Chapter 2
Basic Conservation Equations for Laminar Convection

Abstract In this chapter, the basic conservation equations related to laminar fluid flow conservation equations are introduced. On this basis, the corresponding conservation equations of mass, momentum, and energy for steady laminar forced convection boundary layer are obtained.

2.1 Continuity Equation

The conceptual basis for the derivation of the continuity equation of fluid flow is the mass conservation law. The control volume for the derivation of continuity equation is shown in Fig. 2.1 in which the mass conservation principle is stated as

$$\dot{m}_{\text{increment}} = \dot{m}_{\text{in}} - \dot{m}_{\text{out}}. \tag{2.1}$$

where $\dot{m}_{\text{increment}}$ expresses the mass increment per unit time in the control volume, $\dot{m}_{\text{in}}$ represents the mass flowing into the control volume per unit time, and $\dot{m}_{\text{out}}$ is the mass flowing out of the control volume per unit time. The dot notation signifies a unit time.

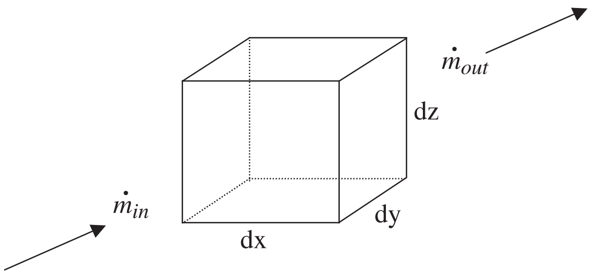

Fig. 2.1 Control volume for derivation of the continuity equations

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In the control volume, the mass of fluid flow is given by $\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, and the mass increment per unit time in the control volume can be expressed as

$$\dot{m}_{\text{increment}} = \frac{\partial \rho}{\partial \tau}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{2.2}$$

The mass flowing per unit time into the control volume in the x direction is given by $\rho w_x \mathrm{d}y\,\mathrm{d}z$. The mass flowing out of the control volume in a unit time in the x direction is given by $\left[\rho w_x + \frac{\partial(\rho w_x)}{\partial x}\mathrm{d}x\right]\mathrm{d}y\,\mathrm{d}z$. Thus, the mass increment per unit time in the x direction in the control volume is given by $\frac{\partial(\rho w_x)}{\partial x}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Similarly, the mass increments in the control volume in the y and z directions per unit time are given by $\frac{\partial(\rho w_y)}{\partial y}\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$ and $\frac{\partial(\rho w_z)}{\partial z}\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$, respectively. We thus obtain

$$\dot{m}_{\text{out}} - \dot{m}_{\text{in}} = \left[\frac{\partial(\rho w_x)}{\partial x} + \frac{\partial(\rho w_y)}{\partial y} + \frac{\partial(\rho w_z)}{\partial z}\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{2.3}$$

With (2.2) and (2.3), (2.1) becomes in Cartesian coordinates:

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial(\rho w_x)}{\partial x} + \frac{\partial(\rho w_y)}{\partial y} + \frac{\partial(\rho w_z)}{\partial z} = 0, \tag{2.4}$$

or in the vector notation

$$\frac{\partial \rho}{\partial \tau} + \nabla \cdot (\rho \vec{W}) = 0, \tag{2.5}$$

where $\vec{W} = i w_x + j w_y + k w_z$ is the fluid velocity.

For steady state, the vector and Cartesian forms of the continuity equation are given by

$$\frac{\partial}{\partial x}(\rho w_x) + \frac{\partial}{\partial y}(\rho w_y) + \frac{\partial}{\partial z}(\rho w_z) = 0, \tag{2.6}$$

and

$$\frac{\partial}{\partial x}(\rho w_x) + \frac{\partial}{\partial y}(\rho w_y) = 0, \tag{2.7}$$

respectively for three- and two- dimensional continuity equations.

2.2 Momentum Equation (Navier–Stokes Equations)

The control volume for derivation of the momentum equation of fluid flow is shown in Fig. 2.2. Take an enclosed surface A that includes the control volume. Meanwhile, take $\vec{F}_{\mathrm{m}}$ as mass force per unit mass fluid, $\vec{F}_{\mathrm{m,v}}$ as the total mass force in the control volume, $\vec{\tau}_n$ as surface force per unit mass fluid flow at per area of surface, $\vec{\tau}_{n,A}$ as surface force in the control volume, $\dot{G}_{\mathrm{increment}}$ as momentum increment of the per unit mass fluid flow at unit time, and $G_{\mathrm{increment}}$ as momentum increment of the fluid flow per unit time in the volume. According to the momentum law, the momentum increment of the fluid flow per unit time in the control volume equals the sum of total mass force and surface forced in the same volume, as

$$G_{\mathrm{increment}} = \vec{F}_{\mathrm{m,v}} + \vec{\tau}_{n,A}. \tag{2.8}$$

$\vec{F}_{\mathrm{m,v}}$, $\vec{\tau}_{n,A}$, and $G_{\mathrm{increment}}$ in the control volume are expressed as, respectively,

$$\vec{F}_{\mathrm{m,v}} = \int_V \rho \vec{F}_{\mathrm{m}} \,\mathrm{d}V, \tag{2.9}$$

$$\vec{\tau}_{n,A} = \int_A \vec{\tau}_n \, dA, \tag{2.10}$$

$$G_{\mathrm{increment}} = \frac{D}{D\tau} \int_V \rho \vec{W} \,\mathrm{d}V. \tag{2.11}$$

According to tensor calculation, the right side of (2.10) is expressed as

$$\int_A \vec{\tau}_n \mathrm{d}A = \int_V \nabla \cdot [\tau] \mathrm{d}V, \tag{2.12}$$

where $\nabla \cdot [\tau]$ is divergence of the shear force tensor.

