

Chapter 1

A Brief Survey of Focal Decomposition

Mauricio M. Peixoto

Abstract We present a brief survey of focal decomposition stressing how this subject relates naturally to some a priori unrelated mathematical and physical subjects.

1.1 Introduction

The concept of focal decomposition is a formalization of the rough geometric idea of focalization. Given that we have a number of “trajectories” passing through a point P will they ever meet again, and how at some subsequent point(s)? The trajectories we will consider will be the trajectories of a second order ordinary differential equation in Euclidean spaces, or else the geodesics of a Riemannian manifold M . Focal decomposition was then introduced in [10], in the context of the simplest instance of focalization namely the 2-point boundary value problem for a second order ordinary differential equation. The concept of focal decomposition was then naturally extended to Riemannian manifolds by Kupka and Peixoto in [6]. Originally and for some time later focal decomposition was called “ σ -decomposition”, a misnomer. We present here a brief survey of focal decomposition stressing how this subject relates naturally to some a priori unrelated mathematical and physical subjects. As shown by the examples below.

1. The semi-classical quantization of the pendulum equation $\ddot{x} + \sin x = 0$ via the Feynman path integral method.
2. The arithmetic of binary quadratic forms.
3. The Brillouin zones of a crystal.
4. The Landau–Ramanujan function associated to binary quadratic forms.

M.M. Peixoto

Instituto de Matemática Pura e Aplicada (IMPA), Estrada Dona Castorina, 110,
22460-320 Rio de Janeiro, RJ, Brazil
e-mail: peixoto@impa.br

5. Inspired by the Mostow Rigidity Theorem, F. Kwakkel in his Groningen thesis [8] introduced, essentially, the concept of Focal Rigidity opening up what is likely to be a fruitful field of research.

1.2 Ordinary Differential Equations

Our starting point is the 2-point boundary value problem for a second order ordinary differential equation

$$\ddot{x} = f(t, x, \dot{x}) \quad x(t_1) = x_1, \quad x(t_2) = x_2, \quad t, x, \dot{x} \in \mathbb{R}. \quad (1.1)$$

This is the simplest and oldest of all boundary value problems introduced by Euler in his work on the foundations of the calculus of variations in the eighteenth century. There is a vast literature on this problem, mostly in the context of applied mathematics and functional analysis. Here we are primarily interested in the number of solutions of the problem (1.1). Let $\mathbb{R}^4(t_1, x_1, t_2, x_2) = \mathbb{R}^2(t_1, x_1) \times \mathbb{R}^2(t_2, x_2)$ be the totality of pairs of points of the (t, x) plane and to each point $(t_1, x_1, t_2, x_2) \in \mathbb{R}^4$ associate its index $i(t_1, x_1, t_2, x_2)$ the number of solutions of the problem (1.1), a non negative number or ∞ . So the possible values of i are $0, 1, 2, \dots, \infty$. When $t_1 = t_2$, the index i is defined to be 0, if $x_1 \neq x_2$, and ∞ , if $x_1 = x_2$. Call $\sum_i \subset \mathbb{R}^4$ the set of points to which the index i has been assigned. Then

$$\mathbb{R}^4 = \Sigma_0 \cup \Sigma_1 \cup \dots \cup \Sigma_\infty \quad (1.2)$$

and (1.2) is called Focal Decomposition associated to the 2-point problem (1.1). Clearly the sets $\sum_i$ are disjoint. Fixing the point (t_1, x_1) and calling

$$\sigma_i = \Sigma_i \cap \{(t_1, x_1)\} \times \mathbb{R}^2(t_2, x_2)$$

we get

$$\mathbb{R}^2 = \sigma_0 \cup \sigma_1 \cup \dots \cup \sigma_\infty \quad (1.3)$$

with the σ_i disjoint sets. Then (1.3) is called the Focal Decomposition relative to the 2-point boundary value problem (1.1) with base point (t_1, x_1) . This is called the restricted problem with base point (t_1, x_1) . So far there is no indication whatsoever of what the sets σ_i and $\sum_i$ might be, and how they decompose $\mathbb{R}^2$ or $\mathbb{R}^4$. Consider the case of the restricted problem (1.3). The equation

$$\ddot{x} = f(t, x, \dot{x})$$

can be written

$$\dot{t} = 1, \dot{x} = u, \dot{u} = f(t, x, u) \quad (1.4)$$

the base point being (t_1, x_1) . Call $\mathcal{F}$ the foliation of $\mathbb{R}^3(t, x, u)$ defined by the trajectories of (1.4).

