About the Pricing Equations in Finance

Stéphane Crépey

Abstract In this article we study a decoupled forward backward stochastic differential equation (FBSDE) and the associated *system of partial integro-differential obstacle problems*, in a flexible Markovian set-up made of a jump-diffusion with regimes.

These equations are motivated by numerous applications in financial modeling, whence the title of the paper. This financial motivation is developed in the first part of the paper, which provides a synthetic view of the theory of pricing and hedging financial derivatives, using backward stochastic differential equations (BSDEs) as main tool.

In the second part of the paper, we establish the well-posedness of reflected BSDEs with jumps coming out of the pricing and hedging problems exposed in the first part. We first provide a construction of a Markovian model made of a jump-diffusion – like component X interacting with a continuous-time Markov chain – like component N. The jump process N defines the so-called *regime* of the coefficients of X, whence the name of *jump-diffusion with regimes* for this model. Motivated by *optimal stopping* and *optimal stopping game* problems (pricing equations of *American or game contingent claims*), we introduce the related *reflected and doubly reflected Markovian BSDEs*, showing that they are *well-posed* in the sense that they have *unique solutions, which depend continuously on their input data.* As an aside, we establish the *Markov property* of the model.

In the third part of the paper we derive the related *variational inequality approach*. We first introduce the systems of partial integro-differential variational inequalities formally associated to the reflected BSDEs, and we state suitable definitions of viscosity solutions for these problems, accounting for jumps and/or systems of equations. We then show that the state-processes (first components Y) of the solutions to the reflected BSDEs can be characterized in terms of the *value functions* of related optimal stopping or game problems, given as *viscosity solutions with polynomial growth* to related integro-differential obstacle problems. We further

S. Crépey (✉)
Département de Mathématiques, Université d'Évry Val d'Essonne, 91025 Évry Cedex, France
e-mail: stephane.crepey@univ-evry.fr

A.R. Carmona et al. (eds.), *Paris-Princeton Lectures on Mathematical Finance 2010*, Lecture Notes in Mathematics 2003, DOI 10.1007/978-3-642-14660-2_2,

establish a *comparison principle* for semi-continuous viscosity solutions to these problems, which implies in particular the *uniqueness* of the viscosity solutions. This comparison principle is subsequently used for proving the convergence of *stable, monotone and consistent* approximation schemes to the value functions.

Finally in the last part of the paper we provide various extensions of the results needed for applications in finance to pricing problems involving *discrete dividends* on a financial derivative or on the underlying asset, as well as various forms of *discrete path-dependence*.

1 Introduction

In this article, we establish the well-posedness of a decoupled *forward backward stochastic differential equation* and of the associated *system of partial integro-differential obstacle problems*, in a rather flexible Markovian set-up made of a jump-diffusion model with regimes.

These equations are motivated by numerous applications in financial modeling, whence the title of the paper. This financial motivation is developed in Part I, where we essentially reduce the problem of pricing and hedging financial derivatives to that of solving (typically reflected) backward stochastic differential equations (BSDEs), or, equivalently in the Markovian case, partial integro-differential equations or variational inequalities (PIDEs or PDEs for short).

In Parts II–IV, we tackle the resulting Markovian BSDE and PDE problems. In Crépey and Matoussi [38], a priori estimates and comparison principles were derived for reflected or doubly reflected BSDEs in the general, non-Markovian set-up of a model driven by a continuous local martingale and an integer-valued random measure. In Part II we use these results to establish the well-posedness of *Markovian reflected BSDEs*, which is used in Part III for studying the associated partial integro-differential systems of obstacle problems, in a rather flexible Markovian set-up made of a jump-diffusion model with regimes. As an aside we prove the convergence of any *stable, monotone and consistent* approximation scheme for these problems. Part IV provides various extensions of the previous results needed for applications in finance to pricing problems involving *discrete dividends* on a financial derivative or on an underlying asset, as well as various forms of *discrete path-dependence*.

The main results are summed-up in Propositions 30 and 31, which synthesize the major findings of Part II and III, respectively.

This paper lays the mathematical foundation of a large body of work in credit risk and financial modeling [15, 16, 20, 39]. Even if rather expected in their statement, many of the mathematical results derived in Parts II–IV are innovative. In particular, doubly reflected BSDEs with a delayed or an even more general intermittent upper barrier (RDBSDEs and RIBSDEs, see Definitions 9(ii) and 16), have not been considered elsewhere in the literature (if not for the preliminary results of Crépey and Matoussi [38]). Also, the Markovian model which is considered in detail in Parts II and III was already considered and some of the results of the present paper were already announced and used in [16, 20, 38]. But the possibility to construct a model with all the required properties was taken for granted there. The mathematical construction of the model in Sect. 7 is non-trivial, and was not done elsewhere before. The treatment of the Markovian BSDEs with jumps and of their PDE interpretation in Parts II and III, including the proof of convergence of a numerical deterministic scheme to the viscosity solution of a system of integro-differential variational inequalities, is quite technical too.

As for Part I, we believe that, beyond providing the motivation for the mathematical results of Parts II–IV, it also has the merit of giving a unified, cross market perspective (see Sects. 3.3.3 and 6.6) on the theory of pricing and hedging financial derivatives, via the use of BSDEs as a main tool.

Part I on one hand, and Parts II–IV on the other hand, can be read essentially independently. The reader who would be mainly interested in the financial applications can thus read Part I first, taking for granted the results of Parts II–IV whenever they are used therein (see Propositions 5, 6, 8, 14 and 16 in particular). Likewise readers mainly interested by the mathematical results of Parts II–IV can skip Part I at first reading.

1.1 Detailed Outline

Section 2 develops the theory of risk-neutral pricing and hedging of financial derivatives, using BSDEs as a main tool (see El Karoui et al. [46] for a general reference on BSDEs in finance). The central result, Proposition 3, can be informally stated as follows: Under the assumption, thoroughly investigated in Part II, that a reflected backward stochastic differential equation (BSDE) related to a financial derivative, relatively to a risk-neutral probability measure $\mathbb{P}$ over a primary market of hedging instruments, admits a solution Π, then Π is the minimal *superhedging price up to a* $\mathbb{P}$ *– local martingale cost process* for the derivative at hand, this cost being equal to 0 in the case of complete markets. This notion of hedge *with local martingale cost* thus establishes a connection between arbitrage prices and hedging, in a rather general, possibly incomplete, market.

In Sect. 3, we consider the specification of these results to the *Markovian set-up*. Using the results of Part III, a complementary *variational inequality* approach may then be developed, and more *explicit and constructive hedging strategies* may be given (see Sect. 3.5 in particular).