With (2.9), (2.10), (2.11), and (2.12), (2.8) is rewritten as

$$\frac{D}{D\tau} \int_V \rho \vec{W} \,\mathrm{d}V = \int_V \rho \vec{F}_{\mathrm{m}} \,\mathrm{d}V + \int_V \nabla \cdot [\tau] \mathrm{d}V, \tag{2.13}$$

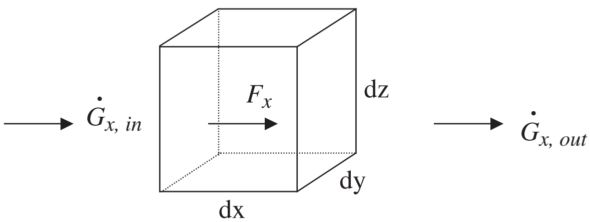

Fig. 2.2 Control volume for derivation of momentum equations

i.e.

$$\int_V \left\{ \frac{D(\rho \vec{W})}{D\tau} - \rho \vec{F} - \nabla \cdot [\tau] \right\} \mathrm{d}V = 0. \tag{2.14}$$

Therefore, we have

$$\frac{D(\rho \vec{W})}{D\tau} = \rho \vec{F} + \nabla \cdot [\tau]. \tag{2.15}$$

This is the Navier–Stokes equations of fluid flow. For Cartesian coordinates, (2.15) can be expressed as

$$\frac{D(\rho w_x)}{D\tau} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x, \tag{2.16}$$

$$\frac{D(\rho w_y)}{D\tau} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y, \tag{2.17}$$

$$\frac{D(\rho w_z)}{D\tau} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho g_z, \tag{2.18}$$

where

$$\tau_{xx} = -\left[p + \frac{2}{3}\mu \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z} \right) \right] + 2\mu \frac{\partial w_x}{\partial x},$$
$$\tau_{yy} = -\left[p + \frac{2}{3}\mu \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z} \right) \right] + 2\mu \frac{\partial w_y}{\partial y},$$
$$\tau_{zz} = -\left[p + \frac{2}{3}\mu \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z} \right) \right] + 2\mu \frac{\partial w_z}{\partial z},$$

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial w_y}{\partial x} + \frac{\partial w_x}{\partial y} \right),$$
$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial w_z}{\partial y} + \frac{\partial w_y}{\partial z} \right),$$
$$\tau_{zx} = \tau_{xz} = \mu \left(\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x} \right),$$

g_x, g_y, and g_z are gravity accelerations in x, y, and z directions, respectively.

Then, (2.16) to (2.18) become

$$\frac{D(\rho w_x)}{D\tau} = -\frac{\partial p}{\partial x} + 2\frac{\partial}{\partial x}\left(\mu \frac{\partial w_x}{\partial x} \right) + \frac{\partial}{\partial y}\left[\mu \left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x} \right) \right] + \frac{\partial}{\partial z}\left[\mu \left(\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x} \right) \right]$$
$$- \frac{\partial}{\partial x}\left[\frac{2}{3}\mu \left((\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z} \right) \right] + \rho g_x, \tag{2.19}$$

$$\frac{D(\rho w_y)}{D\tau} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x}\right)\right] + 2\frac{\partial}{\partial y}\left(\mu\frac{\partial w_y}{\partial y}\right) + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y}\right)\right]$$
$$-\frac{\partial}{\partial y}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z}\right)\right] + \rho g_y, \quad (2.20)$$

$$\frac{D(\rho w_z)}{D\tau} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y}\right)\right] + 2\frac{\partial}{\partial z}\left[\mu\frac{\partial w_z}{\partial z}\right)$$
$$-\frac{\partial}{\partial z}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z}\right)\right] + \rho g_z \quad (2.21)$$

For steady state, the momentum equations (2.19) – (2.21) are given as follows, respectively,

$$\rho\left(\frac{\partial w_x}{\partial x}w_x + \frac{\partial w_x}{\partial y}w_y + \frac{\partial w_x}{\partial z}w_z\right) + w_x\left(w_x\frac{\partial \rho}{\partial x} + w_y\frac{\partial \rho}{\partial y} + w_z\frac{\partial \rho}{\partial z}\right) =$$
$$-\frac{\partial p}{\partial x} + 2\frac{\partial}{\partial x}\left(\mu\frac{\partial w_x}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x}\right)\right]$$
$$-\frac{\partial}{\partial x}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z}\right)\right] + \rho g_x, \quad (2.22)$$

$$\rho\left(\frac{\partial w_y}{\partial x}w_x + \frac{\partial w_y}{\partial y}w_y + \frac{\partial w_y}{\partial z}w_z\right) + w_y\left(w_x\frac{\partial \rho}{\partial x} + w_y\frac{\partial \rho}{\partial y} + w_z\frac{\partial \rho}{\partial z}\right) =$$
$$-\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x}\right)\right] + 2\frac{\partial}{\partial y}\left(\mu\frac{\partial w_y}{\partial y}\right) + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y}\right)\right]$$
$$-\frac{\partial}{\partial y}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z}\right)\right] + \rho g_y, \quad (2.23)$$

$$\rho\left(\frac{\partial w_z}{\partial x}w_x + \frac{\partial w_z}{\partial y}w_y + \frac{\partial w_z}{\partial z}w_z\right) + w_z\left(w_x\frac{\partial \rho}{\partial x} + w_y\frac{\partial \rho}{\partial y} + w_z\frac{\partial \rho}{\partial z}\right) =$$
$$-\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y}\right)\right] + 2\frac{\partial}{\partial z}\left(\mu\frac{\partial w_z}{\partial z}\right)$$
$$-\frac{\partial}{\partial z}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z}\right)\right] + \rho g_z. \quad (2.24)$$

