Chapter 2
Perturbations of Anti de Sitter Black Holes

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Abstract I review perturbations of black holes in asymptotically anti de Sitter space. I show how the quasi-normal modes governing these perturbations can be calculated analytically and discuss the implications on the hydrodynamics of gauge theory fluids per the AdS/CFT correspondence. I also discuss phase transitions of hairy black holes with hyperbolic horizons and the dual superconductors emphasizing the analytical calculation of their properties.

2.1 Introduction

The perturbations of a black hole are governed by quasi-normal modes (QNMs). The latter are typically obtained by solving a wave equation for small fluctuations in the black hole background subject to the conditions that the flux be ingoing at the horizon and outgoing at asymptotic infinity. These boundary conditions in general lead to a discrete spectrum of complex frequencies whose imaginary part determines the decay time of the small fluctuations

$$\Im\omega = \frac{1}{\tau}. \tag{2.1}$$

There is a vast literature on quasi-normal modes and I make no attempt to review it here. Instead, I concentrate on obtaining analytic expressions for QNMs of black hole perturbations in asymptotically AdS space. One can rarely obtain analytic expressions in closed form. Instead, I discuss techniques which allow one to

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calculate the spectrum perturbatively in an asymptotic regime (high or low overtones). For high overtones, the frequencies at leading order are proportional to the radius of the horizon. For low overtones, one in general obtains an additional frequency which is inversely proportional to the horizon radius. Thus for large black holes there is a gap between the lowest frequency and the rest of the spectrum of QNMs. I pay special attention to the lowest frequencies because they govern the behavior of the gauge theory fluid on the boundary according to the AdS/CFT correspondence. The latter may have experimental consequences pertaining to the formation of the quark-gluon plasma in heavy ion collisions. Moreover, I discuss phase transitions to hairy black holes which correspond to a dual superconducting phase. I concentrate on the case of black holes with a hyperbolic horizon because their properties can be understood analytically.

In Sect. 2.2 I discuss scalar, gravitational and electromagnetic perturbations of an AdS Schwarzschild black hole analytically calculating the QNM spectrum in the high frequency regime. In Sect. 2.3 I calculate the QNM spectrum analytically in the low frequency regime and discuss its relevance to the hyrdodynamic behavior of the dual gauge theory fluid on the boundary. In Sect. 2.4 I introduce hairy black holes and discuss their phase transition in the case of a hyperbolic horizon which can be understood analytically. The dual gauge theory corresponds to a superconductor whose properties can be calculated via electromagnetic perturbations. Finally, I conclude with Sect. 2.5.

2.2 Perturbations

In this section I discuss scalar, gravitational and electromagnetic perturbations of an AdS Schwarzschild black hole in d dimensions analytically calculating the QNM spectrum in the high frequency regime. Low overtones will be discussed in the next section.

The metric of an AdS Schwarzschild black hole is

$$ds^2 = -\left(\frac{r^2}{l^2} + K - \frac{2\mu}{r^{d-3}}\right)dt^2 + \frac{dr^2}{\frac{r^2}{l^2} + K - \frac{2\mu}{r^{d-3}}} + r^2 d\Sigma^2_{K,d-2}. \tag{2.2}$$

I shall choose units so that the AdS radius $l = 1$. The horizon radius and Hawking temperature are, respectively,

$$2\mu = r_+^{d-1}\left(1 + \frac{K}{r_+^2}\right), \quad T_H = \frac{(d-1)r_+^2 + K(d-3)}{4\pi r_+}. \tag{2.3}$$

The mass and entropy of the hole are, respectively,

$$M = (d-2)(K + r_+^2)\frac{r_+^{d-3}}{16\pi G} Vol(\Sigma_{K,d-2}), \quad S = \frac{r_+^{d-2}}{4G} Vol(\Sigma_{K,d-2}). \tag{2.4}$$

The parameter K determines the curvature of the horizon and the boundary of AdS space. For $K = 0, +1, -1$ we have, respectively, a flat ($\mathbb{R}^{d-2}$), spherical ($\mathbb{S}^{d-2}$) and hyperbolic ($\mathbb{H}^{d-2}/\Gamma$, topological black hole, where Γ is a discrete group of isometries) horizon (boundary).

The harmonics on $\Sigma_{K,d-2}$ satisfy

$$\left(\nabla^2 + k^2\right)\mathbb{T} = 0. \tag{2.5}$$

For $K = 0$, k is the momentum; for $K = +1$, the eigenvalues are quantized,

$$k^2 = l(l + d - 3) - \delta, \tag{2.6}$$

whereas for $K = -1$,

$$k^2 = \xi^2 + \left(\frac{d-3}{2}\right)^2 + \delta, \tag{2.7}$$

where ξ is discrete for non-trivial Γ. $\delta = 0, 1, 2$ for scalar, vector, or tensor perturbations, respectively.

According to the AdS/CFT correspondence, QNMs of AdS black holes are expected to correspond to perturbations of the dual Conformal Field Theory (CFT) on the boundary. The establishment of such a correspondence is hindered by difficulties in solving the wave equation governing the various types of perturbation. In three dimensions one obtains a hypergeometric equation which leads to explicit analytic expressions for the QNMs [1, 2]. In five dimensions one obtains a Heun equation and a derivation of analytic expressions for QNMs is no longer possible. On the other hand, numerical results exist in four, five and seven dimensions [3–5].

2.2.1 Scalar Perturbations

To find the asymptotic form of QNMs, we need to find an approximation to the wave equation valid in the high frequency regime. In three dimensions the resulting wave equation will be an exact equation (hypergeometric equation). In five dimensions, I shall turn the Heun equation into a hypergeometric equation which will lead to an analytic expression for the asymptotic form of QNM frequencies in agreement with numerical results.

2.2.1.1 AdS$_3$

In three dimensions the wave equation for a massless scalar field is

$$\frac{1}{r}\partial_r\left(r^3\left(1 - \frac{r_+^2}{r^2}\right)\partial_r\Phi\right) - \frac{1}{r^2 - r_+^2}\partial_t^2\Phi + \frac{1}{r^2}\partial_x^2\Phi = 0. \tag{2.8}$$

Writing the wavefunction in the form

$$\Phi = e^{i(\omega t - px)}\Psi(y), \quad y = \frac{r_+^2}{r^2}, \tag{2.9}$$

the wave function becomes

$$y^2(y-1)((y-1)\Psi')' + \hat{\omega}^2 y\Psi + \hat{p}^2 y(y-1)\Psi = 0 \tag{2.10}$$

to be solved in the interval $0 < y < 1$, where

$$\hat{\omega} = \frac{\omega}{2r_+} = \frac{\omega}{4\pi T_H}, \quad \hat{p} = \frac{p}{2r_+} = \frac{p}{4\pi T_H}. \tag{2.11}$$

For QNMs, we are interested in the solution

$$\Psi(y) = y(1-y)^{i\hat{\omega}}{}_2F_1(1 + i(\hat{\omega} + \hat{p}), 1 + i(\hat{\omega} - \hat{p}); 2; y), \tag{2.12}$$

which vanishes at the boundary ($y \to 0$). Near the horizon ($y \to 1$), we obtain a mixture of ingoing and outgoing waves,

$$\Psi \sim A_+(1-y)^{-i\hat{\omega}} + A_-(1-y)^{+i\hat{\omega}}, \quad A_\pm = \frac{\Gamma(\pm 2i\hat{\omega})}{\Gamma(1 \pm i(\hat{\omega} + \hat{p}))\Gamma(1 \pm i(\hat{\omega} - \hat{p}))}.$$

Setting $A_- = 0$, we deduce the quasi-normal frequencies

$$\hat{\omega} = \pm\hat{p} - in, \quad n = 1, 2, \ldots \tag{2.13}$$

which form a discrete spectrum of complex frequencies with $\Im\hat{\omega} < 0$.

2.2.1.2 AdS$_5$

Restricting attention to the case of a large black hole, the massless scalar wave equation reads

$$\frac{1}{r^3}\partial_r(r^5 f(r)\,\partial_r\Phi) - \frac{1}{r^2 f(r)}\partial_t^2\Phi - \frac{1}{r^2}\,\nabla^2\Phi = 0, \quad f(r) = 1 - \frac{r_+^4}{r^4}. \tag{2.14}$$

Writing the solution in the form

$$\Phi = e^{i(\omega t - \boldsymbol{p}\cdot\boldsymbol{x})}\Psi(y), \quad y = \frac{r^2}{r_+^2} \tag{2.15}$$

the radial wave equation becomes

$$(y^2-1)\big(y(y^2-1)\Psi'\big)' + \left(\frac{\hat{\omega}^2}{4}\,y^2 - \frac{\hat{p}^2}{4}\,(y^2-1)\right)\Psi = 0. \tag{2.16}$$

For QNMs, we are interested in the analytic solution which vanishes at the boundary and behaves as an ingoing wave at the horizon. The wave equation contains an additional (unphysical) singularity at $y = -1$, at which the wavefunction behaves as $\Psi \sim (y+1)^{\pm\hat{\omega}/4}$. Isolating the behavior of the wavefunction near the singularities $y = \pm 1$,

$$\Psi(y) = (y-1)^{-i\hat{\omega}/4}(y+1)^{\pm\hat{\omega}/4}F_{\pm}(y), \tag{2.17}$$

we shall obtain two sets of modes with the same $\Im\hat{\omega}$, but opposite $\Re\hat{\omega}$.

