Chapter 2
Condorcet's Paradox and Group Coherence

2.1 Introduction

The possibility that various election paradoxes might exist has been seen to be a potentially significant threat to the stability of election processes, and we have developed a number of different mathematical models that can be used to assess the likelihood that these paradoxes might actually be observed. These basic models have been used to yield some support to the intuitively appealing hypothesis that the likelihood that these voting paradoxes will be observed should tend to decrease with increasing levels of social homogeneity among the preferences of voters in the population, or as the degree of dependence among voters' preferences in the population tends to increase. There is a direct linkage between increases in the measure of dependence among voters' preferences and the degree of social homogeneity that is expected to exist in a voting situation.

An extensive survey of the work that has been performed to investigate the association between the likelihood that voting paradoxes might occur and degrees of social homogeneity is summarized in Gehrlein (2006a). The many different measures of social homogeneity that have been developed in the literature can be categorized as being either Population Specific Measures of Homogeneity or Situation Specific Measures of Homogeneity. As in Chap. 1, we focus on the association between the likelihood that a PMRW exists and degrees of social homogeneity, since this area has received most of the attention in this type of analysis. The extension of this analysis to other voting paradoxes will then be considered later.

2.2 Population Specific Measures of Homogeneity

A *Population Specific Measure of Social Homogeneity* (*PSM*) is related to parameters of the population from which random voter preference profiles or voting situations are generated. For three candidates, $\{A, B, C\}$, these measures are based

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on the p_i's from the $\boldsymbol{p}$ vectors that describe the likelihood that a randomly selected voter will have the i^{th} possible linear preference ranking on the candidates. The measure $H(\boldsymbol{p})$ from (1.59) is one such a PSM, and it was pointed out in Chap. 1 that $P^S_{PMRW}(3, n, DC)$ generally increases as $H(\boldsymbol{p})$ increases for $\boldsymbol{p}$ vectors in the DC subset. However, it was also noted that this relationship deteriorates as n becomes large. To the degree that the level of dependence between voters' preferences is related to social homogeneity, the Parameter α in P–E probability models is also a PSM.

The general conclusion in Gehrlein (2006a) is that studies that have looked for a general connection between $P^S_{PMRW}(m, n, \boldsymbol{p})$ and various PSM's have only found at best a weak relationship. An explanation of this outcome can be based on the fact that any $\boldsymbol{p}$ vector for a population will have only one value for the PSM that is being considered, while it is possible that many voting situations could be generated from that $\boldsymbol{p}$. This leads to the consideration of measures of social homogeneity that are based on characteristics of specific voting situations themselves, rather than on the characteristics of the population from which a voting situation is obtained.

2.3 Situation Specific Measures of Homogeneity

A *Situation Specific Measure of Homogeneity* (*SSM*) does not measure social homogeneity based on $\boldsymbol{p}$ vectors, as the PSM's do. SSM's are based on the n_i's of the particular $\boldsymbol{n}$ vector for a given voting situation, or on the $\boldsymbol{n}$ vector that is obtained by accumulating individual preferences in a voter preference profile. A SSM would use the actual observed proportions, n_i/n, as a substitute for the p_i terms in any PSM. For any particular voting situation, we know with certainty whether or not a PMRW exists. It is therefore quite reasonable to expect to have a stronger correlation between social homogeneity and the probability that a PMRW exists for studies in which social homogeneity is measured by some SSM.

Most simple SSM's still do not lead to a strong general relationship between social homogeneity and the probability that a PMRW exists. However, it was found in Gehrlein (2006a) that when the voters' preferences are formed by a process that imposes some internal structural consistency or some mutual coherence on voter preference profiles or voting situations, much stronger relationships can be found between SSM's and the probability that a PMRW exists. The measures of mutual coherence that have been found to exhibit this tendency are based on some simple extensions of natural underlying conditions on voting situations that require that a PMRW must exist.

Black (1958) found one such condition when voters' preferences are restricted to have the property of *single-peaked preferences*. To describe this property, we define a measure of preference or *utility*, $U^i(C_j)$, that a given i^{th} voter associates with candidate C_j in an m-candidate election on candidates $\{C_1, C_2, \ldots, C_m\}$. Increased measures of $U^i(C_j)$ indicate that a voter has an increased preference, or

Fig. 2.1 An example preference profile with three voters and six candidates

Voter 1: $C_6 \succ C_3 \succ C_5 \succ C_1 \succ C_4 \succ C_2$
Voter 2: $C_4 \succ C_3 \succ C_6 \succ C_2 \succ C_5 \succ C_1$
Voter 3: $C_2 \succ C_4 \succ C_3 \succ C_6 \succ C_5 \succ C_1$.

utility, for the given candidate, so that the given voter's individual preference ranking on candidates will have $C_j \succ C_k$ if, and only if, $U^i\left(C_j\right) > U^i(C_k)$.

Consider a simple example voter preference profile with three voters, where each individual voter has a linear preference ranking on six candidates, as shown in Fig. 2.1.

We can determine if the three voter's preference rankings in the example in Fig. 2.1 meet the definition of single-peaked preferences by trying to find $U^i\left(C_j\right)$ values that are consistent with the preference rankings of the individual voters, while simultaneously meeting an additional restriction. This additional restriction can be established by drawing a graph like the one that is shown in Fig. 2.2.

Values of $U^i\left(C_j\right)$ are displayed on the vertical axis of the graph in Fig. 2.2, and the horizontal axis of the graph represents the sequence of C_j's that corresponds to some linear overall reference ranking. Let $C_i\boldsymbol{O}C_j$ denote the fact that C_i is ranked before C_j in this overall reference ranking. The specific overall reference ranking that is used in Fig. 2.2 is $C_2\boldsymbol{O}C_4\boldsymbol{O}C_3\boldsymbol{O}C_6\boldsymbol{O}C_5\boldsymbol{O}C_1$. Figure 2.2 shows a plot of possible $U^i\left(C_j\right)$ values for each voter, as associated with specific candidates in the sequence of C_j's in the overall reference ranking, such that the given $U^i\left(C_j\right)$ values for a given i would reproduce the linear preference ranking of the associated i^{th} voter in Fig. 2.1. The results that are displayed in Fig. 2.2 have $U^1(C_6) > U^1(C_3) > U^1(C_5) > U^1(C_1) > U^1(C_4) > U^1(C_2)$, to correspond with the linear preference ranking $C_6 \succ C_3 \succ C_5 \succ C_1 \succ C_4 \succ C_2$ for Voter 1. We do not claim that the $U^i\left(C_j\right)$ values in the graph necessarily represent the true utility values that voters have for candidates. The only claim is that they are possible utility values that would result in the voters' preference rankings on candidates.

Any of the possible 720 linear rankings on the six candidates could have been used as an overall reference ranking. However, the specific overall reference ranking used for Fig. 2.2 is of particular interest, since it results in plots of the possible $U^i\left(C_j\right)$

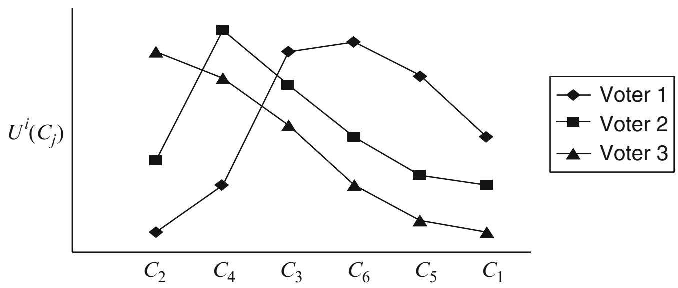

Fig. 2.2 A graph of single-peaked preference curves for three voters

values that have single-peaked preference curves for each voter. Using the definition in Black (1958, p. 7), a "single-peaked (preference) curve is one which changes its direction at most once, from up to down." The logical foundation of the definition for single-peaked preferences is given in Black (1958, pp. 8–9):

> While in practice a (committee) member's preference curve may be of any shape whatsoever, there is reason to expect that, in some important practical problems the (preference) valuations actually carried out will tend to take the form of isolated points on single-peaked curves. This would be particularly likely to happen if the committee were considering different possible sizes of a numerical quantity and choosing one size in preference to the others. It might, for example, be reaching a decision with regard to the price of a product to be marketed by a firm, or to the output for a future period, or the wage rate of labor, or the height of a particular tax, or the legal age of leaving school, and so on.

Buchanan (1970) and Browning (1972) also consider various sets of natural conditions that are likely to lead to the existence of single-peaked preferences for a group of voters. Gaertner (2005) notes that arguments that ultimately lead to the same definition of single-peaked preferences can be found as far back as the work of Pufendorf in the seventeenth century. It can be concluded that the notion of single-peaked preferences is not simply a mathematical artifact, and that it does have a basis in reality for some voting scenarios.

The condition of single-peaked preference curves indicates the existence of a situation in which all voters have preferences that are *mutually coherent*. That is, the presence of such a situation suggests that there is mutual agreement among the voters that some underlying characteristics of candidates exist that allow for the sequencing of the candidates in some natural order from left to right, according to their rankings in an overall reference order. Each voter would then have some particular most preferred candidate in the sequence, with decreasing preferences on candidates as they are ranked farther away, to the left or to the right, from their most preferred candidate within the sequence of candidates in the overall reference order.

List (2002) discusses the notion of having different levels of group coherence of preference, such that voters' preferences might reflect a *substantive level agreement*, to the extent that their preferences, or views, tend to have some degree of consistency or homogeneity. However, voters might go beyond that and have some degree of *meta-level agreement*, to the extent that they can agree on a common dimension on which issues can be conceptualized. The voters might be largely in agreement as to what this common dimension is, while being in great disagreement as to what the optimal position on the dimension is. Positioning issues along such a dimension is perfectly consistent with the notion of single-peaked preferences. List (2002) argues that agreement at the meta-level is more likely to reduce occurrences of paradoxical results like PMR cycles than is agreement on a substantive level.

Dryzek and List (2003) extend this notion, by pointing out that two or more individuals can agree on a substantive level to the extent that their preferences are the same. However, these individuals might instead disagree on any common ranking of alternatives that would reflect their own preferences, while they could still agree on some ranking of alternatives along a common dimension. This second scenario is agreement on a meta-level. As described above, agreement on a meta-level would imply a condition like single-peakedness. The introduction of issue complexity might rule out any common agreement on any single dimension, but multiple relevant issue dimensions coupled with individual voter's preference rankings of alternatives on the issue dimensions might lead to some "intra-dimensional single-peakedness".

Grofman and Uhlaner (1985) previously proposed a similar concept regarding the existence of "meta-preferences" that would result when voters have preferences for characteristics of broadly defined processes that might be involved in determining their individual preferences on candidates, rather than simply having preferences for candidates. They suggest that the additional structure that results from processes that are based on such meta-preferences would lead to an increased level of overall understanding of the entire decision process, and therefore to more overall stability. This increased stability would therefore suggest that paradoxical voting outcomes should be less likely to be observed.

All of this is supported by the work of Black (1958), where arguments are developed to show that PMR must be transitive for odd n if *any* overall reference order and possible $U^i(C_j)$ values that are consistent with voters' preference rankings can be found to result in single-peaked preference curves for all voters. That is, all voters' preference curves must be single-peaked relative to the same overall reference order. However, the assumption of perfectly single-peaked preferences forces some very strict requirements on voters' preferences, particularly when there are many voters in the electorate.