Let us compare term $\rho\left(\frac{\partial w_x}{\partial x}w_x + \frac{\partial w_x}{\partial y}w_y + \frac{\partial w_x}{\partial z}w_z\right)$ with term $w_x\left(w_x\frac{\partial \rho}{\partial x} + w_y\frac{\partial \rho}{\partial y} + w_z\frac{\partial \rho}{\partial z}\right)$ in (2.22). In general, derivatives $\frac{\partial w_x}{\partial x}$, $\frac{\partial w_x}{\partial y}$, and $\frac{\partial w_x}{\partial z}$ are much larger than the derivatives $\frac{\partial \rho_x}{\partial x}$, $\frac{\partial \rho_x}{\partial y}$, and $\frac{\partial \rho_x}{\partial z}$, respectively. In this case, the term $w_x\left(w_x\frac{\partial \rho}{\partial x} + w_y\frac{\partial \rho}{\partial y} + w_z\frac{\partial \rho}{\partial z}\right)$ is omitted, and (2.22) is rewritten as generally

$$\rho\left(\frac{\partial w_x}{\partial x}w_x+\frac{\partial w_x}{\partial y}w_y+\frac{\partial w_x}{\partial z}w_z\right)=-\frac{\partial p}{\partial x}+2\frac{\partial}{\partial x}\left(\mu\frac{\partial w_x}{\partial x}\right)$$
$$+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w_x}{\partial z}+\frac{\partial w_z}{\partial x}\right)\right]$$
$$-\frac{\partial}{\partial x}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}+\frac{\partial w_z}{\partial z}\right)\right]+\rho g_x. \tag{2.25}$$

Similarly, in general, (2.23) and (2.24) are rewritten as respectively

$$\rho\left(\frac{\partial w_y}{\partial x}w_x+\frac{\partial w_y}{\partial y}w_y+\frac{\partial w_y}{\partial z}w_z\right)=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x}\right)\right]$$
$$+2\frac{\partial}{\partial y}\left(\mu\frac{\partial w_y}{\partial y}\right)+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w_y}{\partial z}+\frac{\partial w_z}{\partial y}\right)\right]$$
$$-\frac{\partial}{\partial y}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}+\frac{\partial w_z}{\partial z}\right)\right]+\rho g_y, \tag{2.26}$$

$$\rho\left(\frac{\partial w_z}{\partial x}w_x+\frac{\partial w_z}{\partial y}w_y+\frac{\partial w_z}{\partial z}w_z\right)=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w_x}{\partial z}+\frac{\partial w_z}{\partial x}\right)\right]$$
$$+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w_y}{\partial z}+\frac{\partial w_z}{\partial y}\right)\right]+2\frac{\partial}{\partial z}\left(\mu\frac{\partial w_z}{\partial z}\right)$$
$$-\frac{\partial}{\partial z}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}+\frac{\partial w_z}{\partial z}\right)\right]+\rho g_z \tag{2.27}$$

2.3 Energy Equation

The control volume for derivation of the energy equation of fluid flow is shown in Fig. 2.3. Take an enclosed surface A that includes the control volume. According to the first law of thermodynamics, we have the following equation:

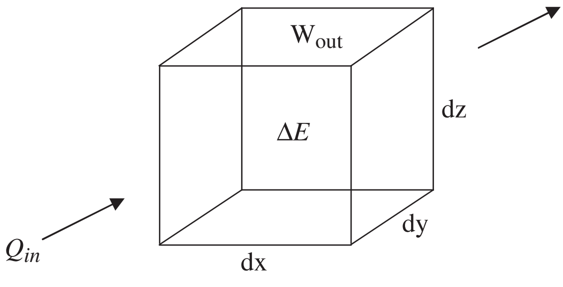

Fig. 2.3 Control volume for derivation of the energy equations of fluid flow

$$\Delta\dot{E} = \dot{Q} + \dot{W}_{\text{out}}, \tag{2.28}$$

where $\Delta\dot{E}$ is energy increment in the system per unit time, $\dot{Q}$ is heat increment in the system per unit time, and $\dot{W}_{\text{out}}$ denotes work done by the mass force and surface force on the system per unit time.

The energy increment per unit time in the system is described as

$$\Delta\dot{E} = \frac{D}{D\tau}\int_V \rho\left(e + \frac{W^2}{2}\right)\mathrm{d}V, \tag{2.29}$$

where τ denotes time, $\frac{W^2}{2}$ is the fluid kinetic energy per unit mass, W is fluid velocity, and the symbol e represents the internal energy per unit mass.

The work done by the mass force and surface force on the system per unit time is expressed as

$$\dot{W}_{\text{out}} = \int_V \rho\,\vec{F}\cdot\vec{W}\,\mathrm{d}V + \int_A \vec{\tau_n}\cdot\vec{W}\,\mathrm{d}A, \tag{2.30}$$

where $\vec{F}$ is the mass force per unit mass and $\vec{\tau_n}$ is surface force acting on unit area.

The heat increment entering into the system per unit time through thermal conduction is described by using Fourier's law as follows:

$$\dot{Q} = \int_A \lambda\frac{\partial t}{\partial n}\mathrm{d}A, \tag{2.31}$$

where n is normal line of the surface, and here the heat conduction is considered only.