Section 4 presents various extensions of the previous results. Section 4.1 generalizes the previous risk-neutral approach to a martingale modeling approach relatively to an arbitrary *numeraire* B (positive primary asset price process) which may be used for discounting other price processes, rather than a savings account (riskless asset) in the risk-neutral approach. This extension is particularly important for dealing with interest-rate derivatives. Section 4.2, which is based on Bielecki et al. [20], refines the risk-neutral martingale modeling approach of Sects. 2 and 3 to the specific case, important for equity-to-credit applications, of *defaultable derivatives*, with all cash flows killed at the default time θ of a reference entity. Finally in Sect. 4.3 we deal with the issue of callability and *call protection* (*intermittent* call protection vs. call protection *before a stopping time*).

In Part I, well-posedness of the pricing BSDEs and PDEs is taken for granted. The following sections of the paper (Parts II–IV) are devoted to the mathematics of these pricing equations.

In Sect. 5 we recall the general set-up of [38] and the general form of the BSDEs we are interested in.

In Sect. 6, we present a versatile Markovian specification of this general set-up, made of a jump-diffusion X interacting with a pure jump process N (which in the simplest case reduces to a Markov chain in continuous time). The interaction between X and N is materialized by the fact that the coefficients of the dynamics of X depend on N, and also, by a mutual dependence of the jump intensity of either process on the other one. Such coupled dependence is motivated by applications like modeling *frailty* and *contagion* in *portfolio credit risk* (see [16]).

But the construction of a model with such mutual dependence is a non-trivial issue, and we treat it in detail in Sect. 7, resorting to a suitable *Markovian change of probability measure*.

This model may also be viewed as a generalization of the interacting Itô process and point process model considered by Becherer and Schweizer in [10]. Yet as opposed to the set-up of [10] where linear reaction-diffusion systems of parabolic equations (pricing equations of *European contingent claims*, from the point of view of the financial interpretation) are considered from the point of view of *classical solutions*, here the application one has in mind consists of more general *optimal stopping* or *optimal stopping game* problems (pricing equations of *American or game contingent claims*, see Part I) for which the related reaction-diffusion systems typically do not have classical solutions. This leads us to study in Sect. 8 the related *reflected and doubly reflected Markovian BSDEs* (see [20, 46, 47]), showing that they are *well-posed* in the sense that they have *unique solutions, which depend continuously on their input data.*

In Sect. 9 we derive the associated *Markov and flow properties*.

In Sect. 10 we introduce the systems of partial integro-differential variational inequalities formally associated to our reflected BSDEs, and we state suitable definitions of semi-continuous viscosity solutions and solutions for these problems.

In Sect. 11 we show that the state-processes (first components Y) of the solutions to our reflected BSDEs can be characterized in terms of the *value functions* to related optimal stopping or game problems, given as *viscosity solutions with polynomial growth* to the related obstacle problems.

We establish in Sect. 12 a *semi-continuous viscosity solutions comparison principle*, which implies in particular *uniqueness* of viscosity solutions for these problems.

This comparison principle is subsequently used in Sect. 13 for proving the convergence of *stable, monotone and consistent* approximation schemes (cf. Barles and Souganidis; see also [8] Briani et al. [28], Cont and Voltchkova [36] or Jakobsen et al. [64]) to the viscosity solutions of the equations. These results thus extend to models with regimes (whence *systems* of PDEs [9, 60]) the results of [8, 28], among others.

In Sects. 14–16 we provide extensions of the previous results to a factor process model (X, N) possibly involving further *deterministic jumps* at some fixed times T_ls. This is required for applications to pricing problems involving *discrete dividends* on a financial derivative or on an underlying asset, and also, to be able to deal with the issue of *discrete path-dependence*.

Part I
Martingale Modeling in Finance

In this part (see Sect. 1 for a detailed outline), we show how the task of pricing and hedging financial derivatives can generically be reduced to that of solving (typically reflected) BSDEs, or, equivalently in the Markovian case, PDEs. These equations are called *pricing equations* in this paper. Well-posedness of these equations in suitable spaces of solutions will be taken for granted whenever needed in this part, and will then be thoroughly studied in the remaining three parts of the paper.

2 General Set-Up

The evolution of a financial market model is given throughout this part in terms of stochastic processes defined on a continuous time stochastic basis $(\Omega, \mathbb{F}, \widehat{\mathbb{P}})$, where $\widehat{\mathbb{P}}$ denotes the *objective* (also called statistical, historical, physical..) probability measure. We may and do assume that the filtration $\mathbb{F}$ satisfies the usual completeness and right-continuity conditions, and that all semimartingales are càdlàg (i.e., almost surely right continuous with left limits). Finally, since we are always in the context of pricing contingent claims with a fixed maturity T, we further assume that $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ with $\mathcal{F}_0$ trivial and $\mathcal{F}_T = \mathcal{F}$. Moreover, we declare that a *process* on $[0,T]$ (resp. *a random variable*) has to be $\mathbb{F}$-adapted (resp. $\mathcal{F}$-measurable), by definition.

We shall typically work under a *risk-neutral* probability measure $\mathbb{P} \sim \widehat{\mathbb{P}}$, or more generally, under a martingale probability measure $\mathbb{P}$ relative to a suitable *numeraire* (see Sect. 4.1), such that the prices of primary assets, once properly discounted and adjusted for dividends, are $\mathbb{P}$ – local martingales.

As we shall now see, under mild technical conditions, existence of such a martingale measure $\mathbb{P}$ is equivalent to a suitable notion of no-arbitrage.

2.1 Pricing by Arbitrage

2.1.1 Primary Market Model

To model a financial derivative with maturity T, we consider a primary market composed of the savings account B and of d primary risky assets. The discount factor β is supposed to be absolutely continuous with respect to the Lebesgue measure, and given by

$$\beta_t = \exp(-\int_0^t r_u\, du) \tag{1}$$

(so $\beta_0 = 1$ and $\beta = B^{-1}$), for a bounded from below *short-term interest rate* process r.

The primary risky assets, with $\mathbb{R}^d$-valued price process P, may pay dividends, whose cumulative value process, denoted by $\mathcal{D}$, is assumed to be an $\mathbb{R}^d$-valued process of finite variation. Given the price process P, we define the *cumulative price* $\widehat{P}$ of the asset as

$$\widehat{P}_t = P_t + \beta_t^{-1} \int_{[0,t]} \beta_u \, d\mathcal{D}_u. \tag{2}$$

In the financial interpretation, the last term in (2) represents the current value at time t of all dividend payments of the asset over the period $[0, t]$, under the assumption that all dividends are immediately reinvested in the savings account.

For technical reasons we assume that $\widehat{P}$ is a locally bounded semimartingale.

We assume that the primary market model is free of arbitrage opportunities (though presumably incomplete), in the sense that the so-called no free lunch with vanishing risk (NFLVR) condition is satisfied. This NFLVR condition is a specific no arbitrage condition involving wealth processes of admissible self-financing primary trading strategies (see Delbaen and Schachermayer [42]). We do not reproduce here the full definition of arbitrage price, since it is rather technical and will not be explicitly used in the sequel. It will be enough for us to recall the related notions of trading strategies in the primary market.