Niemi (1969) proposed the notion of using some measure to the proximity of a voting situation to having perfectly single-peaked preferences as a SSM, since it might be overly restrictive to assume that all voters in a large electorate will have preferences that are single-peaked. Given Black's result, it seems very reasonable to assume that the probability that PMR is transitive will remain high as long as the preferences of most voters in a voting situation are consistent with the restriction of single-peaked preferences. Niemi proposed that the proximity of a voting situation to having perfectly single-peaked preferences could effectively be measured as the minimum proportion of voters in the electorate who must have their preferences ignored so that the preferences on the remaining candidates will be perfectly single-peaked. As this necessary proportion of voters decreases, the closer the preferences in the original voting situation are to being perfectly single-peaked. Niemi (1970) performs an empirical study of seven three-candidate elections in which complete preference rankings were reported by voters, to find that only one case resulted in the existence of a PMR cycle, and that this case was the one that was farthest removed from the condition of perfect single-peakedness with this measure. One difficulty of using this measure as a SSM is that it can be difficult to calculate this proportion, but results of Arrow (1963) can be applied to obtain a proxy for this measure very easily in the case of three-candidate elections.

Arrow (1963) approaches the concept of single-peaked preferences in a very different manner, by considering only the ordinal relationships between candidates in rankings, without using Black's $U^i(C_j)$ values. Arrow's findings lead to an alternative definition of single-peaked preferences, such that voters' preferences are perfectly single-peaked if for every triple of candidates, at least one candidate is never ranked as least preferred among the three candidates by any voter. Arrow's definition lacks the conceptual appeal of Black's utility based definition, but it is a completely equivalent definition of single-peaked preferences.

2.3.1 Weak Measures of Group Coherence

The ideas that were proposed above by Black, Niemi and Arrow are all combined in Gehrlein (2004b) to develop a SSM, *Parameter b*, that measures the minimum number of times that some candidate is bottom ranked, or is least preferred, in the preferences of the n voters in a voting situation, to serve as a simple measure of the proximity of a voting situation to representing perfectly single-peaked preferences in a three-candidate election, where

$$b = Min\{n_1 + n_3, n_2 + n_4, n_5 + n_6\}. \tag{2.1}$$

Here, the n_i terms are defined for a voting situation from Fig. 1.1, which is reproduced here for convenience in Fig. 2.3.

If b is equal to zero for a voting situation with three candidates, some candidate is never ranked as least preferred, so the voting situation represents the condition in which voters have perfectly single-peaked preferences. This would happen, for example if $n_1 + n_3 = 0$, where the definitions from Fig. 2.3 indicate that this requires that Candidate C is never the least preferred candidate for any voter in the associated voting situation. When b is maximized at $n/3$, a voting situation reflects very disperse preferences of voters over candidates to reflect a situation that is very far removed from perfect single-peakedness.

As Parameter b increases in voting situations, the preferences of voters in a voting situation become more removed from the condition of perfect single-peakedness. Another perspective on this issue is that a voting situation with a small Parameter b reflects a situation in which there is some candidate that very few voters think is the worst of the three candidates. The electorate would be somewhat united by their *weak* support of, or lack of complete opposition to, the election of such a candidate. In that sense, this candidate can be viewed as a *Weak Positively Unifying Candidate* that voters would not generally think of as reflecting the worst possible outcome if that candidate were to be elected.

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
n_1	n_2	n_3	n_4	n_5	n_6

Fig. 2.3 The six possible linear preference rankings on three candidates

Vickery (1960) considers the well known condition of *single-troughed preferences*, and proves that the imposition of this assumption on voting situations will also lead to the necessary existence of a PMRW. This condition is also known as *single-dipped preferences* in the literature, but we use the term single-troughed preferences since that term is the originally used by Vickery. The condition of single-troughed preferences is equivalent to the condition of single-peaked preferences, since every single-peaked voting situation corresponds to a single troughed-voting situation in which all voters' preference rankings are inverted. For a three-candidate election, it follows from Arrow (1963) that a voting situation with perfectly single-troughed preferences is one in which at least one candidate is never ranked as most preferred by any voter.

Following the development of Parameter b above, *Parameter* t measures the proximity of a voting situation to meeting the condition of perfectly single-troughed preferences, with

$$t = Min\{n_1 + n_2, n_3 + n_5, n_4 + n_6\}. \tag{2.2}$$

The definition of n_i's in Fig. 2.3 are used to define Parameter t as the minimum number of times that some candidate is top-ranked as the most preferred candidate in the voters' preference rankings, so that a voting situation is perfectly single-troughed if $t = 0$, and the value of t then reflects the relative proximity of a voting situation to the condition of perfect single-troughedness. Any candidate that very few voters rank as the most preferred candidate in a voting situation can be viewed as a *Weak Negatively Unifying Candidate* since none of the voters would generally think of the election of this candidate as reflecting the best possible outcome. The electorate would be weakly unified by their opposition to, or lack in complete support of, the election of such a candidate.

Ward (1965) develops another restriction on voting situations that leads to the conclusion that a PMRW must exist in a three-candidate election. This condition requires that some candidate must be *perfectly polarizing*, in the sense that this candidate is never middle ranked, or ranked at the center, of any voter's preference ranking. That is, every voter will either consider this candidate to be either the most preferred or the least preferred. The definition of n_i's in Fig. 2.3 are used to define *Parameter* c to reflect the proximity of a voting situation to the condition of perfect polarization, with

$$c = Min\{n_3 + n_4, n_1 + n_6, n_2 + n_5\}. \tag{2.3}$$

If $c = 0$, some candidate is perfectly polarizing, since all voters will rank that candidate as either least preferred or most preferred, and the value of c measures the proximity of a voting situation to the condition of perfect polarization. Any candidate that very few voters rank in the middle of their preference ranking can generally be viewed as a *Weak Polarizing Candidate*.

Parameters b and t are combined in Gehrlein (2008) to obtain another measure of group coherence. By ignoring the distinction between positively unifying and negatively unifying candidates, *Parameter u* measures the presence of an overall unifying candidate in a voting situation with

$$u = Minimum\{b, t\}. \tag{2.4}$$

A small value of Parameter u for a voting situation indicates that some candidate is close to being either positively or negatively unifying, and Parameter u measures the proximity of a voting situation to having a *Weak Overall Unifying Candidate*.

2.3.2 Strong Measures of Group Coherence

Stronger measures of group coherence are developed in Gehrlein (2009), and each of these measures is a more restrictive variation of Parameters b, t, c and u. A Weak Positively Unifying Candidate was defined as some candidate that is ranked as least preferred by a small proportion of voters in a voting situation, and the proximity of a voting situation to having a perfect Weak Positively Unifying Candidate is measure by Parameter b. A candidate would more strongly reflect the notion of being a positively unifying candidate by being ranked as most preferred by a large proportion of the voters in a voting situation. *Parameter* t^* is defined accordingly from the definition of the n_i's in Fig. 2.3, with

$$t^* = Max\{n_1 + n_2, n_3 + n_5, n_4 + n_6\}. \tag{2.5}$$

If $t^* = n$, the same candidate is ranked as most preferred by all voters, making it a perfect *Strong Positively Unifying Candidate*, and Parameter t^* is used as a measure of the proximity of a voting situation to this condition.

The same basic logic can be used to strengthen the definition the proximity of a voting situation to having perfect Weak Negatively Unifying Candidate, as measured by Parameter t. *Parameter* b^* is defined accordingly by

$$b^* = Max\{n_5 + n_6, n_2 + n_4, n_1 + n_3\}. \tag{2.6}$$

If $b^* = n$, the same candidate is ranked as least preferred by all voters, making it a perfect *Strong Negatively Unifying Candidate*, and Parameter b^* is used as a measure of the proximity of a voting situation to this condition.

Parameter c measured the proximity of a voting situation to the condition of perfect weak polarization. The strong measure that is associated with this parameter is *Parameter* c^*, with

$$c^* = Max\{n_3 + n_4, n_1 + n_6, n_2 + n_5\}. \tag{2.7}$$

If $c^* = n$, the same candidate is middle-ranked in the preferences of all voters, so that this candidate is neither extremely liked nor extremely disliked by any voter, making it a perfect *Strong Centrist Candidate*, and Parameter c^* is used as a measure of the proximity of a voting situation to this condition.

Parameters b^* and t^* are combined as above, by ignoring the distinction between positively unifying and negatively unifying candidates, and *Parameter* u^* measures the presence of a *Strong Overall Unifying Candidate* in a voting situation with

$$u^* = Max\{b^*, t^*\}. \tag{2.8}$$

A large value of Parameter u^* therefore indicates that a voting situation has some candidate that is close to representing either a strong positively or a strong negatively unifying candidate.

2.4 Obtaining Probability Representations

In order to determine the impact that these measures of group coherence have on the probability that a PMRW exists, attention is focused to the development of representations for the conditional probability that a PMRW exists, given that voting situations have specified values of these SSM's. These probability representations are based on a direct extension of the assumption of IAC. For any particular $X \in \{b, t, c, u, b^*, t^*, c^*, u^*\}$, the *Conditional Impartial Anonymous Culture Condition* $(IAC_X(k))$ is used to develop probability representations for election outcomes, conditional on the assumption that only voting situations for which Parameter X has a specified value of k can be observed, and that each of these possible voting situations is equally likely to be observed.

The conditional probability that a strict PMRW exists for n voters with three candidates, given the assumption of $IAC_X(k)$ for $X \in \{b, t, c, u, b^*, t^*, c^*, u^*\}$, is denoted by $P^S_{PMRW}(3, n \,|\, IAC_X(k))$. The logic that led to (1.27) is easily generalized to

$$P^S_{PMRW}(3, n \,|\, IAC_X(k)) = \frac{3N^{\{A\}}_{PMRW}(3, n, IAC_X(k))}{K(3, n, IAC_X(k))}. \tag{2.9}$$

Here, $N^{\{A\}}_{PMRW}(3, n, IAC_X(k))$ and $K(3, n, IAC_X(k))$ are defined in the obvious fashion, following the development of (1.27).

Gehrlein (2004b) derived a representation for $P^S_{PMRW}(3, n \,|\, IAC_b(k))$ with the subspace partitioning process that was described in the development of a representation for $N^{\{A\}}_{PMRW}(3, L, MC)$ in Chap. 1. An eight subspace partition is required to remove all *Max* and *Min* arguments that are required in the summation limits to have Candidate A as the PMRW with $b = k$, while obtaining a representation for $N^{\{A\}}_{PMRW}(3, n, IAC_b(k))$. The resulting representation for odd $n \geq 7$ is given by

$$
\begin{aligned}
&P^{S}_{PMRW}(3, n \mid IAC_b(k)) \\
&\quad = \frac{-k(17 - 21k - 11k^2) + (5 - 26k - 4k^2)n + 3(2 - k)n^2 + n^3}{(n - 3k)[(n + 1)(n + 5) - 3k(2 + k)]}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0 \leq k \leq (n - 1)/4 \\
&\quad \frac{3(3 - 2k - 6k^3) + (11 + 18k^2)n + 3(1 - 2k)n^2 + n^3}{2(k + 1)[(n + 1)(n + 5) - 3k(2 + k)]}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } (n + 1)/4 \leq k \leq (n - 1)/3 \\
&\frac{3}{4}, \text{ for } k = n/3.
\end{aligned} \tag{2.10}
$$

The subspace partitioning procedure is further complicated in this situation with the addition of Parameter b to the required summation limits in such probability representations. In order to facilitate the process of obtaining these representations, Gehrlein (2005, 2006b) develops an extension of EUPIA that obtains representations for the conditional probability that voting outcomes are observed, given that voting situations are constrained to have some specified value of a measurable parameter.