Definition 1.1. The star of the base point (t_1, x_1) , $\Omega(t_1, x_1)$ is the union of all the leaves $\mathcal{F}(t_1, x_1, u)$, $-\infty < u < \infty$.

Now consider the projection

$$\prod : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \quad (1.5)$$

defined by $\prod(t, x, u) = (t, x)$. It is straightforward that

$$(t_2, x_2) \in \sigma_i \Leftrightarrow \text{card} \left\{ \left(\prod | \Omega \right)^{-1} (t_2, x_2) \right\} = i.$$

This shows then that the whole focal decomposition (1.3) can be obtained by applying the projection $\prod$ on the star. Thanks to a theorem by Hironaka [5], pp. 40–43, we have the

Theorem 1.1 (Existence Theorem (Peixoto - Thom)). *If f in (1.1) is analytic,*

$$\left(\prod | \Omega \right) (t_1, x_1)$$

is proper on $\mathbb{R}^2 \setminus \delta$, where $\delta = \{(t, x) | t = t_1\}$ then there exists an analytical Whitney stratification of $\mathbb{R}^2 \setminus \delta$ such that each $\sigma_i \setminus \delta$ is the locally finite union of strata of this stratification.

For a proof of the above see [13]. For a more general situation see [12].

1.2.1 Focal Decomposition of the Pendulum Equation

The pendulum equation $\ddot{x} + \sin x = 0$ can be formally integrated by Jacobi elliptic functions so that we have the equation of the star at the origin, $\Omega(0, 0)$. To begin with we determine the focal decomposition restricted to the t -axis, which follows from the very definitions of the Jacobian elliptic functions involved. Local sections of $\Omega(0, 0)$ by the planes $t = \pm\pi, \pm2\pi, \pm3\pi, \dots$ give local sections which are then glued together and projected on the plane $u = 0$ producing the focal decomposition below. See [13] for the justification of Fig. 1.1.

1.3 Semiclassical Quantization and the Pendulum Equation

Consider a second order equation (1.6) which is the Euler equation of a certain action integral

$$S = \int_{P_1}^{P_2} L(t, x, \dot{x}) dt, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0. \quad (1.6)$$