Definition 1. A *primary trading strategy* (ζ^0, ζ) built on the primary market is an $\mathbb{R} \times \mathbb{R}^{1 \otimes d}$-valued process, with ζ predictable and locally bounded, where ζ^0 and the row-vector ζ represent the number of units held in the savings account and in each of the primary risky assets. The related *wealth process* $\mathcal{W}$ is thus given by:

$$\mathcal{W}_t = \zeta_t^0 B_t + \zeta_t P_t, \tag{3}$$

for $t \in [0, T]$. Accounting for dividends, we say that the strategy is *self-financing* if

$$d\mathcal{W}_t = \zeta_t^0 dB_t + \zeta_t \left(dP_t + d\mathcal{D}_t \right)$$

or, equivalently[1]

$$d(\beta_t \mathcal{W}_t) = \zeta_t \, d(\beta_t \widehat{P}_t). \tag{4}$$

If, moreover, the discounted wealth process $\beta\mathcal{W}$ is bounded from below, the strategy is said to be *admissible*.

Given the initial wealth w of a self-financing primary trading strategy and the strategy ζ in the primary risky assets, the related wealth process is thus given by, for $t \in [0, T]$:

$$\beta_t \mathcal{W}_t = w + \int_0^t \zeta_u \, d(\beta_u \widehat{P}_u) \tag{5}$$

[1] This equivalence is very general (cf. Sect. 4.1), and it is an easy exercise in the present context where β, given by (1), is a finite variation and continuous process.

and the process ζ^0 (number of units held in the savings account) is then uniquely determined as

$$\zeta^0_t = \beta_t(\mathcal{W}_t - \zeta_t P_t).$$

In the sequel we restrict ourselves to self-financing trading strategies. We thus *re-define* a (self-financing) primary trading strategy as a pair (w, ζ), made of an initial wealth $w \in \mathbb{R}$ and an $\mathbb{R}^{1\otimes d}$-valued predictable locally bounded primary strategy in the risky assets ζ, with related wealth process $\mathcal{W}$ defined by (5).

2.1.2 Financial Derivatives

In the sequel we are going to extend the financial market by introducing a financial *derivative* relative to the primary market. A derivative is a financial claim between an investor (or *holder* of a claim) and a financial institution (or *issuer*), involving in a sense made precise in Definition 2 below, some or all of the following cash flows (or payoffs):

- A bounded variation cumulative *dividend process* $D = (D_t)_{t\in[0,T]}$,
- Terminal cash flows, consisting of:

– A *payment* ξ *at maturity* T, where ξ denotes a bounded from below real-valued random variable,
– And, in the case of American or game products with early exercice features, *put and/or call payment processes* $L = (L_t)_{t\in[0,T]}$ and $U = (U_t)_{t\in[0,T]}$, given as real-valued, bounded from below, càdlàg processes such that $L \le U$ and $L_T \le \xi \le U_T$.

The put payment L_t corresponds to a payment made by the issuer to the holder of the claim at time t, in case the holder of the claim would decide to terminate ("put") the contract at time t. Likewise, the call payment U_t corresponds to a payment made by the issuer to the holder of the claim at time t, in case the issuer of the claim would decide to terminate ("call") the contract at time t.

Of course, there is also the initial cash flow (only null in the case of a swapped derivative with initial value equal to zero, by construction), namely the purchasing price of the contract paid at the initiation time by the holder and received by the issuer.

The terminology "derivative" comes from the fact that all the above cash flows are typically given as functions of the "primary" asset price processes P. More generally, the price Π of a derivative and the prices P of the primary assets may be given as functions of a common set of *factors* (traded or not) X (cf. Sect. 3). One may then consider the issue of *factor hedging* the claim with price process Π by the primary assets with price process P, via the common dependence of Π and P on X.

Here and henceforth all the financial cash flows are seen from the point of view of the *holder* of the claim. In this perspective, the assumption above that all the cash flows are bounded from below, which from the mathematical point of view ensures their integrability in $\mathbb{R} \cup \{+\infty\}$, is indeed satisfied by a vast majority of real-life financial derivatives.

Remark 1. Usually in the derivative pricing and hedging literature, dividends are implicitly set to zero, or equivalently, implicitly amalgamated with the terminal cash flows L, U and ξ. The related notion of price thus effectively corresponds to a *cum-dividend price* (present value of future cash flows plus already perceived dividends reinvested in the savings account), as opposed to the market notion of *ex-dividend price*. Since an important proportion of financial derivatives (starting with all swapped derivatives) only entails dividends (terminal cash flows $L = U = \xi = 0$), it is our opinion that it is better to make the dividends appear explicitly. This is in fact a necessity for the study of defaultable derivatives in Sect. 4.2, where we shall see that the specific structure of the products' cash flows and their distribution between dividends (in the sense of coupons and recovery) and terminal payoffs, is fruitfully exploited in the so-called reduced form approach to these problems.

We are now in a position to introduce the formal definition of a financial derivative, distinguishing more specifically European claims, American claims and game claims. It will soon become apparent that European claims can be considered as special cases of American claims, which are themselves included in game claims, so that we shall eventually be able to reduce attention to game claims.

In the following definitions, the put time (*put or maturity* time, to be precise) τ, and the call (or maturity) time σ, represent stopping times at the holder's and at the issuer's convenience, respectively.

Definition 2. **(i)** An *European claim* is a financial claim with dividend process D, and with payment ξ at maturity T.
(ii) An *American claim* is a financial claim with dividend process D, and with payment at the terminal (put or maturity) time τ given by,

$$\mathbb{1}_{\{\tau<T\}} L_\tau + \mathbb{1}_{\{\tau=T\}} \xi. \tag{6}$$

(iii) A *game claim* is a financial claim with dividend process D, and with payment at the terminal (call, put or maturity) time $\nu = \tau \wedge \sigma$ given by,[2]

$$\mathbb{1}_{\{\nu=\tau<T\}} L_\tau + \mathbb{1}_{\{\sigma<\tau\}} U_\sigma + \mathbb{1}_{\{\nu=T\}} \xi. \tag{7}$$

Moreover, there may be a *call protection* modeled in the form of a stopping time $\bar{\sigma}$ such that calls are not allowed to occur before $\bar{\sigma}$.

Example 1. In the simplest case of an European vanilla call/put option with maturity T and strike K on $S = P^1$, the first primary risky asset, one has $D = 0$ and $\xi = (S_T - K)^{\pm}$.

Note 1. **(i)** The above classification, which is good enough for the purpose of this article, is by no means exhaustive. For instance Bermudan products corresponding

[2] With priority of a put over a call, here, though this happens to be rather immaterial in terms of pricing and hedging the claim.

to constrained put policies might also be introduced. Note however that Bermudan products can be included in the above set-up by considering a suitably adjusted put payoff process L. This is indeed a consequence of Proposition 1(ii) below, in conjunction with our boundedness from below assumption on all the cash flows at hand.