$F_{\pm}(y)$ satisfies the Heun equation

$$\begin{aligned} &y(y^2-1)F\pm'' + \left\{\left(3 - \frac{i\pm 1}{2}\,\hat{\omega}\right)y^2 - \frac{i\pm 1}{2}\,\hat{\omega}y - 1\right\}F'_{\pm} \\ &+ \left\{\frac{\hat{\omega}}{2}\left(\pm\frac{i\hat{\omega}}{4} \mp 1 - i\right)y - (i\mp 1)\frac{\hat{\omega}}{4} - \frac{\hat{p}^2}{4}\right\} F_{\pm} = 0 \end{aligned} \tag{2.18}$$

to be solved in a region in the complex y-plane containing $|y| \geq 1$ which includes the physical regime $r > r_+$.

For large $\hat{\omega}$, the constant terms in the polynomial coefficients of F' and F are small compared with the other terms, therefore they may be dropped. The wave equation may then be approximated by a hypergeometric equation

$$(y^2-1)F''_{\pm} + \left\{\left(3 - \frac{i\pm 1}{2}\,\hat{\omega}\right)y - \frac{i\pm 1}{2}\,\hat{\omega}\right\}F'_{\pm} + \frac{\hat{\omega}}{2}\left(\pm\frac{i\hat{\omega}}{4} \mp 1 - i\right) F_{\pm} = 0, \tag{2.19}$$

in the asymptotic limit of large frequencies $\hat{\omega}$. The acceptable solution is

$$F_0(x) = {}_2F_1(a_+, a_-; c; (y+1)/2), \quad a_{\pm} = 1 - \tfrac{i\pm 1}{4}\,\hat{\omega} \pm 1, \quad c = \tfrac{3}{2} \pm \tfrac{1}{2}\hat{\omega}. \tag{2.20}$$

For proper behavior at the boundary ($y \to \infty$), we demand that F be a *polynomial*, which leads to the condition

$$a_+ = -n, \quad n = 1, 2, \ldots \tag{2.21}$$

Indeed, it implies that F is a polynomial of order n, so as $y \to \infty$, $F \sim y^n \sim y^{-a_+}$ and $\Psi \sim y^{-i\hat{\omega}/4}y^{\pm\hat{\omega}/4}y^{-a_+} \sim y^{-2}$, as expected.

We deduce the quasi-normal frequencies [6]

$$\hat{\omega} = \frac{\omega}{4\pi T_H} = 2n(\pm 1 - i) \tag{2.22}$$

in agreement with numerical results.

It is perhaps worth mentioning that these frequencies may also be deduced by a simple monodromy argument [6]. Considering the monodromies around the singularities, if the wavefunction has no singularities other than $y = \pm 1$, the contour around $y = +1$ may be unobstructedly deformed into the contour around $y = -1$, which yields

$$\mathcal{M}(1)\mathcal{M}(-1) = 1. \tag{2.23}$$

Since the respective monodromies are $\mathcal{M}(1) = e^{\pi\hat{\omega}/2}$ and $\mathcal{M}(-1) = e^{\mp i\pi\hat{\omega}/2}$, using $\Im\hat{\omega} < 0$, we deduce $\hat{\omega} = 2n(\pm 1 - i)$, in agreement with our result above.

2.2.2 Gravitational Perturbations

Next I consider gravitational perturbations. For definiteness, I concentrate on the case of spherical black holes ($K = +1$). I shall derive analytic expressions for QNMs [7] including first-order corrections [8]. The results are in good agreement with results of numerical analysis [9]. Extension to other forms of the horizon is straightforward [10].

The radial wave equation for gravitational perturbations in the black-hole background (2.2) can be cast into a Schrödinger-like form,

$$-\frac{d^2\Psi}{dr_*^2} + V[r(r_*)]\Psi = \omega^2\Psi \ , \tag{2.24}$$

in terms of the tortoise coordinate defined by

$$\frac{dr_*}{dr} = \frac{1}{f(r)} \ . \tag{2.25}$$

The potential V for the various types of perturbation has been found by Ishibashi and Kodama [11]. For tensor, vector and scalar perturbations, one obtains, respectively,

$$V_{\mathsf{T}}(r) = f(r)\left\{\frac{\ell(\ell+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} + \frac{(d-2)f'(r)}{2r}\right\} \tag{2.26}$$

$$V_{\mathsf{V}}(r) = f(r)\left\{\frac{\ell(\ell+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} - \frac{rf'''(r)}{2(d-3)}\right\} \tag{2.27}$$

$$\begin{aligned}
V_{\mathsf{S}}(r) = \frac{f(r)}{4r^2}&\left[\ell(\ell+d-3)-(d-2)+\frac{(d-1)(d-2)\mu}{r^{d-3}}\right]^{-2} \\
&\times\left\{\frac{d(d-1)^2(d-2)^3\mu^2}{R^2r^{2d-8}} - \frac{6(d-1)(d-2)^2(d-4)[\ell(\ell+d-3)-(d-2)]\mu}{R^2r^{d-5}}\right. \\
&+\frac{(d-4)(d-6)[\ell(\ell+d-3)-(d-2)]^2r^2}{R^2} + \frac{2(d-1)^2(d-2)^4\mu^3}{r^{3d-9}} \\
&+\frac{4(d-1)(d-2)(2d^2-11d+18)[\ell(\ell+d-3)-(d-2)]\mu^2}{r^{2d-6}} \\
&+\frac{(d-1)^2(d-2)^2(d-4)(d-6)\mu^2}{r^{2d-6}} \\
&-\frac{6(d-2)(d-6)[\ell(\ell+d-3)-(d-2)]^2\mu}{r^{d-3}} \\
&-\frac{6(d-1)(d-2)^2(d-4)[\ell(\ell+d-3)-(d-2)]\mu}{r^{d-3}} \\
&\left.+4[\ell(\ell+d-3)-(d-2)]^3 + d(d-2)[\ell(\ell+d-3)-(d-2)]^2\right\},
\end{aligned}$$

Near the black hole singularity ($r \sim 0$),

$$V_{\mathsf{T}} = -\frac{1}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{T}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \cdots, \quad \mathcal{A}_{\mathsf{T}} = \frac{(d-3)^2}{2(2d-5)} + \frac{\ell(\ell+d-3)}{d-2}, \tag{2.28}$$

$$V_{\mathsf{V}} = \frac{3}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{V}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \cdots, \quad \mathcal{A}_{\mathsf{V}} = \frac{d^2-8d+13}{2(2d-15)} + \frac{\ell(\ell+d-3)}{d-2} \tag{2.29}$$

and

$$V_{\mathsf{S}} = -\frac{1}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{S}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \cdots, \tag{2.30}$$

where

$$\mathcal{A}_{\mathsf{S}} = \frac{(2d^3-24d^2+94d-116)}{4(2d-5)(d-2)} + \frac{(d^2-7d+14)[\ell(\ell+d-3)-(d-2)]}{(d-1)(d-2)^2}. \tag{2.31}$$

I have included only the terms which contribute to the order I am interested in. The behavior of the potential near the origin may be summarized by

$$V = \frac{j^2-1}{4r_*^2} + \mathcal{A}\, r_*^{-\frac{d-1}{d-2}} + \cdots \tag{2.32}$$

where $j = 0$ (2) for scalar and tensor (vector) perturbations.

On the other hand, near the boundary (large r),

$$V = \frac{j_\infty^2-1}{4(r_*-\bar{r}_*)^2} + \cdots, \quad \bar{r}_* = \int_0^\infty \frac{dr}{f(r)}, \tag{2.33}$$

where $j_\infty = d-1$, $d-3$ and $d-5$ for tensor, vector and scalar perturbations, respectively.

After rescaling the tortoise coordinate ($z = \omega r_*$), the wave equation to first order becomes

$$\left(\mathcal{H}_0 + \omega^{-\frac{d-3}{d-2}} \mathcal{H}_1\right)\Psi = 0, \tag{2.34}$$

where

$$\mathcal{H}_0 = \frac{d^2}{dz^2} - \left[\frac{j^2-1}{4z^2} - 1\right], \quad \mathcal{H}_1 = -\mathcal{A}\, z^{-\frac{d-1}{d-2}}. \tag{2.35}$$

By treating $\mathcal{H}_1$ as a perturbation, one may expand the wave function

$$\Psi(z) = \Psi_0(z) + \omega^{-\frac{d-3}{d-2}}\Psi_1(z) + \cdots \tag{2.36}$$

and solve the wave equation perturbatively.

The zeroth-order wave equation,

$$\mathcal{H}_0\Psi_0(z) = 0, \tag{2.37}$$

may be solved in terms of Bessel functions,

$$\Psi_0(z) = A_1\sqrt{z}\,J_{\frac{j}{2}}(z) + A_2\sqrt{z}\,N_{\frac{j}{2}}(z). \tag{2.38}$$

For large z, it behaves as

$$\begin{aligned}\Psi_0(z) &\sim \sqrt{\frac{2}{\pi}}[A_1\cos(z-\alpha_+) + A_2\sin(z-\alpha_+)]\\ &= \frac{1}{\sqrt{2\pi}}(A_1 - iA_2)e^{-i\alpha_+}e^{iz} + \frac{1}{\sqrt{2\pi}}(A_1 + iA_2)e^{+i\alpha_+}e^{-iz},\end{aligned}$$

where $\alpha_\pm = \frac{\pi}{4}(1 \pm j)$.

