2

Knowledge Structures and Learning Spaces

Suppose that some complex system is assessed by an expert, who checks for the presence or absence of some revealing features. Ultimately, the state of the system is described by the subset of features, from a possibly large set, which are detected by the expert. This concept is very general, and becomes powerful only on the background of specific assumptions, in the context of some applications. We begin with the combinatoric underpinnings of the theory formalizing this idea.

2.1 Fundamental Concepts

2.1.1 Example. (*Knowledge structures in education.*) A teacher is examining a student to determine, for instance, which mathematics courses would be appropriate at this stage of the student's career, or whether the student should be allowed to graduate. The teacher will ask one question, then another, chosen as a function of the student's response to the first one. After a few questions, a picture of the student's knowledge state will emerge, which will become increasingly more precise in the course of the examination. By 'knowledge state' we mean here the set of all problems that the student is capable of solving in ideal conditions[1]. The next few definitions provides a rigorous framework.

2.1.2 Definition. A *knowledge structure* is a pair $(Q, \mathcal{K})$ in which Q is a nonempty set, and $\mathcal{K}$ is a family of subsets of Q, containing at least Q and the empty set $\varnothing$. The set Q is called the *domain* of the knowledge structure. Its elements are referred to as *questions* or *items* and the subsets in the family $\mathcal{K}$ are labeled *(knowledge) states*. Occasionally, we shall say that $\mathcal{K}$ is a *knowledge structure on* a set Q to mean that $(Q, \mathcal{K})$ is a knowledge structure. The specification of the domain can be omitted without ambiguity since we have $\cup\mathcal{K} = Q$.

[1] We assume, for the time being, that there are no careless errors or lucky guesses.

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2.1.3 Example. Consider the domain $U = \{a, b, c, d, e, f\}$ equipped with the knowledge structure

$$\begin{aligned}\mathcal{H} = \big\{\varnothing, \{d\}, \{a, c\}, \{e, f\}, \{a, b, c\}, \{a, c, d\}, \{d, e, f\},\\ \{a, b, c, d\}, \{a, c, e, f\}, \{a, c, d, e, f\}, U\,\big\}.\end{aligned} \tag{2.1}$$

As illustrated by this example, we do not assume that all subsets of the domain are states. The knowledge structure $\mathcal{H}$ contains eleven states out of sixty-four possible subsets of U.

2.1.4 Definition. Let $\mathcal{F}$ be a family of sets. We denote by $\mathcal{F}_q$ the collection of all sets in $\mathcal{F}$ containing q. In the knowledge structure $\mathcal{H}$ of Example 2.1.3, we have, for instance

$$\begin{aligned}\mathcal{H}_a &= \big\{\{a, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}, \{a, c, e, f\}, \{a, c, d, e, f\}, U\,\big\},\\ \mathcal{H}_e &= \big\{\{e, f\}, \{d, e, f\}, \{a, c, e, f\}, \{a, c, d, e, f\},\ U\,\big\}.\end{aligned}$$

Items a and c carry the same information relative to $\mathcal{H}$ in the sense that they are contained in the same states: any state containing a also contains c, and vice versa. In other terms, we have $\mathcal{H}_a = \mathcal{H}_c$. From a practical viewpoint, any individual whose state contains item a has necessarily mastered item c, and vice versa. Thus, in testing the acquired knowledge of a subject, only one of these two questions must be asked. Similarly, we also have $\mathcal{H}_e = \mathcal{H}_f$.

2.1.5 Definition. In a knowledge structure $(Q, \mathcal{K})$, the set of all the items contained in the same states as item q is denoted by q^* and is called a *notion*; we thus have

$$q^* = \{r \in Q \,|\, \mathcal{K}_q = \mathcal{K}_r\}.$$

The collection Q^* of all notions is a partition of the set Q of items. When two items belong to the same notion, we shall sometimes say that they are *equally informative*. In such a case, the two items form a pair in the equivalence relation on Q associated to the partition Q^*.

In Example 2.1.3, we have the four notions

$$a^* = \{a, c\}, \quad b^* = \{b\}, \quad d^* = \{d\}, \quad e^* = \{e, f\},$$

forming the partition $U^* = \big\{\{a, c\}, \{b\}, \{d\}, \{e, f\}\big\}$.

A knowledge structure in which each notion contains a single item is called *discriminative*. A discriminative knowledge structure can always be manufactured from any knowledge structure $(Q, \mathcal{K})$ by forming the notions, and constructing the knowledge structure $\mathcal{K}^*$ induced by $\mathcal{K}$ on Q^* via the definitions

$$\begin{aligned}K^* &= \{q^* \,|\, q \in K\} && (K \in \mathcal{K})\\ \mathcal{K}^* &= \{K^* \,|\, K \in \mathcal{K}\}.\end{aligned}$$

Note that as $\varnothing$, $Q \in \mathcal{K}$ and $\varnothing^* = \varnothing$, we have $\varnothing, Q^* \in \mathcal{K}^*$.

The knowledge structure $(Q^*, \mathcal{K}^*)$ is called the *discriminative reduction* of $(Q, \mathcal{K})$. Since this construction is straightforward, we shall often simplify matters and suppose that a particular knowledge structure under consideration is discriminative.

2.1.6 Example. We construct the discriminative reduction of the knowledge structure $(U, \mathcal{H})$ of Example 2.1.3 by setting

$$a^* = \{a, c\}, \quad b^* = \{b\}, \quad d^* = \{d\}, \quad e^* = \{e, f\};$$
$$U^* = \{a^*, b^*, d^*, e^*\};$$
$$\mathcal{H}^* = \big\{\varnothing, \{d^*\}, \{a^*\}, \{e^*\}, \{a^*, b^*\}, \{a^*, d^*\}, \{d^*, e^*\},$$
$$\{a^*, b^*, d^*\}, \{a^*, e^*\}, \{a^*, d^*, e^*\}, U^*\big\}.$$

Thus, $(U^*, \mathcal{H}^*)$ is formed by aggregating equally informative items from U. The graph of the discriminative reduction $\mathcal{H}^*$ is displayed in Figure 2.1. (The graph of a knowledge structure was introduced in 1.1.3.)

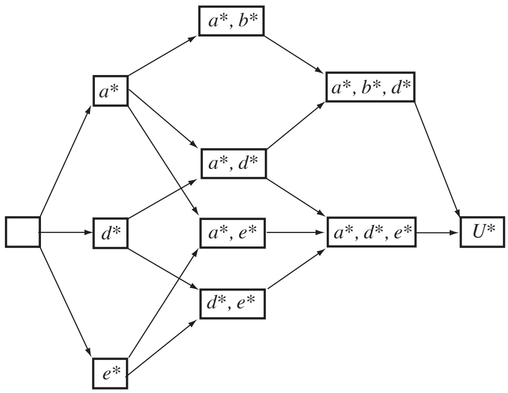

Figure 2.1. The discriminative reduction of the knowledge structure $\mathcal{H}$ of Eq. (2.1).

2.1.7 Definition. A knowledge structure $(Q, \mathcal{K})$ is called *finite* (respectively *essentially finite*), if Q (respectively $\mathcal{K}$) is finite. A similar definition holds for *countable* (respectively *essentially countable*) knowledge structures.

