Preface

This book is a much enlarged second edition of “Knowledge Spaces”, by Jean-Paul Doignon and Jean-Claude Falmagne, which appeared in 1999. Chapters 2, 10, 16, 17 and 18 are new, and several of the other chapters have been extensively revised[1]. The reasons for the change of title and the extended content are explained below. As much of our earlier preface remains valid, we reproduce the useful parts here.

The work reported in these pages began during the academic year 1982–83. One of us (JCF) was on sabbatical leave at the University of Regensburg. For various reasons, the time was ripe for a new joint research subject. Our long term collaboration was thriving, and we could envisage an ambitious commitment. We decided to build an efficient machine for the assessment of knowledge—for example, that of students learning a scholarly subject. We began at once to work out the theoretical components of such a machine. Until then, we had been engaged in topics dealing mostly with geometry, combinatorics, psychophysics, and especially measurement theory. The last subject has some bearing on the content of this book. A close look at the foundations of measurement, in the sense that this term has in the physical sciences, may go a long way toward convincing you of its limited applicability. It seemed to us that in many scientific areas, from chemistry to biology and especially the behavioral sciences, theories must often be built on a very different footing than that of classical physics. Evidently, the standard physical scales such as time, mass, or length may always be used in measuring aspects of phenomena. But the substrate proper to these other sciences may very well be of a fundamentally different nature. In short, nineteenth century physics is a bad example. This is not always understood. There was in fact a belief, shared by many nineteenth century scientists, that for an academic endeavour to be called a ‘science’, it had to resemble classical physics in critical ways. In particular, its basic observations had to be quantified in terms of measurement scales in the exact sense of classical physics.

Prominent advocates of that view were Francis Galton, Karl Pearson and William Thomson Kelvin. Because that position is still influential today, with a detrimental effect on fields such as ‘psychological measurement’, which is relevant to our subject, it is worth quoting some opinions in detail. In Pearson’s biography of Galton (Pearson, 1924, Vol. II, p. 345), we can find the following definition:

> “**Anthropometry**, *or the art of measuring the physical and mental faculties of human beings, enables a shorthand description of any individual by measuring a small sample of his dimensions and qualities. These will sufficiently define his bodily proportions, his massiveness,*

[1] The content of the book is summarized in Section 1.5 on page 12.

strength, agility, keenness of sense, energy, health, intellectual capacity and mental character, and will substitute concise and exact **numerical**[2] *values for verbose and disputable estimates*[3]*.”*

For scientists of that era, it was hard to imagine a non-numerical approach to precise study of an empirical phenomenon. Karl Pearson himself, for instance—commenting on a piece critical of Galton’s methods by the editor of the *Spectator*[4]—, wrote

> *“There might be difficulty in ranking Gladstone and Disraeli for “Candour”, but few would question John Morley’s position relative to both of them in this quality. It would require an intellect their equal to rank truly in scholarship Henry Bradshaw, Robertson Smith and Lord Acton, but most judges would place all three above Sir John Seeley, as they would place Seeley above Oscar Browning. After all, there are such things as brackets, which only makes the statistical theory of ranking slightly less simple in the handling*” (Pearson, 1924, Vol. II, p. 345).

In other words, measuring a psychical attribute such as ‘candour’ only requires fudging a bit around the edges of the order relation of the real numbers, making it either, in current terminology, a ‘weak order’ (cf. 1.6.7 in Chapter 1) or perhaps a ‘semiorder’ (cf. Problems 9 and 10 in Chapter 4).

As for Kelvin, his position on the subject is well-known, and often summarized in the form: *“If you cannot measure it, then it is not science.”* (In French: *“Il n’y a de science que du mesurable.”*) The full quotation is:

> *“When you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meager and unsatisfactory kind: it may be the beginning of knowledge, but you are scarcely, in your thoughts, advanced to the stage of* **science**, *whatever the matter may be*” (Kelvin, 1889).

Such a position, which equates precision with the use of numbers, was not on the whole beneficial to the development of mature sciences outside of physics. It certainly had a costly impact on the assessment of mental traits. For instance, for the sake of scientific precision, the assessment of mathematical

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[2] Our emphasis.