With (2.29) to (2.31), (2.28) is rewritten as

$$\frac{D}{D\tau}\int_V \rho\left(e + \frac{W^2}{2}\right)\mathrm{d}V = \int_V \rho\,\vec{F}\cdot\vec{W}\,\mathrm{d}V + \int_A \vec{\tau_n}\cdot\vec{W}\,\mathrm{d}A + \int_A \lambda\frac{\partial t}{\partial n}\mathrm{d}A, \tag{2.32}$$

where

$$\frac{D}{D\tau}\int_V \rho\left(e + \frac{W^2}{2}\right)\mathrm{d}V = \int_V \frac{D}{D\tau}\left[\rho\left(e + \frac{W^2}{2}\right)\right]\mathrm{d}V, \tag{2.33}$$

$$\int_A \vec{\tau_n}\cdot\vec{W}\,\mathrm{d}A = \int_A \vec{n}\,[\tau]\cdot\vec{W}\,\mathrm{d}A = \int_A \vec{n}\,([\tau]\cdot\vec{W})\mathrm{d}A = \int_v \nabla\cdot([\tau]\cdot\vec{W})\mathrm{d}V, \tag{2.34}$$

$$\int_A \lambda\frac{\partial t}{\partial n}\mathrm{d}A = \int_V \nabla\cdot(\lambda\nabla t)\mathrm{d}V. \tag{2.35}$$

With (2.33), (2.34), and (2.35), (2.32) is rewritten as

$$\int_V \frac{D}{D\tau}\left[\rho\left(e+\frac{W^2}{2}\right)\right]\mathrm{d}V = \int_V \rho\,\vec{F}\cdot\vec{W}\,\mathrm{d}V + \int_v \nabla\cdot([\tau]\cdot\vec{W})\mathrm{d}V + \int_V \nabla\cdot(\lambda\nabla t)\mathrm{d}V. \tag{2.36}$$

Then

$$\frac{D}{D\tau}\left[\rho\left(e+\frac{W^2}{2}\right)\right] = \rho\,\vec{F}\cdot\vec{W} + \nabla\cdot([\tau]\cdot\vec{W}) + \nabla\cdot(\lambda\nabla t), \tag{2.37}$$

where $[\tau]$ denotes tensor of shear force.

Equation (2.37) is the energy equation.

Through tensor and vector analysis, (2.37) can be further derived into the following form:

$$\frac{D(\rho e)}{D\tau} = [\tau]\cdot[\varepsilon] + \nabla\cdot(\lambda\nabla t). \tag{2.38}$$

Equation (2.38) is another form of the energy equation. Here, $[\tau]\cdot[\varepsilon]$ is the scalar quantity product of force tensor $[\tau]$ and deformation rate tensor $[\varepsilon]$, and represents the work done by fluid deformation surface force. The physical significance of (2.38) is that the internal energy increment of fluid with unit volume during the unit time equals the sum of the work done by deformation surface force of fluid with unit volume and the heat entering the system.

With the general Newtonian law, the tensor of shear force is expressed as

$$[\tau] = 2\mu[\varepsilon] - \left(p + \frac{2}{3}\mu\nabla\cdot\vec{W}\right)[I], \tag{2.39}$$

where $[I]$ is unit tensor.

According to (2.39) the following equation can be obtained:

$$[\tau]\cdot[\varepsilon] = -p\nabla\cdot\vec{W} - \frac{2}{3}\mu(\nabla\cdot\vec{W})^2 + 2\mu[\varepsilon]^2. \tag{2.40}$$

Then, (2.38) can be rewritten as

$$\frac{D(\rho e)}{D\tau} = -p\nabla\cdot\vec{W} + \Phi + \nabla\cdot(\lambda\nabla t), \tag{2.41}$$

where $\Phi = -\frac{2}{3}\mu(\nabla\cdot\vec{W})^2 + 2\mu[\varepsilon]^2$ is viscous thermal dissipation, which is further described as

$$\begin{aligned}\Phi = \mu\Bigg\{&2\left(\frac{\partial w_x}{\partial x}\right)^2 + 2\left(\frac{\partial w_y}{\partial y}\right)^2 + 2\left(\frac{\partial w_z}{\partial z}\right)^2 + \left(\frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x}\right)^2\\ &+\left(\frac{\partial w_y}{\partial z}+\frac{\partial w_z}{\partial y}\right)^2 + \left(\frac{\partial w_z}{\partial x}+\frac{\partial w_x}{\partial z}\right)^2 - \frac{2}{3}\left[\mathrm{div}(\vec{W})\right]^2\Bigg\},\end{aligned} \tag{2.42}$$

where

$$\mathrm{div}(\vec{W}) = \frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z}.$$

According to continuity equation (2.5), we have

$$\nabla \cdot \vec{W} = -\frac{1}{\rho}\frac{D\rho}{D\tau} = \rho\frac{D}{D\tau}\left(\frac{1}{\rho}\right).$$

With the above equation, (2.41) is changed into the following form:

$$\left[\frac{D(\rho e)}{D\tau} + p\rho\frac{D}{D\tau}\left(\frac{1}{\rho}\right)\right] = \Phi + \nabla \cdot (\lambda \nabla t). \tag{2.43}$$

According to thermodynamics equation of fluid, we have

$$\frac{D(\rho h)}{D\tau} = \frac{D(\rho e)}{D\tau} + p\rho\frac{D}{D\tau}\left(\frac{1}{\rho}\right) + \frac{Dp}{D\tau}. \tag{2.44}$$

Then, Equation (2.43) can be expressed as the following enthalpy form:

$$\frac{D(\rho h)}{D\tau} = \frac{Dp}{D\tau} + \Phi + \nabla \cdot (\lambda \nabla t), \tag{2.45}$$

or

$$\frac{D(\rho c_p \cdot t)}{D\tau} = \frac{Dp}{D\tau} + \Phi + \nabla \cdot (\lambda \nabla t), \tag{2.46}$$

where $h = c_p \cdot t$, while c_p is specific heat.

