Part I
Analysis of Dimensions of Physical Quantities

A few introductory words

An abstract number, for example, 1 or $2\frac{2}{3}$, and the arithmetic of abstract numbers, for example, that $2 + 3 = 5$ irrespective of whether one is adding apples or elephants, is a great achievement of civilization comparable with the invention of writing. We have become so used to this that we are no longer aware of the miracle lying somewhere at the very foundation of the amazing effectiveness of mathematics.

If you know what object a number refers to, then, as a rule, at the same time there immediately arise additional possibilities and constraints. We recall the nursery rhyme: "And I have obtained in my answer two navvies and two thirds". Yes, the number $2\frac{2}{3}$ is permissible in arithmetic but is not permissible in this concrete situation.

How does one make use of the fact that in concrete problems we do not deal with abstract numbers but rather with dimensional quantities?

Is there a science taking account of this? Not a great one, but yes. Every qualified natural scientist knows it (as well as the dangers of the unskillful use of it). And this is what we shall be talking about.[4]

[4] For the information of the reader: the numbering of the formulae is continuous within each chapter, but starts afresh in subsequent chapters.

Chapter 1
Elements of the theory

1.1 Dimension of a physical quantity (preliminaries)

1.1.1 Measurement, unit of measurement, measuring process

All the above concepts are fundamental and have been subjected to analysis by the best representatives of science, primarily physicists and mathematicians. This entails the analysis of the concepts of space, solid body, motion, time, causality, and so on.

At this stage we do not intend to get too deeply involved in this; however, we note that each theory provides a good model of some sphere of phenomena only in certain scales. Unfortunately, sometimes we only know what these scales are when (and only when) the theory ceases to conform to reality. At those moments we usually return to the fundamentals of the theory again, subjecting it to a thorough analysis and appropriate reconstruction.

So let us now start with the accumulation of the useful concrete material available to us.

1.1.2 Basic and derived units

In life we constantly use certain units of measurement of length, mass, time, velocity, energy, power, and so on. We single out some of them as basic units, while others turn out to be derived units.

Examples of basic units are the unit L of length, which is the *metre* (denoted by m); the unit M of mass, which is the *kilogramme* (denoted by kg) and the unit T of time, which is the *second* (denoted by s).

Examples of derived units are v velocity (m/s) $[v] = LT^{-1}$; V volume (m^3) $[V] = L^3$; a acceleration (m/s^2) $[a] = LT^{-2}$; l light year $[l] = [cT] = L$; F force, $F = Ma$, $[F] = [Ma] = MLT^{-2}$.

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In all these examples we observe the formula $L^{d_1}M^{d_2}T^{d_3}$ for the dimension of a mechanical physical quantity; $\{d_1, d_2, d_3\}$ is the dimension vector in the basis $\{L, M, T\}$.

We shall develop this vectorial algebro-geometric analogy.

1.1.3 *Dependent and independent units*

Example. The units of the quantities v, a and F are independent and can also be taken as the basic units; this is because $[L] = v^2a^{-1}$, $[M] = Fa^{-1}$, $[T] = va^{-1}$.

We may guess analogies with vector space, its basis and systems of independent vectors. (A deeper sense of this analogy will be revealed later.)

1.2 A formula for the dimension of a physical quantity

1.2.1 *Change of the numerical values of a physical quantity under a change of the sizes of the basic units.*

Example. If distance is measured in kilometres (that is, instead of 1 metre as the unit of measurement of lengths one takes 1 kilometre as the unit, which is 1000 times greater), then the same physical length will have two different numerical values with respect to these two units of length, namely, $1\text{km} = 10^3\text{m}$, $L\text{km} = 10^3 L\text{m}$, $1\text{m} = 10^{-3}\text{km}$, $L\text{m} = 10^{-3}L\text{km}$. Thus, a change of the unit of length by α times leads to a change in the numerical value of Lm of all lengths being measured by α^{-1} times, that is, the value L is replaced by $\alpha^{-1}L$.

This also relates to possible changes of the units of mass and time (tonne, gramme, milligramme; hour, day, year, millisecond, and so on).

Hence, if a physical quantity has dimensions $\{L^{d_1}M^{d_2}T^{d_3}\}$ in the basis $\{L, M, T\}$, that is, $\{d_1, d_2, d_3\}$ is its dimension vector, then the change in the units of measurement of length, mass and time by α_1, α_2, α_3 times, respectively, must entail a change in the numerical value of this quantity by $\alpha_1^{-d_1}\alpha_2^{-d_2}\alpha_3^{-d_3}$ times.

1.2.2 *Postulate of the invariance of the ratio of the values of physical quantities with the same name.*

Example. The area of a triangle is a function $y = f(x_1, x_2, x_3)$ of the lengths of its three sides. We take another triangle and calculate its area $\tilde{y} = f(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$.

