0$ is a scheme, and the projections to the two factors are étale. There are natural morphisms $s, t : X_1 \to X_0$ (the projections to the two factors), $u : X_0 \to X_1$ (the diagonal), $i : X_1 \to X_1$ (interchanging the factors), plus a composition morphism $m : X_1 \,{}_s{\times}_t\, X_1 \to X_1$ defined as the projection onto the first and third factors in

$$X_1 \times_{X_0} X_1 = X_0 \times_{\mathcal{M}} X_0 \times_{\mathcal{M}} X_0 \to X_0 \times_{\mathcal{M}} X_0 = X_1 .$$

These morphisms satisfy the scheme-theoretic analogues of the equalities (4.2) defining an orbifold structure, which, for convenience, we state again:

$$\begin{aligned} &(s,t)u = \Delta\,, \quad (s,t)i = \eta(s,t)\,, \quad (t,s)m = (t \times s)\,, \\ &m \circ (ut, \mathrm{id}_{X_1}) = \mathrm{id}_{X_1}\,, \quad m \circ (\mathrm{id}_{X_1}, us) = \mathrm{id}_{X_1}\,, \\ &m \circ (\mathrm{id}_{X_1}, m) = m \circ (m, \mathrm{id}_{X_1})\,. \end{aligned} \tag{8.9}$$

In addition, the two projections s and t are étale, as we observed. It can be shown that the datum of the schemes X_0 and X_1, together with the morphisms s, t, u, i, m, completely determines the stack $\mathcal{M}$.

Conversely, suppose that we are given schemes and étale morphisms

$$X_1 \underset{t}{\overset{s}{\rightrightarrows}} X_0\,, \tag{8.10}$$

plus morphisms s, t, u, i, m subject to conditions (8.9), such that $(s,t) : X_1 \to X_0 \times X_0$ is quasicompact and separated. Then one can define a "quotient Deligne–Mumford stack" $\mathcal{M}$ with an étale surjective map $X_0 \to \mathcal{M}$ such that $X_1 = X_0 \times_{\mathcal{M}} X_0$. Proving these facts would require more descent theory than is at our disposal. For a proof, we therefore refer to [298], or to [671], p. 668, and references therein. We may notice, however, that a "baby" version of the argument is the proof of Proposition (2.9) in Chapter XIII and that the latter proposition is in fact sufficient to fully prove the existence of the stack $\mathcal{M}$ in special instances (cf. specifically Remark (2.10) in Chapter XIII). Here we shall limit ourselves to giving an idea of a possible construction of the stack

$\mathcal{M}$ starting from (8.10). We will just describe the category $\mathcal{M}(T)$ for a given scheme T. Objects of this category should correspond to morphisms from T to $\mathcal{M}$. Thus, it is natural to look for "morphisms from T to $X_1 \rightrightarrows X_0$." To put T and $X_1 \rightrightarrows X_0$ on the same footing, we consider a groupoid presentation of T, that is, a surjective étale map $T' \to T$. We then define a category

$$\mathcal{M}(T' \to T) = \operatorname{Hom}\left(T' \times_T T' \rightrightarrows T', X_1 \rightrightarrows X_0\right),$$

where of course an object of $\mathcal{M}(T' \to T)$ consists of a pair of morphisms $\varphi : T' \to X_0$ and $\Phi : T' \times_T T' \to X_1$ satisfying obvious compatibility conditions. Freeing T from the choice of a groupoid presentation, one arrives at the following definition of $\mathcal{M}(T)$:

$$\mathcal{M}(T) = \lim_{\substack{T' \to T \\ \text{ét, surj}}} \mathcal{M}(T' \to T).$$

The datum of (8.10) and of the morphisms s, t, u, i, m is called a *groupoid presentation of the Deligne–Mumford stack* $\mathcal{M}$.

Among other things, the axioms (8.9) give, for each point $x \in X_0$, a group structure to the fiber

$$G_x = (s,t)^{-1}(x,x)\,, \tag{8.11}$$

where $(s,t) : X_1 \to X_0 \times X_0$. This is the isotropy group which we encountered in (4.4), in the orbifold context.

Groupoid presentations come particularly handy in performing various constructions on stacks and particularly in defining the notions of normalization of a stack and of quotient of a stack modulo the action of a finite group.

Example (8.12) (Substacks). Let $\mathcal{M}$ be a Deligne–Mumford stack. A representable morphism $\mathcal{N} \to \mathcal{M}$ is a *closed* (resp., *open*) *immersion* if, for any morphism $S \to \mathcal{M}$ with S a scheme, $S \times_{\mathcal{M}} \mathcal{N} \to S$ is a closed (resp., open) immersion of schemes. It can be shown that under these circumstances $\mathcal{N}$ is necessarily a Deligne–Mumford stack; the easy proof is left to the reader. A *closed* (resp., *open*) *substack* of $\mathcal{M}$ is an equivalence class of closed (resp., open) immersions in $\mathcal{M}$ modulo isomorphism over $\mathcal{M}$. In other words, two immersions $\mathcal{N} \to \mathcal{M}$ and $\mathcal{A} \to \mathcal{M}$ define the same substack if and only if there is an isomorphism $\mathcal{N} \to \mathcal{A}$ such that the diagram

$$\begin{tikzcd} \mathcal{N} \arrow[rr, "\simeq"] \arrow[dr] & & \mathcal{A} \arrow[dl] \\ & \mathcal{M} & \end{tikzcd}$$

commutes. Let $X \to \mathcal{M}$ be an atlas for the Deligne–Mumford stack $\mathcal{M}$, and let $\pi_1, \pi_2 : X \times_{\mathcal{M}} X \to X$ be the projections to the two factors. By definition, a closed (resp., open) substack of $\mathcal{M}$ yields a closed (resp., open) subscheme Y of X with the property that $\pi_1^{-1}(Y) = \pi_2^{-1}(Y)$. In fact, it can be shown (cf. Remark (2.10) in Chapter XIII) that giving a closed (resp., open) substack of $\mathcal{M}$ is equivalent to giving a closed (resp., open) subscheme of X with this property. More precisely, if we are given such a subscheme Y, the substack corresponding to it is the one defined by the groupoid presentation

$$\pi_1^{-1}(Y) = \pi_2^{-1}(Y) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} Y .$$

Example (8.13) (Normalization). Let $\mathcal{M}$ be a Deligne–Mumford stack with groupoid presentation (8.10). The normalization of $\mathcal{M}$ is obtained by normalizing both “space” (i.e., X_0) and “relations” (i.e., X_1). If $\widehat{X}_1$ and $\widehat{X}_0$ denote the normalizations of X_1 and X_0, respectively, the structure maps s, t, u, i, m lift to morphisms $\widehat{s}, \widehat{t} : \widehat{X}_1 \to \widehat{X}_0$ $\widehat{u} : \widehat{X}_0 \to \widehat{X}_1$, $\widehat{i} : \widehat{X}_1 \to \widehat{X}_1$, and $\widehat{m} : \widehat{X}_1 \,{}_s\times_t \widehat{X}_1 \to \widehat{X}_1$. These liftings satisfy the identities (4.2) necessary to make

$$\widehat{X}_1 \underset{\widehat{t}}{\overset{\widehat{s}}{\rightrightarrows}} \widehat{X}_0 \tag{8.14}$$

into a groupoid presentation of a Deligne–Mumford stack, since these identities are satisfied on open dense subsets of the various domains of definition. Moreover, the diagonal $\widehat{X}_1 \to \widehat{X}_0 \times \widehat{X}_0$ is separated. Thus, (8.14) indeed defines a Deligne–Mumford stack $\widehat{\mathcal{M}}$ which, by definition, is *the normalization of* $\mathcal{M}$. Of course, a posteriori, we have that

$$\widehat{X}_1 = \widehat{X}_0 \times_{\widehat{\mathcal{M}}} \widehat{X}_0 .$$

Example (8.15) (Quotient modulo a finite group action). We consider actions of a finite group G on a Deligne–Mumford stack $\mathcal{M} = (\mathcal{C}, p)$ in the following restrictive sense. We assume that to each element of G there corresponds a morphism of $\mathcal{M}$ into itself and that the following properties are satisfied. First of all, the identity element of G corresponds to the identity on $\mathcal{M}$. Secondly, the product in G corresponds to the composition of morphisms of $\mathcal{M}$, *in the strict sense*, and not just up to isomorphism of functors. From now on we assume that the category $\mathcal{C}$ has coproducts. To form the quotient Deligne–Mumford stack $[\mathcal{M}/G]$, we proceed as follows. First assume that there is a groupoid presentaton of the stack $\mathcal{M}$ such that the action of G lifts to G-actions μ_0 on X_0 and μ_1 on X_1 for which all structural morphisms are equivariant. In this case we proceed exactly as in the orbifold case, that is, we define the stack $[\mathcal{M}/G]$ as the one given by the groupoid presentation

$$Y_1 \underset{t_G}{\overset{s_G}{\rightrightarrows}} Y_0, \tag{8.16}$$

where

$$Y_0 = X_0\,, \quad Y_1 = G \times X_1\,,$$
$$s_G = s \circ pr_{X_1}\,, \quad t_G = t \circ \mu_1\,,$$

the unit u_G and the inverse i_G are the obvious ones, and the composition map m_G is defined, in the scheme-theoretic context, exactly as we did in (4.7) in the orbifold context. The remaining problem is then to find a groupoid presentation of $\mathcal{M}$ to which we can lift the given G-action on $\mathcal{M}$. For this, start with an arbitrary presentation $Z_1 \underset{t}{\overset{s}{\rightrightarrows}} Z_0$ of $\mathcal{M}$. The étale morphism $f : Z_0 \to \mathcal{M}$ corresponds to an object ξ in $\mathcal{M}(Z_0)$. As G acts on $\mathcal{M}$, for every $\sigma \in G$, we may consider the object ξ^σ in $\mathcal{M}(Z_0)$ and its moduli morphism $f^\sigma : Z_0 \to \mathcal{M}$ which is also étale (cf. Exercise A-4). We then set

$$X_0 = \coprod_{\sigma \in G} Z_0 \cong G \times Z_0 \tag{8.17}$$

and define a new étale surjective morphism

$$f^G = \coprod_{\sigma \in G} f^\sigma \,:\, X_0 \to \mathcal{M}. \tag{8.18}$$

Then we set

$$X_1 = X_0 \times_{\mathcal{M}} X_0\,. \tag{8.19}$$

As an exercise, the reader should verify that $X_1 \rightrightarrows X_0$ is a groupoid presentation of $\mathcal{M}$ to which the action of G lifts.