In Cartesian form, the energy (2.46) can be rewritten as

$$\begin{aligned} &\frac{\partial(\rho c_p \cdot t)}{\partial \tau} + w_x\frac{\partial(\rho c_p \cdot t)}{\partial x} + w_y\frac{\partial(\rho c_p \cdot t)}{\partial y} + w_z\frac{\partial(\rho c_p \cdot t)}{\partial z} \\ &= \frac{DP}{D\tau} + \frac{\partial}{\partial x}\left(\lambda\frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda\frac{\partial t}{\partial z}\right) + \Phi. \end{aligned} \tag{2.47}$$

For steady state, the three-dimensional Cartesian form of the energy equation (2.47) is changed into

$$\begin{aligned} w_x\frac{\partial(\rho c_p t)}{\partial x} + w_y\frac{\partial(\rho c_p t)}{\partial y} + w_z\frac{\partial(\rho c_p t)}{\partial z} &= \frac{\partial}{\partial x}\left(\lambda\frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial t}{\partial y}\right) \\ &\quad + \frac{\partial}{\partial z}\left(\lambda\frac{\partial t}{\partial z}\right) + \Phi. \end{aligned} \tag{2.48}$$

Above equation is usually approximately rewritten as

$$\rho\left[w_x\frac{\partial(c_p\cdot t)}{\partial x}+w_y\frac{\partial(c_p\cdot t)}{\partial y}+w_z\frac{\partial(c_p\cdot t)}{\partial z}\right]=\frac{\partial}{\partial x}\left(\lambda\frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial t}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial t}{\partial z}\right)+\Phi, \tag{2.49}$$

where the temperature-dependence of density is ignored.

For steady state, the two-dimensional Cartesian form of the energy equation (2.48) is changed into

$$w_x\frac{\partial(\rho c_p t)}{\partial x}+w_y\frac{\partial(\rho c_p t)}{\partial y}=\frac{\partial}{\partial x}\left(\lambda\frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial t}{\partial y}\right)+\Phi. \tag{2.50}$$

Above equation is usually approximately rewritten as

$$\rho\left[w_x\frac{\partial(c_p t)}{\partial x}+w_y\frac{\partial(c_p t)}{\partial y}\right]=\frac{\partial}{\partial x}\left(\lambda\frac{\partial t}{\partial x}t\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial t}{\partial y}\right)+\Phi. \tag{2.51}$$

With the two-dimensional form, the viscous thermal dissipation is

$$\Phi=\mu\left\{2\left(\frac{\partial w_x}{\partial x}\right)^2+2\left(\frac{\partial w_y}{\partial y}\right)^2+\left(\frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x}\right)^2-\frac{2}{3}\left[div(\vec{W})\right]^2\right\}, \tag{2.52}$$

where

$$div(\vec{W})=\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}.$$

2.4 Governing Partial Differential Equations of Laminar Forced Convection Boundary Layers with Consideration of Variable Physical Properties

2.4.1 Principle of the Quantitative Grade Analysis

The general governing partial differential equations can be transformed to the related boundary layer equations. In fact, the only difference between the governing equations of free and forced convection is that buoyant force is only considered in the related momentum equation of free convection. Before the quantitative grade analysis, it is necessary to define its analytical standard. A normal quantitative grade is regarded as $\{1\}$, i.e. unit quantity grade, a very small quantitative grade is regarded as $\{\delta\}$, even very small quantitative grade is regarded as $\{\delta^2\}$, and so on. Then, $\frac{\{1\}}{\{1\}} = \{1\}$, $\frac{\{\delta\}}{\{\delta\}} = \{1\}$, $\frac{\{1\}}{\{\delta\}} = \{\delta^{-1}\}$, $\frac{\{1\}}{\{\delta^2\}} = \{\delta^{-2}\}$.

Furthermore, according to the theory of laminar forced boundary layer, the quantities of the velocity component w_x and the coordinate x can be regarded as unity, i.e. $\{w_x\} = \{1\}$ and $\{x\} = \{1\}$. The quantities of the velocity component w_y and the coordinate y should be regarded as δ, i.e. $\{w_y\} = \{\delta\}$ and $\{y\} = \{\delta\}$. However, the quantity of temperature t can be regarded as $\{t\} = \{1\}$, the quantity grade of the pressure gradient $\frac{\partial p}{\partial x}$ can be regarded as unity, i.e. $\left\{\frac{\partial p}{\partial x}\right\} = \{1\}$, but the quantity grade of the pressure gradient $\frac{\partial p}{\partial y}$ is only regarded as small quantity grade, i.e. $\left\{\frac{\partial p}{\partial y}\right\} = \{\delta\}$.

According to the quantitative grade of the general fluid, the quantitative grade of density ρ, thermal conductivity λ, absolute viscosity μ, and specific heat c_p are $\{\rho\} = \{1\}$, $\{\lambda\} = \{\delta^2\}$, $\{\mu\} = \{\delta^2\}$, and $\{c_p\} = \{1\}$. In addition, for gravity acceleration, we have $\{g_x\} = \{0\}$, and $\{g_y\} = \{1\}$ for forced convection.

2.4.2 Continuity Equation

Based on the (2.7), the steady-state two-dimensional continuity equation is given by

$$\frac{\partial}{\partial x}(\rho w_x) + \frac{\partial}{\partial y}(\rho w_y) = 0 \tag{2.53}$$

where variable fluid density is considered.