On the opposite the explicit introduction of call protections appears to be a useful modeling ingredient. Such protections are actually quite typical in the case of real-life callable products like, for instance, convertible bonds (see Sect. 4.2.1), with the effect of making the product cheaper to the investor (holder of the claim). The introduction of such call protections also allows one to consider an American claim as a game claim with call protection $\bar{\sigma} = T$.

(ii) In Sect. 4.3, building on the mathematical results of Sect. 16, we consider products with more general, hence potentially more realistic forms of *intermittent* call protection, namely call protection *whenever a certain condition* is satisfied, rather than more specifically call protection *before a stopping time* above.

By classic arbitrage theory (see, e.g., [18, 32, 42]), the NFLVR condition in a perfect market (without transaction costs, in particular) is equivalent to the existence of a *risk-neutral measure* $\mathbb{P} \in \mathcal{M}$, where $\mathcal{M}$ denotes the set of probability measures $\mathbb{P} \sim \widehat{\mathbb{P}}$ such that $\beta\widehat{P}$ is a $\mathbb{P}$ – local martingale.

In the sequel, the statement $(\Pi_t)_{t\in[0,T]}$ *is an arbitrage price for a derivative* is to be understood as $(P_t, \Pi_t)_{t\in[0,T]}$ *is an arbitrage price for the extended market consisting of the primary market and the derivative*. The notion of arbitrage price process of a financial derivative referred to in the next result is the classical notion of No Free Lunch with Vanishing Risk condition of Delbaen and Schachermayer [42] in the case of European claims, subsequently extended to game (including American) claims by Kallsen and Kühn [67]. The proof of this result is based on a rather straightforward application of Theorem 2.9 in Kallsen and Kühn [67] (see Bielecki et al. [18] for the details).

Let $\mathcal{T}_t$ and $\bar{\mathcal{T}}_t$ (or simply $\mathcal{T}$ and $\bar{\mathcal{T}}$, in case $t = 0$) denote the set of $[t, T]$-valued and $[t \vee \bar{\sigma}, T]$-valued stopping times. Let also ν stand for $\sigma \wedge \tau$, for any $(\sigma, \tau) \in \bar{\mathcal{T}}_t \times \mathcal{T}_t$.

Proposition 1. **(i)** *For any* $\mathbb{P} \in \mathcal{M}$, *the process* $\Pi = (\Pi_t)_{t\in[0,T]}$ *defined by*

$$\beta_t \Pi_t = \mathbb{E}_{\mathbb{P}}\Big\{ \int_t^T \beta_u \, dD_u + \beta_T \xi \,\Big|\, \mathcal{F}_t \Big\}, \quad t \in [0, T] \tag{8}$$

is an arbitrage price of the related European claim. Moreover, any arbitrage price of the claim is of this form provided

$$\sup_{\mathbb{P}\in\mathcal{M}} \mathbb{E}_{\mathbb{P}}\Big\{ \int_{[0,T]} \beta_u \, dD_u + \beta_T \xi \Big\} < \infty; \tag{9}$$

(ii) *For any* $\mathbb{P} \in \mathcal{M}$, *the process* $\Pi = (\Pi_t)_{t\in[0,T]}$ *defined by*

$$\beta_t \Pi_t = \operatorname{esssup}_{\tau\in\mathcal{T}_t} \mathbb{E}_{\mathbb{P}}\Big\{ \int_t^\tau \beta_u \, dD_u + \beta_\tau \big(\mathbb{1}_{\{\tau<T\}} L_\tau + \mathbb{1}_{\{\tau=T\}} \xi \big) \,\Big|\, \mathcal{F}_t \Big\}, \quad t \in [0, T] \tag{10}$$

is an arbitrage price of the related American claim as soon as it is a semimartingale. Moreover, any arbitrage price of the claim is of this form provided

$$\sup_{\mathbb{P}\in\mathcal{M}} \mathbb{E}_{\mathbb{P}} \mathrm{sup}_{t\in[0,T]} \Big\{ \int_{[0,t]} \beta_u \, dD_u + \beta_t \big(\mathbb{1}_{\{t<T\}} L_t + \mathbb{1}_{\{t=T\}} \xi \big) \Big\} < \infty \ ; \quad (11)$$

(iii) *For any $\mathbb{P} \in \mathcal{M}$, the process $\Pi = (\Pi_t)_{t\in[0,T]}$ defined by*

$$\begin{aligned} \mathrm{esssup}_{\tau\in\mathcal{T}_t} \mathrm{essinf}_{\sigma\in\bar{\mathcal{T}}_t} \mathbb{E}_{\mathbb{P}} \Big\{ \int_t^{\nu} \beta_u \, dD_u + \beta_\nu \big(\mathbb{1}_{\{\nu=\tau<T\}} L_\tau \\ + \mathbb{1}_{\{\sigma<\tau\}} U_\sigma + \mathbb{1}_{\{\nu=T\}} \xi \big) \, \big| \, \mathcal{F}_t \Big\} = \beta_t \Pi_t \\ = \mathrm{essinf}_{\sigma\in\bar{\mathcal{T}}_t} \mathrm{esssup}_{\tau\in\mathcal{T}_t} \mathbb{E}_{\mathbb{P}} \Big\{ \int_t^{\nu} \beta_u \, dD_u + \beta_\nu \big(\mathbb{1}_{\{\nu=\tau<T\}} L_\tau \\ + \mathbb{1}_{\{\sigma<\tau\}} U_\sigma + \mathbb{1}_{\{\nu=T\}} \xi \big) \, \big| \, \mathcal{F}_t \Big\}, \quad t \in [0,T] \end{aligned} \quad (12)$$

is an arbitrage price of the related game claim as soon as it is a well-defined semimartingale (which supposes in particular that equality indeed holds between the left hand side and the right hand side in (12)). Moreover, any arbitrage price of the claim is of this form assuming (11).

In view of these results, one may interpret an European claim as an American claim with a fictitious put payment process L defined by $\beta L = -c$, where $-c$ is a strict minorant of $\int_t^T \beta_u \, dD_u + \beta_T \xi$. Indeed, in view of Propositions 1(ii), for this specification of L, exercise of the put before maturity is always sub-optimal to the holder of the claim. It is thus equivalent for a process Π to be an arbitrage price of the European claim with the cash flows D, ξ, or to be an arbitrage price of the American claim with the cash flows D, L, ξ, with L thus specified.

Henceforth by default, by "financial derivative" or "game option," we shall mean game claim, possibly with a call protection $\bar{\sigma}$, including American claim (case $\bar{\sigma} = T$, in particular European claim with L as specified above) as a special case. Arbitrage prices of the form (8), (10) or (12) will be called $\mathbb{P}$-*prices* in the sequel.