Remark (8.20). It is important to remark that, by the way in which the quotient of a Deligne–Mumford stack $\mathcal{M}$ modulo a finite group G is defined, and in sharp contrast with the case, say, of affine schemes, the quotient morphism $\mathcal{M} \to [\mathcal{M}/G]$ is always étale. In a sense, since isotropy groups of points are part of the structure, the action of a finite group is always free.

As we have observed, the definition of Deligne–Mumford stack via groupoid presentations is the translation, in the algebraic context, of the definition of orbifold via orbifold groupoids. In fact, given a groupoid presentation $X_1 \rightrightarrows X_0$ of a *smooth* Deligne–Mumford stack, the underlying analytic object $(X_1)_{an} \rightrightarrows (X_0)_{an}$ is clearly an analytic orbifold groupoid. This is the sense in which we shall sometimes treat smooth Deligne–Mumford stacks, and in particular moduli stacks, as orbifolds.

9. Back to algebraic spaces.

The definition of Deligne–Mumford stack via groupoid presentations is obviously very close to the one of algebraic space. In fact, for a presentation $X_1 \rightrightarrows X_0$ to define a separated algebraic space, it suffices to add the requirement that $X_1 \to X_0 \times X_0$ be a closed immersion. Therefore, separated algebraic spaces can be regarded as a particular kind of Deligne–Mumford stacks. It turns out that separated algebraic spaces are precisely those Deligne–Mumford stacks $\mathcal{A}$ such that $\mathcal{A}(T)$ is a set for every scheme T (meaning that the only morphisms in the category $\mathcal{A}(T)$ are the identities). In this case, $T \mapsto \mathcal{A}(T)$ becomes a functor $Sch \to Sets$, and $\mathcal{A}$ is the stack associated to this functor. This is just the translation in the language of stacks of Artin's original definition of (separated) algebraic space.

Regarding an algebraic space as a stack has the considerable advantage of freeing us from relying on an accidental presentation and in particular makes it possible to define a morphism of algebraic spaces to be simply a morphism of stacks. In this sense we can make sense out of the assertion that a morphism of algebraic spaces is an open immersion, finite, proper, étale, and so on. On the other hand, as in the case of stacks, the groupoid presentation of an algebraic space comes in handy when defining the concept of normalization. It is clear that the normalization of an algebraic space, as defined in the previous section, is an algebraic space.

In this section we are going to give a proof of the following result.

THEOREM (9.1). *Let X be a reduced, separated algebraic space. Then there exist a normal scheme Z and a finite group G acting on Z such that X is isomorphic to the quotient Z/G.*

As an application, we shall then prove Theorem (2.9), which we restate here for the convenience of the reader.

THEOREM (9.2). *There exists a family of stable n-pointed genus g curves $\eta : \mathcal{X} \to Z$, parameterized by a normal scheme Z, whose moduli map*

$$m : Z \to \overline{M}_{g,n}$$

is finite and surjective. Moreover, we may choose Z so that $\overline{M}_{g,n}$ is the quotient of Z modulo the action of a finite group.

The proof of Theorem (9.1) relies on the concept of normalization of an irreducible algebraic space X in a finite field extension L of its field of rational function $K(X)$. To explain how this is defined, we need a basic fact about algebraic spaces, namely Zariski's connectedness theorem, whose proof can be found in [428], p. 233, Theorem 4.1.

THEOREM (9.3) (ZARISKI'S CONNECTEDNESS THEOREM). *Let $f : Y \to X$ be a proper morphism of separated algebraic spaces. Then there exists a factorization*

(9.4)
$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, "f"'] & T \arrow[d, "h"] \\ & X \end{tikzcd}$$

where h is a finite morphism, and g is a proper morphism with connected fibers.

Now let X be an irreducible separated algebraic space, and let L be a finite extension of $K(X)$. We want to define the *normalization X^L of X in L*. We proceed as follows. As we know, there exists an open subspace U in X which is an affine scheme (cf. (3.4)). Let U^L be the normalization of U in L, and let V be a completion of U^L. Denote by Y the closure in $V \times X$ of the graph of $U^L \to X$. There is a proper morphism $f : Y \to X$, to which we apply Zariski's connectedness theorem to get a diagram (9.4). The morphism $U^L \to T$ is birational. We define X^L to be the normalization of T. Notice that the injective morphism $U \to X$ lifts to an open immersion $U^L \hookrightarrow X^L$, since both spaces are normal analytic varieties. The same argument proves the following slightly more general fact.

LEMMA (9.5). *Let X be an irreducible separated algebraic space. Let U be an irreducible scheme, let $f : U \to X$ be a quasi-finite dominant morphism, and let L be a finite extension of $K = K(U)$. Then f lifts to an open immersion $U^L \hookrightarrow X^L$.*

Now let us prove (9.1). We may as well assume that X is irreducible. Let $\coprod U_i \to X$ be an étale cover, where the U_i are irreducible and affine. Let L be a finite Galois extension of $K = K(X)$ containing all the $K(U_i)$. Let G be the Galois group of L over K. By the preceding lemma, the schemes U_i^L are openly immersed in X^L. We can use the action of G on X^L to move the U_i^L around; the result is a covering of X^L by open affine schemes. Thus, $Z = X^L$ is a scheme, and $X = Z/G$. This proves Theorem (9.1).

We now turn to Theorem (9.2). As we already mentioned, this result will be crucial in our proof of the projectivity of $\overline{M}_{g,n}$, and this is the only place where it will be used. We already know that $\overline{M}_{1,1}$ and $\overline{M}_2$

are projective, and hence, for our purposes, it is sufficient to give a simplified proof of Theorem (2.9) which applies to all cases except those of $\overline{M}_{1,1}$ and $\overline{M}_2$. The simplification is made possible by the fact that, as shown by Proposition (2.5), a general stable n-pointed curve of genus g has no nontrivial automorphisms except when $g = n = 1$, or $g = 2$, $n = 0$.

We then assume that $(g, n) \neq (2, 0), (1, 1)$. The argument is a slight variant of the proof of (9.1). Notice that, in the latter, one does not need the full strength of the fact that the U_i are étale over X, since all we need is that thery are quasi-finite over it. We then consider a finite number of standard algebraic Kuranishi families $\mathcal{Y}_i \to X_i$ such that $\coprod X_i \to \overline{M}_{g,n}$ is onto. We let L be a Galois extension of $K = K(\overline{M}_{g,n})$ containing all the $K(X_i)$, and we set $Z = X^L$. As in the proof of (9.1), we can say that each one of the X_i embeds as an open subset in Z. Moreover, the translates of the X_i under the action of $\mathrm{Gal}(L/K)$ cover Z. Let $U_1, \ldots, U_N$ be this cover. Each one of these open sets carries a family $\eta_i : \mathcal{W}_i \to U_i$ of stable n-pointed curves whose moduli map is the composition of $U_i \hookrightarrow Z$ and $Z \to \overline{M}_{g,n}$. We wish to patch together the various families $\eta_i : \mathcal{W}_i \to U_i$ to form the desired family $\eta : \mathcal{X} \to Z$. Since we are assuming that $(g, n) \neq (2, 0), (1, 1)$, this patching can be easily performed. In fact, it follows from Proposition (2.5) that, in this case, the curves equipped with nontrivial automorphisms are parameterized, in any Kuranishi family, by a proper analytic subset. In fact, the locus in question is just the projection of the Hilbert scheme of automorphisms of fibers, which, as we know from Theorem (5.1) in Chapter X, is proper over the base. Now suppose that $U_i \cap U_j \neq \emptyset$. For u outside a proper analytic subvariety Σ of $U_i \cap U_j$, there is a unique isomorphism between $\eta_i^{-1}(u)$ and $\eta_j^{-1}(u)$; this yields a canonical identification

$$\begin{tikzcd}
\mathcal{W}_i|_{U_i \cap U_j \smallsetminus \Sigma} \arrow[rr, "\sim"] \arrow[dr] & & \mathcal{W}_j|_{U_i \cap U_j \smallsetminus \Sigma} \arrow[dl] \\
& U_i \cap U_j \smallsetminus \Sigma &
\end{tikzcd}$$

Theorem (5.1) of Chapter X shows that this identification extends uniquely to all of $U_i \cap U_j$. The family $\eta : \mathcal{X} \to Z$ is thus constructed.

Exercise (9.6). Modify the proof of Theorem (2.9) so that it also covers the cases $(g, n) = (1, 1)$ and $(g, n) = (2, 0)$. (Hint: replace the original Z with a suitable finite cover.)

10. The universal curve, projections and clutchings.

In this section we rephrase the geometrical constructions discussed in Sections 6, 7, and 8 of Chapter X in the language of stacks and orbifolds.