2.4.1 EUPIA2

With the assumption of either IAC or MC, EUPIA was developed to obtain a representation for the number of voting situations with n voters, $E^A(n)$, such that the n_i's meet the necessary conditions for Candidate A to meet the requirements of Event F. With the assumption of $IAC_b(k)$, *EUPIA2* obtains a representation for the number of voting situations, $E^A(n,k)$, such that the n_i's meet the necessary conditions for Candidate A to meet the requirements of Event F and simultaneously meet the necessary conditions for some defined parameter of the voting situation, like b, to match a specified integer value k.

The basic requirements of the conditions that are needed for EUPIA to work are expressed in the discussion that followed Axiom 1.1, where the simple linear form restriction is imposed on the *Max* and *Min* arguments in the summation bounds that are required for Event F to be observed in a voting situation. The extension of this logic to EUPIA2 relies on an extension of the simple linear form restriction. The *extended linear form restriction* requires that each upper and lower summation bound on the representation to obtain $E^A(n,k)$ is expressible as the *Max* or *Min* of some set of simple linear functions of n, a specified k for some defined parameter and n_i's that are previously defined in the series of summation indexes. As with the definition of a simple linear form restriction, the coefficients in these simple linear functions must be rational numbers. Given the nature of identities for sums of powers of integers, it is very simple to show that:

Axiom 2.1 If the restrictions on the n_i's in a three-candidate voting situation that are necessary for Event F to be observed and to simultaneously meet the necessary conditions for some defined Parameter $X \in \{b, t, c, u, b^*, t^*, c^*, u^*\}$ to have a specified integer value k meet the extended linear form restriction, then

$$E^A(n,k) = \sum_{i=0}^{5} \sum_{j=0}^{5-i} \tau_{ij} n^i k^j, \tag{2.11}$$

for some integer sequence $n = \psi + p\upsilon$, with $\upsilon = 0, 1, 2, \ldots$.

As in Axiom 1.1, the τ_{ij} coefficients in (2.11) must be rational numbers, and these arguments can easily be extended to representations with MC by replacing n with L in the definition of the extended linear form restriction.

It is then a trivial extension of a result proved in Gehrlein (2006a) that:

Axiom 2.2 If the necessary conditions that are required to obtain $E^A(n)$ for some Event F in a three-candidate election meet the simple linear form restriction, then $E^A(n,k)$ must result in a functional form as specified in (2.11), if Parameter $X \in \{b, t, c, u, b^*, t^*, c^*, u^*\}$ is simultaneously required to have a specified integer value k.

2.4.1.1 Obtaining a Representation for $P^S_{PMRW}(3, n \,|\, IAC_b(k))$ with EUPIA2

We illustrate the procedure for obtaining representations with EUPIA2 by developing a representation for $P^S_{PMRW}(3, n \,|\, IAC_b(k))$. The first step is to obtain a representation for the number of voting situations, $K(3, n, IAC_b(k))$, with n voters that have a specified value, k, for Parameter b, as defined in (2.1). The representation for $K(3, n, IAC)$ in (1.25) is clearly consistent with the simple linear form restriction, so Axiom 2.2 requires that the representation for $K(3, n, IAC_b(k))$ must have the general form of (2.11).

The process is initiated by fixing k at some specified numerical value and then using computer enumeration procedures to obtain values of $NVS^A(\psi + pj \,|\, k)$ for each value of $j = 0(1)7$. In this case, $NVS^A(\psi + pj \,|\, k)$ is a count of the number of voting situations with $\psi + pj$ voters for which Parameter b is equal to the specified value of k. Since k can be treated as a constant in (2.11), the k^j term can be absorbed into the τ_{ij} term and the general form can be reduced to a linear function with a single variable, n, as in (1.44), for that specified k.

EUPIA is then used directly to find the conditional representation for $K(3, n, IAC_b(k))$, denoted as $K(3, n, IAC_b(k) \,|\, k)$, for the k value has been specified, and

$$K(3, n, IAC_b(k) \,|\, k) = \sum_{i=0}^{5} C_i^k n^i, \tag{2.12}$$

for some integer sequence $n = \psi + pj$, with $j = 0, 1, 2, \ldots$

The process is then repeated for each integer k value with $0 \leq k < n/3$, and the C_i^k terms that are obtained for these $K(3, n, IAC_b(k) \mid k)$ representations will typically be different for each given k. For the process to work effectively, we need to start the search process in EUPIA2 with a relatively large value of ψ.

Table 2.1 summarizes the C_i^k values that were obtained for $0 \leq i \leq 3$ for each $0 \leq k \leq 11$ when EUPIA2 was run while arbitrarily setting $\psi = 35$ in all cases. The results give $C_i^k = 0$, for all $i \geq 4$, and the periodicity for all cases is found to have $p = 1$. Furthermore, additional EUPIA2 runs were performed to verify that the relevant entries in Table 2.1 remain valid for all integer values of $\psi \geq 1$.

A representation for $K(3, n, IAC_b(k) \mid k)$ can be obtained very easily for any specified k in the range $0 \leq k \leq (n-2)/3$ by using the known form of the representation in (2.12) along with the C_i^k entries in Table 2.1.

When the general form of the representations that are given in (1.44) and (2.11) are considered along with the representation for $K(3, n, IAC_b(k) \mid k)$ that is given in (2.12), we are led directly to the conclusion that each C_i^k coefficient must be obtainable as a function of k, with

$$C_i^k = \sum_{j=0}^{5-i} \partial_{ij} k^j \text{ for some rational } \partial_{ij} \text{ coefficients for a specified } i. \tag{2.13}$$

The earlier logic of the development of EUPIA and the known values of C_i^k that are given in Table 2.1 for a specified i can be used for $k = 0, 1, 2, \ldots, 6-i$ to establish a set of $6-i$ simultaneous equations, following the format of (2.13), with $6-i$ unknowns. The solution of the $6-i$ simultaneous equations will then give the $6-i$ values of the ∂_{ij} coefficients in the general representation for C_i^k. When the particular case with $i = 0$ is considered, six variables $\{\partial_{00}, \partial_{01}, \partial_{02}, \partial_{03}, \partial_{04}, \partial_{05}\}$ are defined. Using the associated entries for C_0^k that are listed in Table 2.1, the six simultaneous equations are given in (2.14).

Table 2.1 Computed C_i^k values with the specified k for $\psi = 35$ and $p = 1$

k	C_0^k	C_1^k	C_2^k	C_3^k
0	0	5/2	3	1/2
1	12	−22	3	1
2	171	−165/2	0	3/2
3	720	−188	−6	2
4	2010	−695/2	−15	5/2
5	4500	−570	−27	3
6	8757	−1729/2	−42	7/2
7	15456	−1240	−60	4
8	25380	−3411/2	−81	9/2
9	39420	−2270	−105	5
10	58575	−5885/2	−132	11/2
11	83952	−3732	−162	6

$$\begin{aligned}
&\partial_{00} + \partial_{01}0 + \partial_{02}0^2 + \partial_{03}0^3 + \partial_{04}0^4 + \partial_{05}0^5 = 0 \\
&\partial_{00} + \partial_{01}1 + \partial_{02}1^2 + \partial_{03}1^3 + \partial_{04}1^4 + \partial_{05}1^5 = 12 \\
&\partial_{00} + \partial_{01}2 + \partial_{02}2^2 + \partial_{03}2^3 + \partial_{04}2^4 + \partial_{05}2^5 = 171 \\
&\partial_{00} + \partial_{01}3 + \partial_{02}3^2 + \partial_{03}3^3 + \partial_{04}3^4 + \partial_{05}3^5 = 720 \\
&\partial_{00} + \partial_{01}4 + \partial_{02}4^2 + \partial_{03}4^3 + \partial_{04}4^4 + \partial_{05}4^5 = 2010 \\
&\partial_{00} + \partial_{01}5 + \partial_{02}5^2 + \partial_{03}5^3 + \partial_{04}5^4 + \partial_{05}5^5 = 4500.
\end{aligned} \tag{2.14}$$

Algebraic techniques are then used to solve the six simultaneous equations in (2.14) for the six unknown variables, with:

$$\begin{aligned}
&\partial_{00} = 0 \qquad \partial_{01} = \frac{-15}{2} \quad \partial_{02} = \frac{3}{2} \\
&\partial_{03} = \frac{27}{2} \qquad \partial_{04} = \frac{9}{2} \qquad \partial_{05} = 0.
\end{aligned} \tag{2.15}$$

Given these results, it follows that

$$C_0^k = \frac{-15}{2}k + \frac{3}{2}k^2 + \frac{27}{2}k^3 + \frac{9}{2}k^4 = \frac{3k(k+1)(3k^2+6k-5)}{2}. \tag{2.16}$$

Similar analysis is used to obtain the representations for the remaining C_i^k terms for $i = 1,2,3,4$ and:

$$\begin{aligned}
C_1^k &= -\frac{1}{2}(k+1)\left(3k^2+24k-5\right) \\
C_2^k &= -\frac{3}{2}(k+1)(k-2) \\
C_3^k &= \frac{(k+1)}{2}.
\end{aligned} \tag{2.17}$$

It is easily verified that these functional forms will generate the values that appear in the associated columns of Table 2.1 for any specified k.

After substitution the C_i^k terms from (2.16) and (2.17) into (2.12) and performing the necessary algebraic reduction, we obtain

$$K(3,n,IAC_b(k)) = \frac{(k+1)(n-3k)[(n+1)(n+5)-3k(2+k)]}{2}, \quad \text{for } n \geq 1 \text{ and } k \leq (n-2)/3. \tag{2.18}$$

The result that is given in (2.18) is exactly the same as the representation for $K(3,n,IAC_b(k))$ in Gehrlein (2004b).

For the special case that $k = n/3$ when n is a multiple of three, it is easily shown that

$$K\left(3, n, IAC_b\left(\frac{n}{3}\right)\right) = \left(\frac{n+3}{3}\right)^3. \tag{2.19}$$

A representation for $N_{PMRW}^{\{A\}}(3, n, IAC_b(k))$ can be obtained in the same fashion that was used to obtain the representation for $K(3, n, IAC_b(k))$ in (2.19). The conditions on n_i's that result in Candidate A being the strict PMRW for odd n in (1.5) clearly meet the simple linear form restriction. Axiom 2.2 then requires that the representation for $N_{PMRW}^{\{A\}}(3, n, IAC_b(k))$ must have the form of (2.11).