2.2 Connection with Hedging

We adopt a definition of hedging of a game option stemming from successive developments, starting from the hedging of American options examined by Karatzas [68], and subsequently followed by El Karoui and Quenez [45], Kifer [69], Ma and Cvitanić [76], Hamadène [55], and, in the context of defaultable derivatives examined in Sect. 4.2, Bielecki et al. [20, 23] (see also Schweizer [85]). This definition will be later shown to be consistent with the concept of arbitrage pricing of Proposition 1(iii) for a game option (which encompasses American and European options as special cases).

We first introduce a (very large, to be specified later) class of hedges *with semimartingale cost process* Q. The issuer of a financial derivative immediately sets up a primary hedging strategy such that the corresponding wealth process $\mathcal{W}$ reduces to a *cost* or *hedging error* Q, after accounting for the "dividend cost" $-D$ and for the "terminal loss" given by $-L$, $-U$ or $-\xi$. The initial wealth w may then be used as a safe issuer price, up to the hedging error Q, for the derivative at hand. Recall that we denote $\nu = \tau \wedge \sigma$.

Definition 3. An *hedge with semimartingale cost process* Q (*issuer hedge starting at time 0*) for a game option is represented by a triplet (w, ζ, σ) such that:
- (w, ζ) is a primary trading strategy,
- The call time σ belongs to $\bar{\mathcal{T}}$,
- The wealth process $\mathcal{W}$ of the strategy (w, ζ) satisfies for every put time τ in $\mathcal{T}$, almost surely,

$$\beta_\nu \mathcal{W}_\nu + \int_0^\nu \beta_u dQ_u \geq \int_0^\nu \beta_u dD_u + \beta_\nu \Big(\mathbb{1}_{\{\nu=\tau<T\}} L_\tau + \mathbb{1}_{\{\sigma<\tau\}} U_\sigma + \mathbb{1}_{\{\tau=\sigma=T\}} \xi \Big). \tag{13}$$

In the special case of European derivatives, in which case $\bar{\sigma} = T$, and if moreover equality holds in (13) at $t = T$, then, almost surely,

$$\beta_T \mathcal{W}_T + \int_0^T \beta_u dQ_u = \int_0^T \beta_u dD_u \; + \; \beta_T \xi. \tag{14}$$

In this case one effectively deals with a *replicating strategy with cost* Q.

Note 2. **(i)** The process Q is to be interpreted as the cumulative *financing cost*, that is, the amount of cash added to (if $dQ_t \geq 0$) or withdrawn from (if $dQ_t \leq 0$) the hedging portfolio in order to get a perfect, but no longer self-financing, hedge.
(ii) Hedges *at no cost* (that is, with $Q = 0$) are thus in effect *super-hedges*.
(iii) In relation with admissibility issues (see the end of Definition 1), note that the left hand side of (13) (discounted wealth process with financing costs included) is bounded from below, for any hedge (w, ζ, σ) with cost Q.

This class of hedges with cost Q is obviously too large for any practical purpose, so we will restrict our attention to hedges with a *local martingale* cost Q under a particular risk-neutral measure $\mathbb{P}$ (cf. the related notions of *risk-minimizing strategy* in Föllmer and Sondermann [50] and *mean self-financing hedge* in Schweizer [85]). *Henceforth in this part, we thus work under a fixed but arbitrary risk-neutral measure* $\mathbb{P}$, with $\mathbb{P}$-expectation denoted by $\mathbb{E}$. All the measure-dependent notions, like *martingale*, or *compensator*, implicitly refer to this probability measure $\mathbb{P}$. In practical applications, it is convenient to think of $\mathbb{P}$ as "the pricing measure chosen by the market" to price a contingent claim. For pricing and hedging purposes this measure is typically estimated by calibration of a model to market data.

2.2.1 BSDE Modeling

We shall now postulate suitable integrability and regularity conditions embedded in the standing assumption that a related reflected backward stochastic differential equation (BSDE, see El Karoui et al. [46] for a general reference in connection with finance and El Karoui et al. [47] for a seminal reference on reflected BSDEs) has a solution. We shall thus introduce a reflected BSDE (15) under the probability measure $\mathbb{P}$, with data defined in terms of those of a derivative. Assuming that (15) has a solution (for which various sets of sufficient regularity and integrability conditions are known in the literature, see Part II and [38, 56, 57]), we shall be in a position to deduce explicit hedging strategies with minimal initial wealth for the related derivative.

We assume further for the sake of simplicity that $dD_t = C_t dt$ for some progressively measurable time-integrable coupon rate process C.

Remark 2. It is important to note for applications that it is also possible to deal with discrete dividends: see [20] and Sect. 14 in Part IV.

We then consider the following *reflected BSDE* with data $\beta, C, \xi, L, U, \bar{\sigma}$:

$$\begin{cases} \beta_t \Pi_t = \beta_T \xi + \int_t^T \beta_u C_u du + \int_t^T \beta_u (dK_u - dM_u), \quad t \in [0, T] \\ L_t \leq \Pi_t \leq \bar{U}_t, \quad t \in [0, T] \\ \int_0^T (\Pi_u - L_u)\, dK_u^+ = \int_0^T (\bar{U}_u - \Pi_u)\, dK_u^- = 0 \end{cases} \tag{15}$$

where, with the convention that $0 \times \pm\infty = 0$ in the last line above,

$$\bar{U}_t = \mathbb{1}_{\{t < \bar{\sigma}\}} \infty + \mathbb{1}_{\{t \geq \bar{\sigma}\}} U_t. \tag{16}$$

Definition 4. *(See Part II for more formal definitions, including in particular the specification of* spaces *for the inputs and outputs to (15)).* By a $\mathbb{P}$*-solution* to (15), we mean a triplet (Π, M, K) such that all conditions in (15) are satisfied, where:
- The *state-process* Π is a real valued, càdlàg process,
- M is a $\mathbb{P}$-martingale vanishing at time 0,
- K is a non-decreasing continuous process null at time 0, and $K^{\pm}$ denote the components of the Jordan decomposition of K.

By *the Jordan decomposition* of K in the last bullet point, we mean the unique decomposition $K = K^+ - K^-$ into the difference of non-decreasing processes $K^{\pm}$ null at 0, defining mutually singular random measures on $[0, T]$.

Remark 3. The first line of (15) can be interpreted as giving the Doob–Meyer decomposition $\int_0^t \beta_u (dK_u - dM_u)$ of the special semimartingale

$$\beta_t \widehat{\Pi}_t := \beta_t \Pi_t + \int_0^t \beta_u C_u du. \tag{17}$$

So an equivalent definition of a solution to (15) would be that of a special semimartingale Π (rather than a triplet of processes (Π, M, K)) such that all conditions in (15) are satisfied, where M and K therein are to be understood as the canonical local martingale and finite variation predictable components of process $\int_{[0,\cdot]} \beta_t^{-1} d(\beta_t \widehat{\Pi}_t)$.