The universal curve

We denote by $\overline{\mathcal{C}}_{g,P}$ the category whose objects are families of stable P-pointed curves of genus g

$$\xi : X \to S\,, \qquad \sigma_p : S \to X\,, \quad p \in P\,,$$

which are equipped with an extra section

$$\delta : S \to X$$

(on which we make no extra requirements, as, for instance, not intersecting the other sections or not hitting the nodes of the fibers). The morphisms in $\overline{\mathcal{C}}_{g,P}$ are defined in the usual way as cartesian squares

$$\begin{tikzcd} X' \arrow[r, "F"] \arrow[d, "\xi'"'] & X \arrow[d, "\xi"] \\ S \arrow[r, "f"] & S' \end{tikzcd}$$

such that $F \circ \sigma'_y = \sigma_y \circ f$, $y \in P$, $F \circ \delta' = \delta \circ f$, the notation being self-explanatory. As usual, a functor $q : \overline{\mathcal{C}}_{g,P} \to Sch/\mathbb{C}$ is defined by assigning to a family of curves its parameter space. In this way the pair $(\overline{\mathcal{C}}_{g,P}, q)$ becomes a groupoid, which, for brevity, we simply denote with the symbol $\overline{\mathcal{C}}_{g,P}$. Theorem (8.21) in Chapter X shows that there is an isomorphism of groupoids

$$\lambda : \overline{\mathcal{C}}_{g,P} \to \overline{\mathcal{M}}_{g,P\cup\{x\}}\,,$$

so that, in particular, $\overline{\mathcal{C}}_{g,P}$ is a Deligne–Mumford stack. Explicitly, the definition of λ is as follows. Let $(\xi : X \to S,\ \sigma_p,\ p \in P,\ \delta)$ be an object in $\overline{\mathcal{C}}_{g,P}$. The stabilization procedure described in the above-mentioned section yields, in a functorial way, a family $\xi' : X' \to S$ of stable $(P \cup \{x\})$-pointed genus g curves. The assignement $\xi \mapsto \xi'$ defines λ. Theorem (8.21) in Chapter X says that assigning to a family $(\eta : Y \to S, \{\sigma_q : q \in P \cup \{x\}\})$ its xth contraction gives an inverse of λ up to isomorphism of functors.

The Deligne–Mumford stack $\overline{\mathcal{C}}_{g,P}$ has a scheme incarnation or, as one says, a coarse moduli space $\overline{C}_{g,P}$. As a set, $\overline{C}_{g,P}$ is the set of isomorphism classes of triples $(C, \{x_p\}_{p\in P}, x)$, where $(C, \{x_p\}_{p\in P})$ is a stable P-pointed curve of genus g, and x is a point on C. The analytic structure of $\overline{C}_{g,P}$ is given as follows. Let $[C, \{x_p\}_{p\in P}, x] \in \overline{C}_{g,P}$, and denote by G the automorphism group of $(C, \{x_p\}_{p\in P})$. Let $\xi : X \to B$ be a Kuranishi family for C. The group G acts on X. If U is a G-invariant neighborhood of $x \in X$, then a local patch for the analytic structure of $\overline{C}_{g,P}$ near $[C, \{x_p\}_{p\in P}, x]$ is given by U/G. By universality, the isomorphism λ drops to an isomorphism between moduli spaces

$$\lambda : \overline{C}_{g,P} \to \overline{M}_{g,P\cup\{x\}}.$$

The projectivity of $\overline{M}_{g,P\cup\{x\}}$ implies, in particular, that $\overline{C}_{g,P}$ is a scheme and that λ is an isomorphism of schemes.

Projections

In the language of stacks, the projection operation described in Section 6 of Chapter X gives morphisms

$$\mathrm{Pr}_x : \overline{\mathcal{M}}_{g,P\cup\{x\}} \longrightarrow \overline{\mathcal{M}}_{g,P}$$

for any finite set P and any $x \notin P$. This morphism is naturally called the xth *projection morphism.* As we observed in Lemma (6.10) of Chapter X, it makes good sense to ignore any number of sections so that, given a finite set L disjoint from P, we may define a projection morphism

$$\mathrm{Pr}_L : \overline{\mathcal{M}}_{g,P\cup L} \longrightarrow \overline{\mathcal{M}}_{g,P}.$$

An important property of the projection morphisms is the following.

LEMMA (10.1). $\mathrm{Pr}_x : \overline{\mathcal{M}}_{g,P\cup\{x\}} \longrightarrow \overline{\mathcal{M}}_{g,P}$ *is representable.*

Proof. To see why the lemma is true, it is convenient to identify the morphism Pr_x with

$$\overline{\mathcal{C}}_{g,P} \to \overline{\mathcal{M}}_{g,P}.$$

We must show that, given a scheme S and a morphism $\alpha : S \to \overline{\mathcal{M}}_{g,P}$, the stack $\overline{\mathcal{C}}_{g,P} \times_{\overline{\mathcal{M}}_{g,P}} S$ is (isomorphic to) a scheme. The morphism α corresponds to a family $X \to S$ of P-pointed stable curves. It is then essentially obvious that $\overline{\mathcal{C}}_{g,P} \times_{\overline{\mathcal{M}}_{g,P}} S$ is just X. Q.E.D.

The projection morphisms Pr_L can also be defined at the level of moduli spaces. In fact, since $\overline{M}_{g,P\cup L}$ is a coarse moduli space for $\overline{\mathcal{M}}_{g,P\cup L}$, the composition of Pr_L with the moduli map $m_L : \overline{\mathcal{M}}_{g,P} \to \overline{M}_{g,P}$ necessarily factors through the moduli map $m : \overline{\mathcal{M}}_{g,P\cup L} \to \overline{M}_{g,P\cup L}$. We denote the factoring morphism again with the symbol Pr_L:

$$\mathrm{Pr}_L : \overline{M}_{g,P\cup L} \longrightarrow \overline{M}_{g,P}$$

Clutchings

Our next objective is to describe the boundary $\partial M_{g,P} = \overline{M}_{g,P} \smallsetminus M_{g,P}$ of $\overline{M}_{g,P}$. Fix a stable P-marked, genus g dual graph Γ. We adopt the notation introduced at the beginning of Section 7 of Chapter X and in Definition (2.16) in the same chapter. We set

$$\overline{\mathcal{M}}_\Gamma = \prod_{v\in V} \overline{\mathcal{M}}_{g_v,L_v}, \tag{10.2}$$

and we define a morphism

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,P} \tag{10.3}$$

as follows. An object η in $\overline{\mathcal{M}}_\Gamma(S)$ is the datum of a family

$$\eta_v : X_v \to S \tag{10.4}$$

of stable L_v-pointed curves of genus g_v for each $v \in V$. The morphisms in $\overline{\mathcal{M}}_\Gamma(S)$ are the isomorphisms between these families. The groupoid $\overline{\mathcal{M}}_\Gamma$ is obviously a Deligne–Mumford stack. The clutching procedure described in Section 7 of Chapter X yields a family

$$\xi_\Gamma(\eta) : X' \to S$$

of stable P-pointed genus g curves. The functoriality of this construction exactly says that ξ_Γ is a morphism of Deligne–Mumford stacks. It is in fact a finite morphism. We may also introduce a closed substack

$$\mathcal{D}_\Gamma \subset \overline{\mathcal{M}}_{g,P} \tag{10.5}$$

parameterizing the curves which are in the image of $\overline{\mathcal{M}}_\Gamma$ under ξ_Γ. An object in $\mathcal{D}_\Gamma(S)$ is the datum of a family

$$\sigma : X \to S$$

of stable P-pointed curves of genus g whose fibers have dual graphs which are specializations of Γ, in a sense to be made precise below. It is implicit in (8.12) that a closed substack of a Deligne–Mumford stack is itself a Deligne–Mumford stack. The codimension of $\mathcal{D}_\Gamma$ in $\overline{\mathcal{M}}_{g,P}$ is equal to the number $|E(\Gamma)|$ of edges of Γ. We denote by

$$\Delta_\Gamma \subset \overline{M}_{g,P} \tag{10.6}$$

the coarse moduli space of $\mathcal{D}_\Gamma$. If we assume for a moment what is going to be proved in the next chapter, namely that $\overline{M}_{g,P}$ is a projective scheme, then we see that Δ_Γ is a closed subscheme of $\overline{M}_{g,P}$. It will follow from the irreducibility of $\overline{M}_{g,n}$ (cf. Corollary (4.2) in Chapter XV or Corollary (11.9) in Chapter XXI) that Δ_Γ is always irreducible.

We shall often refer to the $\mathcal{D}_\Gamma$ (or the Δ_Γ) as the *boundary strata* of $\overline{\mathcal{M}}_{g,P}$ (or of $\overline{M}_{g,P}$). The simplest boundary strata are those of codimension 1, which correspond to the stable graphs with a single edge. These are of two kinds and are illustrated in Figure 4. First of all, there is the graph Γ_{irr} with only one vertex and one edge. In addition to this, there are graphs $\Gamma_\mathcal{P}$ attached to stable bipartitions $\mathcal{P} = \{(a, A), (b, B)\}$

of (g, P). These have two vertices, one of genus a and A-marked, the other of genus $b = g - a$ and B-marked, where $B = P \smallsetminus A$.

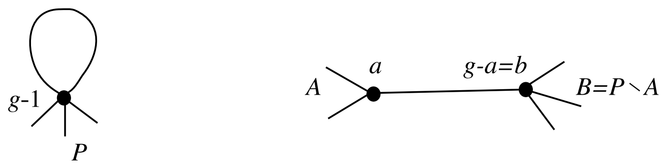

Figure 4.