Typically, knowledge structures encountered in education are essentially finite. They may not be finite however: at least conceptually, some notions may contain a potentially infinite number of equally informative questions. Problems 3 and 4 require the reader to show that any knowledge structure $\mathcal{K}$ has the same cardinality as its discriminative reduction $\mathcal{K}^*$, and that the knowledge structure $(Q, \mathcal{K})$ is essentially finite if and only if Q^* is finite.

As suggested by these first few definitions, our choice of terminology is primarily guided by Example 2.1.1, which has also motivated many of our theoretical developments. However, as illustrated by Examples 1.4.1 to 1.4.4 our results are potentially applicable to very different fields.

An important special case of a knowledge structure arises when the family of states is a learning space.

2.2 Axioms for Learning Spaces

2.2.1 Definition. A knowledge structure $(Q, \mathcal{K})$ is called a *learning space* if it satisfies the two following conditions.

[L1] LEARNING SMOOTHNESS. For any two states K, L such that $K \subset L$, there exists a finite chain of states

$$K = K_0 \subset K_1 \subset \cdots \subset K_p = L \tag{2.2}$$

such that $|K_i \setminus K_{i-1}| = 1$ for $1 \leq i \leq p$ and so $|L \setminus K| = p$.

Intuitively, in pedagogical language: *If the state K of the learner is included in some other state L then the learner can reach state L by mastering the missing items one at a time.*

In the sequel, we refer to a chain (2.2) as an *L1–chain* from K to L.

[L2] LEARNING CONSISTENCY. If K, L are two states satisfying $K \subset L$ and q is an item such that $K + \{q\} \in \mathcal{K}$, then $L \cup \{q\} \in \mathcal{K}$.

In short: *Knowing more does not prevent learning something new.*

Notice that any learning space is finite. Indeed, Condition [L1] applied to the two states $\varnothing$ and Q implies that Q is a finite set.

From the pedagogical standpoint of Example 2.1.1, both of these axioms seem sensible. This mathematical object occurs in another field, however. In the combinatoric literature, a learning space is sometimes referred to as an 'antimatroid', which is then typically defined by different (but equivalent) axioms (e.g. Korte, Lovász, and Schrader, 1991). As we shall see in Definition 2.2.2, this structure is a family of sets closed under union and satisfying a particular 'accessibility' condition. Originally, however, the label 'antimatroid' was attached to their dual structures, namely, to families closed under intersection (see in particular Edelman and Jamison, 1985; Welsh, 1995; Björner, Las Vergnas, Sturmfels, White, and Ziegler, 1999). As the dates of the last two references suggest, this dual usage is still current. For an overview of the origins and the many avatars of the 'antimatroid' concept, we refer the reader to Monjardet (1985).

Our next definition formalizes various properties ensuing from Axioms [L1] and [L2]. We have encountered two of them earlier, namely, closure under union and wellgradedness (cf. 1.1.6 and 1.1.7). Another one is that of a '(union-closed) antimatroid.' Theorem 2.2.4 spells out the relationships among all these concepts.

2.2.2 Definition. A family $\mathcal{K}$ is *closed under union* when $\cup\mathcal{F} \in \mathcal{K}$ whenever $\mathcal{F} \subseteq \mathcal{K}$. This implies $\varnothing \in \mathcal{K}$, because by convention the union of the empty subfamily is the empty set. When a family is closed under union, we will sometimes say for short that it is *union-closed*, or even $\cup$-*closed*. When the family $\mathcal{K}$ of a knowledge structure $(Q, \mathcal{K})$ is union-closed, we call $(Q, \mathcal{K})$ a *knowledge space*; we may also say equivalently, that $\mathcal{K}$ is a *knowledge space*.

On occasion, we also use the phrase *closed under finite union*. When applied to a family $\mathcal{K}$, it means that for any K and L in $\mathcal{K}$, the set $K \cup L$ is also in $\mathcal{K}$. Note that, in such a case, the empty set does not necessarily belong to the family $\mathcal{K}$.

The *dual* of a knowledge structure $\mathcal{K}$ on Q is the knowledge structure $\overline{\mathcal{K}}$ containing all the complements of the states of $\mathcal{K}$, that is, the family

$$\overline{\mathcal{K}} = \{K \in 2^Q \,|\, Q \setminus K \in \mathcal{K}\}.$$

Thus, $\mathcal{K}$ and $\overline{\mathcal{K}}$ have the same domain. It is clear that if $\mathcal{K}$ is a knowledge space, then $\overline{\mathcal{K}}$ is an *intersection-closed* knowledge structure, that is, $\cap\mathcal{F} \in \overline{\mathcal{K}}$ whenever $\mathcal{F} \subseteq \overline{\mathcal{K}}$, with $\varnothing, Q \in \overline{\mathcal{K}}$.

We recall (from 1.6.12) that the canonical distance d between two finite sets A and B is defined by counting the number of elements in their symmetric difference $A \triangle B$:

$$d(A, B) = |A \triangle B| = |(A \setminus B) \cup (B \setminus A)|. \tag{2.3}$$

A family of sets $\mathcal{F}$ is *well-graded* or a *wg-family* if, for any two distinct sets $K, L \in \mathcal{F}$, there is a finite sequence of states $K = K_0, K_1, \ldots, K_p = L$ such that $d(K_{i-1}, K_i) = 1$ for $1 \leq i \leq p$ and moreover $p = d(K, L)$. We call the sequence of sets (K_i) a *tight path* from K to L. It is clear that a well-graded knowledge structure is finite and discriminative (Problem 2).

A family $\mathcal{K}$ of subsets of a finite set $Q = \cup\mathcal{K}$ is an *antimatroid* if it is closed under union and satisfies the following axiom:

[MA] If K is a nonempty subset of Q belonging to the family $\mathcal{K}$, then there is some q in K such that $K \setminus \{q\} \in \mathcal{K}$.

We call the sets in $\mathcal{K}$ states, and we also say then that the pair $(Q, \mathcal{K})$ is an antimatroid. It is clear that $(Q, \mathcal{K})$ is then a discriminative knowledge structure. A family $\mathcal{K}$ satisfying Axiom [MA] is said to be *accessible* or *downgradable* (see Doble, Doignon, Falmagne, and Fishburn, 2001, for the latter term).

The following result allows us to trim some proofs.

2.2.3 Lemma. *The two following conditions are equivalent for a $\cup$-closed family of sets $\mathcal{K}$:*

(i) *$\mathcal{K}$ is well-graded;*
(ii) *for any two sets K and L such that $K \subset L$, there is a tight path from K to L.*

Proof. It is clear that (i) implies (ii). Suppose that (ii) is true. For any two distinct sets K and L, there exists a tight path $K = K_0 \subset K_1 \subset \cdots \subset K_q = K \cup L$ and another tight path $L = L_0 \subset L_1 \subset \cdots \subset L_p = K \cup L$. Reversing the order of the sets in the latter tight path and redefining $K_{q+1} = L_{p-1}$, $K_{q+2} = L_{p-2}$, ..., $K_{q+p} = L = L_0$, we get the tight path $K = K_0$, K_1, ..., $K_{q+p} = L$, with $|K \triangle L| = q + p$. □

When applied to knowledge structures, the wellgradedness property is a strengthening of [L1]: any L1–chain is a special kind of tight path.

All three of the conditions introduced in Definition 2.2.2—$\cup$-closure, well-gradedness, and accessibility—hold in any learning space. In fact, we have the following result.