According to the above principle for the quantity analysis, now, we take the steady-state two-dimensional continuity equation (2.53) as example to do the quantity analysis as follows:

$$\begin{array}{lcccccl} & \frac{\partial}{\partial x}(\rho w_x) & + & \frac{\partial}{\partial y}(\rho w_y) & = 0 & & \\ & \frac{\{1\}}{\{1\}} & + & \frac{\{\delta\}}{\{\delta\}} & = 0 & & \\ \text{i.e.} & \{1\} & + & \{1\} & = 0 & & (2.53\text{a}) \end{array}$$

The above ratios of quantity grade related to the terms of (2.53) shows that both of the two terms of (2.53) should be kept, and (2.53) can be regarded as the continuity equation of the two-dimensionless steady-state forced convection with laminar two-dimensional boundary layer.

2.4.3 Momentum Equations (Navier–Stokes Equations)

The momentum equations (2.25) and (2.26) are rewritten as for steady two-dimensional convection

$$\rho\left(w_x\frac{\partial w_x}{\partial x}+w_y\frac{\partial w_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+2\frac{\partial}{\partial x}\left(\mu\frac{\partial w_x}{\partial x}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x}\right)\right]$$
$$-\frac{\partial}{\partial x}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}\right)\right]+\rho g_x, \tag{2.54}$$

$$\rho\left(w_x\frac{\partial w_y}{\partial x}+w_y\frac{\partial w_y}{\partial y}\right)=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x}\right)\right]+2\frac{\partial}{\partial y}\left(\mu\frac{\partial w_y}{\partial y}\right)$$
$$-\frac{\partial}{\partial y}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}\right)\right]+\rho g_y. \tag{2.55}$$

The quantity grades of the terms of (2.54) and (2.55) are expressed as follows, respectively:

$$\rho\left(w_x\frac{\partial w_x}{\partial x}+w_y\frac{\partial w_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+2\frac{\partial}{\partial x}\left(\mu\frac{\partial w_x}{\partial x}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x}\right)\right]$$
$$-\frac{\partial}{\partial x}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}\right)\right]+\rho g_x$$
$$\{1\}(\{1\}\frac{\{1\}}{\{1\}}+\{\delta\}\frac{\{1\}}{\{\delta\}})=\{1\}+\frac{\{1\}}{\{1\}}\{\delta^2\}\frac{\{1\}}{\{1\}}+\frac{\{1\}}{\{\delta\}}\{\delta^2](\frac{\{1\}}{\{\delta\}}+\frac{\{\delta\}}{\{1\}})$$
$$-\frac{\{1\}}{\{1\}}\delta^2(\frac{\{1\}}{\{1\}}+\frac{\{\delta\}}{\{\delta\}})+\{1\}\{0\} \tag{2.54a}$$

$$\rho\left(w_x\frac{\partial w_y}{\partial x}+w_y\frac{\partial w_y}{\partial y}\right)=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x}\right)\right]+2\frac{\partial}{\partial y}\left(\mu\frac{\partial w_y}{\partial y}\right)$$
$$-\frac{\partial}{\partial y}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}\right)\right]+\rho g_y$$
$$\{1\}(\{\delta\}+\{\delta\})=\{\delta\}+(\{\delta\}+\{\delta^3\})+\{\delta\}-(\{\delta\}(\{1\}+\{1\}))+\{1\}\{0\} \tag{2.55a}$$

Here, for forced convection, the volume forces ρg_x and ρg_y are regarded as zero.

Then, the quantity grades of (2.54a) is further simplified as follows, respectively:

$$\rho\left(w_x\frac{\partial w_x}{\partial x}+w_y\frac{\partial w_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+2\frac{\partial}{\partial x}\left(\mu\frac{\partial w_x}{\partial x}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x}\right)\right]$$
$$-\frac{\partial}{\partial x}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}\right)\right]$$

$$\{1)(\{1\} + \{1\}) = \{1\} + \{\delta^2\} + \{1\} + \{\delta^2\} - (\{\delta^2\} + \{\delta^2\}) \tag{2.54b}$$

Observing the quantity grades in (2.54a) it is found that the terms $2\frac{\partial}{\partial x}\left(\mu\frac{\partial w_x}{\partial x}\right)$, $\frac{\partial w_y}{\partial x}$ in term $\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x}\right)\right]$, $\frac{\partial}{\partial x}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y}\right)\right]$ are very small and can be ignored. Then, (2.54a) is simplified as follows:

$$\rho\left(w_x\frac{\partial w_x}{\partial x} + w_y\frac{\partial w_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w_x}{\partial y}\right)\right]. \tag{2.56}$$

Comparing the quantity grades of (2.55a) with those of (2.54b), it is found that the quantity grades of (2.55a) are very small. Then, (2.55) can be ignored, and only (2.56) is taken as the momentum equation of steady-state forced convection with two-dimensional boundary layer.