In accordance with the conventions established in Section 2 of Chapter X, we usually write $\Gamma_{a,A}$ for $\Gamma_{\mathcal{P}}$. We normally write $\mathcal{D}_{irr}$ for $\mathcal{D}_{\Gamma_{irr}}$ and $\mathcal{D}_{a,A}$ for $\mathcal{D}_{\Gamma_{a,A}}$. The coarse counterparts of these substacks are the divisors Δ_{irr} and $\Delta_{a,A}$ introduced in Section 2. The clutching morphisms $\xi_{\Gamma_{irr}}$ and $\xi_{\Gamma_{a,A}}$ are usually written ξ_{irr} and $\xi_{a,A}$, respectively:
(10.7)
$$\xi_{irr} : \overline{\mathcal{M}}_{g-1,P\cup\{x,y\}} \to \overline{\mathcal{M}}_{g,P}\,, \qquad \xi_{a,A} : \overline{\mathcal{M}}_{a,A\cup\{x\}} \times \overline{\mathcal{M}}_{g-a,A^c\cup\{y\}} \to \overline{\mathcal{M}}_{g,P}\,.$$

We shall use the same symbols also to denote the corresponding morphisms between coarse moduli spaces.

We need to introduce some notions regarding dual graphs of stable curves and prove a number of auxiliary results about them. The reader should go back to the terminology introduced in the definition (2.16) of Chapter X.

Definition (10.8). *Let Γ be a graph. A subgraph $\Gamma' \subset \Gamma$ is a graph Γ' such that $L(\Gamma') \subset L(\Gamma)$, $V(\Gamma') \subset V(\Gamma)$, $\iota_{\Gamma'} = \iota_{\Gamma|L(\Gamma')}$, and $L_v(\Gamma') = L_v(\Gamma) \cap L(\Gamma')$ for every $v \in V(\Gamma')$.*

Suppose that we are given a P-marked dual graph Γ and a subgraph $I \subset \Gamma$ having no legs and containing all the vertices of Γ. Equivalently, we may think of I as being obtained from Γ by removing a subset from $E(\Gamma)$, together with all the legs. We want to construct a new graph Γ_I which is obtained from Γ by contracting to a point each connected component of I. Formally, we proceed as follows. Let W be the set of connected components of I. When we want to view $w \in W$ as a subgraph of Γ, we denote it by I_w; thus $I_w = w$. We let Γ_I be the P-marked dual graph defined as follows:

$$L(\Gamma_I) = L(\Gamma) \smallsetminus L(I)\,, \quad V(\Gamma_I) = W\,, \quad L_w(\Gamma_I) = \bigcup_{v\in V(I_w)} (L_v(\Gamma) \smallsetminus L_v(I_w))\,,$$
$$\iota_{\Gamma_I} = \iota_{\Gamma|L(\Gamma_I)}\,, \quad g_w(\Gamma_I) = g(I_w) \qquad \text{for} \quad w \in W\,,$$

while the indexing of the legs by P is the same as for Γ. Clearly, there is a continuous map $c_I : |\Gamma| \to |\Gamma_I|$ which contracts $|I_w|$ to the vertex $|w|$

and which induces a homeomorphism between $|\Gamma|\smallsetminus|I|$ and $|\Gamma_I|\smallsetminus W$, and we have

$$c_I^{-1}(E(|\Gamma_I|)) = E(|\Gamma|)\smallsetminus E(|I|).$$

A P-marked dual graph Γ' is said to be a *specialization of* Γ if Γ is isomorphic to Γ'_I, for some $I\subset\Gamma'$. We call an *I-contraction* or simply a *contraction* any map $c:|\Gamma'|\to|\Gamma|$ which is the composition of c_I with the map $|\Gamma'_I|\to|\Gamma|$ induced by an isomorphism of graphs. In the following figure the graph I which is contracted is the one with bold edges.

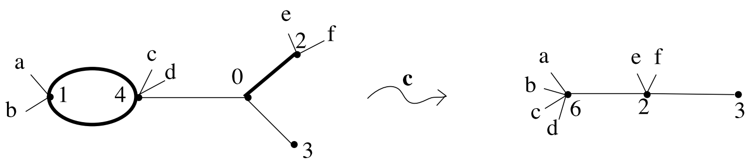

Figure 5.

Let Γ and Γ' be P-marked dual graphs and suppose that Γ is isomorphic to Γ'_I for some subgraph I of Γ'. Let σ' be an automorphism of Γ' which induces an automorphism of I. Then σ' descends to an automorphism σ of Γ, and we say that σ' *is a specialization of* σ.

We now come to stable curves. Let

$$\pi:\mathcal{C}\to S\,,\quad \{\tau_p:S\to X\}_{p\in P} \tag{10.9}$$

be a family of stable P-pointed genus g curves parameterized by a scheme S. In what follows, to keep the notation simple, we will usually not mention the marked sections. Suppose that there exists a subvariety

$$\Sigma\subset\mathrm{Sing}(\mathcal{C}) \tag{10.10}$$

which is proper and étale over S. This subvariety cuts on each fiber C_s of π a finite set of nodes Σ_s, and we may form the graph $\mathrm{Graph}^{\Sigma_s}(C_s)$ associated to the P-pointed curve C_s and to the set Σ_s of nodes, as explained in Chapter X below definition (2.16). Let Γ be a fixed P-marked genus g graph. We shall say that Σ is a *weak* Γ*-marking* if each graph $\mathrm{Graph}^{\Sigma_s}(C_s)$ is isomorphic to Γ.

Families of stable P-pointed, genus g curves endowed with a weak Γ-marking form a stack, which we denote by $\mathcal{E}_\Gamma$.

Now suppose that the family (10.9) has been obtained from an object $X\to S$ of $\overline{\mathcal{M}}_\Gamma$ via clutching and that Σ is precisely the locus of nodes produced by the clutching operation. Then, by construction, the family (10.9) comes with a weak Γ-marking. However, in this case, there is more structure. First of all, the locus Σ is clearly a union of sections, and the same is true for its preimage in X, that is, in the partial normalization of $\mathcal{C}$ along Σ. At this point, one can attach to the family a graph $\mathrm{Graph}^{\Sigma}(\mathcal{C})$ repeating, word by word, the construction that led to the graph associated to a single curve and a choice of nodes on it. In effect, here we are treating $\mathcal{C}$ as a single curve and Σ as a set of nodes. Notice that there is a canonical isomorphism between Γ and $\mathrm{Graph}^{\Sigma}(\mathcal{C})$ (and also between $\mathrm{Graph}^{\Sigma}(\mathcal{C})$ and $\mathrm{Graph}^{\Sigma_s}(\mathcal{C}_s)$ for each s in S). One expresses all of this by saying that the family (10.9) is endowed with a Γ-*marking*. It is evident that giving an object in $\mathcal{M}_\Gamma(S)$ is equivalent to giving a family of stable P-pointed genus g curves over S together with a Γ-marking.

By what we have said, the morphism $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,P}$ can be viewed as the composition of two forgetful morphisms

$$\overline{\mathcal{M}}_\Gamma \to \mathcal{E}_\Gamma \to \mathcal{D}_\Gamma \subset \overline{\mathcal{M}}_{g,P},$$

where the first one forgets the Γ-marking while still keeping the corresponding weak one, and the second forgets the extra Γ-structure altogether.

It is also clear that the automorphism group $\mathrm{Aut}(\Gamma)$ acts on $\overline{\mathcal{M}}_\Gamma$ in the sense of Example (8.15). In the next proposition we will see that the morphism $\overline{\mathcal{M}}_\Gamma \to \mathcal{E}_\Gamma$ comes from taking the quotient of $\overline{\mathcal{M}}_\Gamma$ by $\mathrm{Aut}(\Gamma)$, while $\mathcal{E}_\Gamma \to \mathcal{D}_\Gamma$ is the normalization morphism.

Proposition (10.11).

i) *$\mathcal{E}_\Gamma$ is the normalization of the substack $\mathcal{D}_\Gamma \subset \overline{\mathcal{M}}_{g,P}$.*
ii) *The morphism $\overline{\mathcal{M}}_\Gamma \to \mathcal{E}_\Gamma$ can be identified with the quotient morphism $\overline{\mathcal{M}}_\Gamma \to [\overline{\mathcal{M}}_\Gamma / \mathrm{Aut}(\Gamma)]$.*

Before proving this proposition, in order to form an intuition of what is going on, let us consider the clutching morphism at the more concrete level of moduli spaces. Since

$$\overline{M}_\Gamma = \prod_{v \in V} \overline{M}_{g_v, L_v}$$

is a coarse moduli space for $\overline{\mathcal{M}}_\Gamma$, the same reasoning we used in the context of the projection morphisms tells us that the clutching morphisms ξ_Γ descend to scheme morphisms

$$\xi_\Gamma : \overline{M}_\Gamma \to \overline{M}_{g,P} \tag{10.12}$$

between the corresponding coarse moduli spaces and that the image of $\overline{M}_\Gamma$ under ξ_Γ is Δ_Γ. In general, the morphisms ξ_Γ are not injective. Let us look at Figure 6 below. In the first row we consider

$$\xi_\Gamma : \overline{M}_{g-1,\{a,b\}} \to \overline{M}_g,$$

and we see that $\xi_\Gamma[C;x_a,x_b]=\xi_\Gamma[C;x_b,x_a]$. Therefore, in this case, ξ_Γ is generically two-to-one. But, as the second row shows, the cardinality of the fiber may jump. In fact,

$$\xi_\Gamma[C;x_a,x_b]=\xi_\Gamma[C';x_a,x_b]=\xi_\Gamma[C;x_b,x_a]=\xi_\Gamma[C';x_b,x_a],$$

where $(C;x_a,x_b)$ (resp. $(C';x'_a,x'_b)$) is obtained from a curve $(C_0;x_a,x_b,x'_a,x'_b)$ identifying x'_a with x'_b (resp x_a with x_b). We next consider the map

$$\xi_\Gamma:\ \overline{M}_{h,\{a\}}\times\overline{M}_{h+1,\{b\}}\to\overline{M}_{2h+1},$$

where Γ consists of two vertices joined by one edge. This morphism is generically injective, but, as we see in the third row of Figure 6, it is not injective. In fact, if E is elliptic and D is the curve obtained from C and E by identifying x_c with x_d, and if D' is the curve obtained from C' and E by identifying x_a with x_b, we get

$$\xi_\Gamma([D,x_a],[C',x_b])=\xi_\Gamma([D',x_d],[C,x_c])\,.$$

What these pictures suggest, and what the reader will easily verify, is that the morphism ξ_Γ factors through a generically injective morphism

$$\overline{\xi_\Gamma}:\overline{M}_\Gamma/\operatorname{Aut}(\Gamma)\longrightarrow\overline{M}_{g,P}\,. \tag{10.13}$$