Note that the first line of (15) is equivalent to

$$\Pi_t = \xi + \int_t^T (C_u - r_u \Pi_u) du + (K_T - K_t) - (M_T - M_t), \quad t \in [0, T]. \quad (18)$$

As established in [38, 56, 57], existence and uniqueness of a solution to (15) (under suitable L_2-integrability conditions on the data and the solution) are essentially equivalent to the so-called *Mokobodski condition*, namely, the existence of a *quasimartingale* Y (special semimartingale with additional integrability properties, Sect. 16.2.2) such that $L \leq Y \leq U$ on $[0, T]$. Existence and uniqueness of a solution to (15) thus hold when one of the barriers is a quasimartingale and, in particular, when one of the barriers is given as $S \vee c$, where S is a square-integrable Itô process and c is a constant in $\mathbb{R} \cup \{-\infty\}$ (see [38] as well as Note 8(v) and Proposition 30 in Part II). This covers, for instance, the put payment process L of an American vanilla option, or of a convertible bond (see Definition 7 and Bielecki et al. [18, 19]). Moreover one typically has $K = 0$ in the case of an European derivative.

We thus work henceforth in this part under the following hypothesis.

Assumption 1 Equation (15) admits a solution (Π, M, K), with K equal to zero in the special case of an European derivative.

Proposition 2. *Π is the $\mathbb{P}$-price process of the derivative.*

Proof. If (Π, M, K) is a solution to (15), then Π is a (special) semimartingale (see (18)), and, by a standard verification principle (cf. Proposition 18 in Part II), Π satisfies (12), which in the special cases of American (resp. European) options reduces to (10) (resp. (8)). One thus concludes by an application of Proposition 1. □

We are now ready to interpret the $\mathbb{P}$-price Π, thus defined via (15), in terms of the notion of hedging introduced in Sect. 2.2. Let us set

$$\sigma^* = \inf \left\{ u \in [t \vee \bar{\sigma}, T] \, ; \, \Pi_u \geq U_u \right\} \wedge T. \quad (19)$$

Using the minimality condition (third line) in (15) and the continuity of $K^{\pm}$, one thus has,

$$K^- = 0 \text{ and } K = K^+ \geq 0 \text{ on } [0, \sigma^*] \,, \; \Pi_{\sigma^*} = U_{\sigma^*} \text{ on } \{\sigma^* < T\}. \quad (20)$$

Note that for any primary strategy ζ, the issuer's *Profit and Loss* (or *Tracking Error*) process $(e_t)_{t \in [0,T]}$ relative to the price process Π of Proposition 2 is given for $t \in [0, T]$ by:

$$\beta_t e_t = \Pi_0 - \int_0^t \beta_u C_u du + \int_0^t \zeta_u d(\beta_u \widehat{P}_u) - \beta_t \Pi_t = \int_0^t \Big(-d(\beta_u \widehat{\Pi}_u) + \zeta_u d(\beta_u \widehat{P}_u)\Big) \tag{21}$$

where $\widehat{\Pi}$ is defined by (17), so that, in view of Proposition 2, $\widehat{\Pi}$ can be interpreted as the $\mathbb{P}$ – *cumulative price* of the option (cf. (2)). Observe in view of (18) that the tracking error process e is a special semimartingale. Let the $\mathbb{P}$ – local martingale $\rho = \rho(\zeta)$ be such that $\rho_0 = 0$ and $\int_0^\cdot \beta_t d\rho_t$ is the local martingale component of the special semimartingale βe, so (cf. (21), (18))

$$d\rho_t = dM_t - \zeta_t \beta_t^{-1} d(\beta_t \widehat{P}_t) \tag{22}$$

$$\beta_t e_t = \int_0^t \beta_u dK_u - \int_0^t \beta_u d\rho_u. \tag{23}$$

The arguments underlying the following result are classical, and already present for instance in Lepeltier and Maingueneau [75] (in the specific contexts of the Cox–Ross–Rubinstein or Black–Scholes models, analogous results can also be found in Kifer [69]).

Proposition 3. **(i)** *For any primary strategy ζ, (Π_0, ζ, σ^*), is an hedge with $\mathbb{P}$ – local martingale cost $\rho(\zeta)$;*
(ii) *Π_0 is the minimal initial wealth of an hedge with $\mathbb{P}$ – local martingale cost;*
(iii) *In the special case of an European derivative with $K = 0$, then (Π_0, ζ) is a replicating strategy with $\mathbb{P}$ – local martingale cost ρ. Π_0 is thus also the minimal initial wealth of a replicating strategy with $\mathbb{P}$ – local martingale cost.*

Proof. **(i)** One must show that for any $\tau \in \mathcal{T}$, almost surely:

$$\begin{aligned} &\Pi_0 + \int_0^{\sigma^* \wedge \tau} \zeta_u d(\beta_u \widehat{P}_u) + \int_0^{\sigma^* \wedge \tau} \beta_u d\rho_u \\ &\geq \int_0^{\sigma^* \wedge \tau} \beta_u C_u du + \beta_{\sigma^* \wedge \tau} \Big(\mathbb{1}_{\{\sigma^* \wedge \tau = \tau < T\}} L_t + \mathbb{1}_{\{\sigma^* < \tau\}} U_{\sigma^*} + \mathbb{1}_{\{\sigma^* = \tau = T\}} \xi\Big) \end{aligned} \tag{24}$$

or equivalently, using (22):

$$\begin{aligned} &\Pi_0 + \int_0^{\sigma^* \wedge \tau} \beta_u dM_u \\ &\geq \int_0^{\sigma^* \wedge \tau} \beta_u C_u du \,+\, \beta_{\sigma^* \wedge \tau} \Big(\mathbb{1}_{\{\sigma^* \wedge \tau = \tau < T\}} L_\tau + \mathbb{1}_{\{\sigma^* < \tau\}} U_{\sigma^*} + \mathbb{1}_{\{\sigma^* = \tau = T\}} \xi\Big) \end{aligned} \tag{25}$$

where by the first line in (15):

$$\Pi_0 + \int_0^{\sigma^* \wedge \tau} \beta_u dM_u = \beta_{\sigma^* \wedge \tau} \Pi_{\sigma^* \wedge \tau} + \int_0^{\sigma^* \wedge \tau} \beta_u C_u du + \int_0^{\sigma^* \wedge \tau} \beta_u dK_u.$$