[3] This excerpt is from an address on “Anthropometry at Schools” given in 1905 by Galton at the London Congress of the Royal Institute for Preventive Medicine. The text was published in the *Journal of Preventive Medicine*, Vol. XIV, pp. 93–98, London, 1906.

[4] The *Spectator*, May 23, 1874. The editor of the *Spectator* was taking Galton to task for his method of ranking applied to psychical character. He used ‘candour’ and ‘power of repartee’ as examples.

knowledge was superseded in the US by the measurement of mathematical aptitude using instruments directly inspired from Galton via Alfred Binet in France. They are still used today in such forms as the S.A.T.[5], the G.R.E. (*Graduate Record Examination*), and other similar tests.

In the mind of Galton and his followers, the numerical measurement of mental traits was to be a prelude to the establishment of sound, predictive scientific theories in the spirit of those used so successfully in classical physics. The planned constructions, however, never went much beyond the measurement stage[6].

Of course, we are enjoying the benefits of hindsight. In all fairness, there were important mitigating circumstances affecting those who uphold the cause of numerical measurement as a prerequisite to science. For one thing, the appropriate mathematical tools were not yet available for different conceptions. More importantly, the 'Analytical Engine' of Charles Babbage was still a dream, and close to another century had to pass before the appearance of computing machines capable of handling the symbolic manipulations that would be required.

The material of this book represents a sharp departure from other approaches to the assessment of knowledge. Its mathematics is in the spirit of current research in combinatorics. No attempt is made to obtain a numerical representation[7]. We start from the concept of a possibly large but essentially discrete set of 'units of knowledge.' In the case of elementary algebra, for instance, one such unit might be a particular type of algebra problem. The full set of questions may contain several hundred such problems. Two key concepts are: the 'knowledge state', a subset of problems that some individual is capable of solving correctly, and the 'knowledge structure', which is a distinguished collection of knowledge states. For beginning algebra, a useful knowledge structure may contain several million feasible knowledge states.

An important difference between the psychometric approach and that expounded in this book concerns the choice of problems representing a particular curriculum, such as beginning algebra. In our case, there is no essential restriction. The problems involved in any assessment can be chosen in a pool covering the entire curriculum. By contrast, the problems selected in the construction of a psychometric test must satisfy a criterion of homogeneity: they

[5] Interestingly, the meaning of the acronym S.A.T. was changed a few years ago by Educational Testing Service from '*Scholastic Aptitude Test*' to '*Scholastic Assessment Test*', suggesting that a different philosophy on the part of the test makers was considered. Today, it appears that 'S.A.T.' has become an abbreviation, a mnemonic without any intended meaning.

[6] Sophisticated mathematical theories can certainly be found in some areas of the behavioral sciences, but they do not generally rely on 'psychological measurement.'

[7] For example, in the form of one or more measurement scales quantifying some 'aptitudes.'

must contribute to the estimation of a numerical score. Any problem that does not satisfy this criterion—for which there is a technical statistical definition—may be rejected, even though it may be an essential part of the curriculum.

If a test is used as part of the assessment of students competence, with potential consequences not only for the students but also for the teacher or the school, the practice of teaching-to-the-test is unavoidable. In that respect, the difference between the two approaches is critical. With assessments based on the knowledge space methodology, teaching-to-the-test is beneficial[8] because the questions of the tests are essential parts of the curriculum. When psychometric tests are used, the practice may result in a distortion of the educational process, as is often argued and criticized.

The title "Knowledge Spaces" given to the first edition was consonant with our initial goal of building a machine for assessing knowledge. It later turned out that the resulting instrument could form the core component of a teaching engine, for the sensible reason that ascertaining the exact knowledge state of a student in a scholarly subject is the essential step toward educating the student in that subject. However, the change of focus from assessing to teaching prompted the development of a particular kind of knowledge space, called a 'learning space', hence the title of this new edition of our book. The difference between the names is not merely skin-deep. It is motivated by a re-axiomatization of the concept. Learning spaces are specified by two simple, pedagogically inescapable axioms. They are presented and discussed in Chapter 2. (Chapter 1 contains a general, non technical introduction to our subject.)