2.2.4 Theorem. *For any knowledge structure $(Q, \mathcal{K})$, the following three conditions are equivalent.*

(i) $(Q, \mathcal{K})$ *is a learning space.*
(ii) $(Q, \mathcal{K})$ *is an antimatroid.*
(iii) $(Q, \mathcal{K})$ *is a well-graded knowledge space.*

The equivalence of (i) and (iii) was established by Cosyn and Uzun (2009). Clearly, under each of the three conditions, the knowledge structure $(Q, \mathcal{K})$ is discriminative. Note in passing that this result also holds under a substantially weaker form of Axiom [MA] (cf. Condition (iii) in Theorem 5.4.1)[2].

Proof. (i) $\Rightarrow$ (ii). Suppose that $(Q, \mathcal{K})$ is a learning space. Thus, Q is necessarily finite. Axiom [MA] results immediately from the fact that, for any state K, there is an L1-chain from $\varnothing$ to K. Turning to the $\cup$-closure, we take any two states K and L in $\mathcal{K}$ and suppose that neither of them is empty or a subset of the other (otherwise $\cup$-closure holds trivially). Since $\varnothing \subset L$, Axiom [L1] implies the existence of an L1–chain $\varnothing \subset \{q_1\} \subset \cdots \subset \{q_1, \ldots, q_n\} = L$.

Let $j \in \{1, \ldots, n\}$ be the first index with $q_j \notin K$. If $j > 1$; we have

$$\{q_1, \ldots, q_{j-1}\} \subset K, \text{ and } \{q_1, \ldots, q_{j-1}\} + \{q_j\} \in \mathcal{K}. \tag{2.4}$$

By Axiom [L2] and with $q_j \in L$, we get $K + \{q_j\} \in \mathcal{K}$ with $K + \{q_j\} \subseteq K \cup L$. A similar argument applies with $\varnothing \subset K$ in (2.4) if $j = 1$. Applying induction yields $K \cup L \in \mathcal{K}$.

(ii) $\Rightarrow$ (iii). Only the wellgradedness must be established. We use Lemma 2.2.3. Take any two states K, L with $K \subset L$ (with possibly $K = \varnothing$). Repeated applications of Axiom [MA] to state L gives us a sequence of states $L_0 = L, L_1, \ldots, L_k = \varnothing$ such that $q_{i-1} \in L_{i-1}$ and $L_i = L_{i-1} \setminus \{q_{i-1}\}$ for $i = 1, \ldots, k$. Let j be the largest index such that $q_j \notin K$ (there must be such an index since $K \subset L$). We obtain $K \subset K \cup \{q_j\} = K \cup L_j \subseteq L$. Replacing K with $K \cup \{q_j\}$ and using induction we see that the condition in Lemma 2.2.3 is satisfied, and so the wellgradedness of $(Q, \mathcal{K})$.

[2] We give yet another characterization of learning spaces in Theorem 11.5.3.

(iii) $\Rightarrow$ (i). Axiom [L1] results from the wellgradedness condition. Suppose that $K \subset L$ for two states K and L and that $K + \{q\}$ is also a state. By $\cup$–closure, the set $(K + \{q\}) \cup L = L \cup \{q\}$ is also a state; so, [L2] holds. $\square$

2.2.5 Remarks. The concept of a well-graded knowledge space was investigated by Falmagne and Doignon (1988b). In the early stage of our work, knowledge spaces were at the focus of our developments. From a pedagogical standpoint, they were motivated by the following argument.

Consider the case of two students engaged in extensive interactions for a long time, and suppose that their initial knowledge states with respect to a particular body of information are K and L. At some point, one of these students could conceivably have acquired the joint knowledge of both. The knowledge state of this student would then be $K \cup L$. Obviously, there is no certainty that this will happen. However, requiring the existence of a state in the structure to cover this case may be reasonable.

Some may find such an argument only moderately convincing. As for the wellgradedness condition, its *a priori* justification is far from obvious. Yet, the two conditions are equivalent to [L1]-[L2]. In fact, both the $\cup$-closed and wellgradedness conditions do play critical roles, but their pedagogical imports are subtle. We will see in Chapter 3 that the $\cup$-closed condition makes it possible to summarize any knowledge space by its 'base'[3], which is a typically much smaller subfamily of the knowledge space. In view of the very large size of the knowledge structures encountered in practice, this feature is precious because it facilitates computation. As for the wellgradedness condition, it guarantees that any state can be faithfully represented by its two 'fringes' which are also comparatively much smaller sets[4] (cf. Theorem 4.1.7).

We devote Chapters 3 and 4 to a detailed discussion of knowledge spaces and well-graded knowledge structures, respectively.

As a preparation for our next section, a weakening of some of the concepts of Theorem 2.2.4 is in order.

2.2.6 Definition. A family $\mathcal{F}$ of subsets of a nonempty set Q is a *partial knowledge structure* if it contains the set $Q = \cup\mathcal{F}$. The discriminative concept introduced in Definition 2.1.5 also applies in the partial case. We do not assume that $|\mathcal{F}| \geq 2$. We also call the sets in $\mathcal{F}$ *states*. A partial knowledge structure $\mathcal{F}$ is a *partial learning space* if it satisfies Axioms [L1] and [L2]. A family $\mathcal{F}$ is *partially $\cup$-closed* if for any nonempty subfamily $\mathcal{G}$ of $\mathcal{F}$, we have $\cup\mathcal{G} \in \mathcal{F}$. (Contrary to the $\cup$-closure condition, partial $\cup$-closure does not imply that the empty set belongs to the family.) A *partial knowledge space* $\mathcal{F}$ is a partial knowledge structure that is partially $\cup$-closed.

[3] This concept is defined and investigated in Section 3.4.

[4] In practical applications of the concept of a learning space, the fringes summarize—without loss of information—a knowledge state in a way that may be more meaningful, for the teacher and the student, than a full listing of all the items mastered.

The equivalence (i) $\Leftrightarrow$ (iii) in Theorem 2.2.4 ceases to hold in the case of partial structures. Indeed, we have the following result.

2.2.7 Lemma. *Any well-graded partially $\cup$-closed family is a partial learning space. The converse implication is false.*

PROOF. Let $\mathcal{K}$ be a well-graded partially $\cup$-closed family. Axiom [L1] is a special case of the wellgradedness condition. If $K \subset L$ for two sets K and L in $\mathcal{K}$ and $K + \{q\}$ is in $\mathcal{K}$, then the set $(K + \{q\}) \cup L = L \cup \{q\}$ is in $\mathcal{K}$ by partial $\cup$–closure, and so [L2] holds. The example below disproves the converse. $\square$

2.2.8 Example. The family of sets

$$\mathcal{L} = \{\{a\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$$

is a partial learning space. It is the union of the two chains

$$\{a\} \subset \{a, b\} \subset \{a, b, c\}, \qquad \{c\} \subset \{b, c\} \subset \{a, b, c\}$$

with $\cup\mathcal{L}$ as the only common state. However, $\mathcal{L}$ is neither $\cup$-closed nor well-graded. The knowledge structure $\mathcal{L}' = \{\varnothing\} \cup \mathcal{L}$ does not satisfy [L1] since we have $\varnothing \subset \{a\}$ with $\varnothing + \{c\}$ as a state of $\mathcal{L}'$, but $\{a\} \cup \{c\}$ is not a state.