Under a change of the size of the unit of length the numerical values of y and $\tilde{y}$ change. However, their ratio $y/\tilde{y}$ remains unaltered.

Suppose now that we have two physical quantities $y = f(x_1, x_2, \ldots, x_m)$, $\tilde{y} = f(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m)$, depending not merely on three quantities, but now on a finite collection of quantities involving just length, mass or time. Then we get the following fundamental postulate of dimension theory.

POSTULATE OF THE ABSOLUTENESS OF RATIOS

$$\frac{\tilde{y}}{y} = \frac{f(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m)}{f(x_1, x_2, \ldots, x_m)} = \frac{f(\alpha\tilde{x}_1, \alpha\tilde{x}_2, \ldots, \alpha\tilde{x}_m)}{f(\alpha x_1, \alpha x_2, \ldots, \alpha x_m)}. \tag{1.1}$$

In other words, it is postulated that under a change of scale of a basic unit (length, mass, time, ...) physical quantities y, $\tilde{y}$ of the same type (all areas, all volumes, all velocities, all forces, ...) change their numerical values in the same proportion (its own for each type of quantity).

1.2.3 *Function of dimension and a formula for the dimension of a physical quantity in a given basis.*

Equation 1.1 shows that the ratio

$$\frac{f(\alpha x_1, \alpha x_2, \ldots, \alpha x_m)}{f(x_1, x_2, \ldots, x_m)} =: \varphi(\alpha)$$

depends only on α. This enables one to indicate the form of the function $\varphi(\alpha)$. First we note that

$$\frac{\varphi(\alpha_1)}{\varphi(\alpha_2)} = \varphi\left(\frac{\alpha_1}{\alpha_2}\right). \tag{1.2}$$

Indeed,

$$\frac{\varphi(\alpha_1)}{\varphi(\alpha_2)} = \frac{f(\alpha_1 x_1, \alpha_1 x_2, \ldots, \alpha_1 x_m)}{f(\alpha_2 x_1, \alpha_2 x_2, \ldots, \alpha_2 x_m)} = \frac{f(\frac{\alpha_1}{\alpha_2} x_1, \frac{\alpha_1}{\alpha_2} x_2, \ldots, \frac{\alpha_1}{\alpha_2} x_m)}{f(x_1, x_2, \ldots, x_m)} = \varphi\left(\frac{\alpha_1}{\alpha_2}\right).$$

Here the left and right equalities follow immediately from the definition of the function φ; and, taking into account the independence of φ of the variables $(x_1, x_2, \ldots, x_m)$, one can go over to the variables $\alpha_2^{-1}(x_1, x_2, \ldots, x_m)$ and obtain the inner equality.

Assuming that φ is a regular function we differentiate (1.2) with respect to α_1 and then setting $\alpha_1 = \alpha_2$ we obtain the equation

$$\frac{1}{\varphi(\alpha)}\frac{d\varphi}{d\alpha} = \frac{1}{\alpha}\varphi'(1).$$

Since $\varphi(1) = 1$, the solution of this equation with this initial condition has the form

$$\varphi(\alpha) = \alpha^d.$$

Thus, the above form of the function φ is a consequence of our postulate of the theory of dimensions of physical quantities. In our formulation of the postulate we have supposed (to keep our initial considerations simple) that all the variables $(x_1, x_2, ..., x_m)$ have the same character (length or mass or time or velocity, ...). But, clearly, any functional dependence of a physical quantity on a collection of physical variables can be represented in the form

$$y = f(x_1, x_2, ..., x_{m_1}, y_1, y_2, ..., y_{m_2}, ..., z_1, z_2, ..., z_{m_k}),$$

where physical variables of the same type are collected in a group and denoted by a common symbol (we did not want to introduce a numbering of the groups). This means that

$$\frac{f(\alpha_1 x_1, \alpha_1 x_2, ..., \alpha_1 x_{m_1}, ..., \alpha_k z_1, \alpha_k z_2, ..., \alpha_k z_{m_k})}{f(x_1, x_2, ..., x_{m_1}, ..., z_1, z_2, ..., z_{m_k})} = \alpha_1{}^{d_1} \cdot ... \cdot \alpha_k{}^{d_k},$$

if the variables in different groups are allowed to have independent changes of scale. In this case, within the scope of the postulate we have found the law

$$\begin{aligned} &f(\alpha_1 x_1, \alpha_1 x_2, ..., \alpha_1 x_{m_1}, ..., \alpha_k z_1, \alpha_k z_2, ..., \alpha_k z_{m_k}) = \\ &= \alpha_1{}^{d_1} \cdot ... \cdot \alpha_k{}^{d_k}\, f(x_1, x_2, ..., x_{m_1}, ..., z_1, z_2, ..., z_{m_k}) \end{aligned} \tag{1.3}$$

for the change of the numerical value of the physical quantity

$$y = f(x_1, x_2, ..., x_{m_1}, ..., z_1, z_2, ..., z_{m_k})$$

under a change of the scales of the units of physically independent variables.