Inequality (25) then follows from (20) and from the following relations, which are valid by the terminal and put conditions in (15):

$$\Pi_T = \xi\,,\ \Pi_\tau \geq L_\tau.$$

(ii) There exists an hedge with initial wealth Π_0 and $\mathbb{P}$ – local martingale cost, by (i) applied with, for instance, $\zeta = 0$. Moreover, for any hedge (w, ζ, σ) with $\mathbb{P}$ – local martingale cost Q, one has for every $t \in [0, T]$:

$$\begin{aligned} w &+ \int_0^{\sigma\wedge t} \zeta_u d(\beta_u \widehat{P}_u) + \int_0^{\sigma\wedge t} \beta_u dQ_u \\ &\geq \int_0^{\sigma\wedge t} \beta_u C_u du \;+\; \beta_{\sigma\wedge t}\Big(\mathbb{1}_{\{\sigma\wedge t = t < T\}} L_t + \mathbb{1}_{\{\sigma < t\}} U_\sigma + \mathbb{1}_{\{\sigma = t = T\}} \xi\Big) \end{aligned} \tag{26}$$

The left hand side is thus a bounded from below local martingale, hence it is a supermartingale. Moreover, (26) also holds with a stopping time $\tau \in \mathcal{T}$ instead of t therein. So, by taking expectations in (26) with τ instead of t therein:

$$w \geq \mathbb{E}\left\{\int_0^{\sigma\wedge\tau} \beta_u C_u du + \beta_{\sigma\wedge\tau}\Big(\mathbb{1}_{\{\sigma\wedge=\tau<T\}} L_\tau + \mathbb{1}_{\{\sigma<\tau\}} U_\tau + \mathbb{1}_{\{\sigma=\tau=T\}} \xi\Big)\right\}.$$

Hence $w \geq \Pi_0$ follows, by (12).
(iii) In the special case of an European derivative, the stated results follow by setting $K = 0$ in the previous points of the proof. □

Note 3. **(i)** Proposition 3 thus *characterizes* the $\mathbb{P}$-price (*arbitrage price* relative to the risk-neutral measure $\mathbb{P}$) of a derivative as the *smallest initial wealth of a hedge* with $\mathbb{P}$ – local martingale cost, under the assumption that the related reflected BSDE (15) has a solution. For related results, see also Föllmer and Sondermann [50] or Schweizer [85].
(ii) The special case $\rho = 0$ in the previous results corresponds to a suitable form of model completeness (replicability of European options, cf. point (iii) of the proposition), in which the issuer of the option *wishes to* hedge all the risks embedded in the option.

The case $\rho \neq 0$ corresponds to either model incompleteness, or a situation of model completeness in which the issuer *wishes not to* hedge all the risks embedded in the product at hand, for instance because she wants to limit transaction costs, or because she *wishes to take some bets* in specific risk directions.
(iii) In case where ρ may be taken equal to 0 in Proposition 3, the minimality statements in this proposition can be used to prove uniqueness of the related arbitrage prices.
(iv) Analogous definitions and results hold for holder hedges.
(v) It is also easy to see that one could state analogous definitions and results regarding hedging a defaultable game option starting at any date $t \in [0, T]$, rather than at time 0 above.

3 Markovian Set-Up

3.1 Markovian FBSDE Approach

In order to be usable in practice, a dynamic pricing model needs to be constructive, or *Markovian* in some sense, relatively to a given derivative. This will be achieved by assuming that the related BSDE (15) is *Markovian* (see Sect. 4 of [46] and Part II).

Definition 5. We say that the BSDE (15) is a *Markovian backward stochastic differential equation* if the input data r, C, ξ, L and U of (15) are given by Borel-measurable functions of some $\mathbb{R}^q$-valued $(\mathbb{F}, \mathbb{P})$-Markov *factor* process X, so

$$r_t = r(t, X_t)\,,\ C_t = C(t, X_t)\,,\ \xi = \xi(X_T)\,,\ L_t = L(t, X_t)\,,\ U_t = U(t, X_t)\,, \tag{27}$$

and is $\bar{\sigma}$ is the first time of entry, capped at T, of the process (t, X), into a given closed subset of $[0, T] \times \mathbb{R}^q$.

Remark 4. By a slight abuse of notation, the related functions are thus denoted in (27) by the same symbols as the corresponding processes or random variables.

In particular, the system made of the specification of a forward dynamics for X, together with the BSDE (15), constitutes a decoupled *Markovian forward backward system of equations* in (X, Π, M, K). The system is decoupled in the sense that the forward component of the system serves as an input for the backward component (X is an input to (15), cf. (27)), but not the other way round. See Definition 11 in Part II for more complete and formal statements.

From the point of view of interpretation, the components of X are observable *factors*. These are intimately, though non-trivially, related with the primary risky asset price process P, as follows:

- Most factors are typically given as primary price processes. The components of X that are not included in P (if any) are to be understood as simple factors that may be required to "Markovianize" the payoffs of the derivative at hand, such as factors accounting for path dependence in the derivative's payoff, and/or non-traded factors such as stochastic volatility in the dynamics of the assets underlying the derivative;
- Some of the primary price processes may not be needed as factors, but are used for hedging purposes.

Note that observability of the factor process X in the mathematical sense of $\mathbb{F}$-adaptedness is not sufficient in practice. In order for a factor process model to be usable in practice, a constructive *mapping* from a collection of meaningful and directly observable economic variables to X is needed. Otherwise, the model will be useless.

3.2 Factor Process Dynamics

Under a rather generic specification for the Markov factor process X, we now derive a *variational inequality approach* for pricing and hedging a financial derivative. We thus assume that the factor process X is an $(\mathbb{F} = \mathbb{F}^{W,N}, \mathbb{P})$-solution of the following Markovian (forward) stochastic differential equation in $\mathbb{R}^q$:

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t + \delta(t, X_{t-})dN_t, \tag{28}$$

where:
- W is a q-dimensional Brownian motion, and
- N is a *compensated* integer-valued random measure with finite jump intensity measure $\lambda(t, X_t, dx)$, for some deterministic function λ.

In particular $\delta(t, X_{t-})dN_t$ in (28) is a short-hand for $\int_{\mathbb{R}^q} \delta(t, X_{t-}, x)N(dt, dx)$, where the integration is with respect to the x variable. The *response jump size function* δ and the *intensity measure* λ, like the other model coefficients b and σ of X, are to be specified depending on the application at hand: see Sect. 3.3 for specific examples and Definition 10 in Part II for more precise statements.

Remark 5. The generic and "abstract" jump-diffusion (28) will be made precise and specified in Part II in the form of a process $\mathcal{X} = (X, N)$ in which a jump-diffusion – like component X interacts with a continuous-time Markov chain – like component N; so the process $\mathcal{X}$ in Part II corresponds to X here.