2.3 The nondiscriminative case*

The axiomatics given in the previous section for learning spaces and well-graded knowledge spaces imply that their models are always discriminative knowledge structures (in the sense of Definition 2.1.5). It is straightforward to adapt the axioms so as to cover nondiscriminative structures. We state here the modified axioms and review some of their consequences, without going into much detail.

2.3.1 Definition. In the case of structures that may not be discriminative, we must modify the concept of distance between two states in the structure. Rather than counting the number of items by which two states differ, we count here the number of notions. Remember from 2.1.5 that q^* denotes the notion containing the item q, and that for any state K, we set $K^* = \{q^* \,|\, q \in K\}$.

Suppose that $(Q, \mathcal{K})$ is an essentially finite knowledge structure. Let K and L be two states in $(Q, \mathcal{K})$. The *essential distance* between K and L is defined by

$$e(K, L) = |K^* \triangle L^*|.$$

We can verify that the function $e : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ is a distance in the usual sense (cf. 1.6.12).

2.3.2 Definition. A knowledge structure $(Q, \mathcal{K})$ is called a *quasi learning space* if it satisfies the two following conditions.

[L1*] QUASI LEARNING SMOOTHNESS. For any two states K, L such that $K \subset L$ there exists a chain of $1 + p$ states

$$K = K_0 \subset K_1 \subset \cdots \subset K_p = L \tag{2.5}$$

with $p = e(K, L)$ and $K_i = K_{i-1} + \{q_i^*\}$ for some $q_i \in Q$, $1 \leq i \leq p$.

In the sequel, we refer to a chain (2.2) as a *quasi L1–chain* from K to L.

[L2*] QUASI LEARNING CONSISTENCY. If K, L are two states satisfying $K \subset L$ and q is an item such that $K + \{q^*\} \in \mathcal{K}$, then $L \cup \{q^*\} \in \mathcal{K}$.

Our next definition introduces a nondiscriminative variant of the wellgradedness condition.

2.3.3 Definition. A family of sets $\mathcal{F}$ is *quasi well-graded* or a *qwg-family* if, for any two distinct states $K, L \in \mathcal{F}$, there exists a finite sequence of states $K = K_0$, K_1, $\ldots$, $K_p = L$ such that $e(K_{i-1}, K_i) = 1$ for $1 \leq i \leq p$ and moreover $p = e(K, L)$. We call the sequence of sets (K_i) a *quasi tight path* from K to L. A quasi well-graded knowledge structure is essentially finite (Problem 16).

We leave to the reader the verification of the following result, which extends the equivalence (i) $\Leftrightarrow$ (iii) in Theorem 2.2.4 (see Problem 9) .

2.3.4 Theorem. *For any knowledge structure* $(Q, \mathcal{K})$*, the following two conditions are equivalent.*

(i) $(Q, \mathcal{K})$ *is a quasi learning space.*
(ii) $(Q, \mathcal{K})$ *is a quasi well-graded knowledge space.*

We will not pursue here the extension of the theory to structures which may not be discriminative. In any event, discriminative reduction is always at hand to generate a discriminative structure from a nondiscriminative one.

2.4 Projections

As we argued before, an empirical learning space can be very large, numbering millions of states. The concept of a 'projection' discussed in this section provides a way of parsing such a large structure into meaningful components. Moreover, when the learning space concerns a scholarly curriculum such as high school algebra, a projection may provide a convenient instrument for the programming of a placement test.

The key idea is that if $\mathcal{K}$ is a learning space on a domain Q, then any proper subset Q' of Q defines a learning space $\mathcal{K}_{|Q'}$ on Q' which is in some sense consonant with $\mathcal{K}$. We call $\mathcal{K}_{|Q'}$ a 'projection' of $\mathcal{K}$ on Q', a terminology consistent with that used for media by Cavagnaro (2008) and Eppstein, Falmagne, and Ovchinnikov (2008). (We discuss the relationship between media and learning spaces in Chapter 10.) Moreover, this construction defines a partition of $\mathcal{K}$ such that each equivalence class is a subfamily of $\mathcal{K}$ satisfying two key properties of a learning space, namely wellgradedness and $\cup$-closure. Actually, it is possible to choose Q' so that each of these classes is essentially (via a trivial transformation) either a learning space consistent with $\mathcal{K}$ or the singleton $\{\varnothing\}$. These results, which are mostly due to Falmagne (2008), are presented in this section.

2.4.1 Definition. Suppose that $(Q, \mathcal{K})$ is a partial knowledge structure with $|Q| \geq 2$, and let Q' be any proper nonempty subset of Q. Define a relation $\sim_{Q'}$ on $\mathcal{K}$ by

$$K \sim_{Q'} L \iff K \cap Q' = L \cap Q' \tag{2.6}$$

$$\iff K \triangle L \subseteq Q \setminus Q'. \tag{2.7}$$

Thus, $\sim_{Q'}$ is an equivalence relation on $\mathcal{K}$. When the context specifies the subset Q', we sometimes use the shorthand $\sim$ for $\sim_{Q'}$ in the sequel. The equivalence between the right hand sides of (2.6) and (2.7) is easily verified (cf. Problem 11). We denote by $[K]$ the equivalence class of $\sim$ containing K, and by $\mathcal{K}_\sim = \{[K] \,|\, K \in \mathcal{K}\}$ the partition of $\mathcal{K}$ induced by $\sim$. We may also say for short that such a partition is *induced* by the set Q'. In the sequel, we always assume that $|Q| \geq 2$, so that $|Q'| \geq 1$.

2.4.2 Definition. Let $(Q, \mathcal{K})$ be a partial knowledge structure and take any nonempty proper subset Q' of Q. The family

$$\mathcal{K}_{|Q'} = \{W \subset Q' \,|\, W = K \cap Q' \text{ for some } K \in \mathcal{K}\} \tag{2.8}$$

is called the *projection* of $\mathcal{K}$ on Q'. We thus have $\mathcal{K}_{|Q'} \subseteq 2^{Q'}$. Depending on the context, we may also refer to $\mathcal{K}_{|Q'}$ as a *substructure* of $\mathcal{K}$. Each set $W = K \cap Q'$ with $K \in \mathcal{K}$ is called the *trace* of the state K on Q'. Example 2.4.3 shows that the sets in $\mathcal{K}_{|Q'}$ may not be states of $\mathcal{K}$. For any state K in $\mathcal{K}$ and with $[K]$ as in Definition 2.4.1, we define the family

$$\mathcal{K}_{[K]} = \{M \subseteq Q \,|\, M = L \setminus \cap [K] \text{ for some } L \sim K\}. \tag{2.9}$$

(If $\varnothing \in \mathcal{K}$, we thus have $\mathcal{K}_{[\varnothing]} = [\varnothing]$.) The family $\mathcal{K}_{[K]}$ is called a Q'-*child*, or simply a *child* of $\mathcal{K}$ when the set Q' is made clear by the context. As shown by our next example, a child of $\mathcal{K}$ may take the form of the singleton $\{\varnothing\}$ and we may have $\mathcal{K}_{[K]} = \mathcal{K}_{[L]}$ even when $K \not\sim L$. The set $\{\varnothing\}$ is called the *trivial* child. We refer to $\mathcal{K}$ as the *parent* structure.