At the boundary ($r \to \infty$), the wavefunction ought to vanish, therefore the acceptable solution is

$$\Psi_0(r_*) = B\sqrt{\omega(r_* - \bar{r}_*)}\;J_{\frac{j_\infty}{2}}(\omega(r_* - \bar{r}_*)). \tag{2.39}$$

Indeed, $\Psi \to 0$ as $r_* \to \bar{r}_*$, as desired.

Asymptotically (large z), it behaves as

$$\Psi(r_*) \sim \sqrt{\frac{2}{\pi}}B\cos[\omega(r_* - \bar{r}_*) + \beta], \quad \beta = \frac{\pi}{4}(1 + j_\infty). \tag{2.40}$$

This ought to be matched to the asymptotic form of the wavefunction in the vicinity of the black-hole singularity along the Stokes line $\Im z = \Im(\omega r_*) = 0$. This leads to a constraint on the coefficients A_1, A_2,

$$A_1\tan(\omega\bar{r}_* - \beta - \alpha_+) - A_2 = 0. \tag{2.41}$$

By imposing the boundary condition at the horizon

$$\Psi(z) \sim e^{iz}, \quad z \to -\infty, \tag{2.42}$$

one obtains a second constraint. To find it, one needs to analytically continue the wavefunction near the black hole singularity ($z = 0$) to negative values of z. A rotation of z by $-\pi$ corresponds to a rotation by $-\frac{\pi}{d-2}$ near the origin in the complex r-plane. Using the known behavior of Bessel functions

$$J_\nu(e^{-i\pi}z) = e^{-i\pi\nu}J_\nu(z), \quad N_\nu(e^{-i\pi}z) = e^{i\pi\nu}N_\nu(z) - 2i\cos\pi\nu\,J_\nu(z), \tag{2.43}$$

for $z<0$ the wavefunction changes to

$$\Psi_0(z) = e^{-i\pi(j+1)/2}\sqrt{-z}\left\{\left[A_1 - i(1+e^{i\pi j})A_2\right] J_{\frac{j}{2}}(-z) + A_2 e^{i\pi j} N_{\frac{j}{2}}(-z)\right\} \quad (2.44)$$

whose asymptotic behavior is given by

$$\Psi \sim \frac{e^{-i\pi(j+1)/2}}{\sqrt{2\pi}}\left[A_1 - i(1+2e^{j\pi i})A_2\right] e^{-iz} + \frac{e^{-i\pi(j+1)/2}}{\sqrt{2\pi}}\left[A_1 - iA_2\right] e^{iz}. \quad (2.45)$$

Therefore one obtains a second constraint

$$A_1 - i(1+2e^{j\pi i})A_2 = 0. \quad (2.46)$$

The two constraints are compatible provided

$$\begin{vmatrix} 1 & -i(1+2e^{j\pi i}) \\ \tan(\omega\bar{r}_* - \beta - \alpha_+) & -1 \end{vmatrix} = 0, \quad (2.47)$$

which yields the quasi-normal frequencies [7]

$$\omega\bar{r}_* = \frac{\pi}{4}(2 + j + j_\infty) - \tan^{-1}\frac{i}{1+2e^{j\pi i}} + n\pi. \quad (2.48)$$

The first-order correction to the above asymptotic expression may be found by standard perturbation theory [8]. To first order, the wave equation becomes

$$\mathcal{H}_0\Psi_1 + \mathcal{H}_1\Psi_0 = 0. \quad (2.49)$$

The solution is

$$\Psi_1(z) = \sqrt{z}N_{\frac{j}{2}}(z)\int_0^z dz'\frac{\sqrt{z'}\,J_{\frac{j}{2}}(z')\mathcal{H}_1\Psi_0(z')}{\mathcal{W}} - \sqrt{z}J_{\frac{j}{2}}(z)\int_0^z dz'\frac{\sqrt{z'}\,N_{\frac{j}{2}}(z')\mathcal{H}_1\Psi_0(z')}{\mathcal{W}}. \quad (2.50)$$

where $\mathcal{W} = 2/\pi$ is the Wronskian.

The wavefunction to first order reads

$$\Psi(z) = \{A_1[1 - b(z)] - A_2a_2(z)\}\sqrt{z}J_{\frac{j}{2}}(z) + \{A_2[1 + b(z)] + A_1a_1(z)\}\sqrt{z}N_{\frac{j}{2}}(z), \quad (2.51)$$

where

$$a_1(z) = \frac{\pi\mathcal{A}}{2}\,\omega^{-\frac{d-3}{d-2}}\int_0^z dz'\, z'^{-\frac{1}{d-2}}J_{\frac{j}{2}}(z')J_{\frac{j}{2}}(z'),$$

$$a_2(z) = \frac{\pi\mathcal{A}}{2}\,\omega^{-\frac{d-3}{d-2}}\int_0^z dz'\, z'^{-\frac{1}{d-2}}N_{\frac{j}{2}}(z')N_{\frac{j}{2}}(z'),$$

$$b(z) = \frac{\pi\mathcal{A}}{2}\,\omega^{-\frac{d-3}{d-2}}\int_0^z dz'\, z'^{-\frac{1}{d-2}}J_{\frac{j}{2}}(z')N_{\frac{j}{2}}(z'),$$

and $\mathcal{A}$ depends on the type of perturbation.

Asymptotically, it behaves as

$$\Psi(z) \sim \sqrt{\frac{2}{\pi}}[A_1' \cos(z-\alpha_+) + A_2' \sin(z-\alpha_+)], \tag{2.52}$$

where

$$A_1' = [1-\bar{b}]A_1 - \bar{a}_2 A_2, \quad A_2' = [1+\bar{b}]A_2 + \bar{a}_1 A_1 \tag{2.53}$$

and I introduced the notation

$$\bar{a}_1 = a_1(\infty), \quad \bar{a}_2 = a_2(\infty), \quad \bar{b} = b(\infty). \tag{2.54}$$

The first constraint is modified to

$$A_1' \tan(\omega \bar{r}_* - \beta - \alpha_+) - A_2' = 0. \tag{2.55}$$

Explicitly,

$$[(1-\bar{b})\tan(\omega\bar{r}_* - \beta - \alpha_+) - \bar{a}_1]A_1 - [1 + \bar{b} + \bar{a}_2 \tan(\omega\bar{r}_* - \beta - \alpha_+)]A_2 = 0. \tag{2.56}$$

To find the second constraint to first order, one needs to approach the horizon. This entails a rotation by $-\pi$ in the z-plane. Using

$$\begin{aligned}
a_1(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}} e^{-i\pi j} a_1(z), \\
a_2(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}} \left[e^{i\pi j} a_2(z) - 4\cos^2\frac{\pi j}{2} a_1(z) - 2i(1+e^{i\pi j})b(z) \right], \\
b(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}} \left[b(z) - i(1+e^{-i\pi j})a_1(z) \right],
\end{aligned}$$

in the limit $z \to -\infty$ one obtains

$$\Psi(z) \sim -ie^{-ij\pi/2} B_1 \cos(-z-\alpha_+) - ie^{ij\pi/2} B_2 \sin(-z-\alpha_+), \tag{2.57}$$

where

$$\begin{aligned}
B_1 = {} & A_1 - A_1 e^{-i\pi\frac{d-3}{d-2}} [\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] \\
& - A_2 e^{-i\pi\frac{d-3}{d-2}} \left[e^{+i\pi j}\bar{a}_2 - 4\cos^2\frac{\pi j}{2}\bar{a}_1 - 2i(1+e^{+i\pi j})\bar{b} \right] \\
& - i(1+e^{i\pi j}) \left[A_2 + A_2 e^{-i\pi\frac{d-3}{d-2}} [\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] + A_1 e^{-i\pi\frac{d-3}{d-2}} e^{-i\pi j}\bar{a}_1 \right] \\
B_2 = {} & A_2 + A_2 e^{-i\pi\frac{d-3}{d-2}} [\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] + A_1 e^{-i\pi\frac{d-3}{d-2}} e^{-i\pi j}\bar{a}_1 .
\end{aligned}$$

Therefore the second constraint to first order reads

$$[1 - e^{-i\pi\frac{d-3}{d-2}}(i\bar{a}_1 + \bar{b})]A_1 - [i(1 + 2e^{i\pi j}) + e^{-i\pi\frac{d-3}{d-2}}((1 + e^{i\pi j})\bar{a}_1 + e^{i\pi j}\bar{a}_2 - i\bar{b})]A_2 = 0 \tag{2.58}$$

Compatibility of the two first-order constraints yields

$$\begin{vmatrix} 1+\bar{b}+\bar{a}_2\tan(\omega\bar{r}_* - \beta - \alpha_+) & i(1+2e^{i\pi j}) + e^{-i\pi\frac{d-3}{d-2}}((1+e^{i\pi j})\bar{a}_1 + e^{i\pi j}\bar{a}_2 - i\bar{b}) \\ (1-\bar{b})\tan(\omega\bar{r}_* - \beta - \alpha_+) - \bar{a}_1 & 1 - e^{-i\pi\frac{d-3}{d-2}}(i\bar{a}_1 + \bar{b}) \end{vmatrix} = 0, \tag{2.59}$$

leading to the first-order expression for quasi-normal frequencies,

$$\begin{aligned} \omega\bar{r}_* = {} & \frac{\pi}{4}(2 + j + j_\infty) + \frac{1}{2i}\ln 2 + n\pi \\ & - \frac{1}{8}\left\{6i\bar{b} - 2ie^{-i\pi\frac{d-3}{d-2}}\bar{b} - 9\bar{a}_1 + e^{-i\pi\frac{d-3}{d-2}}\bar{a}_1 + \bar{a}_2 - e^{-i\pi\frac{d-3}{d-2}}\bar{a}_2\right\}, \end{aligned}$$

where

$$\begin{aligned} \bar{a}_1 &= \frac{\pi\mathcal{A}}{4}\left(\frac{n\pi}{2\bar{r}_*}\right)^{-\frac{d-3}{d-2}} \frac{\Gamma(\frac{1}{d-2})\Gamma(\frac{j}{2} + \frac{d-3}{2(d-2)})}{\Gamma^2(\frac{d-1}{2(d-2)})\Gamma(\frac{j}{2} + \frac{d-1}{2(d-2)})} \\ \bar{a}_2 &= \left[1 + 2\cot\frac{\pi(d-3)}{2(d-2)}\cot\frac{\pi}{2}\left(-j + \frac{d-3}{d-2}\right)\right]\bar{a}_1 \\ \bar{b} &= -\cot\frac{\pi(d-3)}{2(d-2)}\bar{a}_1. \end{aligned}$$

Thus the first-order correction is $\sim \mathcal{O}(n^{-\frac{d-3}{d-2}})$.