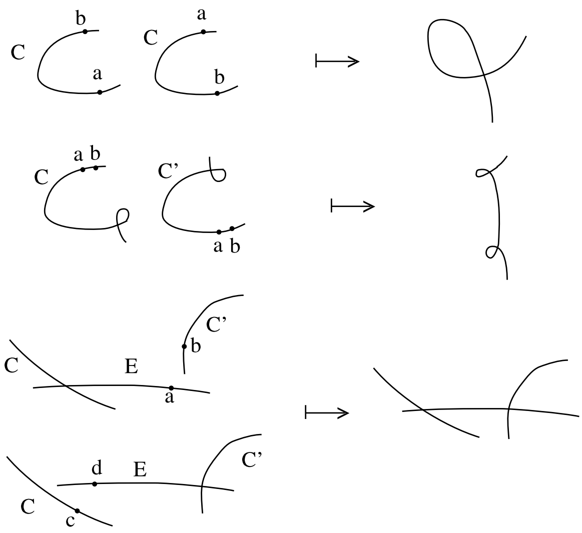

Figure 6.

In the above two examples we see that the subvariety Δ_Γ folds into itself forming a double point p.

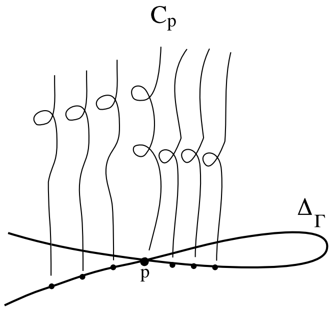

Figure 7.

The two branches of Δ_Γ at p correspond to the smoothing of one of the two nodes of C_p.

Proof of Proposition (10.11). Let us start by examining the substack $\mathcal{D}_\Gamma \subset \overline{\mathcal{M}}_{g,P}$. To construct an étale surjective morphism $X_\Gamma \to \mathcal{D}_\Gamma$ from a scheme X_Γ (i.e., an atlas for $\mathcal{D}_\Gamma$), we start from an atlas $X \to \overline{\mathcal{M}}_{g,P}$ consisting of the union of smooth bases of algebraic Kuranishi families (the "slices" of the Hilbert scheme $\mathrm{H}_{\nu,g,n}$ of ν-log-canonical stable P-pointed curves of genus g), and we look at the corresponding family of curves $\pi : \mathcal{C} \to X$. We then let X_Γ be the subvariety of X defined by

$$X_\Gamma = \{s \in X : \mathrm{Graph}(C_s) \text{ is a specialization of } \Gamma\},$$

and we denote by $\eta : \mathcal{C}_\Gamma \to X_\Gamma$ the restriction of π to X_Γ. We then get a surjective morphism $X_\Gamma \to \mathcal{D}_\Gamma$. Let us look more closely at the morphism

$$X_\Gamma \times_{\mathcal{D}_\Gamma} X_\Gamma = \mathbf{Isom}_{X_\Gamma \times X_\Gamma}(p_1^*\eta, p_2^*\eta) \longrightarrow X_\Gamma, \tag{10.14}$$

where p_1 and p_2 are the natural projections. The reason why this is an étale cover stems from the analogue, in the present setting, of Proposition (3.10) and therefore, in the final analysis, from the analogue of Lemma (3.11). Using the notation of that lemma, where $U \subset X$ denotes the basis of a local Kuranishi family, we set $U_\Gamma = U \cap X_\Gamma$ and let $\alpha : \mathcal{C}_{U_\Gamma} \to U_\Gamma$ be the restriction of the family η to U_Γ. If C is the central fiber of this family and $H = \mathrm{Aut}(C)$, then, also in this case, H acts on U_Γ, and we have an isomorphism between $H \times U_\Gamma$ and $\mathbf{Isom}_{U_\Gamma \times U_\Gamma}(p_1^*\alpha, p_2^*\alpha)$, under which the natural projection

$$U_\Gamma \times_{\mathcal{D}_\Gamma} U_\Gamma = \mathbf{Isom}_{U_\Gamma \times U_\Gamma}(p_1^*\alpha, p_2^*\alpha) \longrightarrow U_\Gamma \tag{10.15}$$

is just the projection

$$H \times U_\Gamma \to U_\Gamma\,. \tag{10.16}$$

This is the reason why (10.14) and, therefore, $X_\Gamma \to \mathcal{D}_\Gamma$ are étale. We next describe the normalization $\widehat{\mathcal{D}}_\Gamma$ of $\mathcal{D}_\Gamma$. Let us go back to Section 8 and in particular to the subsection on the orbifold-like definition of Deligne–Mumford stacks, where we explained what is the normalization of a stack. According to that definition, $\widehat{\mathcal{D}}_\Gamma$ is given by the groupoid presentation

$$\widehat{X_\Gamma \times_{\mathcal{D}_\Gamma} X_\Gamma} \rightrightarrows \widehat{X}_\Gamma\,,$$

where the hat stands for normalization. To analyze this normalization, the relevant local picture is given by the normalization of (10.15), that is, the normalization of (10.16). But the picture of U_Γ is the one of a certain number k of $(3g-3+n-\delta)$-dimensional linear subspaces in $\mathbb{C}^{3g-3+n}$ meeting transversally at the origin. If $|\operatorname{Sing}(C)| = \delta'$, the number k is the number of subsets I of $E(\operatorname{Graph}(C))$ consisting of $\delta' - \delta$ edges, contracting which, we obtain a graph isomorphic to Γ. Let then

$$U_\Gamma = U_1 \cup \cdots \cup U_k \tag{10.17}$$

be the decomposition of U_Γ in linear branches. The normalization $\widehat{U}_\Gamma$ is just the disjoint union of $U_1, \ldots, U_k$, so that the normalization of (10.15), that is, the normalization of (10.16), is nothing but the projection

$$H \times \widehat{U}_\Gamma \to \widehat{U}_\Gamma\,. \tag{10.18}$$

But now, by virtue of the local description of $\widehat{U}_\Gamma$, the pullback $\widehat{\eta}$ to $\widehat{X}_\Gamma$ of the family η over X_Γ is a family of curves with weak Γ-marking, and we have

$$\widehat{U_\Gamma \times_{\mathcal{D}_\Gamma} U_\Gamma} = \widehat{H \times U_\Gamma} = H \times \widehat{U}_\Gamma = \mathbf{Isom}^{\mathcal{E}_\Gamma}_{\widehat{U}_\Gamma \times \widehat{U}_\Gamma}(p_1^*\widehat{\alpha}, p_2^*\widehat{\alpha}) \longrightarrow \widehat{U}_\Gamma, \tag{10.19}$$

where $\widehat{\alpha}$ is the pullback of α to $\widehat{U}_\Gamma$, and $\mathbf{Isom}^{\mathcal{E}_\Gamma}$ stands for isomorphisms respecting the weak Γ-marking. This tells us that the two projections

$$\begin{aligned} &\widehat{X_\Gamma \times_{\mathcal{D}_\Gamma} X_\Gamma} \longrightarrow \widehat{X}_\Gamma\,, \\ &\widehat{X}_\Gamma \times_{\mathcal{E}_\Gamma} \widehat{X}_\Gamma = \mathbf{Isom}^{\mathcal{E}_\Gamma}_{\widehat{X}_\Gamma \times \widehat{X}_\Gamma}(p_1^*\widehat{\eta}, p_2^*\widehat{\eta}) \longrightarrow \widehat{X}_\Gamma \end{aligned} \tag{10.20}$$

may be identified. The fact that the second projection is étale tells us that the morphism $\widehat{X}_\Gamma \to \mathcal{E}_\Gamma$ is étale. On the other hand, the fact that the two projections in (10.20) can be identified tells us that $\mathcal{E}_\Gamma$ and the normalization $\widehat{\mathcal{D}}_\Gamma$ of $\mathcal{D}_\Gamma$ have the same groupoid presentation and are therefore isomorphic.