2.4.4 Energy Equations

The quantity grades of the terms of (2.51) for laminar two-dimensional energy equation is expressed as

$$\begin{gathered}\rho\left[w_x\frac{\partial(c_p t)}{\partial x} + w_y\frac{\partial(c_p t)}{\partial y}\right] = \frac{\partial}{\partial x}\left(\lambda\frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial t}{\partial y}\right) + \Phi \\ \{1\}\left[\{1\}\cdot\frac{\{1\}}{\{1\}} + \{\delta\}\cdot\frac{\{1\}}{\{\delta\}}\right] = \frac{\{1\}}{\{1\}}\left(\delta^2\frac{\{1\}}{\{1\}}\right) + \frac{\{1\}}{\{\delta\}}\delta^2\frac{\{1\}}{\{\delta\}}\end{gathered} \tag{2.51a}$$

The quantity grade of (2.51a) is simplified to

$$\begin{gathered}\rho\left[w_x\frac{\partial(c_p t)}{\partial x} + w_y\frac{\partial(c_p t)}{\partial y}\right] = \frac{\partial}{\partial x}\left(\lambda\frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial t}{\partial y}\right) + \Phi \\ \{1\}[\{1\} + \{1\}\cdot] = \{1\}(\delta^2\} + \{1\})\end{gathered} \tag{2.51b}$$

With the quantity grade analysis from (2.51b), it is seen that the term $\frac{\partial}{\partial x}\left(\lambda\frac{\partial t}{\partial x}\right)$ is very small compared with other terms, and then can be omitted from the equation. Then, the energy equation for steady-state forced convection with two-dimensional laminar boundary layer is simplified to

$$\rho\left[w_x\frac{\partial(c_p t)}{\partial x} + w_y\frac{\partial(c_p t)}{\partial y}\right] = \frac{\partial}{\partial y}\left(\lambda\frac{\partial t}{\partial y}\right) + \Phi. \tag{2.57}$$

Now, we analyze the viscous thermal dissipation Φ. According to (2.52), Φ is the following equation for the laminar stead state convection:

$$\Phi = \mu \left\{ 2\left(\frac{\partial w_x}{\partial x}\right)^2 + 2\left(\frac{\partial w_y}{\partial y}\right)^2 + \left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x}\right)^2 - \frac{2}{3}\left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y}\right)^2 \right\}. \tag{2.52a}$$

With quantity grade analysis, the right side of (2.52a) is expressed as

$$\Phi = \mu \left\{ 2\left(\frac{\partial w_x}{\partial x}\right)^2 + 2\left(\frac{\partial w_y}{\partial y}\right)^2 + \left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x}\right)^2 - \frac{2}{3}\left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y}\right)^2 \right\}$$

$$\{\delta^2\}\left(\left(\frac{\{1\}}{\{1\}}\right)^2 + \left(\frac{\{\delta\}}{\{\delta\}}\right)^2 + \left(\frac{\{1\}}{\{\delta\}} + \frac{\{\delta\}}{\{1\}}\right)^2 - \left(\frac{\{1\}}{\{1\}} + \frac{\{\delta\}}{\{\delta\}}\right)^2\right)$$

The quantity grade of the above equation is equivalent to

$$\{\delta^2\}\left(\{1\} + \{1\} + \left(\frac{\{1\}}{\{\delta^2\}} + \{\delta^2\}\right) - (\{1\} + \{1\})^2\right) \tag{2.52b}$$

With the quantity grade comparison, it is found that only the term $\mu\left(\frac{\partial w_x}{\partial y}\right)^2$ may be kept. Then, for the stead state forced convection with two-dimensional boundary layer, the viscous thermal dissipation is simplified to

$$\Phi = \mu\left(\frac{\partial w_x}{\partial y}\right)^2. \tag{2.52c}$$

With (2.52c), (2.57) is changed to the following equation

$$\rho\left[w_x\frac{\partial(c_p t)}{\partial x} + w_y\frac{\partial(c_p t)}{\partial y}\right] = \frac{\partial}{\partial y}\left(\lambda\frac{\partial t}{\partial y}\right) + \mu\left(\frac{\partial w_x}{\partial y}\right)^2 \tag{2.58}$$

as the steady-state energy equation with laminar forced convection for two-dimensional boundary layer.

In a later chapter, I will investigate the effect of the viscous thermal dissipation on heat transfer of laminar forced convection.

Now the basic governing partial differential equations for description of mass, momentum, and energy conservation of two-dimensional laminar steady-state forced convection boundary layers are shown as follows with consideration viscous thermal dissipation and variable physical properties:

$$\frac{\partial}{\partial x}(\rho w_x) + \frac{\partial}{\partial y}(\rho w_y) = 0, \tag{2.53}$$

$$\rho\left(w_x\frac{\partial w_x}{\partial x} + w_y\frac{\partial w_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w_x}{\partial y}\right)\right], \tag{2.56}$$

$$\rho\left[w_x\frac{\partial(c_p t)}{\partial x} + w_y\frac{\partial(c_p t)}{\partial y}\right] = \frac{\partial}{\partial y}\left(\lambda\frac{\partial t}{\partial y}\right) + \mu\left(\frac{\partial w_x}{\partial y}\right)^2. \tag{2.58}$$

Suppose a bulk flow with the velocity $w_{x,\infty}$ beyond the boundary layer, (2.56) is simplified to the following form:

$$w_{x,\infty}\frac{\partial w_{x,\infty}}{\partial x} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x},$$

i.e.

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -\rho w_{x,\infty}\frac{\partial w_{x,\infty}}{\partial x}.$$

Then (2.56) is changed to

$$\rho\left(w_x\frac{\partial w_x}{\partial x} + w_y\frac{\partial w_x}{\partial y}\right) = \rho w_{x,\infty}\frac{\partial w_{x,\infty}}{\partial x} + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial w_x}{\partial y}\right)\right) \tag{2.56a}$$

Obviously, if the main stream velocity $w_{x,\infty}$ is constant, i.e. $\frac{\partial w_{x,\infty}}{\partial x} = 0$, (2.56a) is identical to the following equations:

$$\rho\left(w_x\frac{\partial w_x}{\partial x} + w_y\frac{\partial w_x}{\partial y}\right) = \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w_x}{\partial y}\right)\right]. \tag{2.56b}$$

For rigorous solutions of the governing equations, the fluid temperature-dependent properties, such as density ρ, absolute viscosity μ, specific heat c_p, and thermal conductivity λ will be considered in the successive chapters of this book.