The above analytic results are in good agreement with numerical results [9] (see Ref. [8] for a detailed comparison).

2.2.3 Electromagnetic Perturbations

The electromagnetic potential in four dimensions is

$$V_{\mathsf{EM}} = \frac{\ell(\ell+1)}{r^2}f(r). \tag{2.60}$$

Near the origin,

$$V_{\mathsf{EM}} = \frac{j^2 - 1}{4r_*^2} + \frac{\ell(\ell+1)r_*^{-3/2}}{2\sqrt{-4\mu}} + \cdots, \tag{2.61}$$

where $j = 1$. Therefore a vanishing potential to zeroth order is obtained. To calculate the QNM spectrum one needs to include first-order corrections from the outset. Working as with gravitational perturbations, one obtains the QNMs

$$\omega \bar{r}_* = n\pi - \frac{i}{4}\ln n + \frac{1}{2i}\ln\left(2(1+i)\mathcal{A}\sqrt{\bar{r}_*}\right), \quad \mathcal{A} = \frac{\ell(\ell+1)}{2\sqrt{-4\mu}}. \tag{2.62}$$

Notice that the first-order correction behaves as $\ln n$, a fact which may be associated with gauge invariance.

As with gravitational perturbations, the above analytic results are in good agreement with numerical results [9] (see Ref. [8] for a detailed comparison).

2.3 Hydrodynamics

There is a correspondence between $\mathcal{N} = 4$ Super Yang–Mills (SYM) theory in the large N limit and type-IIB string theory in $AdS_5 \times S^5$ (AdS/CFT correspondence). In the low energy limit, string theory is reduced to classical supergravity and the AdS/CFT correspondence allows one to calculate all gauge field-theory correlation functions in the strong coupling limit leading to non-trivial predictions on the behavior of gauge theory fluids. For example, the entropy of $\mathcal{N} = 4$ SYM theory in the limit of large ’t Hooft coupling is precisely 3/4 its value in the zero coupling limit.

The long-distance (low-frequency) behavior of any interacting theory at finite temperature must be described by fluid mechanics (hydrodynamics). This leads to a universality in physical properties because hydrodynamics implies very precise constraints on correlation functions of conserved currents and the stress-energy tensor. Their correlators are fixed once a few transport coefficients are known.

2.3.1 Vector Perturbations

I start with vector perturbations and work in the d-dimensional Schwarzschild background (2.2) with $K = +1$ (spherical horizon and boundary). It is convenient to introduce the coordinate [12]

$$u = \left(\frac{r_+}{r}\right)^{d-3}. \tag{2.63}$$

The wave equation becomes

$$-(d-3)^2 u^{\frac{d-4}{d-3}} \hat{f}(u)\left(u^{\frac{d-4}{d-3}}\hat{f}(u)\Psi'\right)' + \hat{V}_{\mathsf{V}}(u)\Psi = \hat{\omega}^2\Psi, \quad \hat{\omega} = \frac{\omega}{r_+}, \tag{2.64}$$

where prime denotes differentiation with respect to u and I have defined

$$\hat{f}(u) \equiv \frac{f(r)}{r^2} = 1 - u^{\frac{2}{d-3}}\left(u - \frac{1-u}{r_+^2}\right) \tag{2.65}$$

$$\begin{aligned}\hat{V}_{\mathsf{V}}(u) \equiv \frac{V_{\mathsf{V}}}{r_+^2} = \hat{f}(u)\Bigg\{&\hat{L}^2 + \frac{(d-2)(d-4)}{4}u^{-\frac{2}{d-3}}\hat{f}(u)\\ &-\frac{(d-1)(d-2)\left(1+\frac{1}{r_+^2}\right)}{2}u\Bigg\},\end{aligned} \tag{2.66}$$

where $\hat{L}^2 = \frac{\ell(\ell+d-3)}{r_+^2}$.

First I consider the large black hole limit $r_+ \to \infty$ keeping $\hat{\omega}$ and $\hat{L}$ fixed (small). Factoring out the behavior at the horizon ($u = 1$)

$$\Psi = (1-u)^{-i\frac{\hat{\omega}}{d-1}}F(u), \tag{2.67}$$

the wave equation simplifies to

$$\mathcal{A}F'' + \mathcal{B}_{\hat{\omega}}F' + \mathcal{C}_{\hat{\omega},\hat{L}}F = 0, \tag{2.68}$$

where

$$\begin{aligned}\mathcal{A} &= -(d-3)^2u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})\\ \mathcal{B}_{\hat{\omega}} &= -(d-3)[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}} - 2(d-3)^2\frac{i\hat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u}\\ \mathcal{C}_{\hat{\omega},\hat{L}} &= \hat{L}^2 + \frac{(d-2)[d-4-3(d-2)u^{\frac{d-1}{d-3}}]}{4}u^{-\frac{2}{d-3}}\\ &\quad-\frac{\hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}} + (d-3)^2\frac{\hat{\omega}^2}{(d-1)^2}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2}\\ &\quad-(d-3)\frac{i\hat{\omega}}{d-1}\frac{[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}}}{1-u} - (d-3)^2\frac{i\hat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2}.\end{aligned}$$

One may solve this equation perturbatively by separating

$$(\mathcal{H}_0 + \mathcal{H}_1)F = 0, \tag{2.69}$$

where

$$\begin{aligned}\mathcal{H}_0F &\equiv \mathcal{A}F'' + \mathcal{B}_0F' + \mathcal{C}_{0,0}F\\ \mathcal{H}_1F &\equiv (\mathcal{B}_{\hat{\omega}} - \mathcal{B}_0)F' + (\mathcal{C}_{\hat{\omega},\hat{L}} - \mathcal{C}_{0,0})F.\end{aligned}$$

Expanding the wavefunction perturbatively,

$$F = F_0 + F_1 + \cdots \tag{2.70}$$

at zeroth order the wave equation reads

$$\mathcal{H}_0 F_0 = 0 \tag{2.71}$$

whose acceptable solution is

$$F_0 = u^{\frac{d-2}{2(d-3)}}, \tag{2.72}$$

being regular at both the horizon ($u = 1$) and the boundary ($u = 0$, or $\Psi \sim r^{-\frac{d-2}{2}} \to 0$ as $r \to \infty$). The Wronskian is

$$\mathcal{W} = \frac{1}{u^{\frac{d-4}{d-3}}(1 - u^{\frac{d-1}{d-3}})} \tag{2.73}$$

and another linearly independent solution is

$$\check{F}_0 = F_0 \int \frac{\mathcal{W}}{F_0^2}, \tag{2.74}$$

which is unacceptable because it diverges at both the horizon ($\check{F}_0 \sim \ln(1-u)$ for $u \approx 1$) and the boundary ($\check{F}_0 \sim u^{-\frac{d-4}{2(d-3)}}$ for $u \approx 0$, or $\Psi \sim r^{\frac{d-4}{2}} \to \infty$ as $r \to \infty$).