2.4.3 Example. Consider the learning space

$$\begin{aligned}\mathcal{F} = \{\varnothing, &\{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}, \{b,c,e\},\\ &\{b,d,f\}, \{a,b,c,d\}, \{a,b,c,e\}, \{b,c,d,e\}, \{b,c,d,f\}, \{b,c,e,f\},\\ &\{a,b,d,f\}, \{a,b,c,d,e\}, \{a,b,c,d,f\}, \{a,b,c,e,f\}, \{b,c,d,e,f\},\\ &\{a,b,c,d,e,f\}, \{a,b,c,d,e,f,g\}\}.\end{aligned} \tag{2.10}$$

The domain of this learning space is thus the set $Q = \{a,b,c,d,e,f\}$. The inclusion graph of $\mathcal{F}$ is pictured by the grey parts of the diagram of Figure 2.2.

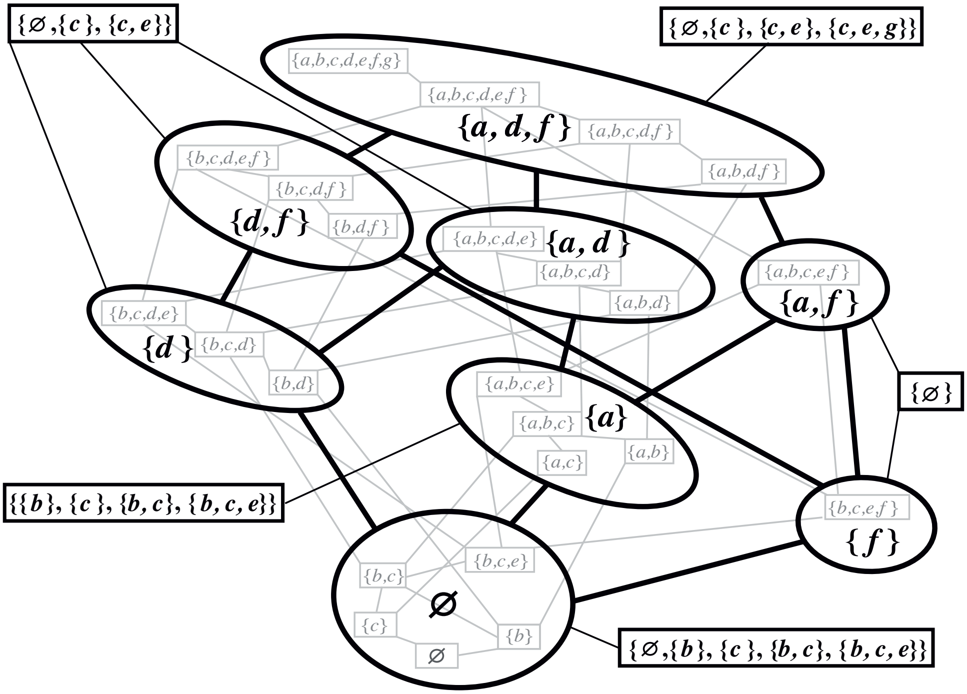

Figure 2.2. In grey, the inclusion graph of the learning space $\mathcal{F}$ of Equation (2.10). Each oval surrounds an equivalence class $[K]$ (in grey) and a particular state (in black) of the projection $\mathcal{F}_{|\{a,d,f\}}$ of $\mathcal{F}$ on $Q' = \{a,d,f\}$, signaling a 1-1 correspondence $\mathcal{F}_{\sim} \to \mathcal{F}_{|\{a,d,f\}}$ (cf. Lemma 2.4.5(ii)). Via the defining equation (2.9), the eight equivalence classes produce five children of $\mathcal{F}$, which are represented in the five black rectangles at the edge of the figure. One of these children is the singleton set $\{\varnothing\}$ (thus, the trivial child), and the others are learning spaces or partial learning spaces (cf. the projection Theorems 2.4.8 and 2.4.12).

The sets marked in black in the eight ovals of the figure represent the states of the projection $\mathcal{F}_{|\{a,d,f\}}$ of $\mathcal{F}$ on the set $\{a,d,f\}$. It is clear that $\mathcal{F}_{|\{a,d,f\}}$ is a learning space[5]. Each of these ovals also surrounds the inclusion subgraph corresponding to an equivalence class of the partition $\mathcal{F}_{\sim}$. This is consistent with Lemma 2.4.5(ii) below, according to which there is a 1-1 correspondence between $\mathcal{F}_{\sim}$ and $\mathcal{F}_{|\{a,d,f\}}$. In this example, the 'learning space' property is transmitted to the children: not only is $\mathcal{F}_{|\{a,d,f\}}$ a learning space, but also any child of $\mathcal{F}$ is a learning space or a partial learning space. Indeed, we have

$$\begin{gathered}
\mathcal{F}_{[\{b,c,e\}]} = \{\varnothing, \{b\}, \{c\}, \{b,c\}, \{b,c,e\}\},\\
\mathcal{F}_{[\{a,b,c,e\}]} = \{\{b\}, \{c\}, \{b,c\}, \{b,c,e\}\},\\
\mathcal{F}_{[\{b,c,d,e\}]} = \mathcal{F}_{[\{b,c,d,e,f\}]} = \mathcal{F}_{[\{a,b,c,d,e\}]} = \{\varnothing, \{c\}, \{c,e\}\},\\
\mathcal{F}_{[\{a,b,c,d,e,f,g\}]} = \{\varnothing, \{c\}, \{c,e\}, \{c,e,g\}\}\\
\mathcal{F}_{[\{b,c,e,f\}]} = \mathcal{F}_{[\{a,b,c,e,f\}]} = \{\varnothing\}.
\end{gathered}$$

These five children are represented in the five black rectangles in Figure 2.2.

Theorem 2.4.8 shows that wellgradedness is inherited by the children of a learning space. These children are also partially $\cup$-closed. In the particular case of this example, just adding the set $\varnothing$ to the child not containing it already, that is, to the child $\mathcal{F}_{[\{a,b,c,e\}]}$, would result in all the children being learning spaces or trivial. This is **not** generally true. The situation is clarified by Theorem 2.4.12.

2.4.4 Remark. The concept of projection for learning spaces is closely related to the concept bearing the same name for media introduced by Cavagnaro (2008). The Projection Theorems 2.4.8 and 2.4.12, the main results of this section, could be derived via similar results concerning the projections of media (cf. Theorem 2.11.6 in Eppstein et al., 2008). This would be a detour, however. The route followed here is direct.

In the next two lemmas, we derive a few consequences of Definition 2.4.2.

2.4.5 Lemma. *The following two statements are true for any partial knowledge structure* $(Q,\mathcal{K})$.

(i) *The projection* $\mathcal{K}_{|Q'}$, *with* $Q' \subset Q$, *is a partial knowledge structure. If* $(Q,\mathcal{K})$ *is a knowledge structure, then so is* $\mathcal{K}_{|Q'}$.

(ii) *The function* $h : [K] \mapsto K \cap Q'$ *is a well defined bijection of* $\mathcal{K}_{\sim}$ *onto* $\mathcal{K}_{|Q'}$.

Proof. (i) Both statements follow from $\varnothing \cap Q' = \varnothing$ and $Q \cap Q' = Q'$.

(ii) That h is a well defined function is due to (2.6). Clearly, $h(\mathcal{K}_{\sim}) = \mathcal{K}_{|Q'}$ by the definitions of h and $\mathcal{K}_{|Q'}$. Suppose that, for some $[K],[L] \in \mathcal{K}_{\sim}$, we have $h([K]) = K \cap Q' = h([L]) = L \cap Q' = X$. Whether or not $X = \varnothing$, this entails $K \sim L$ and so $[K] = [L]$. □

---

[5] This property holds in general. Notice that we have here the special case in which $\mathcal{F}_{|\{a,d,f\}}$ is the power set of $\{a,d,f\}$. However, it is not generally true that for any learning space $(Q,\mathcal{K})$ and $Q' \subset Q$, we have $\mathcal{K}_{|Q'} = 2^{Q'}$.