The laminar forced convection with two-dimensional boundary layer belongs to two-point boundary value problem, which is the basis of three-point boundary value problem for film condensation.

Obviously, the basic governing partial differential equations for mass, momentum, and energy conservation of two-dimensional laminar steady-state forced convection boundary layers with considering viscous thermal dissipation but ignoring the variable physical properties are obtained as based on (2.53), (2.56), and (2.58):

$$\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} = 0 \tag{2.59}$$

$$w_x\frac{\partial w_x}{\partial x} + w_y\frac{\partial w_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 w_x}{\partial y^2} \tag{2.60}$$

with $-\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} = w_{x,\infty}\frac{\partial w_{x,\infty}}{\partial x}$

$$\left[w_x\frac{\partial t}{\partial x} + w_y\frac{\partial t}{\partial y}\right] = \frac{\nu}{\mathrm{Pr}}\frac{\partial^2 t}{\partial^2} + (\nu/c_p)\left(\frac{\partial w_x}{\partial y}\right)^2. \tag{2.61}$$

Now, the three-dimensional basic conservation equations for laminar convection and Two-dimensional basic conservation equations for laminar forced convection boundary layer can be summarized in Tables 2.1 and 2.2, respectively.

2.5 Summary

Table 2.1 Three-dimensional basic conservation equations for laminar convection (with consideration of variable physical properties)

Mass equation	$\frac{\partial}{\partial x}(\rho w_x) + \frac{\partial}{\partial y}(\rho w_y) + \frac{\partial}{\partial z}(\rho w_z) = 0$	(2.6)
Momentum equation	$\rho\left(\frac{\partial w_x}{\partial x}w_x + \frac{\partial w_x}{\partial y}w_y + \frac{\partial w_x}{\partial z}w_z\right) = -\frac{\partial p}{\partial x} + 2\frac{\partial}{\partial x}\left(\mu\frac{\partial w_x}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x}\right)\right] - \frac{\partial}{\partial x}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z}\right)\right] + \rho g_x$	(2.25)
	$\rho\left(\frac{\partial w_y}{\partial x}w_x + \frac{\partial w_y}{\partial y}w_y + \frac{\partial w_y}{\partial z}w_z\right) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}[\mu]\left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x}\right) + 2\frac{\partial}{\partial y}\left(\mu\frac{\partial w_y}{\partial y}\right) + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y}\right)\right] - \frac{\partial}{\partial y}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z}\right)\right] + \rho g_y$	(2.26)
	$\rho\left(\frac{\partial w_z}{\partial x}w_x + \frac{\partial w_z}{\partial y}w_y + \frac{\partial w_z}{\partial z}w_z\right) = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y}\right)\right] + 2\frac{\partial}{\partial z}\left(\mu\frac{\partial w_z}{\partial z}\right) - \frac{\partial}{\partial z}\left[\frac{2}{3}\mu\left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z}\right)\right] + \rho g_z$	(2.27)
Energy equation	$\rho\left[w_x\frac{\partial(c_p \cdot t)}{\partial x} + w_y\frac{\partial(c_p \cdot t)}{\partial y} + w_z\frac{\partial(c_p \cdot t)}{\partial z}\right] = \frac{\partial}{\partial x}\left(\lambda\frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda\frac{\partial t}{\partial z}\right) + \Phi$	(2.49)
	$\Phi = \mu\left\{2\left(\frac{\partial w_x}{\partial x}\right)^2 + 2\left(\frac{\partial w_y}{\partial y}\right)^2 + 2\left(\frac{\partial w_z}{\partial z}\right)^2 + \left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x}\right)^2 + \left(\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y}\right)^2 + \left(\frac{\partial w_z}{\partial x} + \frac{\partial w_x}{\partial z}\right)^2 - \frac{2}{3}\left[div(\vec{W})\right]^2\right\},$	(2.42)

Table 2.2 Two-dimensional basic conservation equations for laminar forced convection boundary layer

With consideration of variable physical properties		
Mass equation	$\frac{\partial}{\partial x}(\rho w_x) + \frac{\partial}{\partial y}(\rho w_y) = 0$	(2.7)
Momentum equation	$\rho\left(w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w_x}{\partial y}\right)\right]$ $-\frac{\mathrm{d}p}{\mathrm{d}x} = \rho w_{x,\infty} \frac{\partial w_{x,\infty}}{\partial x}$	(2.56)
Energy equation	$\rho\left[w_x \frac{\partial (c_p t)}{\partial x} + w_y \frac{\partial (c_p t)}{\partial y}\right] = \frac{\partial}{\partial y}\left(\lambda \frac{\partial t}{\partial y}\right) + \mu\left(\frac{\partial w_x}{\partial y}\right)^2$	(2.58)
	(for consideration of viscous thermal dissipation $\mu\left(\frac{\partial w_x}{\partial y}\right)^2$)	
With ignoring variable physical properties		
Mass equation	$\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} = 0$	(2.59)
Momentum equation	$w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 w_x}{\partial y^2}$ $-\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} = w_{x,\infty} \frac{\partial w_{x,\infty}}{\partial x}$	(2.60)
Energy equation	$\left[w_x \frac{\partial t}{\partial x} + w_y \frac{\partial t}{\partial y}\right] = \frac{\nu}{\mathrm{Pr}} \frac{\partial^2 t}{\partial^2} + (\nu/c_p)\left(\frac{\partial w_x}{\partial y}\right)^2$	(2.61)
	(for consideration of viscous thermal dissipation $\mu\left(\frac{\partial w_x}{\partial y}\right)^2$)	

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