Following the same logic that led to the development of Table 2.1 that ultimately led to representations for $K(3, n, IAC_b(k) \mid k)$ with specified values of k, we use EUPIA to find coefficients D_i^k for specified k values for Parameter b that give representations for $N_{PMRW}^{\{A\}}(3, n, IAC_b(k) \mid k)$, with

$$N_{PMRW}^{\{A\}}(3, n, IAC_b(k) \mid k) = \sum_{i=0}^{3} D_i^k n^i. \tag{2.20}$$

The EUPIA computations were performed with $\psi = 91$, and attempts were made to obtain D_i^k coefficients for all k with $0 \le k \le 30$, and the results are summarized in Table 2.2 for all $0 \le k \le 22$. The periodicity for the representation was found to be $p = 2$ for all k entries.

Coefficients for the representations for $N_{PMRW}^{\{A\}}(3, n, IAC_b(k) \mid k)$ in (2.20) were found for all $0 \le k \le 22$ in Table 2.2, with $p = 2$ and $\psi = 91$. However, no such representation was found with $k = 23$. The reason for this is that representations to obtain $N_{PMRW}^{\{A\}}(3, n, IAC_b(k))$ have one functional form for $k \le \frac{n-3}{4}$ and a second functional form for $k \ge \frac{n+1}{4}$.

EUPIA2 began this process by using computer enumeration techniques to count the number of voting situations, $NVS_{PMRW}^{A}(n \mid k)$ for which Candidate A is the PMRW with a specified value of k for Parameter b, for a series of n values with $n = \psi + jp$ for $j = 0(1)7$. The first term in the series has $n = \psi + 0p = 91$. With $k = 23$ and $n = 91$, $k \ge \frac{n+1}{4}$ so the second functional form should be used to obtain the observed value of $NVS_{PMRW}^{A}(91 \mid 23)$. The third enumerated value that is listed in the series has $n = \psi + 2\,p = 95$. With $k = 23$ and $n = 95$, $k \le \frac{n-3}{4}$ so the first functional form should be used to obtain the observed value of $NVS_{PMRW}^{A}(95 \mid 23)$. This conflict explains why a single functional form is not obtained as a representation for $N_{PMRW}^{\{A\}}(3, n, IAC_b(23) \mid 23)$ when $\psi = 91$ is used to start the series of n values to get the values in Table 2.2. The exact break point of this type in such series can be precisely determined as a function of n by running EUPIA2 with a number of ψ values, to look for consistency in terms of the value of ψ where the first functional form stops working for each ψ. As a result, we find that the first functional form for $N_{PMRW}^{\{A\}}(3, n, IAC_b(k))$ holds over the range of k values with $0 \le k \le (n-1)/4$.

Table 2.2 Computed D_i^k values with the specified k for $\psi = 91$ and $p = 2$

k	D_0^k	D_1^k	D_2^k	D_3^k
0	0	5/6	1	1/6
1	5	−25/3	1	1/3
2	69	−63/2	0	1/2
3	290	−218/3	−2	2/3
4	810	−815/6	−5	5/6
5	1815	−225	−9	1
6	3535	−2065/6	−14	7/6
7	6244	−1492/3	−20	4/3
8	10260	−1377/2	−27	3/2
9	15945	−2765/3	−35	5/3
10	23705	−7205/6	−44	11/6
11	33990	−1530	−54	2
12	47294	−11479/6	−65	13/6
13	64155	−7063/3	−77	7/3
14	85155	−5715/2	−90	5/2
15	110920	−10280/3	−104	8/3
16	142120	−24395/6	−119	17/6
17	179469	−4779	−135	3
18	223725	−33421/6	−152	19/6
19	275690	−19330/3	−170	10/3
20	336210	−14805/2	−189	7/2
21	406175	−25355/3	−209	11/3
22	486519	−57569/6	−230	23/6

A representation for $N_{PMRW}^{\{A\}}(3, n, IAC_b(k))$ for the range of k values with $0 \le k \le (n-1)/4$ is obtained in the same fashion that was used to develop the representation for $K(3, n, IAC_b(k))$ in (2.18). Using the data from Table 2.2, with the necessary functional form like that in (2.13), we obtain

$$D_0^k = \frac{k(k+1)}{6}\left(11k^2 + 21k - 17\right) \quad D_1^k = -\frac{(k+1)}{6}\left(4k^2 + 26k - 5\right)$$
$$D_2^k = -\frac{(k+1)(k-2)}{2} \quad D_3^k = \frac{(k+1)}{6}. \tag{2.21}$$

By using the identity that is given in (2.9) along with the representation for $N_{PMRW}^{\{A\}}(3, n, IAC_b(k) \mid k)$ that follows from (2.20) and (2.21), substitution and algebraic reduction lead to the identical representation for $P_{PMRW}^{S}(3, n, IAC_b(k))$ with $0 \le k \le (n-1)/4$ that was obtained by algebraic methods in (2.10).

The determination of an appropriate representation for $P_{PMRW}^{S}(3, n, IAC_b(k))$ with $k \ge (n+1)/4$ requires some additional manipulation of EUPIA2. Computer enumeration values for $NVS_{PMRW}^{A}(n \mid k)$ were obtained in the last phase for each $n = \psi + pj$ with $j = 0(1)7$ for each $k = 0(1)22$ to obtain the entries in Table 2.2. To obtain the associated representation for $N_{PMRW}^{\{A\}}(3, n, IAC_b(k))$ over the range of k values with $\frac{n+1}{4} \le k \le \frac{n}{3}$, we start by obtaining computer enumeration values for $NVS_{PMRW}^{A}\left(n \mid \frac{n+1}{4} + k'\right)$ for each $n = \psi + pj$ with $j = 0(1)7$, for each value of $k' = 0(1)7$, with $\psi = 91$.

Table 2.3 Computed $F_i^{k'}$ values with the specified k' for $\psi = 91$ and $p = 4$

k'	$F_0^{k'}$	$F_1^{k'}$	$F_2^{k'}$	$F_3^{k'}$	$F_4^{k'}$
0	−231/512	−59/128	17/768	5/128	11/1536
1	5385/512	−751/128	−343/768	−7/128	11/1536
2	60345/512	−2883/128	−415/768	−19/128	11/1536
3	261417/512	−7607/128	−199/768	−31/128	11/1536
4	760665/512	−16075/128	305/768	−43/128	11/1536
5	1765449/512	−29439/128	1097/768	−55/128	11/1536
6	3538425/512	−48851/128	2177/768	−67/128	11/1536
7	6397545/512	−75463/128	3545/768	−79/128	11/1536

Table 2.3 summarizes the resulting $F_i^{k'}$ values such that

$$N_{PMRW}^{\{A\}}\left(3, n, IAC_b\left(\frac{n+1}{4} + k'\right) \mid \frac{n+1}{4} + k'\right) = \sum_{i=0}^{4} F_i^{k'} n^i. \tag{2.22}$$

The entries in Table 2.3 all have periodicity with $p = 4$.

A representation for $N_{PMRW}^{\{A\}}\left(3, n, IAC_b\left(\frac{n+1}{4} + k'\right)\right)$ is then obtained for this range of k values with $\frac{n+1}{4} \le k < \frac{n}{3}$ in the same fashion that was used to developed the representation for the range of k values $0 \le k \le (n-1)/4$ in (2.10). Using the data from Table 2.3, with the necessary functional form like that in (2.13), we obtain

$$\begin{aligned}
F_0^{k'} &= \frac{3}{512}(4k'+1)\left(192k'^3 + 144k'^2 + 100k' - 77\right)\\
F_1^{k'} &= \frac{-1}{128}\left(59 + 356k' + 144k'^2 + 192k'^3\right)\\
F_2^{k'} &= \frac{1}{768}\left(17 - 504k' + 144k'^2\right) \quad F_3^{k'} = \frac{5-12k'}{128}.
\end{aligned} \tag{2.23}$$

A representation for $N_{PMRW}^{\{A\}}(3, n, IAC_b(k))$ can be obtained for the range of k values with $\frac{n+1}{4} \le k < \frac{n}{3}$ by substituting $k - \frac{n+1}{4}$ for k' in the representations for $F_i^{k'}$ in (2.22) and (2.23), with

$$\begin{aligned}
&N_{PMRW}^{\{A\}}(3, n, IAC_b(k))\\
&= \frac{(n-3k)\{3(3-2k-6k^3) + (11+18k^2)n + 3(1-2k)n^2 + n^3\}}{12},\\
&\qquad\qquad \text{for } (n+1)/4 \le k < n/3.
\end{aligned} \tag{2.24}$$

Additional runs with $p = 4$ verify that this representation is valid for all $n = 7(4)\ldots$. By repeating this procedure with $\psi = 93$, this representation is found to be valid for all odd $n \ge 7$ with $(n+1)/4 \le k \le (n-1)/3$.

By using the identity in (2.9) along with the representations from (2.18) and (2.24), substitution and algebraic reduction lead to the same representation for $P_{PMRW}^{S}(3, n, IAC_b(k))$ with $(n+1)/4 \le k \le (n-1)/3$ that was obtained by algebraic methods in (2.10). The case of $k = n/3$ when n is an odd multiple of three must be handled as a special case, and it is quite easy to show that $P^S_{PMRW}(3,n,IAC_b(n/3)) = 3/4$.

By conducting a similar analysis for even values of n, a representation for $P^S_{PMRW}(3,n,IAC_b(k))$ with even $n \geq 8$ is obtained as:

$$\begin{aligned}
&P^S_{PMRW}(3,n\,|\,IAC_b(k))\\
&= \frac{2k(6+31k+11k^2) - 4(2+13k+2k^2)n + 3(3-2k)n^2 + 2n^3}{2(n-3k)[(n+1)(n+5) - 3k(2+k)]},\\
&\qquad\qquad\qquad\qquad \text{for } 0 \leq k \leq (n-4)/4\\
&\frac{2(2-3k+18k^2-9k^3) + 2(1-12k+9k^2)n + (5-6k)n^2 + n^3}{2(k+1)[(n+1)(n+5) - 3k(2+k)]},\\
&\qquad\qquad\qquad\qquad \text{for } n/4 \leq k \leq (n-1)/3\\
&\frac{3n^2}{4(n+3)^2}, \text{ for } k = \frac{n}{3}.
\end{aligned} \tag{2.25}$$

Table 2.4 gives a list of computed values for $P^S_{PMRW}(3,91\,|\,IAC_b(k))$ and $P^S_{PMRW}(3,92\,|\,IAC_b(k))$ from (2.10) and (2.25), for each value over the bounds of possible b values from $0 \leq k \leq 30$. These probabilities decrease as k increases, yielding strong support to the general hypothesis that the likelihood that paradoxical voting outcomes will be observed is expected to decrease as voters' preferences reflect greater degrees of mutual coherence. Similar to observations that were made in earlier analyses, the rate of convergence of $P^S_{PMRW}(3,n\,|\,IAC_b(k))$ to the limiting value of 3/4 occurs much faster for odd n than it does for even n.

The most important observation that can be made from Table 2.4 is that voting situations that are at all close to the condition of having a perfect weak positively unifying candidate, with $b = 0$, have a significantly increased probability that a PMRW will be present. This observation is clearly evident from the fact that $P^S_{PMRW}(3,91\,|\,IAC_b(k)) > 0.99$ for all values of $k \leq 7$. Moreover, voting situations that are farthest removed from this condition have a significantly reduced probability that a PMRW will exist, with $P^S_{PMRW}(3,91\,|\,IAC_b(k)) < 0.80$ for all $k \geq 25$.