2.4.6 Lemma. *Let $\mathcal{K}$ be any $\cup$-closed family, with $Q = \cup\mathcal{K}$ not necessarily in $\mathcal{K}$, and take any $Q' \subset Q$. The following three statements are then true.*

(i) $K \sim_{Q'} \cup[K]$ *for any* $K \in \mathcal{K}$*;*

(ii) $\mathcal{K}_{|Q'}$ *is a $\cup$-closed family. If $\mathcal{K}$ is a knowledge space, so is $\mathcal{K}_{|Q'}$.*

(iii) *The children of $\mathcal{K}$ are also partially $\cup$-closed.*

For knowledge spaces, Lemma 2.4.6(ii) was obtained by Doignon and Falmagne (1999, Theorem 1.16 on p. 25, in which the term 'substructure is used instead of 'projection').

Proof. (i) As $\cup[K]$ is the union of states of $\mathcal{K}$, we get $\cup[K] \in \mathcal{K}$. We must have $K \cap Q' = (\cup[K]) \cap Q'$ because $K \cap Q' = L \cap Q'$ for all $L \in [K]$; so $K \sim \cup[K]$.

(ii) Any subfamily $\mathcal{H}' \subseteq \mathcal{K}_{|Q'}$ is associated to the family

$$\mathcal{H} = \{H \in \mathcal{K} \,|\, H' = H \cap Q' \text{ for some } H' \in \mathcal{H}'\}.$$

As $\mathcal{K}$ is $\cup$-closed, we have $\cup\mathcal{H} \in \mathcal{K}$, yielding $Q' \cap (\cup\mathcal{H}) = \cup\mathcal{H}' \in \mathcal{K}_{|Q'}$. If $\mathcal{K}$ is a knowledge space, then $Q \in \mathcal{K}$, which implies $Q' \in \mathcal{K}_{|Q'}$. Thus $\mathcal{K}_{|Q'}$ is a knowledge space.

(iii) Take $K \in \mathcal{K}$ arbitrarily. We must show that $\mathcal{K}_{[K]}$ is $\cup$-closed. For any $\mathcal{H} \subseteq \mathcal{K}_{[K]}$ we define the associated family

$$\mathcal{H}^\dagger = \{H^\dagger \in \mathcal{K} \,|\, H^\dagger \sim K,\ H^\dagger \setminus \cap[K] \in \mathcal{H}\}.$$

So, $\mathcal{H}^\dagger \subseteq [K]$, which gives $L \cap Q' = K \cap Q'$ for any $L \in \mathcal{H}^\dagger$. Since $\mathcal{K}$ is $\cup$-closed, we have $\cup\mathcal{H}^\dagger \in \mathcal{K}$. We thus get $(\cup\mathcal{H}^\dagger) \cap Q' = K \cap Q'$ and $\cup\mathcal{H}^\dagger \sim K$.

The $\cup$-closure of $\mathcal{K}_{[K]}$ follows from the string of equalities

$$\cup\mathcal{H} = \cup_{H^\dagger \in \mathcal{H}^\dagger}(H^\dagger \setminus \cap[K]) = \cup_{H^\dagger \in \mathcal{H}^\dagger}(H^\dagger \cap (\overline{\cap[K]}) = (\cup_{H^\dagger \in \mathcal{H}^\dagger} H^\dagger) \setminus \cap[K]$$

which gives $\cup\mathcal{H} \in \mathcal{K}_{[K]}$ because $K \sim \cup\mathcal{H}^\dagger \in \mathcal{K}$.

Example 2.4.7 shows that the reverse implications in (ii) and (iii) do not hold. □

2.4.7 Example. Consider the projection of the knowledge structure

$$\mathcal{G} = \big\{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\big\},$$

on the subset $\{c\}$. We thus have the two equivalence classes $[\{a, b\}]$ and $[\{a, b, c\}]$, with the projection $\mathcal{G}_{|\{c\}} = \{\varnothing, \{c\}\}$. The two $\{c\}$-children are $\mathcal{G}_{[\varnothing]} = \big\{\varnothing, \{a\}, \{b\}, \{a, b\}\big\}$ and $\mathcal{G}_{[\{c\}]} = \big\{\varnothing, \{b\}\big\}$. Both $\mathcal{G}_{[\varnothing]}$ and $\mathcal{G}_{[\{c\}]}$ are well-graded and $\cup$-closed, and so is $\mathcal{G}_{|\{c\}}$. However, $\mathcal{G}$ is not $\cup$-closed since $\{b, c\}$ is not a state.

We state the first of our two projection theorems.

2.4.8 Theorem. *Let $(Q, \mathcal{K})$ be a learning space, with $|Q| = |\cup \mathcal{K}| \geq 2$. The following two properties hold for any proper nonempty subset Q' of Q.*

(i) *The projection $\mathcal{K}_{|Q'}$ of $\mathcal{K}$ on Q' is a learning space.*
(ii) *The children of $\mathcal{K}$ are well-graded and partially $\cup$-closed families.*

Note that we may have $\mathcal{K}_{[K]} = \{\varnothing\}$ in (ii) (cf. Example 2.4.3).

PROOF. (i) Since is a learning space, $\mathcal{K}_{|Q'}$ is a knowledge structure by Lemma 2.4.5(i). We prove that Axiom [L1] holds for $\mathcal{K}_{|Q'}$. Assume K, $L \in \mathcal{K}_{|Q'}$ with $K \subset L$. Then, there exist $\tilde{K}$ and $\tilde{L}$ in $\mathcal{K}$ such that $K = \tilde{K} \cap Q'$ and $L = \tilde{L} \cap Q'$. As $\mathcal{K}$ is a learning space, there is a $L1$–chain from $\tilde{K}$ to $\tilde{K} \cup \tilde{L}$, say $\tilde{K} = K_0, K_1, \ldots, K_q = \tilde{K} \cup \tilde{L}$. Then $K = K_0 \cap Q', K_1 \cap Q', \ldots, K_q \cap Q' = L$ yields a $L1$–chain from K to L in $\mathcal{K}_{[K]}$ after deleting from the sequence any set identical to a previous set. Axiom [L2] also holds for $\mathcal{K}_{|Q'}$. Indeed, take K, $L \in \mathcal{K}_{|Q'}$ and $q \in Q'$ with $K \subset L$ and $K \cup \{q\} \in \mathcal{K}_{|Q'}$. There exist $\tilde{K}$, $\tilde{L}$, M in $\mathcal{K}$ such that $K = \tilde{K} \cap Q'$, $L = \tilde{L} \cap Q'$ and $K \cup \{q\} = M \cap Q'$. So, we have $L \cup \{q\} = (\tilde{L} \cup M) \cap Q'$, thus $L \cup \{q\} \in \mathcal{K}_{|Q'}$.