It now remains to show that $[\overline{\mathcal{M}}_\Gamma/\operatorname{Aut}(\Gamma)] \cong \mathcal{E}_\Gamma$. We set $G = \operatorname{Aut}(\Gamma)$. We will prove that $[\overline{\mathcal{M}}_\Gamma/G]$ is isomorphic to $\mathcal{E}_\Gamma$ by showing that $[\overline{\mathcal{M}}_\Gamma/G]$

and $\mathcal{E}_\Gamma$ have a common groupoid presentation. By the definition of $\overline{\mathcal{M}}_\Gamma$, there is a Γ-marked family of curves

$$\eta : \mathcal{C} \to Y_0 \tag{10.21}$$

for which the moduli morphism $m : Y_0 \to \overline{\mathcal{M}}_\Gamma$ is étale and surjective. As we explained in Section 8, we may assume that it is G-equivariant, and we get a groupoid presentation $Y_1 \rightrightarrows Y_0$ of $[\overline{\mathcal{M}}_\Gamma/G]$ by setting

$$Y_1 = G \times Y_0 \times_{\overline{\mathcal{M}}_\Gamma} Y_0.$$

On the other hand, we have

$$G \times Y_0 \times_{\overline{\mathcal{M}}_\Gamma} Y_0 = G \times \mathbf{Isom}^\Gamma_{Y_0 \times Y_0}(p_1^*\eta, p_2^*\eta)\,,$$

where p_1 and p_2 are the natural projections, and $\mathbf{Isom}^\Gamma$ stands for isomorphisms respecting the Γ-marking. We now look at $\mathcal{E}_\Gamma$. Remembering only the weak Γ-marking, the family (10.21) may be considered as an object of $\mathcal{E}_\Gamma(Y_0)$, yielding a morphism $Y_0 \to \mathcal{E}_\Gamma$, which is readily seen to be étale. In fact, locally, Y_0 looks like a product $\prod_{v\in V} U_v$, where U_v is the basis of a standard Kuranishi family for a curve C_v in $\overline{\mathcal{M}}_{g_v,L_v}$, and this product, in turn, is isomorphic to one of the branches U_i of $U_\Gamma \subset U$, where U is the basis for a Kuranishi family of the curve C obtained via the clutching ξ_Γ, starting from the C_v. Finally, we have

$$Y_0 \times_{\mathcal{E}_\Gamma} Y_0 = \mathbf{Isom}^{\mathcal{E}_\Gamma}_{Y_0 \times Y_0}(p_1^*\eta, p_2^*\eta) = G \times \mathbf{Isom}^\Gamma_{Y_0 \times Y_0}(p_1^*\eta, p_2^*\eta)\,.$$

Q.E.D.

Corollary (10.22). *Let Γ be a stable P-marked dual graph of genus g. Assume that* $\mathrm{Aut}(\Gamma) = \{\mathrm{id}_\Gamma\}$. *Furthermore, assume that, for every graph Γ' which is a specialization of Γ, all the elements in* $\mathrm{Aut}(\Gamma')$ *are specializations of* id_Γ. *Then $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,P}$ is a closed immersion.*

Proof. The fact that $\mathrm{Aut}(\Gamma) = \{\mathrm{id}_\Gamma\}$ tells us that $\overline{\mathcal{M}}_\Gamma$ is the normalization of the closed substack $\mathcal{D}_\Gamma$. Let us then prove that $\mathcal{D}_\Gamma$ is a smooth Deligne–Mumford stack. Going back to the local picture (10.17), we must prove that $k = 1$. But now the various branches U_i of U_Γ correspond to the various contractions $c_I : |\,\mathrm{Graph}(C)| \to |\Gamma|$, where $[C] \in \overline{M}_\Gamma$. As one can easily check, any two contractions c_I and c_J are linked by an automorphism $\sigma \in \mathrm{Graph}(C)$. Since, by assumption, σ is a specialization of id_Γ, we must have $I = J$.

Q.E.D.

In studying the geometry of $\overline{\mathcal{M}}_{g,P}$ it is important to describe how the various boundary strata intersect, not only the divisorial ones. We must therefore understand cartesian diagrams of the type

$$\begin{tikzcd}
\overline{\mathcal{M}}_\Gamma \times_{\mathcal{M}_{g,P}} \overline{\mathcal{M}}_{\Gamma'} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_\Gamma \arrow[d, "\xi_\Gamma"] \\
\overline{\mathcal{M}}_{\Gamma'} \arrow[r, "\xi_{\Gamma'}"] & \mathcal{M}_{g,P}
\end{tikzcd}$$

Set $\overline{\mathcal{M}}_{\Gamma\Gamma'} = \overline{\mathcal{M}}_\Gamma \times_{\mathcal{M}_{g,P}} \overline{\mathcal{M}}_{\Gamma'}$. We let $\mathcal{G}_{\Gamma\Gamma'}$ denote a set of representatives for the isomorphism classes of triples (Λ, c, c') where Λ is a P-marked, genus g, dual graph, and $c : |\Lambda| \to |\Gamma|$ and $c' : |\Lambda| \to |\Gamma'|$ are contractions. We also insist that

$$E(|\Lambda|) = c^{-1}(E(|\Gamma|) \cup {c'}^{-1}(E(|\Gamma'|)\,. \tag{10.23}$$

This simply means that, given a curve C with dual graph equal to Λ, smoothing the nodes corresponding to the edges of ${c'}^{-1}E(|\Gamma'|) \smallsetminus c^{-1}(E(|\Gamma|)$ produces a curve whose graph is Γ, and similarly when the roles of Γ and Γ' are reversed. An isomorphism between triples (Λ, c, c') and $(\Lambda_1, c_1, {c'}_1)$ is an isomorphism between Λ and Λ' commuting with the contractions. Figure 8 below describes $\mathcal{G}_{\Gamma\Gamma'}$ in four examples, where Γ and Γ' are two unpointed graphs (we are assuming that $a, b, g > 1$ and that $a \neq b$).

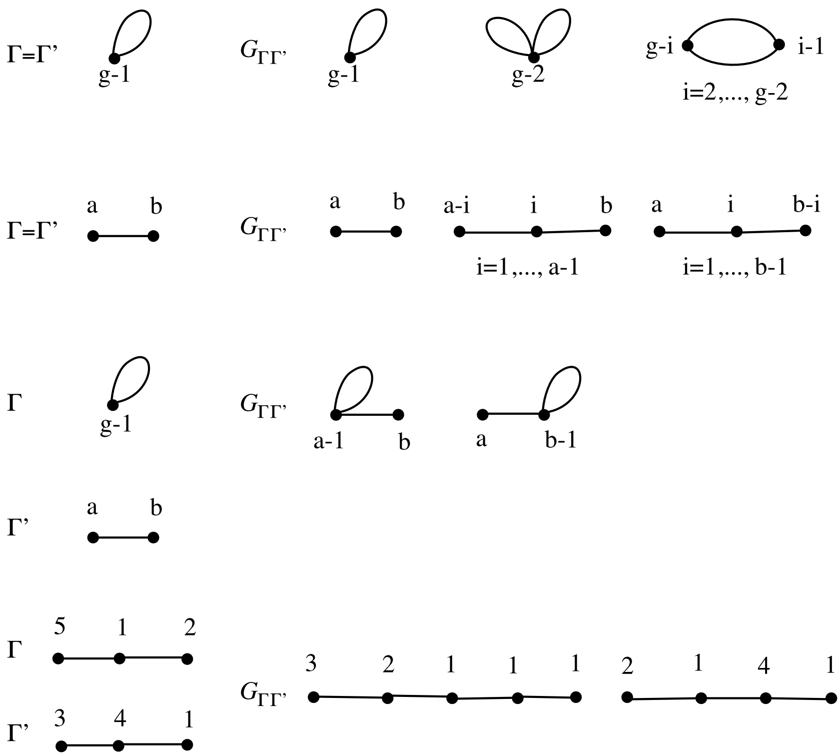

Figure 8.

We will now show how an intersection of two boundary strata decomposes into a disjoint union of boundary strata.

PROPOSITION (10.24). *There is an isomorphism*

$$\overline{\mathcal{M}}_{\Gamma\Gamma'} \cong \coprod_{\Lambda\in\mathcal{G}_{\Gamma\Gamma'}} \overline{\mathcal{M}}_{\Lambda}.$$

Proof. It suffices to give isomorphisms between $\overline{\mathcal{M}}_{\Gamma\Gamma'}(T)$ and $\coprod_{\Lambda\in\mathcal{G}_{\Gamma\Gamma'}} \overline{\mathcal{M}}_{\Lambda}(T)$, where T is a scheme. An object of $\overline{\mathcal{M}}_{\Gamma\Gamma'}(T)$ is a triple (ξ,ξ',φ), where ξ (resp. ξ') is a family of Γ-marked (resp. Γ'-marked) stable P-pointed genus g curves, parameterized by T, and $\varphi : \xi \to \xi'$ is a T-isomorphism. Now suppose $\xi : \mathcal{C} \to T$ is an object in $\overline{\mathcal{M}}_{\Lambda}(T)$. This means that we are given a subvariety Σ of $\mathrm{Sing}(\mathcal{C})$, proper and étale over T, whose inverse image in the partial normalization along Σ itself is a union of sections, plus an isomorphism $\gamma : \mathrm{Graph}^{\Sigma}(\mathcal{C}) \xrightarrow{\sim} \Lambda$. Composing γ with the given contractions $c : \Lambda \to \Gamma$ and $c' : \Lambda \to \Gamma'$ provides two subsets

$$\Sigma_1 = (c\gamma)^{-1}(E(\Gamma)) \quad \text{and} \quad \Sigma_2 = (c'\gamma)^{-1}(E(\Gamma'))$$

such that $\Sigma = \Sigma_1 \cup \Sigma_2$, and isomorphisms $\gamma_1 : \mathrm{Graph}^{\Sigma_1}(\mathcal{C}) \xrightarrow{\sim} \Gamma$ and $\gamma_2 : \mathrm{Graph}^{\Sigma_2}(\mathcal{C}) \xrightarrow{\sim} \Gamma'$. This exhibits ξ as an object in both $\overline{\mathcal{M}}_{\Gamma}$ and $\overline{\mathcal{M}}_{\Gamma'}$ and therefore as an object of $\overline{\mathcal{M}}_{\Gamma\Gamma'}$. Conversely, given an object (ξ,ξ',φ) in $\overline{\mathcal{M}}_{\Gamma\Gamma'}(T)$, the family ξ is endowed with a Γ-marking and, via φ, with a Γ'-marking, that is, with isomorphisms $\gamma : \mathrm{Graph}^{\Sigma_1}(\mathcal{C}) \xrightarrow{\sim} \Gamma$ and $\gamma' : \mathrm{Graph}^{\Sigma_2}(\mathcal{C}) \xrightarrow{\sim} \Gamma'$. From these one gets contractions $\overline{c} : \mathrm{Graph}^{\Sigma_1\cup\Sigma_2}(\mathcal{C}) \to \Gamma$ and $\overline{c}' : \mathrm{Graph}^{\Sigma_1\cup\Sigma_2}(\mathcal{C}) \to \Gamma'$, and therefore a unique element $(\Lambda, c, c') \in \mathcal{G}_{\Gamma\Gamma'}$ and a unique isomorphism $\theta : \mathrm{Graph}^{\Sigma_1\cup\Sigma_2}(\mathcal{C}) \to \Lambda$ such that $c\theta = \overline{c}$ and $c'\theta = \overline{c}'$. The reader will check that these associations carry over to morphisms, thus establishing the desired isomorphism between $\overline{\mathcal{M}}_{\Gamma\Gamma'}(T)$ and $\coprod_{\Lambda\in\mathcal{G}_{\Gamma\Gamma'}} \overline{\mathcal{M}}_{\Lambda}(T)$. Q.E.D.