2.4.1.2 Other $P^S_{PMRW}(3,n\,|\,IAC_X(k))$ Representations for Weak Measures

The EUPIA2 procedure can be used in the same manner to obtain representations for $P^S_{PMRW}(3,n\,|\,IAC_X(k))$ for each $X \in \{t,c,u,b^*,t^*,c^*,u^*\}$. However, this is simplified for Parameter t, based on the following result from Gehrlein (2004b).

Lemma 2.1 $P^S_{PMRW}(3,n\,|\,IAC_b(k)) = P^S_{PMRW}(3,n\,|\,IAC_t(k))$ for odd $n \geq 3$.

Thus, the impact of having voters' preferences reflect some degree of proximity to a perfect weak negatively unifying candidate is identical to the impact of having the same degree of proximity to perfect weak positively unifying candidate. At least this is true with regard to the relationship of these two measures of group mutual coherence to the probability that a PMRW exists.

Table 2.4 Computed values for each of $P^S_{PMRW}(3, 91|IAC_b(k))$, $P^S_{PMRW}(3, 92|IAC_b(k))$, $P^S_{PMRW}(3, 91|IAC_c(k))$ and $P^S_{PMRW}(3, 91|IAC_u(k))$

k	P^S_{PMRW} $(3, 91\|IAC_b(k))$	P^S_{PMRW} $(3, 92\|IAC_b(k))$	P^S_{PMRW} $(3, 91\|IAC_c(k))$	P^S_{PMRW} $(3, 91\|IAC_u(k))$
0	1.0000	0.9837	1.0000	1.0000
1	0.9997	0.9828	0.9920	0.9996
2	0.9991	0.9817	0.9894	0.9990
3	0.9982	0.9803	0.9841	0.9980
4	0.9971	0.9786	0.9810	0.9967
5	0.9957	0.9766	0.9762	0.9951
6	0.9939	0.9743	0.9729	0.9929
7	0.9919	0.9715	0.9683	0.9902
8	0.9894	0.9684	0.9648	0.9870
9	0.9866	0.9649	0.9602	0.9830
10	0.9833	0.9608	0.9565	0.9782
11	0.9795	0.9562	0.9520	0.9724
12	0.9751	0.9509	0.9481	0.9654
13	0.9700	0.9450	0.9435	0.9569
14	0.9641	0.9382	0.9394	0.9466
15	0.9574	0.9304	0.9347	0.9339
16	0.9496	0.9215	0.9304	0.9183
17	0.9404	0.9112	0.9255	0.8987
18	0.9297	0.8993	0.9211	0.8737
19	0.9170	0.8853	0.9160	0.8414
20	0.9017	0.8686	0.9115	0.7985
21	0.8832	0.8485	0.9063	0.7399
22	0.8601	0.8239	0.9016	0.6568
23	0.8325	0.7947	0.8965	0.5427
24	0.8088	0.7693	0.8921	0.4368
25	0.7900	0.7490	0.8875	0.3446
26	0.7754	0.7331	0.8839	0.2637
27	0.7645	0.7211	0.8803	0.1921
28	0.7569	0.7125	0.8779	0.1285
29	0.7523	0.7069	0.8758	0.0722
30	0.7503	0.7040	0.8751	0.0217

A representation for $P^S_{PMRW}(3, n \mid IAC_c(k))$ is obtained in Gehrlein (2005), and the details of how this representation was obtained with EUPIA2 are presented there. The development of this representation was complicated by an additional issue, since the representation has different forms for odd and even values of Parameter c. That is, the representation has periodicity equal to two for the k component. The resulting representation for odd $n \geq 3$ is given by

$$P^S_{PMRW}(3,n|IAC_c(k))$$

$$= \frac{\left[\begin{array}{c}(139k^3+472k^2+146k-244)k-4(7k^3+102k^2+84k-20)n \\ -6(9k^2-6k-16)n^2+16(k+1)n^3+3\delta^2_{k+1}\{(6k^2+24k-1)+4(k-2)n-2n^2\}\end{array}\right]}{16(k+1)(n-3k)\{(n+1)(n+5)-3k(2+k)\}},$$

$$\text{for } 0 \leq k \leq (n-1)/4$$

$$\frac{\left[\begin{array}{c}3(-39k^4+72k^3+38k^2-76k+1)+4(57k^3-54k^2-80k+19)n \\ -2(75k^2+6k-47)n^2+4(8k+5)n^3-n^4+3\delta^2_{k+1}\{(6k^2+24k-1)+4(k-2)n-2n^2\}\end{array}\right]}{16(k+1)(n-3k)\{(n+1)(n+5)-3k(2+k)\}},$$

$$\text{for } (n+1)/4 \leq k \leq (n-1)/3$$

$$\frac{7n^2 + 42n + 27}{8(n+3)^2}, \quad \text{for } k = n/3. \tag{2.26}$$

Here, $\delta_x^y = 1$ if x is an integer multiple of y. Otherwise, $\delta_x^y = 0$. The representation in (2.26) is used to compute the $P^S_{PMRW}(3, 91 \mid IAC_c(k))$ entries that are shown in Table 2.4 over the possible Parameter c values from $0 \le k \le 30$.

The values that are presented in Table 2.4 show some very interesting results, with $P^S_{PMRW}(3, 91 \mid IAC_b(k)) > P^S_{PMRW}(3, 91 \mid IAC_c(k))$ for $0 \le k \le 19$ and with $P^S_{PMRW}(3, 91 \mid IAC_c(k)) > P^S_{PMRW}(3, 91 \mid IAC_b(k))$ for $20 \le k \le 30$. This suggests that proximity to of a voting situation to the condition of having a perfect weak positively unifying candidate has more of an impact on the probability that a PMRW exists than does the proximity to a perfect weak polarizing candidate for small values of k. However, as k increases the reverse situation exists. Moreover, $P^S_{PMRW}(3, 91 \mid IAC_c(k))$ and $P^S_{PMRW}(3, 91 \mid IAC_b(k))$ do not seem to be approaching the same limiting value as $k \to n/3$. This observation is verified if we consider the values of these representations in the limiting case as $n \to \infty$, where $P^S_{PMRW}(3, \infty \mid IAC_c(k)) = 7/8$ from (2.26) while $P^S_{PMRW}(3, \infty \mid IAC_b(k)) = 3/4$ from (2.10).

A representation for $P^S_{PMRW}(3, n \mid IAC_u(k))$ was developed in conjunction with other results that are reported in Gehrlein (2008), with

$$P^S_{PMRW}(3, n \mid IAC_u(k))$$
$$= \frac{19k^3 + 93k^2 + 14k + 6 + 2(6k^2 - 24k - 1)n - 6(2k-1)n^2 + 2n^3}{13k^3 + 81k^2 + 14k + 6 + 2(7k^2 - 22k - 1)n - 6(2k-1)n^2 + 2n^3},$$
$$\text{for } 0 \le k \le (n-1)/4$$
$$\frac{3(n-3k)(9k^2 + 3 - 6kn + n^2)}{81k^3 + 54k^2 + 27k + 12 - (63k^2 + 36k + 5)n + 3(5k+2)n^2 - n^3},$$
$$\text{for } (n+1)/4 \le k \le n/3. \tag{2.27}$$

Some interesting results follow directly from these representations. Since a PMRW must exist if $b = 0$ or $t = 0$, it is obvious that a PMRW must exist if $u = 0$. It is also easy to prove that $P^S_{PMRW}(3, n \mid IAC_u(n/3)) = 0$ when n is an odd multiple of three, and this is also evident from the representation in (2.27). Calculated values of $P^S_{PMRW}(3, 91 \mid IAC_u(k))$ are listed in Table 2.4 for each $0 \le k \le 30$. These results yield some dramatic, but potentially misleading results. The calculated results for $P^S_{PMRW}(3, 91 \mid IAC_u(k))$ show a much stronger relationship between the probability that a PMRW exists and the value of Parameter u than was observed previously with any of the Parameters b, t or c.

The potentially misleading result comes from the very evident observation that $P^S_{PMRW}(3, 91 \mid IAC_b(k)) > P^S_{PMRW}(3, 91 \mid IAC_u(k))$ for all $k > 0$, which might make it appear that Parameter u is not as closely associated with the probability that a

PMRW exists than Parameter b is. However, while a PMRW must exist if either $b = 0$ or $u = 0$, the subset of voting situations for which $u = 0$ includes all of the voting situations for which $b = 0$, along with all of remaining voting situations for which $t = 0$. This difference in the basis of comparison of these probabilities does not therefore allow for a direct evaluation of the relative degree of the connection between these parameters and the probability that a PMRW exists. In order to make a fair comparison of these parameters for weak measures of group mutual coherence, it is necessary to consider some other factors.

2.5 Cumulative Probabilities that a PMRW Exists

Instead of considering representations for the probability $P^S_{PMRW}(3, n \,|\, IAC_X(k))$ that a PMRW exists when all voting situations are equally likely to be observed for which Parameter X has a specific value equal to k, it is more useful to consider cumulative probabilities for Parameter X. For each $X \in \{b, t, c, u\}$ a PMRW must exist when the value of X is equal to zero. The $CIAC_X(k^-)$ assumption is an extension of $IAC_X(k)$ that assumes that all voting situations for which Parameter X has a value of q in the range $0 \le q \le k$ are equally likely to be observed. Thus, as k decreases the set of voting situations that are being considered represents the subset of all of the possible voting situations that are closest to having a perfect weak positively unifying candidate, a perfect weak negatively unifying candidate, a perfect weak polarizing candidate or perfect weak overall unifying candidate.