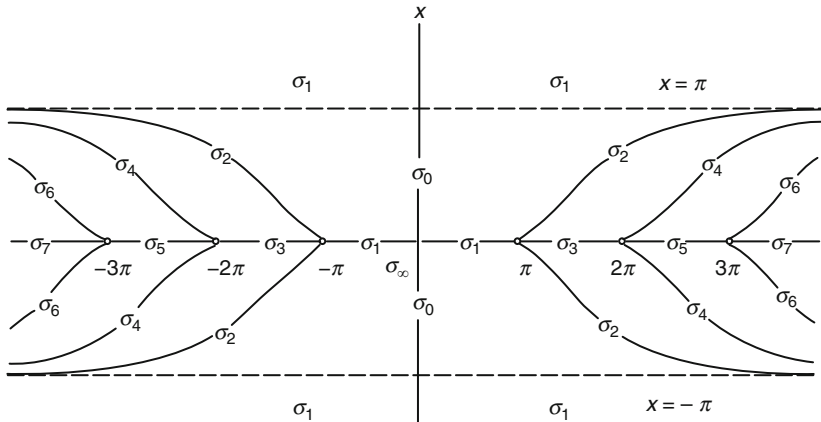


Fig. 1.1 Focal decomposition of the pendulum

Then there is a well defined mathematical operation called the quantization of (1.6) which corresponds to find a certain number of eigenfunctions and the corresponding eigenvalues. One of the more accepted method of quantization is Feynman path integral method. In this method a crucial step is the calculation of a certain amplitude (angle, real number) called the propagator $K(P_1, P_2)$ relative to a pair of points $P_1(t_1, x_1)$ and $P_2(t_2, x_2)$. By Feynman method the calculation of $K(P_1, P_2)$ is a complicated thing: you consider all continuous curves joining P_1 to P_2 . On each such curve one does some operation the result which is some kind of “infinitesimal” dK . Integrating over all continuous curves we get the propagator K . Of course this may be in some cases a good numerical procedure, without a mathematical justification, far from it. Suppose now that we are dealing with the semiclassical quantization. This is the case where the data of the problem such as length, mass, time, etc are such that the action S in (1.6) makes $S \setminus \hbar$ very big. Most important, in the semiclassical case Feynman–Hibbs [4], p. 29, show that in the calculation of the propagator $K(P_1, P_2)$ we need to consider only the curves which are solutions of the Euler equation. Now given a differential equation of the second order such as (1.6) and two points P_1 and P_2 the problem of knowing the number of solutions of this equation passing through P_1 to P_2 is exactly what the focal decomposition is about. True the focal decomposition gives just the number of solutions. But this is just the first step which helps getting the actual solutions through P_1 to P_2 . All this points out to the interest of doing the semiclassical quantization of the pendulum equation. With this and some related problems in mind I joined the physicist C.A.A. de Carvalho and the mathematicians D. Pinheiro and A. A. Pinto. The first paper of a projected series has been accepted for publication [3], another was submitted to a journal.

1.4 Focal Decomposition on Riemannian Manifolds

Let (M, g) be a complete smooth Riemannian manifold with metric g , $p \in M$ and $T_p M$ the tangent plane at p . Since M is complete the exponential map $\exp_p : T_p M \rightarrow M$ is defined everywhere on $T_p M$. Recall that for a vector $v \in T_p M$, $\exp_p v$ is obtained by considering the geodesic passing through p , tangent to v , and mark on it the length $\|v\|$ obtaining the point $\exp_p v$ on that geodesic. Now define the index $I(v)$ as the cardinality of all vector $w \in T_p M$ with the same length and same exponential as v

$$I(v) = \text{card}\{w \in T_p M \mid \|w\| = \|v\| \text{ and } \exp_p w = \exp_p v\}.$$

To every point $p \in M$ we call $\sigma_i(p)$ the totality of the vectors $v \in T_p M$ with index $i = I(v)$. Since every vector of $T_p M$ belongs to one and only one of the $\sigma_i(p)$, we have

$$T_p(M) = \bigcup_i \sigma_i(p) \quad (1.7)$$

and (1.7) is called the focal decomposition of $T_p M$. The tangent bundle has a corresponding focal decomposition

$$TM = \bigcup_i \Sigma_i \quad (1.8)$$

where

$$\Sigma_i = \bigcup_{p \in M} \sigma_i(p). \quad (1.9)$$

All this for the given Riemannian metric g on M . The expression in (1.8) is called the focal decomposition of the tangent bundle. It is clear that the focal decomposition of TM depends only on the metric g . It is also a global concept, all geodesics passing through p have a role in the construction of the sets σ_i and all geodesics of M have a role in the construction of the sets Σ_i .

Theorem 1.2 (Existence Theorem). *If M is analytic then there is an analytical Whitney stratification of TM such that each Σ_i is the locally finite union of strata of this stratification.*

Similarly for the restricted problem with base point $p \in M$: there is an analytical Whitney stratification of $T_p M$ such that each $\sigma_i(p)$ is the locally finite union of strata of this stratification. See [6]. In the case of ordinary differential equations the stratification is a consequence of analyticity plus a properness condition. In the case of a complete Riemannian manifold M , analyticity is enough to imply the stratification. As in the case of o.d.e. no σ_i or Σ_i can be too complicated, say a Cantor set. In any case the natural place to study focal decomposition is a complete real analytical Riemannian manifold. Relating to the variation of the metric, we have in [7] the genericity theorems

Theorem 1.3 (Pointwise Index Theorem). *Given a point p on a manifold M with dimension m , the generic metric g has no more than $m + 1$ geodesics of equal length joining p to some point $q \in M$.*