(ii) Take any child $\mathcal{K}_{[K]}$ of $\mathcal{K}$. By Lemma 2.4.6(iii), $\mathcal{K}_{[K]}$ is a partially $\cup$-closed family. Axiom [L1] and the argument in the proof of Lemma 2.2.3 imply that $[K]$ is well-graded. The wellgradeness of $\mathcal{K}_{[K]}$ follows easily. □

2.4.9 Remark. In Example 2.4.3, we had a situation in which the non trivial children of a learning space were either themselves learning spaces, or would become so by the addition of the set $\{\varnothing\}$. This happens if and only if the subset Q' of the domain defining the projection satisfies the condition spelled out in the next definition.

2.4.10 Example. Take the learning space

$$\mathcal{K} = \big\{\varnothing, \{a\}, \{d\}, \{a,b\}, \{a,d\}, \{c,d\}, \\ \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\big\},$$

with domain $Q = \{a,b,c,d\}$. We set $Q' = \{c\}$ and $K = \{c,d\}$. Then

$$[K] = \big\{\{c,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\big\}$$

and, as $\cap[K] = \{c\}$,

$$\mathcal{K}_{[K]} = \big\{\{d\}, \{a,b\}, \{a,d\}, \{b,d\}, \{a,b,d\}\big\}.$$

Clearly, the child $\mathcal{K}_{[K]}$ is not a learning space and even $\mathcal{K}_{[K]} \cup \{\varnothing\}$ is not: for instance, there is no tight path from $\varnothing$ to $\{a,b\}$. The reason lies in a feature of $[K]$: the element $\{a,b,c\}$ is a minimal element of $[K]$ covering $\cap[K]$ while at the same time $\{a,b,c\} \setminus \cap[K]$ contains more than one element.

2.4.11 Definition. Suppose that $(Q, \mathcal{K})$ is a partial knowledge structure, with $|Q| \geq 2$. A subset $Q' \subset Q$ is *yielding* if for any state L of $\mathcal{K}$ that is minimal for inclusion in some equivalence class $[K]$, we have $|L \setminus \cap[K]| \leq 1$. We recall that $[K]$ is the equivalence class containing K in the partition of $\mathcal{K}$ induced by Q' (cf. Definition 2.4.1). For any non trivial child $\mathcal{K}_{[K]}$ of $\mathcal{K}$, we call $\mathcal{K}^+_{[K]} = \mathcal{K}_{[K]} \cup \{\varnothing\}$ a *plus child* of $\mathcal{K}$.

2.4.12 Theorem. *Suppose that $(Q, \mathcal{K})$ is a learning space with $|Q| \geq 2$, and let Q' be a proper nonempty subset of Q. The two following conditions are then equivalent.*

(i) *The set Q' is yielding.*
(ii) *All the plus children of $\mathcal{K}$ are learning spaces*[6].

Problem 13 asks the reader to investigate whether any learning space always has at least one non trivial child.

PROOF. (i) $\Rightarrow$ (ii). By Lemma 2.4.6(iii), we know that any non trivial child $\mathcal{K}_{[K]}$ is $\cup$-closed. This implies that the associated plus child $\mathcal{K}^+_{[K]}$ is a knowledge space. We use Lemma 2.2.3 to prove that such a plus child is also well-graded. Suppose that L and M are states of $\mathcal{K}^+_{[K]}$, with $\varnothing \subseteq L \subset M$ and, for some positive integer n, $d(L, M) = n$. We have two cases.

CASE 1. Suppose that $L \neq \varnothing$. Then both L and M are in $\mathcal{K}_{[K]}$. As $\mathcal{K}_{[K]}$ is well-graded by Theorem 2.4.8(ii), there exists a tight path

$$L = L_0 \subset L_1 \subset \cdots \subset L_n = M.$$

Since $\varnothing \subset L_0$, this tight path lies entirely in the plus child $\mathcal{K}^+_{[K]}$.

CASE 2. Suppose now that $L = \varnothing$. In view of what we just proved, we only have to show that, for any nonempty $M \in \mathcal{K}^+_{[K]}$, there is a singleton set $\{q\} \in \mathcal{K}^+_{[K]}$ with $q \in M$. By definition of $\mathcal{K}^+_{[K]}$, we have $M = M^\dagger \setminus \cap[K]$ for some $M^\dagger \in [K]$. Take a minimal state N in $[K]$ such that $N \subseteq M^\dagger$ and so $N \setminus \cap[K] \subseteq M$. Since Q' is yielding, we get $|N \setminus \cap[K]| \leq 1$. If $|N \setminus \cap[K]| = 1$, then $N \setminus \cap[K] = \{q\} \subseteq M$ for some $q \in Q$ with $\{q\} \in \mathcal{K}^+_{[K]}$. Suppose that $|N \setminus \cap[K]| = 0$. Thus $N \setminus \cap[K] = \varnothing$ and N must be the only minimal set in $[K]$, which implies that $\cap[K] = N$. By the wellgradedness of $\mathcal{K}$, there exists some $q \in M^\dagger$ such that $M^\dagger \supseteq N + \{q\} \in \mathcal{K}$. We have in fact $N + \{q\} \in [K]$ since $q \in M \setminus N$ implies $N \cap Q' = (N + \{q\}) \cap Q'$. We thus get

$$(N + \{q\}) \setminus \cap[K] = (N + \{q\}) \setminus N = \{q\} \subseteq M \quad \text{with} \quad \{q\} \in \mathcal{K}^+_{[K]}.$$

We have proved that in both cases, the tight path from L to M exists. The plus child $\mathcal{K}^+_{[K]}$ is thus well-graded. Applying Theorem 2.2.4, we conclude that $\mathcal{K}^+_{[K]}$ is a learning space.

[6] Note that we may have $\varnothing \in \mathcal{K}_{[K]}$, in which case $\mathcal{K}^+_{[K]} = \mathcal{K}_{[K]}$ (cf. Example 2.4.3).

(ii) $\Rightarrow$ (i). Let L be a minimal element in the equivalence class $[K]$, where $K \in \mathcal{K}$. Then $\cap[K] \subseteq L$. If equality holds, we have $|L \setminus \cap[K]| = 0$. If $\cap[K] \subset L$ holds, then $\varnothing$ and $L \setminus \cap[K]$ are distinct elements in the plus child $\mathcal{K}^+_{[K]}$. By the wellgradedness of $\mathcal{K}^+_{[K]}$, there is a tight path from $\varnothing$ to $L \setminus \cap[K]$ in $\mathcal{K}^+_{[K]}$. Because L is minimal in $[K]$ and distinct from $\cap[K]$, we see that $L \setminus \cap[K]$ must be a singleton. Hence $|L \setminus \cap[K]| = 1$. $\square$

2.4.13 Remark. The theory of learning spaces provides the combinatoric foundation for various knowledge assessment algorithms. As we discussed in 1.1.12, the goal of an assessment algorithm is to uncover the knowledge state of a student by a sequence of well chosen questions. Two quite different classes of stochastic assessment algorithms are described in Chapters 13 and 14. However, empirically constructed learning spaces are typically so large, with knowledge states numbering millions, that a straightforward application of an assessment algorithm is not always feasible. In such cases, the result of this section may be useful. For example, they pave the way to a two-step assessment in a learning space $(Q, \mathcal{K})$ which is serviceable in those cases in which $\mathcal{K}$ is very large. The first step uses a projection $\mathcal{K}_{|Q'}$ on a suitable—in particular, yielding—subset $Q' \subset Q$. This step ends up with a state $W \subseteq Q'$ of the projection $\mathcal{K}_{|Q'}$, with $W = K \cap Q'$ for some $K \in \mathcal{K}$. The second step is an assessment on the Q'-child $\mathcal{K}_{[K]}$ of $\mathcal{K}$, leading to some state $M = L \setminus \cap[K]$ of $\mathcal{K}_{[K]}$, for some state L of $\mathcal{K}$. The state L can then be taken as the final state obtained for the assessment. If the learning space $\mathcal{K}$ is extremely large, an n-phase assessment along these lines is also feasible in principle.