Let us introduce the following additional notation:
- J_t, a random variable on $\mathbb{R}^q$ with law $\frac{\lambda(t, X_{t-}, dx)}{\lambda(t, X_{t-}, \mathbb{R}^q)}$ conditional on X_{t-}, where x represents the "mark" of the jump of X in $\delta(t, X_{t-}, x)$,
- (t_l), the ordered sequence of the times of jumps of N (note that we deal with a *finite* jump measure λ, so (t_l) is well defined),
- For any vector-valued function u on $\mathbb{R}^q$ and for every $t \in [0, T]$,

$$\delta u(t, x, y) = u(t, x + \delta(t, x, y)) - u(t, x)\,,\ \bar{\delta} u(t, x) = \int_{\mathbb{R}^q} \delta u(t, x, y)\lambda(t, x, dy)$$
$$\delta u_t = \delta u(t, X_{t-}, J_t)\,,\ \bar{\delta} u_t = \bar{\delta} u(t, X_{t-})\,. \tag{29}$$

We apologize to the reader for this admittedly heavy notation, which is motivated by the wish to give intuitive and compact forms below to various expressions of the model's dynamics, generator and Itô formula. Denoting further

$$\bar{\delta}(t, x) := \bar{\delta}\mathrm{Id}_{\mathbb{R}^q}(t, x) = \int_{\mathbb{R}^q} \delta(t, x, y)\lambda(t, x, dy)\,,\ \delta_t = \delta(t, X_{t-}, J_t)\,,$$
$$\bar{\delta}_t = \bar{\delta}(t, X_{t-}),$$

one thus has for instance:

$$\delta(t, X_{t-})dN_t = d\left(\sum_{t_l \le t} \delta_{t_l}\right) - \bar{\delta}_t dt \tag{30}$$

and the dynamics (28) of X may be rewritten as

$$dX_t = \widetilde{b}(t, X_t)dt + \sigma(t, X_t)\, dW_t + d\left(\sum_{t_l \le t} \delta_{t_l}\right) \tag{31}$$

where we set $\widetilde{b}(t, x) = b(t, x) - \bar{\delta}(t, x)$.

3.2.1 Itô Formula and Model Generator

In view of (31), the following variant of the Itô formula holds, for any real-valued function u of class $\mathcal{C}^{1,2}$ on $[0, T] \times \mathbb{R}^q$:

$$du(t, X_t) = \widetilde{\mathcal{G}}u(t, X_t)\, dt + \partial u(t, X_t)\, \sigma(t, X_t)\, dW_t + d\left(\sum_{t_l \le t} \delta u_{t_l}\right) \tag{32}$$

with

$$\widetilde{\mathcal{G}}u(t, x) = \partial_t u(t, x) + \partial u(t, x)\widetilde{b}(t, x) + \frac{1}{2}\mathrm{Tr}[a(t, x)\mathcal{H}u(t, x)] \tag{33}$$

where $a(t, x) = \sigma(t, x)\sigma(t, x)^{\mathsf{T}}$, and where ∇u and $\mathcal{H}u$ denote the *row-gradient* and the *Hessian* of u with respect to x – so in particular

$$\mathrm{Tr}[a(t, x)\mathcal{H}u(t, x)] = \sum_{1 \le i,j,k \le q} \sigma_{i,k}(t, x)\sigma_{j,k}(t, x)\partial^2_{x_i,x_j} u(t, x).$$

Using the short-hand $\delta u(t, X_{t-})dN_t = \int_{x \in \mathbb{R}^q} \delta u(t, X_{t-}, x)N(dt, dx)$, note that one has (cf. (30)),

$$\delta u(t, X_{t-})dN_t = d\left(\sum_{t_l \le t} \delta u_{t_l}\right) - \bar{\delta}u_t dt. \tag{34}$$

The Itô formula (32) may thus be rewritten as

$$du(t, X_t) = \mathcal{G}u(t, X_t)\, dt + \nabla u(t, X_t)\, \sigma(t, X_t)\, dW_t + \delta u(t, X_{t-})dN_t \tag{35}$$

where we set

$$
\begin{aligned}
\mathcal{G}u(t,x) &= \widetilde{\mathcal{G}}u(t,x) + \bar{\delta}u(t,x) \\
&= \partial_t u(t,x) + \nabla u(t,x) b(t,x) + \frac{1}{2}\mathrm{Tr}[a(t,x)\mathcal{H}u(t,x)] \\
&\quad + \bar{\delta}u(t,x) - \nabla u(t,x)\bar{\delta}(t,x). \qquad (36)
\end{aligned}
$$

The process X is thus a Markov process with generator $\mathcal{G}$ (see Proposition 29 in Part III for a more formal derivation).

Remark 6. By a convenient abuse of terminology we call here and henceforth $\mathcal{G}$ the generator of X, whereas strictly speaking $\mathcal{G}$ is the generator of the time-extended process (t, X) (the generator of X does not contain the ∂_t term).

3.2.2 Brackets

Let Π^c and Θ^c, resp. $\Delta\Pi$ and $\Delta\Theta$, denote the continuous local martingale components, resp. the jump processes, of two given real-valued semimartingales Π and Θ. Recall that the quadratic covariation or *bracket* $[\Pi, \Theta]$ is given by

$$
\begin{aligned}
d[\Pi,\Theta]_t &= d(\Pi_t\Theta_t) - \Pi_{t-}d\Theta_t - \Theta_{t-}d\Pi_t \qquad (37)\\
&= d\langle \Pi^c, \Theta^c\rangle_t + d\left(\sum_{s\le t}\Delta\Pi_s\Delta\Theta_s\right) \qquad (38)
\end{aligned}
$$

with the initial condition $[\Pi,\Theta]_0 = 0$. The *sharp bracket* $\langle \Pi, \Theta\rangle$ corresponds to the *compensator* of $[\Pi,\Theta]$, which is well defined provided $[\Pi,\Theta]$ is of locally integrable variation (see, e.g., Protter [84]). Assuming Π and Θ to be defined in terms of the process X of (28) by $\Pi_t = u(t, X_t)$ and $\Theta_t = v(t, X_t)$ for deterministic and "smooth enough" functions u and v, then (38) yields, in view of the Itô formula (35):

$$
d[\Pi,\Theta]_t = \nabla u a (\nabla v)^{\mathsf{T}}(t, X_t)\, dt + d\left(\sum_{t_l\le t}\delta u_{t_l}\delta v_{t_l}\right).
$$

The bracket $[\Pi,\Theta]$ thus admits a compensator $<\Pi,\Theta>$ given as a time-differentiable process with the following Lebesgue-density:

$$
\frac{d{<}\Pi,\Theta{>}_t}{dt} = (u,v)(t, X_t) \qquad (39)
$$

Springer
http://www.springer.com/978-3-642-14659-6

Paris-Princeton Lectures on Mathematical Finance 2010
Cousin, A.; Crépey, S.; Guéant, O.; Hobson, D.;
Jeanblanc, M.; Lasry, J.-M.; Laurent, J.-P.; Lions, P.-L.;
Tankov, P. - Carmona, R.; Çınlar, E.; Ekeland, I.; Jouini,
E.; Scheinkman, J.A.; Touzi, N. (Eds.)
2011, X, 366 p. 45 illus., Softcover
ISBN: 978-3-642-14659-6