Looking at Figure 8, in the first two lines we see the decomposition in strata for the self-intersection of the boundary divisors (in the unpointed case). There we see one "excess intersection" component, while the remaining components are "transverse." In the third example we see a bona fide transverse intersection of two boundary divisors. In the fourth example we are intersecting two codimension 2 boundary strata in $\overline{\mathcal{M}}_8$, and we get a transversal codimension 4 component and an "excess intersection" component of codimension 3.

We end this section by proving the following result.

PROPOSITION (10.25). *The clutching morphisms $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,P}$ are representable.*

Proof. First of all, we are going to show that any clutching morphism can be factored as the composition of a closed embedding with a projection map. Let Γ be the dual graph of a stable P-pointed genus g curve. As usual, we denote by $E = E(\Gamma)$ the set of edges of Γ. We construct a new graph $\widehat{\Gamma}$ in the following way. Fix an edge $\mathbf{l} = \{l, l'\} \in E$. Consider two graphs Γ_l and $\Gamma_{l'}$ as in Figure 9. Split $\mathbf{l}$ in the two halves l and l', then join l with l_∞, l' with l'_∞ and l_0 with l'_0.

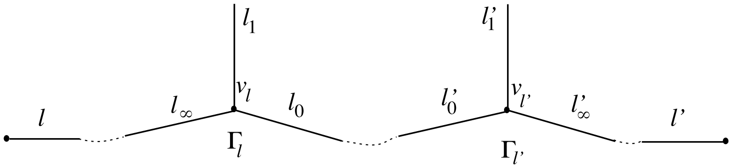

Figure 9.

Repeat this operation for every edge of Γ to obtain $\widehat{\Gamma}$ (see Figure 10 for an example of how to pass from Γ to $\widehat{\Gamma}$).

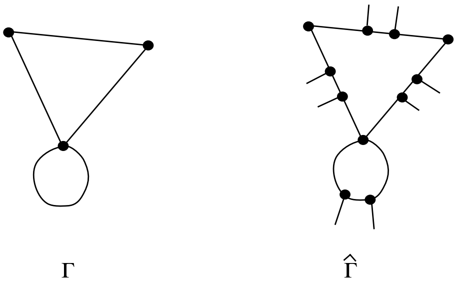

Figure 10.

By definition, $\widehat{\Gamma}$ is marked by the set $P \cup H$, where $H = H(\Gamma)$ is the set of half-edges of Γ which are not legs. From the construction it follows that $\mathrm{Aut}(\widehat{\Gamma}) = \{\mathrm{id}_{\widehat{\Gamma}}\}$. We have a decomposition

$$\xi_\Gamma = \pi_H \circ \xi_{\widehat{\Gamma}} \circ \iota_\Gamma, \tag{10.26}$$

where

$$\iota_\Gamma : \overline{\mathcal{M}}_\Gamma = \prod_{v \in V} \overline{\mathcal{M}}_{g_v, L_v} \longrightarrow$$
$$\overline{\mathcal{M}}_{\widehat{\Gamma}} = \prod_{v \in V} \overline{\mathcal{M}}_{g_v, L_v} \times \prod_{\{l,l'\} \in E} \left(\overline{\mathcal{M}}_{0,\{l_0,l_1,l_\infty\}} \times \overline{\mathcal{M}}_{0,\{l'_0,l'_1,l'_\infty\}} \right)$$

is the natural isomorphism,

$$\xi_{\widehat{\Gamma}} : \overline{\mathcal{M}}_{\widehat{\Gamma}} = \prod_{v \in V} \overline{\mathcal{M}}_{g_v, L_v} \times \prod_{\{l,l'\} \in E} \left(\overline{\mathcal{M}}_{0,\{l_0,l_1,l_\infty\}} \times \overline{\mathcal{M}}_{0,\{l'_0,l'_1,l'_\infty\}} \right) \longrightarrow \overline{\mathcal{M}}_{g, P \cup H}$$

is the morphism defined by $\widehat{\Gamma}$, and π_H is the natural projection from $\overline{\mathcal{M}}_{g,P\cup H}$ to $\overline{\mathcal{M}}_{g,P}$.

Since ι_Γ is an isomorphism, and since we already proved that projections are representable, it is enough to prove that $\xi_{\widehat{\Gamma}}$ is representable. This follows from Corollary (10.22), which shows that $\xi_{\widehat{\Gamma}}$ is a closed immersion and hence is representable.

Q.E.D.

11. Bibliographical notes and further reading.

Our basic references for algebraic spaces are Artin [37,38] and Knutson [428]. A proof of Theorem (9.1) can be found in Laumon and Moret-Bailly [462], Corollaire 16.6.2.

A good introduction to the theory of orbifold is Adem, Leida, and Ruan [3]. Satake's seminal papers on the subject are [612,614].

The theory of stacks was initiated by Deligne and Mumford [167], based in part on ideas of Grothendieck. Good general references are Gillet [298], Vistoli [671], Edidin [190], Laumon and Moret-Bailly [462], Fantechi [242], Canonaco [94], Fantechi–Göttsche–Illusie–Kleiman–Nitsure–Vistoli [243], and the online Stacks Project [397]

A good reference for the theory of descent can be found in [243]. Our treatment of descent for quasicoherent modules is taken directly from Grothendieck's (cf. [326], exposé VIII).

Theorem (2.9) is a special instance of a more general result found in Kollár [439], whose treatment we follow in the proof. A vast generalization of our method of construction of the moduli space of curves as an algebraic space is given in Keel and Mori [410].

As already explained in the bibliographical notes to Chapter X, the projection, clutching, and stabilization constructions were first studied in Knudsen [426]. Our treatment of the intersections of boundary strata is inspired by the one by Graber and Pandharipande in Appendix A of [307].

12. Exercises.

A. **Orbifolds and stacks**

A-1. Let G be a finite group acting holomorphically on a complex manifold M. Suppose that M/G is Hausdorff. Set $X_0 = M$ and $X_1 = G\times M$. Let $s : X_1 = G\times M \to M$ be the projection, and $t : X_1 = G\times M \to M$ be the action. The composition rule, the unit, and the inverse are the obvious ones. Verify that this defines an orbifold structure on M/G. We call the orbifold described in this way, the *quotient orbifold of M modulo G*, and we shall denote it by $[M/G]$.

A-2. Let $f : |\mathbb{X}| \to M$ be an orbifold structure on a complex manifold M. Show that every point in M has a neighborhood with an orbifold structure of the form $[B/G_x]$ where $x \in X_0$, and B is a chart around x.

A-3. Using Lemma (5.1), give a detailed proof of the fact that, given a groupoid (or a Deligne–Mumford stack) $\mathcal{M}$ and a morphism $S \to \mathcal{M}$, the stack $\mathcal{M} \times_{\mathcal{M}} S$ is represented by S.

A-4. Let $h : \mathcal{N} \to \mathcal{N}'$ be an isomorphism of DM stacks. Show that if a representable morphism of Deligne–Mumford stacks $f : \mathcal{M} \to \mathcal{N}$ satisfies a property **P** of morphisms of schemes which is stable under base change, then also fh satisfies **P**.

A-5. Verify that G_x given in (4.4) and in (8.11) is indeed a group.

A-6. Show that (8.17), (8.18), and (8.19), together with appropriate structural maps s_G, t_G, u_G, i_G, and m_G, define a groupoid presentation of $\mathcal{M}$ to which the action of G lifts.

A-7. Show that a closed substack of a Deligne–Mumford stack is a Deligne–Mumford stack.

B. **Genus 0 and 1.**

B-1. Show that $M_{0,n}$ can be described as follows. Let X_n be the n-fold product of $\mathbb{P}^1$, minus the big diagonal. The group $PGL(2)$ acts naturally on X_n and $M_{0,n}$ is just the quotient.

B-2. Show that $\overline{M}_{0,n}$ is smooth.

B-3. Construct $\overline{M}_{0,n+1}$ by blowing up the diagonal in $\overline{M}_{0,n} \times_{\overline{M}_{0,n-1}} \overline{M}_{0,n}$.

B-4. Show that $M_{1,1}$ can be naturally identified, via the period map, with the quotient of the upper half-plane $\mathbb{H}$ modulo the action of $SL(2, \mathbb{Z})$. Deduce that $M_{1,1} = \mathbb{C}$ and $\overline{M}_{1,1} = \mathbb{P}^1$.

B-5. Construct explicitly Kuranishi families for curves in $M_{1,1}$, carefully describing the automorphism groups acting on them.

B-6. Describe the stack $\overline{\mathcal{M}}_{1,1}$.