An objection to this procedure is that it does not feature any mechanism permitting a correction, during Step 2, of any assessment error made in Step 1. The state $W = K \cap Q'$ selected by Step 1 is taken for granted and defines the child $\mathcal{K}_{[K]}$. The assessment in the space $\mathcal{K}_{[K]}$ only amounts to selecting one among the states that are $\sim_{Q'}$ equivalent to K. A more flexible procedure is discussed in Section 13.7[7].

2.5 Original Sources and Related Works

As indicated in Chapter 1, the theory of knowledge spaces was initiated by Doignon and Falmagne (1985). Most of the early work was focused on the axiom of closure under union, in the finite case. From a pedagogical standpoint, there were some weaknesses in this approach. For one thing, the $\cup$-closure condition may not be convincing for an educator, at least *a priori*. For another, the state resulting from an assessment in a knowledge space has no natural,

[7] Another approach would rely on having two or more assessments pursued simultaneously rather that successively, with possible interplay between them. While this possibility is intriguing, we do not expand on this idea here.

economical representation in a style that would be useful to the teacher or the student.

The wellgradedness condition was introduced by Falmagne and Doignon (1988b) to palliate the latter defect. Under this condition, a meaningful representation of any knowledge state is feasible in the form of the two 'fringes' of that state (see Definition 4.1.6). The outer fringe spells out the items that the student is ready to learn, and the inner fringe contains all those items signaling the 'high points' in a student's state. However, the resulting concept of a well-graded knowledge space, although mathematically appropriate, was still suffering from a lack of an immediate pedagogical justification.

The axioms [L1] and [L2] axioms were later proposed by Falmagne to Eric Cosyn and Hasan Uzun as offering a more compelling basis for the theory. In a recent paper, Cosyn and Uzun (2009) proved that for a knowledge structure $\mathcal{K}$, the conditions [L1] and [L2] were in fact equivalent to the hypothesis that $\mathcal{K}$ is a well-graded knowledge space. This result is recalled as the equivalence (i) $\Leftrightarrow$ (iii) of Theorem 2.2.4.

The concept of a projection was already present in our original monograph 'Knowledge Spaces' in the form of a 'substructure' of a knowledge structure (see Doignon and Falmagne, 1999, Theorem 1.16 and Definition 1.17). What is new[8] in Section 2.4, which closely follows Falmagne (2008), is the extension of the results to learning spaces, and, most importantly, the analysis of a knowledge structure $\mathcal{K}$ into a projection $\mathcal{K}_{|Q'}$ and its satellite components in the form of the children $\mathcal{K}_{[K]}$ defined in 2.4.2. These results expand earlier work by Cosyn (2002), who also defined a partition of the knowledge structure, which he called 'coarsening.' However, his partition was chosen arbitrarily and did not arise from an equivalence relation defined by (2.6) via a subset Q' of the domain. By contrast, the definition of a projection given by Cavagnaro (2008) is conceptually quite similar to ours, but applies to media, which are semigroups of transformations rather than families of sets. As we pointed out in Remark 2.4.4, a precise connection between media and learning spaces exists, which will be delineated in Chapter 10.

Problems

1. Construct the discriminative reduction of the knowledge structure

$$\mathcal{K} = \big\{\varnothing, \{a, c, d\}, \{b, e, f\}, \{a, c, d, e, f\}, \{a, b, c, d, e, f\}\big\}.$$

2. Verify that any well-graded knowledge structure is discriminative. Why is a well-graded family of sets not necessarily discriminative?

[8] Besides the name 'projection' replacing 'substructure.'

3. Do we have $|\mathcal{K}| = |\mathcal{K}^*|$ for any knowledge structure $\mathcal{K}$? Prove your answer. (The '$*$' notation is as in Definition 2.1.4.)

4. Show that a knowledge structure $(Q, \mathcal{K})$ is essentially finite if and only if Q^* is finite.

5. Construct a drawing representing the knowledge structure $\mathcal{H}$ of Example 2.1.3 in the style of Figure 2.1 for Example 2.1.6.

6. Consider the following axiom generalizing the closure under union.

   [JS] For any subfamily of states $\mathcal{F}$ in a knowledge structure $(Q, \mathcal{K})$, there exists a unique minimal state $K \in \mathcal{K}$ such that $\cup\mathcal{F} \subseteq K$.

   (Under this axiom, $\mathcal{K}$ is thus a 'join semi lattice' with respect to inclusion.) Construct a finite example in which this axiom is not satisfied.

7. Is it true that if one set is finite in a well-graded family, then all sets of that family are finite? Give a proof or a counterexample.

8. Prove that if a knowledge structure $(Q, \mathcal{K})$ is discriminative, then so is its projection $\mathcal{K}_{|Q'}$ on a subset $Q' \subset Q$, but that the converse does not hold.

9. Prove Theorem 2.3.4.

10. Consider the following modification of Axiom [L1] for a knowledge structure $(Q, \mathcal{K})$.

    [L1'] If K and L are two states with $K \subset L \neq Q$, then there is a chain of states $K = K_0 \subset K_1 \subset \cdots \subset K_n = L$ with $K_i = K_{i-1} + \{q_i\}$ for $1 \leq i \leq n$ and $|L \setminus K| = n$.

    Suppose that $(Q, \mathcal{K})$ satisfies [L1'] and [L2]. Could the domain Q be uncountable? Prove your answer.

11. Let $(\mathcal{Q}, \mathcal{K})$ be a knowledge structure and let Q' be any proper subset of Q. With the equivalence classes $[K]$ defined as in 2.4.1, prove the following two statements:

$$K \triangle L \subseteq Q \setminus Q' \iff K \cap Q' = L \cap Q' \tag{2.11}$$

$$(\cap[K]) \cap Q' = K \cap Q'. \tag{2.12}$$

12. Describe the components $\mathcal{F}_{[\{a,b\}]}$ and $\mathcal{F}_{[\varnothing]}$ in the example of Figure 2.2.

13. Is it true that any learning space has at least one non trivial child, either (i) for some subset of the domain; or (ii) for a given subset of the domain (cf. Theorem 2.4.12).

14. Two knowledge structures $(Q, \mathcal{L})$ and $(Q^\dagger, \mathcal{K}^\dagger)$ are *isomorphic* if there exists a 1-1 correspondence $f : Q \to Q^\dagger$ such that for all $K \subseteq Q$, we have $K \in \mathcal{K}$ if and only if $f(K) \in \mathcal{K}^\dagger$. Prove that $(Q, \mathcal{L})$ is a learning space (resp. knowledge space) if and only if $(Q', \mathcal{K}')$ is a learning space (resp. knowledge space).

15. Is a partial learning space necessarily finite? How about a partial knowledge space?

16. Show that a quasi well-graded knowledge structure is essentially finite (cf. 2.3.3).

17. Prove by two counterexamples that the Axioms [L1] and [L2] independent.

18. What knowledge structures satisfy both [L1] and $\cup$-closure?

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