At first order the wave equation reads

$$\mathcal{H}_0 F_1 = -\mathcal{H}_1 F_0 \tag{2.75}$$

whose solution may be written as

$$F_1 = F_0 \int \frac{\mathcal{W}}{F_0^2} \int \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}}. \tag{2.76}$$

The limits of the inner integral may be adjusted at will because this amounts to adding an arbitrary amount of the unacceptable solution. To ensure regularity at the horizon, choose one of the limits of integration at $u = 1$ rendering the integrand regular at the horizon. Then at the boundary ($u = 0$),

$$F_1 = \check{F}_0 \int_0^1 \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}} + \text{regular terms}. \tag{2.77}$$

The coefficient of the singularity ought to vanish,

$$\int_0^1 \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}} = 0, \tag{2.78}$$

which yields a constraint on the parameters (dispersion relation)

$$\mathbf{a}_0 \hat{L}^2 - i\mathbf{a}_1 \hat{\omega} - \mathbf{a}_2 \hat{\omega}^2 = 0. \tag{2.79}$$

After some algebra, one arrives at

$$\mathbf{a}_0 = \frac{d-3}{d-1}, \quad \mathbf{a}_1 = d-3. \tag{2.80}$$

The coefficient $\mathbf{a}_2$ may also be found explicitly for each dimension d, but it cannot be written as a function of d in closed form. It does not contribute to the dispersion relation at lowest order. E.g., for $d = 4, 5$, one obtains, respectively

$$\mathbf{a}_2 = \frac{65}{108} - \frac{1}{3}\ln 3, \quad \frac{5}{6} - \frac{1}{2}\ln 2. \tag{2.81}$$

Equation (2.79) is quadratic in $\hat{\omega}$ and has two solutions,

$$\hat{\omega}_0 \approx -i\frac{\hat{L}^2}{d-1}, \quad \hat{\omega}_1 \approx -i\frac{d-3}{\mathbf{a}_2} + i\frac{\hat{L}^2}{d-1}. \tag{2.82}$$

In terms of the frequency ω and the quantum number ℓ,

$$\omega_0 \approx -i\frac{\ell(\ell+d-3)}{(d-1)r_+}, \quad \frac{\omega_1}{r_+} \approx -i\frac{d-3}{\mathbf{a}_2} + i\frac{\ell(\ell+d-3)}{(d-1)r_+^2}. \tag{2.83}$$

The smaller of the two, ω_0, is inversely proportional to the radius of the horizon and is not included in the asymptotic spectrum. The other solution, ω_1, is a crude estimate of the first overtone in the asymptotic spectrum, nevertheless it shares two important features with the asymptotic spectrum: it is proportional to r_+ and its dependence on ℓ is $\mathcal{O}(1/r_+^2)$. The approximation may be improved by including higher-order terms. This increases the degree of the polynomial in the dispersion relation (2.79) whose roots then yield approximate values of more QNMs. This method reproduces the asymptotic spectrum derived earlier albeit not in an efficient way.

To include finite size effects, I shall use perturbation theory (assuming $1/r_+$ is small) and replace $\mathcal{H}_1$ by

$$\mathcal{H}_1' = \mathcal{H}_1 + \frac{1}{r_+^2}\mathcal{H}_+ \tag{2.84}$$

where

$$\mathcal{H}_+ F \equiv \mathcal{A}_+ F'' + \mathcal{B}_+ F' + \mathcal{C}_+ F. \tag{2.85}$$

The coefficients may be easily deduced by collecting $\mathcal{O}(1/r_+^2)$ terms in the exact wave equation. One obtains

$$\begin{aligned}
\mathcal{A}_+ &= -2(d-3)^2 u^2 (1-u) \\
\mathcal{B}_+ &= -(d-3)u\left[(d-3)(2-3u) - (d-1)\frac{1-u}{1-u^{\frac{d-1}{d-3}}}u^{\frac{d-1}{d-3}}\right] \\
\mathcal{C}_+ &= \frac{d-2}{2}\left[d-4-(2d-5)u-(d-1)\frac{1-u}{1-u^{\frac{d-1}{d-3}}}u^{\frac{d-1}{d-3}}\right].
\end{aligned}$$

Interestingly, the zeroth order wavefunction F_0 is an eigenfunction of $\mathcal{H}_+$,

$$\mathcal{H}_+ F_0 = -(d-2)F_0, \tag{2.86}$$

therefore the first-order finite-size effect is a simple shift of the angular momentum operator

$$\hat{L}^2 \to \hat{L}^2 - \frac{d-2}{r_+^2}. \tag{2.87}$$

The QNMs of lowest frequency are modified to

$$\omega_0 = -i\frac{\ell(\ell+d-3)-(d-2)}{(d-1)r_+} + \mathcal{O}(1/r_+^2). \tag{2.88}$$

For $d = 4, 5$, we have respectively,

$$\omega_0 = -i\frac{(\ell-1)(\ell+2)}{3r_+}, \quad -i\frac{(\ell+1)^2-4}{4r_+} \tag{2.89}$$

in agreement with numerical results [9, 13].

One deduces from (2.88) the maximum lifetime of the vector modes,

$$\tau_{\max} = \frac{4\pi}{d} T_H. \tag{2.90}$$

In the case of a flat horizon ($K = 0$),

$$\omega_0 = -i\frac{k^2}{(d-1)r_+}, \tag{2.91}$$

which leads to the diffusion constant

$$D = \frac{1}{4\pi T_H}. \tag{2.92}$$

In the case of a hyperbolic horizon ($K = -1$), a similar calculation yields [10]

$$\omega_0 = -i\frac{\xi^2 + \frac{(d-1)^2}{4}}{(d-1)r_+}, \quad \tau = \frac{1}{|\omega_0|} < \frac{16\pi}{(d-1)^2} T_H. \tag{2.93}$$

It follows that for $d = 5$, these modes live longer than their spherical counterparts which is important for plasma behavior.

2.3.2 Scalar Perturbations

Next I consider scalar perturbations which are calculationally more involved but phenomenologically more important because their spectrum contains the lowest frequencies and therefore the longest living modes. For a scalar perturbation we ought to replace the potential $\hat{V}_{\mathsf{V}}$ by

$$\begin{aligned}\hat{V}_{\mathsf{S}}(u) = \frac{\hat{f}(u)}{4}\left[\hat{m} + \left(1+\frac{1}{r_+^2}\right)u\right]^{-2} \\ \times \Bigg\{ d(d-2)\left(1+\frac{1}{r_+^2}\right)^2 u^{\frac{2d-8}{d-3}} - 6(d-2)(d-4)\hat{m}\left(1+\frac{1}{r_+^2}\right)u^{\frac{d-5}{d-3}} \\ + (d-4)(d-6)\hat{m}^2 u^{-\frac{2}{d-3}} + (d-2)^2\left(1+\frac{1}{r_+^2}\right)^3 u^3 \\ + 2(2d^2-11d+18)\hat{m}\left(1+\frac{1}{r_+^2}\right)^2 u^2 \\ + \frac{(d-4)(d-6)\left(1+\frac{1}{r_+^2}\right)^2}{r_+^2}u^2 - 3(d-2)(d-6)\hat{m}^2\left(1+\frac{1}{r_+^2}\right)u \\ - \frac{6(d-2)(d-4)\hat{m}\left(1+\frac{1}{r_+^2}\right)}{r_+^2}u + 2(d-1)(d-2)\hat{m}^3 + d(d-2)\frac{\hat{m}^2}{r_+^2}\Bigg\},\end{aligned} \tag{2.94}$$

where $\hat{m} = 2\frac{\ell(\ell+d-3)-(d-2)}{(d-1)(d-2)r_+^2} = \frac{2(\ell+d-2)(\ell-1)}{(d-1)(d-2)r_+^2}$.

In the large black hole limit $r_+ \to \infty$ with $\hat{m}$ fixed (small), the potential simplifies to

$$\begin{aligned}\hat{V}_{\mathsf{S}}^{(0)}(u) = \frac{1-u^{\frac{d-1}{d-3}}}{4(\hat{m}+u)^2}\Bigg\{ d(d-2)u^{\frac{2d-8}{d-3}} - 6(d-2)(d-4)\hat{m}u^{\frac{d-5}{d-3}}. \\ + (d-4)(d-6)\hat{m}^2u^{-\frac{2}{d-3}} + (d-2)^2u^3 \\ +2(2d^2-11d+18)\hat{m}u^2 - 3(d-2)(d-6)\hat{m}^2u + 2(d-1)(d-2)\hat{m}^3\Bigg\}.\end{aligned} \tag{2.95}$$

The wave equation has an additional singularity due to the double pole of the scalar potential at $u = -\hat{m}$. It is desirable to factor out the behavior not only at the horizon, but also at the boundary and the pole of the scalar potential,

$$\Psi = (1-u)^{-i\frac{\hat{\omega}}{d-1}}\frac{u^{\frac{d-4}{2(d-3)}}}{\hat{m}+u}F(u). \tag{2.96}$$

Then the wave equation reads

$$\mathcal{A}F'' + \mathcal{B}_{\hat{\omega}}F' + \mathcal{C}_{\hat{\omega}}F = 0, \tag{2.97}$$

where

$$\begin{aligned}
\mathcal{A} =& -(d-3)^2 u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}}) \\
\mathcal{B}_{\hat{\omega}} =& -(d-3)u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})\left[\frac{d-4}{u} - \frac{2(d-3)}{\hat{m}+u}\right] \\
& -(d-3)[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}} - 2(d-3)^2\frac{i\hat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u} \\
\mathcal{C}_{\hat{\omega}} =& -u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})\left[-\frac{(d-2)(d-4)}{4u^2} - \frac{(d-3)(d-4)}{u(\hat{m}+u)} + \frac{2(d-3)^2}{(\hat{m}+u)^2}\right] \\
& -\left[\left\{d-4-(2d-5)u^{\frac{d-1}{d-3}}\right\}u^{\frac{d-5}{d-3}} + 2(d-3)\frac{i\hat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u}\right] \\
& \times\left[\frac{d-4}{2u} - \frac{d-3}{\hat{m}+u}\right] - (d-3)\frac{i\hat{\omega}}{d-1}\frac{[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}}}{1-u} \\
& -(d-3)^2\frac{i\hat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2} + \frac{\hat{V}_{\mathsf{S}}^{(0)}(u) - \hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}} \\
& + (d-3)^2\frac{\hat{\omega}^2}{(d-1)^2}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2}
\end{aligned}$$

I shall define the zeroth-order wave equation as $\mathcal{H}_0 F_0 = 0$, where

$$\mathcal{H}_0 F \equiv \mathcal{A}F'' + \mathcal{B}_0 F'. \tag{2.98}$$

The acceptable zeroth-order solution is

$$F_0(u) = 1, \tag{2.99}$$

which is plainly regular at all singular points ($u = 0, 1, -\hat{m}$). It corresponds to a wavefunction vanishing at the boundary ($\Psi \sim r^{-\frac{d-4}{2}}$ as $r \to \infty$).