Theorem 1.4 (Uniform Index Theorem). *The generic metric of a compact manifold M of dimension m has no more than $2m + 2$ geodesics of equal length joining distinct points of M .*

A final comment about focal decomposition on a Riemannian manifold is that as a consequence of the angle lemma in [6] we get that in the focal decomposition of $T_p M$, (1.7), only σ_1 has non empty interior, i.e. contains some open set of $T_p M$. So σ_1 has full measure. This is in complete disagreement with Fig. 1.1 corresponding to the focal decomposition of the pendulum equation. There σ_i has positive measure if and only if i is odd. The theories are different.

1.4.1 Focal Decomposition on the Flat Torus

Consider the flat torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the usual metric determined by the quadratic form $x^2 + y^2$. The exponential map coincides with the covering map in which the coordinates (x, y) are reduced mod 1. Take $(0, 0)$ as the base point. To determine the index $I(x, y)$ of $p = (x, y)$ on the tangent space T_p , draw a circle centred at $(0, 0)$ and then $I(x, y)$ equals the cardinality of the pair of integers (m, n) such that

$$x^2 + y^2 = (x + m)^2 + (y + n)^2. \quad (1.10)$$

If $(x, y) \in \mathbb{Z}^2$ then $I(x, y)$ is exactly the number of integer solutions of the equation

$$X^2 + Y^2 = N, \quad N = x^2 + y^2, \quad (1.11)$$

then $I(x, y) = R(N)$ and there is a formula of Gauss expressing $R(N)$ in terms of the factorization of N in prime factors. If $(x, y) \notin \mathbb{Z}^2$ then the determination of the index $I(x, y)$ becomes more interesting. In fact we invert the problem, assuming that we know (m, n) and want (x, y) . Then (1.10) is

$$2mx + 2ny + m^2 + n^2 = 0. \quad (1.12)$$

But this is the equation of the perpendicular bisector of the lattice point $(-m, -n)$ call it $L(m, n)$. Now we have proved the

Proposition 1.1. *The index $I(x, y)$ equals 1 plus the number of lines $L(m, n)$ that pass through the point (x, y) . The 1 corresponds to the fact that $m = 0, n = 0$ defines no line $L(m, n)$.*

The family of lines $L(m, n)$, $\mathcal{L}$, (Fig. 1.2) determines the focal decomposition associated to the quadratic form $x^2 + y^2$. So for N real and positive, consider the circle