The definitions of the cumulative probability $P^S_{PMRW}(3, n \,|\, CIAC_X(k^-))$ follow accordingly for each $X \in \{b, t, c, u\}$. These representations are found from a direct extension of the identity in (2.9) for each $0 \le k \le n/3$, with:

$$P^S_{PMRW}(3, n \,|\, CIAC_X(k^-)) = \frac{3 \sum_{q=0}^{k} N^{\{A\}}_{PMRW}(3, n, IAC_X(q))}{\sum_{q=0}^{k} K(3, n, IAC_X(q))}. \tag{2.28}$$

The algebraic manipulations that are required to obtain these representations for each $X \in \{b, t, c, u\}$ were performed to obtain results in Gehrlein (2008) for odd n:

$$P^S_{PMRW}(3, n \,|\, CIAC_b(k^-)) = P^S_{PMRW}(3, n \,|\, CIAC_t(k^-))$$
$$= \frac{2[(-41 + 69k + 22k^2)k + 5(5 - 18k - 2k^2)n + 10(3 - k)n^2 + 5n^3]}{(-73 + 117k + 36k^2)k + 5(10 - 33k - 3k^2)n + 20(3 - k)n^2 + 10n^3},$$
$$\text{for } 0 \le k \le (n-1)/4$$

$$\frac{\begin{bmatrix} 195-1968k-720k^2+3840k^3+4320k^4+1728k^5 \\ +(1661-1680k-6000k^2-5760k^3-2880k^4)n+10(165+200k+216k^2+192k^3)n^2 \\ +30(9-8k-24k^2)n^3+5(15+32k)n^4-11n^5 \end{bmatrix}}{16(k+1)(k+2)[(-73+117k+36k^2)k+5(10-33k-3k^2)n+20(3-k)n^2+10n^3]},$$
$$\text{for}(n+1)/4 \le k \le (n-1)/3. \tag{2.29}$$

$$P^S_{PMRW}(3,n\,|\,CIAC_c(k^-))$$
$$= \frac{\begin{bmatrix} (k+1)\begin{bmatrix} 165-783k+1743k^2+1597k^3+278k^4+10(71-233k-143k^2-7k^3)n \\ +30(31+3k-6k^2)n^2+80(k+2)n^3 \end{bmatrix} \\ -15\delta_k^2\{11+30k+6k^2-2(3-2k)n-2n^2\} \end{bmatrix}}{8(k+1)(k+2)[(-73+117k+36k^2)k+5(10-33k-3k^2)n+20(3-k)n^2+10n^3]},$$
$$\text{for } 0 \le k \le (n-1)/4$$
$$\frac{\begin{bmatrix} 435-952k+480k^2+2200k^3-90k^4-468k^5 \\ +(1349-2520k-4160k^2+840k^3+1140k^4)n+10(177+120k-162k^2-100k^3)n^2 \\ +10(39+72k+32k^2)n^3-5(3+4k)n^4+n^5-30\delta_k^2\{11+30k+6k^2-2(3-2k)n-2n^2\} \end{bmatrix}}{16(k+1)(k+2)[(-73+117k+36k^2)k+5(10-33k-3k^2)n+20(3-k)n^2+10n^3]},$$
$$\text{for } (n+1)/4 \le k \le (n-1)/3. \tag{2.30}$$

$$P^S_{PMRW}(3,n\,|\,CIAC_u(k^-))$$
$$= \frac{30+121k+261k^2+38k^3-10(1+15k-3k^2)n+10(3-4k)n^2+10n^3}{2(15+56k+111k^2+13k^3)-5(2+27k-7k^2)n+10(3-4k)n^2+10n^3},$$
$$\text{for } 0 \le k \le (n-1)/4$$
$$\frac{\begin{bmatrix} 27(25+64k+480k^2+1280k^3+1440k^4+576k^5) \\ +9(101-960k-3840k^2-5760k^3-2880k^4)n+90(29+128k+288k^2+192k^3)n^2 \\ -10(85+576k+576k^2)n^3+15(37+64k)n^4-59n^5 \end{bmatrix}}{16(n-2u)\begin{bmatrix} 18(k+1)(13+42k+63k^2+27k^3)-3(35+250k+360k^2+144k^3)n \\ +(25+24k)(5+6k)n^2-3(5+6k)n^3+n^4 \end{bmatrix}}$$
$$\text{for } (n+1)/4 \le k \le (n-1)/3. \tag{2.31}$$

Here, $\delta_x^y = 1$ if x is an integer multiple of y. Otherwise, $\delta_x^y = 0$.

It follows directly from definitions for each $X \in \{b,t,c,u\}$ that

$$P^S_{PMRW}\left(3,n\,|\,CIAC_X\left(\frac{n^-}{3}\right)\right) = P^S_{PMRW}(3,n,IAC) = \frac{15(n+3)^2}{16(n+2)(n+4)}. \tag{2.32}$$

These representations are far too unwieldy to serve as the basis of any useful analysis, so attention will be focused on the potentially most interesting case of large electorates with limiting probability as $n \to \infty$. To do this, k is replaced with

$\alpha_k n$ in the $P^S_{PMRW}(3, n \mid CIAC_X(k^-))$ representations, so that k is expressed as a proportion, α_k, of n, rather than as an integer value. It then follows from definitions that $0 \le \alpha_k \le 1/3$. The limiting representation as $n \to \infty$ is then determined. The resulting representations for the limiting distributions are denoted by $P^S_{PMRW}\left(3, \infty \mid CIAC_X\left(\alpha_k^-\right)\right)$, with:

$$\begin{aligned}
&P^S_{PMRW}\left(3, \infty \mid CIAC_b\left(\alpha_k^-\right)\right) = P^S_{PMRW}\left(3, \infty \mid CIAC_t\left(\alpha_k^-\right)\right)\\
&\quad = \frac{10 - 20\alpha_k - 20\alpha_k^2 + 44\alpha_k^3}{10 - 20\alpha_k - 15\alpha_k^2 + 36\alpha_k^3}, \text{ for } 0 \le \alpha_k \le 1/4\\
&\quad \frac{-11 + 160\alpha_k - 720\alpha_k^2 + 1920\alpha_k^3 - 2880\alpha_k^4 + 1728\alpha_k^5}{16\alpha_k^2(10 - 20\alpha_k - 15\alpha_k^2 + 36\alpha_k^3)}, \quad \text{for } 1/4 \le \alpha_k \le 1/3.
\end{aligned} \tag{2.33}$$

$$\begin{aligned}
&P^S_{PMRW}\left(3, \infty, \mid CIAC_c\left(\alpha_k^-\right)\right)\\
&\quad = \frac{40 - 90\alpha_k - 35\alpha_k^2 + 139\alpha_k^3}{40 - 80\alpha_k - 60\alpha_k^2 + 144\alpha_k^3}, \text{ for } 0 \le \alpha_k \le 1/4\\
&\quad = \frac{1 - 20\alpha_k + 320\alpha_k^2 - 1000\alpha_k^3 + 1140\alpha_k^4 - 468\alpha_k^5}{16\alpha_k^2(10 - 20\alpha_k - 15\alpha_k^2 + 36\alpha_k^3)}, \quad \text{for } 1/4 \le \alpha_k \le 1/3.
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
&P^S_{PMRW}\left(3, \infty \mid CIAC_u\left(\alpha_k^-\right)\right)\\
&\quad = \frac{10 - 40\alpha_k + 30\alpha_k^2 + 38\alpha_k^3}{10 - 40\alpha_k + 35\alpha_k^2 + 26\alpha_k^3}, \text{ for } 0 \le \alpha_k \le 1/4\\
&\quad \frac{-59 + 960\alpha_k - 5760\alpha_k^2 + 17280\alpha_k^3 - 25920\alpha_k^4 + 15552\alpha_k^5}{16(1 - 2\alpha_k)(1 - 18\alpha_k + 144\alpha_k^2 - 432\alpha_k^3 + 486\alpha_k^4)},\\
&\qquad \text{for } 1/4 \le \alpha_k \le 1/3.
\end{aligned} \tag{2.35}$$

These limiting representations as $n \to \infty$ are much more tractable. Following earlier discussion, these limiting representations result in specific values such that $P^S_{PMRW}(3, \infty \mid CIAC_X(0^-)) = 1$ and $P^S_{PMRW}(3, \infty \mid CIAC_X(1/3^-)) = 15/16$ for each $X \in \{b, t, c, u\}$. The cumulative probability representations ultimately will be very helpful in showing the relationship that exists between the probability that a PMRW exists and the degree of group mutual coherence that is present in voters' preferences. However, the original issue regarding the fact that there is a greater proportion of voting situations with $\alpha_k = 0$ for Parameter u than for Parameter b has not yet been resolved. In order to address this problem, attention is turned to the consideration of the proportion of voting situations that have a specified parameter value.

2.6 Proportions of Profiles with Specified Parameters

We want to develop representations for the proportion of all possible voting situations that have a specified value, q, of Parameter X in some given range $0 \le q \le k$. Define this proportion as $P_{VS}(3, n \,|\, CIAC_X(k^-))$ for each $X \in \{b, t, c, u\}$. The representations for $P_{VS}(3, n \,|\, CIAC_X(k^-))$ are obtained from an identity that follows directly from definitions for $0 \le k \le n/3$, with

$$P_{VS}(3, n \,|\, CIAC_X(k^-)) = \frac{\sum_{q=0}^{k} K(3, n, IAC_X(q))}{K(3, n, IAC)}. \tag{2.36}$$

Gehrlein (2008) performs the algebraic reduction of (2.36) to obtain

$$\begin{aligned} &P_{VS}(3, n \,|\, CIAC_b(k^-)) = P_{VS}(3, n \,|\, CIAC_t(k^-)) = P_{VS}(3, n \,|\, CIAC_c(k^-)) \\ &= \frac{\left[3(k+1)(k+2)\left\{\begin{array}{c} (-73 + 117k + 36k^2)k \\ +5(10 - 33k - 3k^2)n + 20(3-k)n^2 + 10n^3 \end{array}\right\}\right]}{(n+1)(n+2)(n+3)(n+4)(n+5)}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0 \le k \le (n-1)/3 \\ &1, \quad \text{for } k = n/3. \end{aligned} \tag{2.37}$$

Attention will be focused on the limiting distribution, $P_{VS}(3, \infty \,|\, CIAC_X(\alpha_k{}^-))$, as $n \to \infty$, and following the procedure that was used in earlier analyses,

$$\begin{aligned} &P_{VS}(3, \infty \,|\, CIAC_b(\alpha_k{}^-)) = P_{VS}(3, \infty \,|\, CIAC_t(\alpha_k{}^-)) = P_{VS}(3, \infty \,|\, CIAC_c(\alpha_k{}^-)) \\ &= 3\alpha_k{}^2(10 - 20\alpha_k - 15\alpha_k{}^2 + 36\alpha_k{}^3), \quad \text{for } 0 \le \alpha_k \le 1/3. \end{aligned} \tag{2.38}$$

The representation in (2.38) can be used as a basis of a search procedure to find specific values of β_b^p such that $P_{VS}\left(n, \infty \,|\, CIAC_b\left(\beta_b^{p^-}\right)\right) = p$ for each proportion $p = 0.00(0.05)1.00$, and the results are listed in Table 2.5. Based on previous discussion, $\beta_b^p = \beta_t^p = \beta_c^p$ for all p. The results in Table 2.5 indicate for example that 65% of all possible voting situations are included in the range of α_k parameter values with $0 \le \alpha_k \le 0.1924$ for Parameter b, t, or c, and 15% of all possible voting situations are included in the range of α_k parameter values with $0 \le \alpha_k \le 0.0564$ for Parameter u.

The results of Table 2.5 can now be used in conjunction with the limiting representations from (2.33) to compute the limiting conditional cumulative probability $P_{PMRW}^S\left(n, \infty \,|\, CIAC_b\left(\beta_b^{p^-}\right)\right)$ that a PMRW exists for the p percent of all voting situations that are closest to having a perfect weak positively unifying candidate. For example, suppose that we wish to consider the 20% of voting situations that are closest to having a perfect weak positively unifying candidate.

Table 2.5 Computed values of β_b^p, β_t^p, β_c^p and β_u^p, for each proportion $p = 0.00(0.05)1.00$

p	$\beta_b^p = \beta_t^p = \beta_c^p$	β_u^p
0.00	0.0000	0.0000
0.05	0.0428	0.0308
0.10	0.0619	0.0449
0.15	0.0772	0.0564
0.20	0.0908	0.0667
0.25	0.1033	0.0763
0.30	0.1150	0.0854
0.35	0.1264	0.0943
0.40	0.1374	0.1031
0.45	0.1483	0.1118
0.50	0.1591	0.1206
0.55	0.1700	0.1296
0.60	0.1811	0.1388
0.65	0.1924	0.1484
0.70	0.2042	0.1585
0.75	0.2166	0.1695
0.80	0.2298	0.1815
0.85	0.2445	0.1951
0.90	0.2614	0.2117
0.95	0.2829	0.2344
1.00	0.3333	0.3333

The results on Table 2.5 show that $\beta_b^{.20} = 0.0908$. This particular value is used with (2.33) to find that $P_{VS}\left(3, \infty \mid CIAC_b\left(\beta_b^{.20^-}\right)\right) = 0.9956$. So, the probability that a PMRW exists for the 20% of voting situations that are closest to having a perfect weak positively unifying candidate is 0.9956.