C. **Hyperelliptic curves**

C-1. Let $H_g \subset M_g$ be the locus of hyperelliptic curves of genus g.

a) Show that $H_g \subset M_g$ is a $(2g-1)$-dimensional subvariety of M_g.
b) Show that H_g can be identified with the quotient of $M_{0,2g+2}$ modulo the action of the symmetric group $\mathfrak{S}_{2g+2}$.

C-2. Define the stack $\mathcal{H}_g$ of smooth hyperelliptic curves of genus g. Show that the hyperelliptic involution defines a nontrivial $\mathbb{Z}_2$-action on $\mathcal{H}_g$ and that H_g is a coarse moduli space for both $\mathcal{H}_g$ and $\mathcal{H}_g/\mathbb{Z}_2$.

D. **Strata of moduli spaces.**

In this series, we will ask the reader simply to list the strata of various moduli spaces $\overline{M}_{g,n}$. In each case we ask to make a chart listing the various possible topological types of stable n-pointed curves of genus g, the dimensions of the corresponding loci in $\overline{M}_{g,n}$, and the inclusion relations among their closures. We would suggest putting the open stratum (smooth curves with n distinct points) at the top; the various codimension 1 strata on a line below that, the codimension 2 strata on the next line, and so on; indicate the specialization relationships by vertical or diagonal lines.

D-1. Describe all the boundary strata of
$\overline{M}_{0,n}$ for $3 \leq n \leq 6$, $\overline{M}_2$, $\overline{M}_3$, $\overline{M}_{1,2}$, $\overline{M}_{2,1}$, $\overline{M}_{2,2}$, $\overline{M}_{3,1}$, $\overline{M}_{3,2}$, $\overline{M}_{3,3}$.

D-2. Consider the projection $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$.

a) Describe $\pi^{-1}(\partial M_{g,n})$.
b) Describe $\pi^{-1}(M_{g,n}) \cap \partial M_{g,n+1}$.

D-3. Describe the intersection of the codimension one boundary components $\Delta_{a,A} \cap \Delta_{irr}$ and $\Delta_{a,A} \cap \Delta_{a',A'}$ in $\overline{M}_{g,P}$

D-4. How many boundary divisors are there in $\overline{M}_g$? In $\overline{M}_{g,n}$?

D-5. Do every pair of boundary divisors in $\overline{M}_g$ intersect? How about in $\overline{M}_{g,n}$?

D-6. Give an example of a pair of boundary divisors in $\overline{M}_g$ whose intersection is reducible. Can you find an example where the intersection is disconnected?

D-7. Let C be a stable curve of genus g with δ nodes. Show that $\delta \leq 3g-3$. Similarly, show that a stable n-pointed curve of genus g has at most $3g-3+n$ nodes.

D-8. Show that the set $R_g \subset \overline{M}_g$ of stable curves with $3g-3$ nodes is finite. What is its cardinality for $g = 2$, 3, and 4?

D-9. Similarly, count the stable n-pointed curves of genus g having $3g-3+n$ nodes for $(g,n) = (2,1), (2,2), (3,1)$, and $(3,2)$.

D-10. Consider now the locus in $\overline{M}_g$ of stable curves of genus g having $3g-4$ nodes. Show that every component of this locus is a rational curve.

E. **Curves in moduli spaces $\overline{M}_{g,n}$.**

In what follows—in particular, when we discuss line bundles on moduli spaces in the next chapter—it will be useful to have some explicitly given curves in $\overline{M}_{g,n}$. These will be given by one-parameter families of stable curves of genus g. In these exercises, we ask you to verify that all members of the specified family are in fact stable.

E-1. A general pencil of plane curves of degree d, that is, we let F and G be general polynomials of degree d and consider curves C_t defined by linear combinations $t_0F + t_1G$ of the two.

E-2. Let $S \subset \mathbb{P}^3$ be a general surface of degree d, and $C_t = S \cap H_t$ a general pencil of plane sections of S.

E-3. Let $S \subset \mathbb{P}^3$ be an arbitrary smooth surface of degree d, and $C_t = S \cap H_t$ a general pencil of plane sections of S. (Note: This is much harder than the preceding problem and in particular requires a hypothesis of characteristic 0)

E-4. Let B be a smooth curve of genus $g - 1 > 0$, $p \in B$ any point, $\{E_t \subset \mathbb{P}^2\}$ a general pencil of plane cubics, and q a base point of the pencil. Let C_t be the curve obtained from $B \cup E_t$ by identifying p with q.

E-5. Let $p_1, \dots, p_{2g+1} \in \mathbb{P}^1$ be distinct points, and for $p \in \mathbb{P}^1$, let C_p be the hyperelliptic curve given as a double cover of $\mathbb{P}^1$ with branch divisor $p + p_1 + \cdots + p_{2g+1}$.

F. **Unirationality of moduli spaces $M_{g,n}$**

We say that a variety X is *unirational* if there exists a dominant rational map $\mathbb{P}^n \to X$ from a projective (or affine) space to X. In particular, if there is an open subset $U \subset \mathbb{P}^n$ and a family of stable curves of genus g over U whose associated map $U \to M_g$ is dominant, we may conclude that M_g is unirational. (Of course, the converse need not be true—a priori, there might be a dominant rational map $\mathbb{P}^n \to M_g$, but not one arising from a family of stable curves—but in practice the only way we have ever shown a space M_g to be unirational is by exhibiting such a family.) In these exercises, we ask you to prove the unirationality of a particular moduli space $M_{g,n}$ by showing that the specified rationally parameterized family of curves is dominant, that is, the general member of the family is indeed stable, and the general curve of genus g does appear among the member of the family.

F-1. M_2: consider the family of curves given by $y^2 = f(x)$, where f ranges over all sextic polynomials.

F-2. M_3: consider the family of all plane quartics.

F-3. $M_{3,n}$ for $n \leq 14$: consider the family of pointed curves $(C, p_1, \ldots, p_n)$ with $p_i \in \mathbb{P}^2$ and C a plane quartic passing through $p_1, \ldots, p_n$. What is the largest n for which you can say that $M_{3,n}$ is unirational?

F-4. $M_{2,n}$: What is the largest n for which you can say that $M_{2,n}$ is unirational?

F-5. M_4: consider the family of curves of bidegree $(3,3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

F-6. M_5: consider the family of complete intersections of three quadrics in $\mathbb{P}^4$.

F-7. M_6: let $S \subset \mathbb{P}^5$ be a (fixed) del Pezzo surface and consider the family of all intersections $S \cap Q$ of S with quadric hypersurfaces in $\mathbb{P}^5$.

F-8. Using Brill–Noether theory and plane sextics curves, show that M_6 is unirational.

F-9. In the same spirit of the preceding exercise, try $g = 7$.

F-10. Prove that the moduli H_g space of hyperelliptic curves is unirational.

G. **Miscellaneous exercises**

G-1. Find n_0 such that $M_{g,n}$ is smooth for $n \geq n_0$. Is there a similar lower bound for the moduli space of stable curves?

G-2. Let $\overline{M}_g^n$ be the set of isomorphism classes of triples $(C, \mathbf{x}, \mathbf{v})$, where C is a smooth genus g curve, $\mathbf{x} = (x_1, \ldots, x_n)$ is an n-tuple of points of C, and $\mathbf{v} = (v_1, \ldots, v_n)$ is an n-tuple of nonzero tangent vectors with $v_i \in T_{x_i}(C)$, $i = 1, \ldots, n$.

a) Show that $\overline{M}_g^n$ has a natural structure of smooth algebraic variety and compute its dimension.
b) Express the tangent spaces at points of $\overline{M}_g^n$ in cohomological terms.

G-3. Let $\overline{M}_{g,n}^0$ be the locus of automorphism-free, stable n-pointed curves of genus g. Show that $\overline{M}_{g,n}^0$ is an open smooth subset of $\overline{M}_{g,n}$. Set $V_g = \overline{M}_{g,n} \smallsetminus \overline{M}_{g,n}^0$

a) Describe the codimension-one components of $V_{g,n}$. Can any one of these be contained in the singular locus of $\overline{M}_{g,n}$?
b) Show that if Y is component of $V_{g,n}$, which is not of codimension equal to one, then

$$\dim Y \leq 2g - 1\,. \tag{12.1}$$

c) Show that equality in (12.1) holds if and only if Y is the hyperelliptic locus.
d) Show that the singular locus of $\overline{M}_{g,n}$ consists precisely of those components of $V_{g,n}$ which are not divisors in $\overline{M}_{g,n}$.

G-4. Describe the infinitesimal behavior of projections and clutching maps.

G-5. Show that, given a point $\{[C_v;\mathbf{x}_v]\}_{v\in V}$ of $\overline{M}_\Gamma$ and denoting by $[C;\mathbf{x}]$ its image via ξ_Γ, there is an exact sequence

$$1 \to \prod_{v\in V} \mathrm{Aut}(C_v;\mathbf{x}_v) \to \mathrm{Aut}(C;\mathbf{x}) \xrightarrow{\alpha} \mathrm{Aut}(\Gamma)\,. \tag{12.2}$$

G-6. Look at the exact sequence (12.2). Give examples of stable curves for which
a) α is surjective,
a) α is injective,
a) α is neither injective nor surjective.

G-7. Give examples of dual graphs Γ' and Γ such that Γ' is a specialization of Γ, but not every element in $\mathrm{Aut}(\Gamma')$ is the specialization of an element in $\mathrm{Aut}(\Gamma)$.

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Geometry of Algebraic Curves
Volume II with a contribution by Joseph Daniel Harris
Arbarello, E.; Cornalba, M.; Griffiths, P.
2011, XXX, 963 p., Hardcover
ISBN: 978-3-540-42688-2