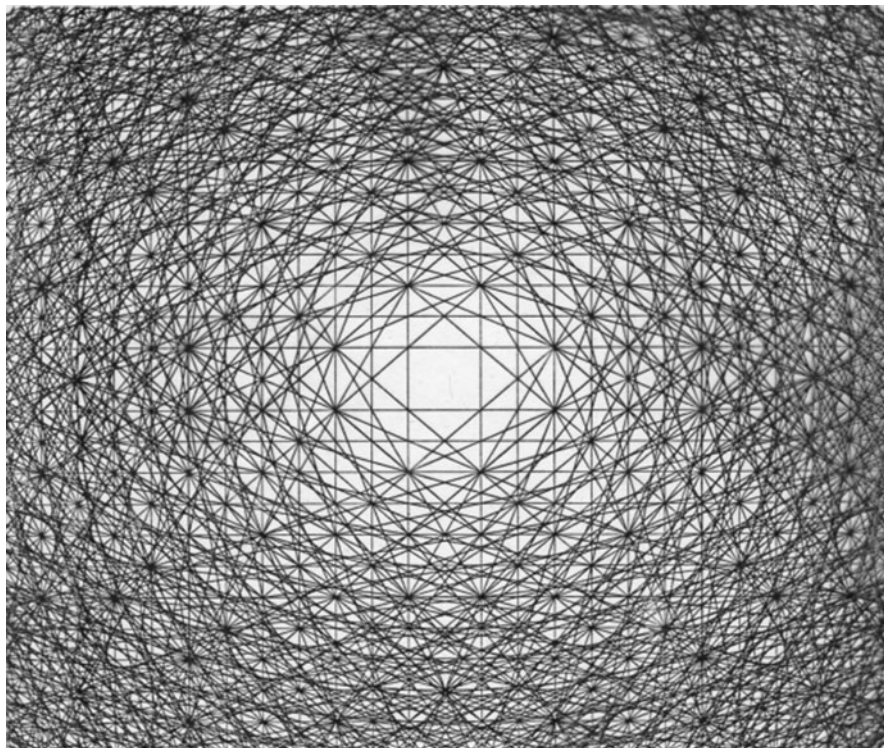


Fig. 1.2 Lines $L(m, n)$

$x^2 + y^2 = N$ and let P be one of its points. Then the number of lines $L(m, n)$ passing through P is exactly the index i for which $P \in \sigma_i$. In this sense the focal decomposition incorporates an extension of all Diophantine equations

$$x^2 + y^2 = N \quad (1.13)$$

for every N and to the whole plane. And one may well say that the index $I(a, b)$ defined for every N is an extension to the whole $\mathbb{R}^2(x, y)$ of the arithmetic function R defined above only for the natural integers.

1.4.2 The Landau–Ramanujan Function

All that was said above about the positive definite quadratic form $x^2 + y^2$ can be translated in a straightforward and appropriate way to any other positive definite quadratic form

$$ax^2 + 2bxy + cy^2, \quad a, b, c \in \mathbb{Z}. \quad (1.14)$$

Going back to the original form $x^2 + y^2$, Figure 1.2 is just the totality of all lines $L(m, n)$. Now every line $L(m, n)$, by definition, is tangent to the circle centred at the origin and passing through the point $(-m/2, -n/2)$. So it is natural to “organize” the lines $L(m, n)$ by the circles to which they are tangent. This is motivation to the following

Definition 1.2. Consider (1.13), $x^2 + y^2 = N$, and call $R(N)$ the number of its integer solutions. The Landau–Ramanujan function associated to it is defined by

$$L(N) = \text{card}\{0 < v \leq N \mid R(v) \neq 0\}$$

$R(v) \neq 0$ means that v is the sum of 2 squares $v = x^2 + y^2$. Then

Theorem 1.5. $R(N) < 2L(4N)$

The proof of this is quite simple. The $4N$ comes from the fact that if $L(m, n)$ is tangent to $x^2 + y^2 = N$, we have $m^2 + n^2 = 4N$. But the most important fact is that if instead of $x^2 + y^2 = N$ we had $ax^2 + 2bxy + cy^2 = N$, the above theorem remains true. The above on the Landau–Ramanujan function is contained on a forthcoming joint work with C. Pugh.

1.5 The Brillouin Zones

The Brillouin zones is a way of organizing the complement of the lines $L(m, n)$ on the plane. Each connected component C of this complement receives an integer number as follows. Join the origin to a point of C by a straight line and count the number i of the lines $L(m, n)$ that this segment meets. All the sets C that have received the same number i constitute the i th zone of Brillouin. See Fig. 1.3.

1.6 Other Works on Focal Decomposition

In [1, 2] and [15] the authors study focal decomposition of linear second order differential equations in $\mathbb{R}^n$. In [1], to such differential equation one can associate a sequence of eigenvalues of the linear operator called the *resonance sequence*. It is shown that two such differential equations are focally equivalent, in the natural topological sense, if and only if the two equations generate the same resonance sequence. In [9] we consider the pair of Diophantine equations