The Wronskian is

$$\mathcal{W} = \frac{(\hat{m}+u)^2}{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})} \tag{2.100}$$

and an unacceptable solution is $\check{F}_0 = \int \mathcal{W}$. It can be written in terms of hypergeometric functions. For $d \geq 6$, it has a singularity at the boundary, $\check{F}_0 \sim u^{-\frac{d-5}{d-3}}$ for $u \approx 0$, or $\Psi \sim r^{\frac{d-6}{2}} \to \infty$ as $r \to \infty$. For $d = 5$, the acceptable wavefunction behaves as $r^{-1/2}$ whereas the unacceptable one behaves as $r^{-1/2}\ln r$. For $d = 4$, the roles of F_0 and $\check{F}_0$ are reversed, however the results still valid because the correct boundary condition at the boundary is a Robin boundary condition [12, 14]. Finally, note that $\check{F}_0$ is also singular (logarithmically) at the horizon ($u = 1$).

Working as in the case of vector modes, one arrives at the first-order constraint

$$\int_0^1 \frac{\mathcal{C}_{\hat{\omega}}}{\mathcal{A}\mathcal{W}} = 0, \tag{2.101}$$

because $\mathcal{H}_1 F_0 \equiv (\mathcal{B}_{\hat{\omega}} - \mathcal{B}_0)F_0' + \mathcal{C}_{\hat{\omega}} F_0 = \mathcal{C}_{\hat{\omega}}$. This leads to the dispersion relation

$$\mathbf{a}_0 - \mathbf{a}_1 i\hat{\omega} - \mathbf{a}_2 \hat{\omega}^2 = 0, \tag{2.102}$$

After some algebra, one obtains

$$\mathbf{a}_0 = \frac{d-1}{2}\,\frac{1+(d-2)\hat{m}}{(1+\hat{m})^2}, \quad \mathbf{a}_1 = \frac{d-3}{(1+\hat{m})^2}, \quad \mathbf{a}_2 = \frac{1}{\hat{m}}\{1+O(\hat{m})\}. \tag{2.103}$$

For small $\hat{m}$, the quadratic equation has solutions

$$\hat{\omega}_0^{\pm} \approx -i\frac{d-3}{2}\,\hat{m} \pm \sqrt{\frac{d-1}{2}\quad \hat{m}} \tag{2.104}$$

related to each other by $\hat{\omega}_0^{+} = -\hat{\omega}_0^{-*}$, which is a general symmetry of the spectrum.

Finite size effects at first order amount to a shift of the coefficient $\mathbf{a}_0$ in the dispersion relation

$$\mathbf{a}_0 \to \mathbf{a}_0 + \frac{1}{r_+^2}\mathbf{a}_+ \tag{2.105}$$

After some tedious but straightforward algebra, we obtain

$$\mathbf{a}_+ = \frac{1}{\hat{m}}\{1+O(\hat{m})\}. \tag{2.106}$$

The modified dispersion relation yields the modes

$$\hat{\omega}_0^{\pm} \approx -i\frac{d-3}{2}\,\hat{m} \pm \sqrt{\frac{d-1}{2}\,\hat{m}+1}. \tag{2.107}$$

In terms of the quantum number ℓ,

$$\omega_0^{\pm} \approx -i(d-3)\,\frac{\ell(\ell+d-3)-(d-2)}{(d-1)(d-2)r_+} \pm \sqrt{\frac{\ell(\ell+d-3)}{d-2}}, \tag{2.108}$$

in agreement with numerical results [13].

Notice that the imaginary part is inversely proportional to r_+, as in vector case. In the scalar case, we also obtained a finite real part independent of r_+.

The maximum lifetime of a gravitational scalar mode is found from (2.108) to be

$$\tau_{\max} = \frac{d-2}{(d-3)d}\,4\pi T_H. \tag{2.109}$$

In the case of a flat horizon ($K = 0$), one obtains

$$\omega = \pm\frac{k}{\sqrt{d-2}} - i\frac{d-3}{(d-1)(d-2)r_+}\quad k^2, \tag{2.110}$$

showing that the speed of sound is

$$v = \frac{1}{\sqrt{d-2}} \tag{2.111}$$

as expected for a CFT and the diffusion constant is

$$D = \frac{d-3}{d-2}\,\frac{1}{4\pi T_H}. \tag{2.112}$$

For a hyperbolic horizon ($K = -1$), a similar calculation yields [10]

$$\omega = \pm\sqrt{\frac{\xi^2 + (\frac{d-3}{2})^2}{d-2}} - i\frac{(d-3)[\xi^2 + \frac{(d-1)^2}{4}]}{(d-1)(d-2)r_+}, \quad \tau < \frac{4(d-2)}{(d-3)(d-1)^2}\,4\pi T_H. \tag{2.113}$$

In the physically relevant case $d = 5$, evidently the $K = -1$ scalar modes live longer than any other modes, which is important for plasma behavior.

2.3.3 Tensor Perturbations

Finally, for completeness I discuss the case of tensor perturbations. Unlike the other two cases of gravitational perturbations, the asymptotic spectrum of tensor perturbations is the entire spectrum. To see this, note that in the large black hole limit, the wave equation reads

$$\begin{aligned}&-(d-3)^2(u^{\frac{2d-8}{d-3}} - u^3)\Psi'' - (d-3)[(d-4)u^{\frac{d-5}{d-3}} - (2d-5)u^2]\Psi' \\ &+ \left\{\hat{L}^2 + \frac{d(d-2)}{4}u^{-\frac{2}{d-3}} + \frac{(d-2)^2}{4}u - \frac{\hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}}\right\}\Psi = 0.\end{aligned}$$

For the zeroth-order equation, we may set $\hat{L} = 0 = \hat{\omega}$. The resulting equation may be solved exactly. Two linearly independent solutions are ($\Psi = F_0$ at zeroth order)

$$F_0(u) = u^{\frac{d-2}{2(d-3)}}, \quad \check{F}_0(u) = u^{-\frac{d-2}{2(d-3)}}\ln\left(1 - u^{\frac{d-1}{d-3}}\right). \tag{2.114}$$

Neither behaves nicely at both ends ($u = 0, 1$). Therefore both are unacceptable which makes it impossible to build a perturbation theory to calculate small frequencies which are inversely proportional to r_0. This negative result is in agreement with numerical results [9, 13] and in accordance with the AdS/CFT correspondence. Indeed, there is no ansatz that can be built from tensor spherical harmonics $\mathbb{T}_{ij}$ satisfying the linearized hydrodynamic equations, because of the conservation and tracelessness properties of $\mathbb{T}_{ij}$.

2.3.4 Hydrodynamics on the AdS Boundary

The above results in the bulk dictate the hydrodynamic behavior of the dual gauge theory fluid on the conformal boundary. To see the correspondence, one needs to understand the hydrodynamics in the linearized regime of a $d-1$ dimensional fluid with dissipative effects. The fluid lives on a space with metric

$$ds^2_{\partial} = -dt^2 + d\Sigma^2_{K,d-2}. \tag{2.115}$$

The hydrodynamic equations are simply the requirement that the stress-energy momentum tensor be conserved,

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.116}$$

As the duality corresponds to a conformal field theory one must also demand scale invariance which implies

$$T^\mu_\mu = 0, \quad \epsilon = (d-2)p, \quad \zeta = 0, \tag{2.117}$$

where ϵ, p and ζ are the energy density, pressure and bulk viscosity of the fluid. In the rest frame of the fluid, the velocity field is $u^\mu = (1, 0, 0, 0)$ and the pressure p_0 is constant. Consider a perturbation

$$u^\mu = (1, u^i), \quad p = p_0 + \delta p, \tag{2.118}$$

Applying the hydrodynamic equations, one obtains

$$\begin{aligned}
&(d-2)\partial_t \delta p + (d-1)p_0 \nabla_i u^i = 0 \\
&(d-1)p_0 \partial_t u^i + \partial^i \delta p - \eta\left[\nabla^j \nabla_j u^i + K(d-3)u^i + \frac{d-4}{d-2}\partial^i(\nabla_j u^j)\right] = 0,
\end{aligned} \tag{2.119}$$

where I used the curvature tensor $R_{ij} = K(d-3)g_{ij}$.

For *vector perturbations*, consider the *ansatz*

$$\delta p = 0, \quad u^i = \mathcal{C}_V e^{-i\omega t}\mathbb{V}^i, \tag{2.120}$$

where $\mathbb{V}^i$ is a vector harmonic.

The hydrodynamic equations imply

$$-i\omega(d-1)p_0 + \eta\left[k_V^2 - K(d-3)\right] = 0. \tag{2.121}$$

Using

$$\frac{\eta}{p_0} = (d-2)\frac{\eta}{s}\frac{S}{M} = \frac{4\pi\eta}{s}\frac{r_+}{K + r_+^2}, \tag{2.122}$$

with ω from the gravity dual, one obtains for large r_+,

$$\frac{\eta}{s} = \frac{1}{4\pi} \tag{2.123}$$

which is the standard value of the ratio in gauge theory fluids with a gravity dual [15].