Computed values from all of the associated representations for $P_{PMRW}^S\left(n, \infty \mid CIAC_X\left(\beta_X^{p^-}\right)\right)$ for each $X \in \{b, t, c, u\}$ are listed in Table 2.6 for each proportion $p = 0.00(0.05)1.00$.

The values in Table 2.6 show some very interesting results. For example, the 50% of all possible voting situations that are closest to having a perfect weak positively or negatively unifying candidate have a PMRW with probability of 0.9857 for large electorates. And, the 15% of all possible voting situations that are closest to having a perfect weak polarizing candidate have a PMRW with probability of 0.9814 for large electorates. Clearly, any significant degree of group mutual coherence among voters' preferences that approaches having a perfect weak positively or negatively unifying candidate leads to a high probability that a PMRW exists. The impact of having voters' preferences that suggest the presence of a candidate approaching a perfect weak polarizing candidate in voting situations is also quite strong, but it is not as significant as the proximity to having a perfect weakly unifying candidate, assuming that there is an equivalence of these factors as they are measured by α_k, since $P_{PMRW}^S\left(n, \infty \mid CIAC_b\left(\beta_b^{p^-}\right)\right) > P_{PMRW}^S\left(n, \infty \mid CIAC_c\left(\beta_c^{p^-}\right)\right)$ for all $0 < p < 1$. Moreover, the results from Table 2.6 show that the 50% of voting situations that are most closely related to having a perfect weak overall unifying candidate have a probability 0.9910 of having a

Table 2.6 Computed values of $P^S_{PMRW}(n, \infty | CIAC_X(\beta_X^{p-}))$ for $X = b, t, c, u$ for each proportion $p = 0.00(0.05)1.00$

p	b, t	c	u
0.00	1.0000	1.0000	1.0000
0.05	0.9991	0.9895	0.9995
0.10	0.9980	0.9850	0.9989
0.15	0.9969	0.9814	0.9983
0.20	0.9956	0.9782	0.9975
0.25	0.9943	0.9753	0.9967
0.30	0.9929	0.9726	0.9958
0.35	0.9913	0.9701	0.9948
0.40	0.9896	0.9676	0.9936
0.45	0.9877	0.9652	0.9924
0.50	0.9857	0.9628	0.9910
0.55	0.9834	0.9605	0.9894
0.60	0.9809	0.9582	0.9876
0.65	0.9781	0.9558	0.9856
0.70	0.9749	0.9535	0.9832
0.75	0.9712	0.9510	0.9804
0.80	0.9669	0.9486	0.9770
0.85	0.9616	0.9460	0.9728
0.90	0.9548	0.9433	0.9671
0.95	0.9466	0.9405	0.9583
1.00	0.9375	0.9375	0.9375

PMRW. This suggests that any voting situation that is relatively close to representing perfect weak overall unifying candidate, as measure by Parameter u, will have a very high probability of yielding a PMRW with large electorates.

2.7 Results with Strong Measures of Group Coherence

The same type of analysis that we have just used with weak measures of group mutual coherence was applied to strong measures in Gehrlein (2009), but there are some differences in how these methods must be applied in that case. Representations are obtained for $P^S_{PMRW}(3, n \,|\, IAC_{X^*}(k))$ for each $X^* \in \{b^*, t^*, c^*, u^*\}$ in exactly the same fashion with EUPIA2. But, a major difference then occurs during the process of obtaining the cumulative probability representations that a PMRW exists with these strong measures of group coherence. The identity in (2.28) was based on the fact that parameter values for the weak measures of group mutual coherence in $X \in \{b, t, c, u\}$ were each closest to the condition of requiring that a PMRW must exist with $X = 0$. However, the parameters for the strong measures of group mutual coherence in $X^* \in \{b^*, t^*, c^*, u^*\}$ are each closest to requiring that a PMRW must exist when $X^* = n$.

For the strong measures of group mutual coherence in $X^* \in \{b^*, t^*, c^*, u^*\}$, the cumulative probability that a PMRW exists is therefore found for a specified range of q values for Parameter X^* in the range $k \leq q \leq n$. The resulting cumulative probability is denoted by $P^S_{PMRW}(3,n\,|\,CIAC_{X^*}(k^+))$. The representations for these cumulative probabilities follow directly from definitions for each possible value of k with $n/3 \le k \le n$, with

$$P^S_{PMRW}(3,n\,|\,CIAC_{X^*}(k^+)) = \frac{3\sum\limits_{q=k}^{n} N^{\{A\}}_{PMRW}(3,n,IAC_{X^*}(q))}{\sum\limits_{q=k}^{n} K(3,n,IAC_{X^*}(q))}. \tag{2.39}$$

The resulting representations are given by:

$$P^S_{PMRW}(3,n|CIAC_{b^*}(k^+)) = P^S_{PMRW}(3,n|CIAC_{t^*}(k^+))$$
$$= \frac{\left[\begin{array}{c} 3(576k^5-1440k^4+1280k^3-1200k^2+784k+65) \\ -(2880k^4-5760k^3+6000k^2-4560k-221)n \\ +10(192k^3-360k^2+344k-11)n^2-30(24k^2-40k+7)n^3+5(32k-17)n^4-11n^5 \end{array}\right]}{16[k(k+1)\{(k-1)(36k^2+45k-154)-5(3k^2+27k-40)n-20(k-4)n^2+10n^3\}]},$$
$$\text{for } (n+1)/3 \le k \le (n-1)/2. \tag{2.40}$$

1, for $(n+1)/2 \le k \le n$

$$P^S_{PMRW}(3,n|CIAC_{c^*}(k^+))$$
$$= \frac{\left[\begin{array}{c} 1476k^5-2610k^4+40k^3+824k+435-(2100k^4-2760k^3+2000k^2-2200k-757)n \\ +10(116k^3-186k^2+216k+25)n^2-10(40k^2-80k+9)n^3+5(20k-11)n^4-7n^5 \\ -30\delta_k^2\{3(10k^2-18k+5)-2(14k-11)n+6n^2\} \end{array}\right]}{16[k(k+1)\{(k-1)(36k^2+45k-154)-5(3k^2+27k-40)n-20(k-4)n^2+10n^3\}]},$$
$$\text{for } (n+1)/3 \le k \le (n-1)/2$$

$$\frac{\left[\begin{array}{c} (n+3-k)(n+1-k)\left\{\begin{array}{c} 34k^3-169k^2+42k+365 \\ -2(31k^2-49k-139)n+(22k+71)n^2+6n^3 \end{array}\right\} \\ -15(1-\delta_k^2)\{2k^2-10k+9-2(2k-5)n+2n^2\} \end{array}\right]}{8(n+1-k)(n+2-k)(n+3-k)(n+4-k)(n+5+4k)},$$
$$\text{for } (n+1)/2 \le k \le n. \tag{2.41}$$

$$P^S_{PMRW}(3,n|CIAC_{u^*}(k^+))$$
$$= \frac{3\left[\begin{array}{c} -9(576k^5-1440k^4+1280k^3-480k^2+64k-25) \\ +3(2880k^4-5760k^3+3840k^2-960k+229)n \\ -30(192k^3-288k^2+128k-29)n^2+30(64k^2-64k+19)n^3-5(64k-37)n^4+23n^5 \end{array}\right]}{16\left[\begin{array}{c} 36k(k-1)(27k^3-63k^2+42k-13)-6(315k^4-810k^3+695k^2-240k+13)n \\ +5(6k-1)(48k^2-82k+37)n^2-5(108k^2-132k+31)n^3+5(20k-11)n^4-7n^5 \end{array}\right]},$$
$$\text{for } n/3 \le k \le (3n-1)/8.$$

$$= \frac{\begin{bmatrix} 8608k^5 - 31760k^4 + 41600k^3 - 23920k^2 + 5892k - 135 \\ -3(5920k^4 - 16960k^3 + 16320k^2 - 6160k + 501)n + 90(160k^3 - 336k^2 + 212k - 33)n^2 \\ -90(64k^2 - 88k + 25)n^3 + 15(76k - 49)n^4 - 87n^5 \end{bmatrix}}{8\begin{bmatrix} 36k(k-1)(27k^3 - 63k^2 + 42k - 13) - 6(315k^4 - 810k^3 + 695k^2 - 240k + 13)n \\ +5(6k-1)(48k^2 - 82k + 37)n^2 - 5(108k^2 - 132k + 31)n^3 + 5(20k - 11)n^4 - 7n^5 \end{bmatrix}},$$
$$\text{for } (3n+1)/8 \le k \le (n-1)/2$$
$$= 1, \text{ for } (n+1)/2 \le k \le n. \tag{2.42}$$

It then follows directly from definitions for each $X^* \in \{b^*, t^*, c^*, u^*\}$ that

$$P^S_{PMRW}\left(3, n \mid CIAC_{X^*}\left(\left(\frac{n}{3}\right)^+\right)\right) = P^S_{PMRW}(3, n, IAC)$$
$$= \frac{15(n+3)^2}{16(n+2)(n+4)}. \tag{2.43}$$

Just as we observed in the case of the representations that were obtained for $P^S_{PMRW}(3, n \mid IAC_X(k))$ in (2.29), (2.30) and (2.31), the resulting representations for $P^S_{PMRW}(3, n \mid CIAC_{X^*}(k^+))$ in (2.40), (2.41) and (2.42) are far too cumbersome for any meaningful analysis. Following earlier analysis, attention therefore is focused on the limiting case for voters as $n \to \infty$, and the resulting representations are defined by $P^S_{PMRW}(3, \infty \mid CIAC_{X^*}(\alpha_k^+))$, for the range $1/3 \le \alpha_k \le 1$, with

$$P^S_{PMRW}(3, \infty \mid CIAC_{b^*}(\alpha_k{}^+)) = P^S_{PMRW}(3, \infty \mid CIAC_{t^*}(\alpha_k{}^+))$$
$$= \frac{1728\alpha_k^5 - 2880\alpha_k^4 + 1920\alpha_k^3 - 720\alpha_k^2 + 160\alpha_k - 11}{16\alpha_k^2(36\alpha_k^3 - 15\alpha_k^2 - 20\alpha_k + 10)}, \quad \text{for } 1/3 \le \alpha_k \le 1/2$$
$$1, \text{ for } 1/2 \le \alpha_k \le 1. \tag{2.44}$$

$$P^S_{PMRW}(3, \infty \mid CIAC_{c^*}(\alpha_k^+))$$
$$= \frac{1476\alpha_k^5 - 2100\alpha_k^4 + 1160\alpha_k^3 - 400\alpha_k^2 + 100\alpha_k - 7}{16\alpha_k^2(36\alpha_k^3 - 15\alpha_k^2 - 20\alpha_k + 10)}, \quad \text{for } 1/3 \le \alpha_k \le 1/2$$
$$= \frac{17\alpha_k + 3}{4(4\alpha_k + 1)}, \text{ for } 1/2 \le \alpha_k \le 1. \tag{2.45}$$

$$P^S_{PMRW}(3, \infty \mid CIAC_{u^*}(\alpha_k^+))$$
$$= \frac{3(-5184\alpha_k^5 + 8640\alpha_k^4 - 5760\alpha_k^3 + 1920\alpha_k^2 - 320\alpha_k + 23)}{16(6\alpha_k - 1)(1{,}62\alpha_k^4 - 288\alpha_k^3 + 192\alpha_k^2 - 58\alpha_k + 7)}, \text{ for } 1/3 \le \alpha_k \le 3/8.$$
$$\frac{8608\alpha_k^5 - 17760\alpha_k^4 + 14400\alpha_k^3 - 5760\alpha_k^2 + 1140\alpha_k - 87}{8(6\alpha_k - 1)(162\alpha_k^4 - 288\alpha_k^3 + 192\alpha_k^2 - 58\alpha_k + 7)}, \text{ for } 3/8 \le \alpha_k \le 1/2$$
$$1, \text{ for } 1/2 \le \alpha_k \le 1. \tag{2.46}$$