$$Ax^2 + Cy^2 = N \tag{i}$$

$$ACx^2 + y^2 = N \tag{ii}$$

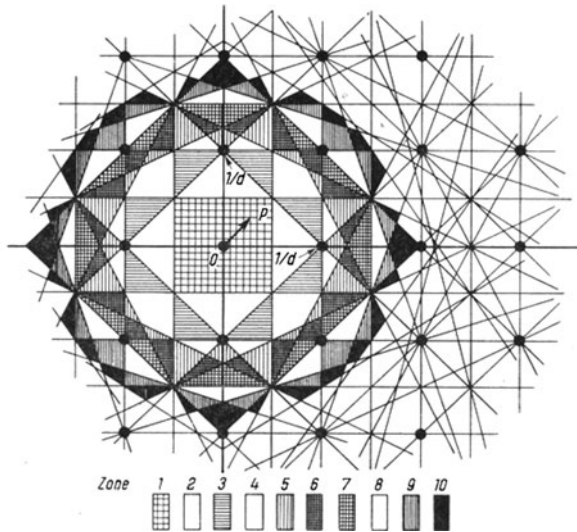


Fig. 1.3 Brillouin zones

with A, C odd, square free, $A \equiv C \pmod{4}$. If $R(N)$ is the number of solutions of (i), $r(N)$ is the number of primitive solutions of (ii) and $\Delta = 4AC$, then we have the convolution formula

$$R(N)^2 = \frac{1}{2} \sum_{d|\Delta N} r(d) R\left(\frac{\Delta N}{d}\right). \quad (1.15)$$

In [14] the same formula is obtained with different conditions on A and C .

References

1. Alvarez, S., Berend, D., Birbrair, L., Girão, D.: Resonance sequences and focal decomposition. *Isr. J. Math.* **170**, 269–284 (2009)
2. Birbrair, L., Sobolevsky, M., Sobolevskii, P.: Focal decompositions for linear differential equations of the second order. *Abstr. Appl. Anal., Estados Unidos*, **14**, 813–821 (2003)
3. Carvalho, C.A.A., Peixoto, M.M., Pinheiro, D., Pinto, A.A.: Focal decomposition, renormalization and semiclassical physics. *J. Differ. Equ. Appl.* (2011)
4. Feynman, R., Hibbs, A.: *Quantum Mechanics and Path Integrals*. McGraw-Hill (1965)
5. Goresky, M., MacPherson, R.: *Stratified Morse Theory Ergebnisse der Mathematik und ihrer Grenzgebiete 3 Folge 14*. Springer (1989)
6. Kupka, I., Peixoto, M.M.: On the enumerative geometry of geodesics. *From Topology to Computation, Proceedings of the Smalefest*, pp. 243–253. Springer (1993)
7. Kupka, I., Peixoto, M.M., Pugh, C.C.: Focal stability of Riemannian Metrics. *J. für die Reine Angewandte Mathe. (Crelle's Journal)* **593**, 31–72 (2006)
8. Kwakkel, F.: *Rigidity of Brillouin zones*, Master Thesis, University of Groningen, Department of Mathematics (2006)

9. Peixoto, M.M.: Sigma décomposition et arithmétique de quelques formes quadratiques définies positives. In: Porte, M. (ed.) *Thom's Festschrift, Passion des Formes*, ENS Editions, pp. 455–479. Fontenay (1994)
10. Peixoto, M.M.: On end point boundary values problem. *J. Differ. Equ.* **44**, 273–280 (1982)
11. Peixoto, M.M.: On a generic theory of end point boundary value problems. *An. Acad. Bras. Cienc.* **41**, 1–6 (1969)
12. Peixoto, M.M., Da Silva, A.R.: Focal decomposition and some results of S.Bernstein on the 2-point boundary value problem. *J. Lond. Math. Soc.* **60**(2), 517–547 (1999)
13. Peixoto, M.M., Thom, R.: Le point de vue énumératif dans les problèmes aux limites pour les équations différentielles ordinaires I, II. *C.R. Acad. Sci.* **303** (1986); Série I, **13**, 629–632; **14**, 693–698; Erratum 307 pp. 197–198
14. Pombo Júnior, D.P.: A convolution formula for certain Diophantine equations. *Acta Sci. Math. (Szeged)* **62**(3-4), 381–390 (1996)
15. Sobolevsky, M.: Decomposição Focal. *Seminário Iberoamericano de Matemáticas. Singularidades en Tordesillas* **3**, 21–27 (2003)

Dynamics, Games and Science I

DYNA 2008, in Honor of Maurício Peixoto and David
Rand, University of Minho, Braga, Portugal, September
8-12, 2008

Peixoto, M.M.; Pinto, A.A.; Rand, D.A.J. (Eds.)

2011, XXXII, 810 p. 168 illus., 66 illus. in color.,

Hardcover

ISBN: 978-3-642-11455-7