For *scalar perturbations*, consider the *ansatz*

$$u^i = \mathcal{A}_S e^{-i\omega t}\partial^i \mathbb{S}, \quad \delta p = \mathcal{B}_S e^{-i\omega t}\mathbb{S}, \tag{2.124}$$

where $\mathbb{S}$ is a scalar harmonic.

The hydrodynamic equations imply the system of equations

$$\begin{aligned} &(d-2)i\omega\mathcal{B}_S + (d-1)p_0 k_S^2 \mathcal{A}_S = 0 \\ &\mathcal{B}_S + \mathcal{A}_S\left[-i\omega(d-1)p_0 - 2(d-3)K\eta + 2\eta k_S^2 \frac{d-3}{d-2}\right] = 0. \end{aligned} \tag{2.125}$$

The determinant must vanish,

$$\begin{vmatrix} (d-2)i\omega & (d-1)p_0 k_S^2 \\ 1 & -i\omega(d-1)p_0 - 2(d-3)K\eta + 2\eta k_S^2\,\frac{d-3}{d-2} \end{vmatrix} = 0. \tag{2.126}$$

Arguing along the same lines as for vector perturbations, we arrive at

$$\frac{\eta}{s} = \frac{1}{4\pi} \tag{2.127}$$

which is the same result as the one obtained with vector QNMs.

2.3.5 Conformal Soliton Flow

The above results have been applied to the study of the quark-gluon plasma which forms in heavy ion collisions (at the Relativistic Heavy Ion Collider (RHIC) and elsewhere). In the case of a spherical horizon ($K = +1$), the boundary of space-time is $S^3 \times \mathbb{R}$. This may be conformally mapped onto a flat Minkowski space. Then by holographic renormalization, the AdS_5-Schwarzschild black hole is dual to a spherical shell of plasma on the four-dimensional Minkowski space which first contracts and then expands (conformal soliton flow) [13].

Quasi-normal modes govern the properties of this plasma with long-lived modes (i.e., of small $\Im\omega$) having the most influence. For example, one obtains the ratio

$$\frac{v_2}{\delta} = \frac{1}{6\pi}\Re\frac{\omega^4 - 40\omega^2 + 72}{\omega^3 - 4\omega}\sin\frac{\pi\omega}{2}, \tag{2.128}$$

where $v_2 = \langle\cos 2\phi\rangle$ evaluated at $\theta = \frac{\pi}{2}$ (mid-rapidity) and averaged with respect to the energy density at late times; $\delta = \frac{\langle y^2 - x^2\rangle}{\langle y^2 + x^2\rangle}$ is the eccentricity at time $t = 0$. Numerically, $\frac{v_2}{\delta} = 0.37$, which compares well with the result from RHIC data, $\frac{v_2}{\delta} \approx 0.323$ [16].

Another observable is the thermalization time which is found to be

$$\tau = \frac{1}{2|\Im\omega|} \approx \frac{1}{8.6 T_{\text{peak}}} \approx 0.08\,\text{fm/c}, \quad T_{\text{peak}} = 300\,\text{MeV} \tag{2.129}$$

not in agreement with the RHIC result $\tau \sim 0.6$ fm/c [17], but still encouragingly small. For comparison, the corresponding result from perturbative QCD is $\tau \gtrsim 2.5$ fm/c [18, 19].

In the case of a hyperbolic horizon (topological black hole; $K = -1$), one needs to work with a conformal map from $\mathbb{H}^{d-2}/\Gamma \times \mathbb{R}$ to a $(d-1)$-dimensional Minkowski space. Finding an explicit form of this map for $d = 5$ involves a considerable amount of numerical work. However, it is important that one consider this case because the modes of hyperbolic black holes live the longest [10].

2.4 Phase Transitions

In this section I discuss hairy black holes in asymptotically AdS space and their duals. At low temperatures, an instability leads to symmetry breaking and the formation of a dual superconductor. Electromagnetic perturbations of the black hole determine the conductivity in the bulk. First I review the case of a flat horizon ($K = 0$) [20] and then I discuss the case of hyperbolic horizon ($K = -1$) where exact analytical results are obtained [21].

2.4.1 $K = 0$

Consider a scalar Ψ of mass $m^2 = -2$, which is above the Breitenlohner-Freedman (BF) bound coupled to an electromagnetic potential A_μ in 3 + 1 dimensions. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \Psi \partial^\mu \Psi + \Psi^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{q^2}{2}\Psi^2(\partial_\mu\theta - A_\mu)(\partial^\mu\theta - A^\mu), \tag{2.130}$$

where θ is a Stückelberg field. q is an arbitrary parameter which can be thought of as the electric charge of the scalar field Ψ (one may instead turn Ψ into a complex scalar field of charge q coupled to an electromagnetic potential in a standard fashion).

The Lagrangian density (2.130) is invariant under the $U(1)$ gauge transformation

$$A_\mu \to A_\mu + \partial_\mu\omega, \quad \theta \to \theta + \omega. \tag{2.131}$$

To fix the gauge, set

$$\theta = 0. \tag{2.132}$$

Working in the probe limit ($q \to \infty$) in which there is no back reaction to the metric, assume that the fields propagate in the black hole background (2.2) with $d = 4$ and $K = 0$. The radius of the horizon and Hawking temperature are, respectively,

$$r_+ = (2\mu)^{1/3}, \quad T = \frac{3r_+}{4\pi}. \tag{2.133}$$

Assuming spherical symmetry and an electrostatic potential $A_0 = \Phi(r)$, the field equations yield two coupled non-linear differential equations [20]

$$\begin{aligned}\Psi'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\Psi' + \left(\frac{\Phi}{f}\right)^2\Psi + \frac{2}{f}\Psi = 0 \\ \Phi'' + \frac{2}{r}\Phi' - \frac{2\Psi^2}{f}\Phi = 0,\end{aligned} \tag{2.134}$$

where I set $q = 1$ and

$$f(r) = r^2 - \frac{2\mu}{r} \tag{2.135}$$

As $r \to \infty$, one obtains the boundary behavior

$$\Psi = \frac{\Psi^{(1)}}{r} + \frac{\Psi^{(2)}}{r^2} + \cdots, \quad \Phi = \Phi^{(0)} + \frac{\Phi^{(1)}}{r} + \cdots \tag{2.136}$$

where one of the $\Psi^{(i)} = 0$ ($i = 1, 2$) for stability, $\Phi^{(0)}$ is the chemical potential and $\Phi^{(1)} = -\rho$ (charge density).

Below a critical temperature T_0 a condensate forms,

$$\langle \mathcal{O}_i \rangle = \sqrt{2}\Psi^{(i)} \tag{2.137}$$

of an operator of dimension $\Delta = i$.

At $T = T_0$, one may set $\Psi = 0$ in the equation for Φ and deduce (using $\Phi(r_+) = 0$)

$$\Phi = \rho\left(\frac{1}{r_+} - \frac{1}{r}\right). \tag{2.138}$$

Then the equation for Ψ turns into an eigenvalue problem which yields

$$T_0 \approx 0.226\sqrt{\rho}, \quad 0.118\sqrt{\rho}$$

depending on the boundary conditions.

To study the properties of the dual CFT, apply an electromagnetic perturbation. It obeys the wave equation

$$A'' + \frac{f'}{f}A' + \left(\frac{\omega^2}{f^2} - \frac{2\Psi^2}{f}\right)A = 0, \tag{2.139}$$

to be solved subject to the boundary conditions that it be ingoing at the horizon, $A \sim f^{-i\omega/(4\pi T)}$, and at the boundary ($r \to \infty$),

$$A = A^{(0)} + \frac{A^{(1)}}{r} + \cdots \tag{2.140}$$

Ohm's law yields the conductivity

$$\sigma(\omega) = \frac{A^{(1)}}{i\omega A^{(0)}}. \tag{2.141}$$

For $T \geq T_0$, $\Psi = 0$, therefore $A \sim e^{i\omega r_*}$ where $r_* = \int dr/f(r)$ is the tortoise coordinate. It follows that

$$\sigma(\omega) = 1. \tag{2.142}$$

At low T, for $\langle \mathcal{O}_1 \rangle \neq 0$, we have

$$\Psi \approx \frac{\langle \mathcal{O}_1 \rangle}{\sqrt{2}\, r}$$

Since $r_+ \to 0$, we obtain $A \sim e^{i\omega' r_*}$, where $\omega' = \sqrt{\omega^2 - \langle \mathcal{O}_1 \rangle^2}$. Therefore, for $\omega < \langle \mathcal{O}_1 \rangle$, $\Re\sigma = 0$, i.e., we obtain a superconductor with a gap.

2.4.2 $K = -1$

Turning to the case of a hyperbolic horizon [21], choose a scalar Ψ of mass $m^2 = -2$, as before, but conformally coupled with potential

$$V(\Psi) = \frac{8\pi G}{3}\Psi^4$$

The system has an exact solution (MTZ black hole [22])

$$ds^2 = -f_{MTZ}(r)dt^2 + \frac{dr^2}{f_{MTZ}(r)} + r^2 d\sigma^2, \quad f_{MTZ} = r^2 - \left(1 + \frac{r_0}{r}\right)^2, \tag{2.143}$$

with

$$\Psi(r) = -\sqrt{\frac{3}{4\pi G}}\frac{r_0}{r + r_0}, \quad \Phi = 0. \tag{2.144}$$

The temperature, entropy and mass are, respectively,

$$T = \frac{1}{\pi}\left(r_+ - \frac{1}{2}\right), \quad S_{MTZ} = \frac{\sigma}{4G}(2r_+ - 1), \quad M_{MTZ} = \frac{\sigma r_+}{4\pi G}(r_+ - 1). \tag{2.145}$$

and the law of thermodynamics $dM = TdS$ holds.