A direct comparison of the cumulative probability values that are obtained from these $P^S_{PMRW}\left(3,\infty \,|\, CIAC_{X^*}\left(\alpha_k^+\right)\right)$ representations for different strong measure of group mutual coherence, as measured by parameters in $X^* \in \{b^*, t^*, c^*, u^*\}$, does not lead to any clear results. The reason for this follows from the fact that the subset of all voting situations for which $b^* = n$ are included in the set of all voting situations with $u^* = n$, along with all other voting situations with $t^* = n$. So the basis of comparison is not the same in all cases. In order to facilitate further analysis, we develop representations for the proportion, $P_{VS}(3, n \,|\, CIAC_{X^*}(k^+))$, of all voting situations that have a specified value, q, for Parameter X^* in the range $k \le q \le n$. These representations are obtained from the identity.

$$P_{VS}(3,n\,|\,CIAC_{X^*}(k^+)) = \frac{\sum\limits_{q=k}^{n} K(3,n,IAC_{X^*}(q))}{K(3,n,IAC)}. \tag{2.47}$$

The necessary algebraic reduction of (2.47) is performed in Gehrlein (2009), to obtain representations for $P_{VS}(3, n \,|\, CIAC_{X^*}(k^+))$ with each $X^* \in \{b^*, t^*, c^*, u^*\}$:

$$\begin{aligned}
&P_{VS}(3,n,CIAC_{b^*}(k^+)) = P_{VS}(3,n,CIAC_{t^*}(k^+)) = P_{VS}(3,n,CIAC_{c^*}(k^+))\\
&= \frac{3k(k+1)[(k-1)(36k^2+45k-154)-5(3k^2+27k-40)n-20(k-4)n^2+10n^3]}{(n+1)(n+2)(n+3)(n+4)(n+5)},\\
&\qquad\qquad\qquad\qquad \text{for } n/3 < k \le (n-1)/2\\
&\frac{3(n+1-k)(n+2-k)(n+3-k)(n+4-k)(n+5+4k)}{(n+1)(n+2)(n+3)(n+4)(n+5)}, \text{ for } [(n+1)/2 \le k \le n.
\end{aligned} \tag{2.48}$$

$$\begin{aligned}
&P_{VS}(3,n,CIAC_{u^*}(k^+))\\
&= \frac{3\begin{bmatrix} 36k(k-1)(27k^3-63k^2+42k-13)-6(315k^4-810k^3+695k^2-240k+13)n \\ +5(6k-1)(48k^2-82k+37)n^2-5(108k^2-132k+31)n^3+5(20k-11)n^4-7n^5 \end{bmatrix}}{(n+1)(n+2)(n+3)(n+4)(n+5)},\\
&\qquad\qquad\qquad\qquad \text{for } n/3 < k \le (n-1)/2\\
&\frac{6(n+1-k)(n+2-k)(n+3-k)(n+4-k)(6k-n)}{(n+1)(n+2)(n+3)(n+4)(n+5)}, \quad \text{for } (n+1)/2 \le k \le n.
\end{aligned} \tag{2.49}$$

The limiting representations as $n \to \infty$ are obtained from (2.48) and (2.49) following previous discussion, with:

$$\begin{aligned}
&P_{VS}\left(3,\infty,CIAC_{b^*}\left(\alpha_k^+\right)\right) = P_{VS}\left(3,\infty,CIAC_{t^*}\left(\alpha_k^+\right)\right) = P_{VS}\left(3,\infty,CIAC_{c^*}\left(\alpha_k^+\right)\right)\\
&\quad = 3\alpha_k^2(36\alpha_k^3 - 15\alpha_k^2 - 20\alpha_k + 10), \quad \text{for } 1/3 \le \alpha_k \le 1/2\\
&\qquad 3(1-\alpha_k)^4(4\alpha_k+1), \text{ for } 1/2 \le \alpha_k \le 1.
\end{aligned} \tag{2.50}$$

$$\begin{aligned} &P_{VS}\left(3,\infty,CIAC_{u^*}\left(\alpha_k^+\right)\right) \\ &\quad = 3(6\alpha_k-1)(162\alpha_k^4-288\alpha_k^3+192\alpha_k^2-58\alpha_k+7), \quad \text{for } 1/3\le\alpha_k\le 1/2 \\ &\qquad 6(1-\alpha_k)^4(6\alpha_k-1), \quad \text{for } 1/2\le\alpha_k\le 1. \end{aligned} \tag{2.51}$$

These results show for example that $P_{VS}(3,\infty,CIAC_{u^*}(0.50^+)) = 0.75$, so that 75% of all voting situations have a value of u^*/n in the range 0.50–1.00 in the limit as $n\to\infty$. A search procedure was then initiated with these representations to find the specific values of $\beta_{X^*}^p$ such that $P_{VS}\left(3,\infty,CIAC_X\left(\beta_{X^*}^{p\ +}\right)\right) = p$ for each $X^*\in\{b^*,t^*,c^*,u^*\}$ with $p = 0.00(0.05)1.00$ and the results are summarized in Table 2.7.

These $\beta_{X^*}^p$ values from Table 2.7 are used in conjunction with the representations from (2.44), (2.45) and (2.46) to obtain the cumulative probability values that a PMRW exists from $P_{PMRW}^S\left(3,\infty\,|\,CIAC_{X^*}\left(\beta_{X^*}^{p\ +}\right)\right)$ for each strong measure of group mutual coherence from $X^*\in\{b^*,t^*,c^*,u^*\}$ with $p = 0.00(0.05)1.00$. The results of these computations are summarized in Table 2.8, and some very interesting and compelling observations directly follow from them.

Just as we observed in the case of the proximity of a voting situation to having a perfect weak polarizing candidate for weak measures of group mutual coherence, the proximity of a voting situation to having a perfect strong centrist candidate has the least amount of impact on the probability that a PMRW will exist. A somewhat surprising result is that the 55% of voting situations that are closest to having a perfect strong positively unifying candidate or perfect strong negatively unifying

Table 2.7 Values of $\beta_{X^*}^p$ for each $X^*\in\{b^*,t^*,c^*,u^*\}$ for each $p = 0.00(0.05)1.00$

p	$\beta_{b^*}^p = \beta_{t^*}^p = \beta_{c^*}^p$	$\beta_{u^*}^p$
0.00	1.0000	1.0000
0.05	0.7456	0.7820
0.10	0.6934	0.7357
0.15	0.6574	0.7032
0.20	0.6289	0.6770
0.25	0.6049	0.6546
0.30	0.5839	0.6347
0.35	0.5651	0.6166
0.40	0.5479	0.5998
0.45	0.5320	0.5840
0.50	0.5173	0.5689
0.55	0.5033	0.5545
0.60	0.4902	0.5405
0.65	0.4773	0.5268
0.70	0.4645	0.5133
0.75	0.4514	0.5000
0.80	0.4376	0.4865
0.85	0.4226	0.4720
0.90	0.4054	0.4551
0.95	0.3838	0.4323
1.00	0.3333	0.3333

Table 2.8 Values of $P^S_{PMRW}\left(\infty|CIAC_{X^*}\left(\beta^{p}_{X^*}{}^{+}\right)\right)$, for each $X^* \in \{b^*, t^*, c^*, u^*\}$ for each $p = 0.00(0.05)1.00$

p	b^*, t^*	c^*	u^*
0.00	1.0000	1.0000	1.0000
0.05	1.0000	0.9840	1.0000
0.10	1.0000	0.9797	1.0000
0.15	1.0000	0.9764	1.0000
0.20	1.0000	0.9736	1.0000
0.25	1.0000	0.9711	1.0000
0.30	1.0000	0.9688	1.0000
0.35	1.0000	0.9667	1.0000
0.40	1.0000	0.9646	1.0000
0.45	1.0000	0.9626	1.0000
0.50	1.0000	0.9607	1.0000
0.55	1.0000	0.9588	1.0000
0.60	0.9988	0.9569	1.0000
0.65	0.9946	0.9544	1.0000
0.70	0.9885	0.9530	1.0000
0.75	0.9812	0.9508	1.0000
0.80	0.9732	0.9485	0.9969
0.85	0.9647	0.9460	0.9891
0.90	0.9558	0.9433	0.9775
0.95	0.9468	0.9405	0.9617
1.00	0.9375	0.9375	0.9375

candidate have a PMRW with certainty. The most compelling observation is that the 75% of voting situations that are closest to having a perfect strong overall unifying candidate will have a PMRW with absolute certainty.

2.8 Conclusion

When voters' preferences in a three-candidate voting situation reflect any significant degree of proximity to having a perfect weak positively or negatively unifying candidate, the probability that a PMRW exists is high. When voters' preferences are at all close to reflecting a situation in which a perfect weak overall unifying candidate exists, the probability that a PMRW exists is very high. An even stronger relationship is shown to exist when voting situations are at all close to having a perfect strong positively or negatively unifying candidate. A PMRW must exist when voting situations are even remotely close to having a perfect strong overall unifying candidate.

It is very important to note that the associated underlying models that lead to any of these measures of mutual group coherence do not actually have to be the basis of the mechanism by which the voters' preference rankings on candidates were actually formed. It is only required that the preferences in a given voting situation could have been obtained by one of these models. As a result, it is easily concluded that Condorcet's Paradox should very rarely be observed in any real elections on a small number of candidates with large electorates, as long as voters' preferences reflect any reasonable degree of group mutual coherence from a number of different possible models, and the observations that have been made from numerous empirical studies should no longer seem surprising.

It can also be concluded from these observations that the use of the *Condorcet Criterion* that voting rules should select the PMRW whenever one exists is a very valid measure of the effectiveness of various voting rules at selecting the alternative that is the overall most preferred candidate. Arguments against the use of the Condorcet Criterion are typically based on the fact that a PMRW does not always exist, so that there might be some confusion over which candidate should be selected as the winner. However, our results indicate that the probability that this confounding issue would ever result is expected to be very small for elections on a small number of candidates with a large number of voters.

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