The two concepts of knowledge state and knowledge structure give rise to various lattice-theoretical developments motivated by the empirical application intended for them. This material is presented in Chapters 2-8. The concept of a learning space has been generalized in the form of an algebraic system called a medium. The connections between media and learning spaces are spelled out in Chapter 10.

The behavioral nature of the typical empirical observations—the responses of human subjects to questions or problems—practically guarantees noisy data. Also, it is reasonable to suppose that all the knowledge states (in our sense) are not equally likely in a population of reference. This means that a probabilistic theory had to be forged to deal with these two kinds of uncertainties. This theory is expounded in Chapters 11 and 12. Chapters 9, 13 and 14 are devoted to various practical schemes for uncovering an individual's knowledge state by sophisticated questioning. Chapters 15 and 16 tackle the complex problems of constructing knowledge spaces and learning spaces.

For a real-life demonstration of a system based on the concepts of this book, we direct the reader to `http://www.aleks.com` where various full-

[8] It is also relevant that, in standard applications of knowledge space theory, such as the `ALEKS` system (see page 10), there are typically no multiple choice problems.

scale programs involving both an assessment module and a learning module are available, covering mathematics and science subjects (see page 10 for a more detailed description). Chapter 17 is devoted to an investigation of the `ALEKS` assessment software from the standpoint of its validity, that is, the accuracy of its predictions. The final chapter 18 contains a list of open problems.

Many worthwhile developments could not be included here. There is much on-going research, especially in two European centers: the University of Padua, with Francesca Cristante, Luca Stefanutti and their colleagues, and the University of Graz by the research team of Dietrich Albert. However, we had to limit our coverage. Further theoretical concepts and results can be found in chapters of two edited volumes, by Albert (1994) and Albert and Lukas (1998). The second one also contains some applications to various domains of knowledge. Current references on knowledge spaces can be obtained at

`http://wundt.kfunigraz.ac.at/hockemeyer/bibliography.html`

thanks to Cord Hockemeyer, who maintains a searchable database.

Our enterprise, from the first idea to the completion of *Knowledge Spaces*, the first edition of this monograph, took 17 years, during which Falmagne benefited from major help in the form of several grants from the National Science Foundation at New York University. JCF also ackowledges a grant from the Army Research Institute (to New York University). He spent the academic year 1987-88 at the Center for Advanced Study in the Behavioral Sciences in Palo Alto. JPD, as a Fulbright grantee, was a visitor at the Center for several months, and substantial progress on our topic was made during that period. Another major grant from the National Science Foundation to JCF at the University of California, Irvine, was instrumental for the development of the educational software `ALEKS` (which belongs to UCI and is licenced to ALEKS Corporation[9]). We thank all these institutions for their financial support.

Numerous colleagues, students and former students were helpful at various stages of our work. Their criticisms and suggestions certainly improved this book. We thank especially Dietrich Albert, Biff Baker, Eric Cosyn, Charlie Chubb, Chris Doble, Nicolas Gauvrit, Cord Hockemeyer, Yung-Fong Hsu, Geoffrey Iverson, Mathieu Koppen, Kamakshi Lakshminarayan, Wil Lampros, Damien Lauly, Arnaud Lenoble, Josef Lukas, Jeff Matayoshi, Bernard Monjardet, Cornelia Müller-Dowling, Louis Narens, Misha Pavel, Michel Regenwetter, Selim Rexhap, Ragnar Steingrimsson, Ching-Fan Seu, Nicolas Thiéry, Vanessa Vanderstappen, Hassan Uzun and Fangyun Yang. We also benefited from the remarks of students from two Erasmus courses given by JPD (Leuven, 1989, and Graz, 1998).

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[9] JCF is the Chairman and a co-founder of ALEKS Corporation.

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We are indebted to Wei Deng for her uncomprimising chase for the typographical errors and solecisms committed in an early draft of the first edition. As the second edition largely relies on the first one, the present work also gained from Wei's pointed remarks. Because this is a human enterprise, the reader will surely uncover some remaining incongruities, for which we accept all responsibilities. We have also benefited from the kind efficiency of Mrs. Glaunsinger, Mrs. Fischer and Dr. Engesser, all from Springer-Verlag who much facilitated the production phase of this work.

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Finally, but not lastly, we thank again our respective spouses, Dina and Monique, for their kind and unwavering support.

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