At $M = 0$, the MTZ black hole coincides with the topological black hole with no hair (Eq. 2.2) with $d = 4$, $K = -1$),

$$ds^2_{\text{AdS}} = -(r^2 - 1)dt^2 + \frac{dr^2}{r^2 - 1} + r^2 d\Sigma^2 \tag{2.146}$$

and an enhanced scaling symmetry (pure AdS space) emerges at the critical temperature

$$T_0 = \frac{1}{2\pi} \tag{2.147}$$

At this point there is a phase transition which can be seen by calculating the difference in free energies,

$$\Delta F = F_{TBH} - F_{MTZ} = -\frac{\sigma}{8\pi G}\ \pi^3 l^3 (T - T_0)^3 + \cdots, \tag{2.148}$$

showing that there is a third-order phase transition at T_0.

Perturbative stability of the MTZ black hole has also been demonstrated for $T < T_0$ ($M < 0$) [21]. Comparing with the flat case, note that here both $\Psi^{(1)}$ and $\Psi^{(2)}$ are non-vanishing, yet the MTZ black hole is stable. However this is true only if the mass is negative which is never the case with a flat horizon. Also, here the condensation of the scalar field has a geometrical origin and is due entirely to its coupling to gravity.

Moreover, the heat capacities in the normal and superconducting (corresponding to the MTZ black hole) phases, respectively, as $T \to 0$ exhibit a power-law behavior

$$C_n \approx \frac{\pi\sigma}{3\sqrt{3}G}\,T, \quad C_s \approx \frac{\pi\sigma}{2G}\,T, \tag{2.149}$$

Since both $\Psi^{(1)}$ and $\Psi^{(2)}$ are non-vanishing, we have a multi-trace deformation of the CFT [23] with a condensate

$$\langle \mathcal{O}_1 \rangle = \sqrt{\frac{3\pi^3}{2G}}\,(T_0^2 - T^2)\,. \tag{2.150}$$

It should be noted that the deformation does not break the global $U(1)$ symmetry because Ψ is a real field (see Eq. 2.130).

To study the conductivity, apply an electromagnetic perturbation. It obeys the wave equation (2.139) which may be solved using first-order perturbation theory in q^2,

$$A = e^{-i\omega r_*} + \frac{q^2}{2i\omega}e^{i_*}\int\limits_{r_+}^{r} dr'\Psi^2(r')e^{-2i_*} - \frac{q^2}{2i\omega}e^{-i_*}\int\limits_{r_+}^{r} dr'\Psi^2(r'). \tag{2.151}$$

The conductivity to first order in q^2 is

$$\sigma(\omega) = \frac{A^{(1)}}{i\omega A^{(0)}} = 1 - \frac{q^2}{i\omega}\int\limits_{r_+}^{\infty} dr\Psi^2(r)e^{-2i\omega r_*}. \tag{2.152}$$

The superfluid density is found from

$$\Re[\sigma(\omega)] \sim \pi n_s \delta(\omega), \quad \Im[\sigma(\omega)] \sim \frac{n_s}{\omega}, \quad \omega \to 0. \tag{2.153}$$

One obtains

$$n_s = q^2\int\limits_{r_+}^{\infty} dr\Psi^2(r) = \frac{3q^2}{4\pi G}\frac{r_0^2}{r_+ + r_0} = \alpha(T_0 - T)^2, \quad \alpha = \frac{3\pi q^2}{4G}. \tag{2.154}$$

Near $T = 0$,

$$n_s(0) - n_s(T) \approx \frac{\alpha}{\pi}T^{\delta}, \quad \delta = 1 \tag{2.155}$$

In Table 2.1, this analytic prediction is compared against exact numerical results for various values of the charge q. Naturally, the agreement is best at small values of q.

Table 2.1 The exponent δ characterizing the low-temperature dependence of the superfluid density n_s

$q/\sqrt{G}$	1	3	5
δ	1.025 ± 0.007	1.52 ± 0.03	1.78 ± 0.03

Table 2.2 Numerical *vs* analytical results for the normal and superfluid densities

$q/\sqrt{G}$	$\gamma_{\text{numerical}}$	$\gamma_{\text{analytical}}$	$\alpha_{\text{numerical}}$	$\alpha_{\text{analytical}}$
0.1	0.0020	0.0024	0.0225	0.024
0.5	0.0538	0.0597	0.552	0.589
1.0	0.187	0.239	2.196	2.356
2.0	0.684	0.955	8.678	9.425
3.0	1.325	2.15	20.35	21.21
5.0	2.522	5.97	52.90	58.90

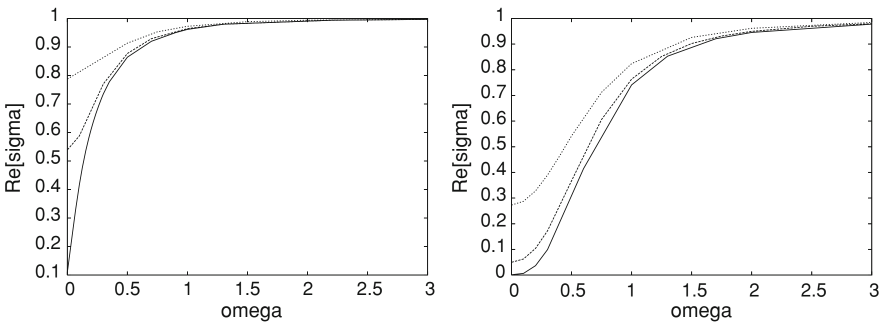

Fig. 2.1 The real part of the conductivity *vs* ω for $q/\sqrt{G} = 2$ (*left*) and $q/\sqrt{G} = 5$ (*right*) and $T = 0.0032, 0.032, 0.064$. The *lowest curve* corresponds to the lowest temperature

The normal, non-superconducting, component of the DC conductivity is

$$n_n = \lim_{\omega \to 0} \Re[\sigma(\omega)]\,. \tag{2.156}$$

Therefore,

$$\ln n_n = 2q^2 \int_{r_+}^{\infty} dr \Psi^2(r) r_*. \tag{2.157}$$

At low T,

$$n_n \sim T^{\gamma}\,, \quad \gamma = \frac{3q^2}{4\pi G}. \tag{2.158}$$

This analytic result and the prediction for the parameter α determining the critical behavior of the superfluid density are compared against exact numerical results in Table 2.2. Again, the agreement is best at small q.

Figures 2.1 and 2.2 show the frequency dependence of the real and imaginary, respectively, parts of the conductivity. The real part of the conductivity becomes smaller as we increase the charge q. Unfortunately, numerical instabilities also increase and we have not been able to produce reliable numerical results above $q/\sqrt{G} = 5$. The superconductor appears to be *gapless*. However, a gap is likely to

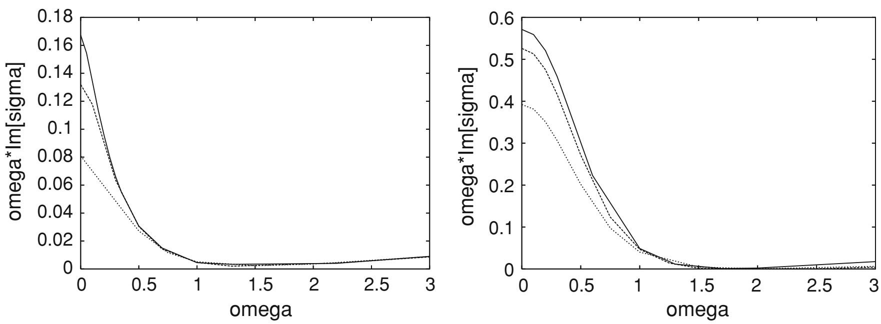

Fig. 2.2 2 The imaginary part of the conductivity multiplied by ω *vs* ω for $q/\sqrt{G} = 2$ (*left*) and $q/\sqrt{G} = 5$ (*right*) and $T = 0.0032, 0.032, 0.064$. The *uppermost curve* corresponds to the lowest temperature

develop above a certain value of the charge q, as indicated by the trend in the graphs as q increases.

2.5 Conclusion

The quasi-normal modes that govern perturbations of black holes in asymptotically AdS space are a powerful tool in understanding the hydrodynamic behavior of a gauge theory fluid at strong coupling. Here I focused on the analytic calculation of QNMs. I discussed both high overtones and low frequencies. I applied the results on gravitational perturbations to the understanding of the quark-gluon plasma produced in heavy ion collisions at RHIC and the LHC. I also considered hairy black holes whose electromagnetic perturbations allow one to analyze the conductivity of the dual conformal field theory and the phase transition to a superconducting state. I reviewed the case of a flat horizon and compared the results with those from black holes with hyperbolic horizon for which exact hairy solutions have been constructed (MTZ black holes [22]). In all these cases, only the low-lying QNMs were needed. It is unclear what physical role high overtones play.

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