The tuple $(d_1, \ldots, d_k)$ is called the *dimension vector* or simply the *dimension* of the quantity y with respect to the distinguished independent physical units of measurement. The function $\varphi(\alpha_1, ..., \alpha_k) = \alpha_1{}^{d_1} \cdot ... \cdot \alpha_k{}^{d_k}$ is called the *dimension function*.

The symbol $[y]$ denotes either the dimension vector or the dimension function depending on the context. A physical quantity is called *dimensionless* if its dimension vector is zero. For example, if $\varphi(\alpha_1, ..., \alpha_k) = \alpha_1{}^{d_1} \cdot ... \cdot \alpha_k{}^{d_k}$ is the dimension function of a physical quantity y with respect to independent physical quantities $\{x_1, \ldots, x_k\}$, then the ratio

$$\Pi := \frac{y}{x_1{}^{d_1} \cdot \ldots \cdot x_k{}^{d_k}}$$

is a dimensionless quantity with respect to $\{x_1, \ldots, x_k\}$.

1.3 Fundamental theorem of dimension theory

1.3.1 The Π*-Theorem.*

We now consider the general case of the dependence

$$y = f(x_1, x_2, \ldots, x_k, \ldots, x_n) \tag{1.4}$$

of the physical quantity y on the variable quantities $x_1, x_2, \ldots, x_n$, among which only the first k are physically (dimensionally) independent in the sense of our theory of the dimensions of physical quantities.

By taking $x_1, x_2, \ldots, x_k$ as the units of measurement of the corresponding quantities, that is, changing the scale by setting $\alpha_1 = x_1{}^{-1}, \ldots, \alpha_k = x_k{}^{-1}$, equation (1.3) yields the relation

$$\Pi = f(1, \ldots 1, \Pi_1, \Pi_2, \ldots, \Pi_{n-k}) \tag{1.5}$$

between the dimensionless quantities

$$\Pi = \frac{y}{x_1{}^{d_1} \cdot \ldots \cdot x_k{}^{d_k}}, \quad \Pi_i = \frac{x_{k+i}}{x_1{}^{d_{i_1}} \cdot \ldots \cdot x_k{}^{d_{i_k}}},$$

where $i = 1, \ldots, n-k$. Equation (1.5) can be rewritten in the form

$$y = x_1{}^{d_1} \cdot \ldots \cdot x_k{}^{d_k} \cdot f(1, \ldots, 1, \Pi_1, \Pi_2, \ldots, \Pi_{n-k}). \tag{1.6}$$

Thus, by using the scale homogeneity of the dependences between physical quantities expressed in the postulate formulated above we can go over from relation (1.4) to the relation (1.5) between dimensionless quantities and, in so doing, reduce the number of variables; or we can go over to the relation (1.6), which is equivalent to relation (1.5), by explicitly selecting the entire dimensional constituent of y with respect to the maximal system $x_1, x_2, \ldots, x_k$ of dimensionally independent variables.

The possibility of such a transition from the general relation (1.4) to the simpler relations (1.5) and (1.6) forms the content of the so-called Π-*Theorem*,[1] which is the fundamental theorem of the theory of dimensions of physical quantities (which we have just proved).

1.3.2 Principle of similarity.

The content, sense and capability of the Π-Theorem, as well as the pitfalls associated with it, will be developed below by way of concrete examples of its use. However, one idea of the effective (and striking) application of the above theorem lies right on the surface and is obvious: without damaging aeroplanes, ships and other such objects, many experiments can be carried out in the laboratory on models, after which the results (for example, the experimentally found dimensionless dependence (1.5)) can be recomputed with the aid of the Π-Theorem for the actual objects of natural size (via formula (1.6)).

[1] It is also called Buckingham's Theorem, in connection with the appearance of the papers by E. Buckingham "On physically similar systems; illustrations of the use of dimensional equations", Phys. Rev. **4** (1914), 345–376, and E. Buckingham "The principle of similitude", Nature **96** (1915), 396–397. In implicit form the Π-Theorem and the Similarity Principle are also contained in the paper J. H. Jeans, Proc. Roy. Soc. **76** (1905), 545. In a sense the laws of similarity were essentially known to Newton and Galileo. A history of this question can be found in the paper [5] along with a reference to the paper [6], where there is everything except the name Π-*Theorem*. In this connection we also mention a slightly curious publication [7].

Chapter 2
Examples of applications

We now consider some examples in which various aspects of the Π-Theorem are explained.

2.1 Period of rotation of a body in a circular orbit (laws of similarity)

A body of mass m is kept in a circular orbit of radius r by a central force F. It is required to find the period of rotation

$$P = f(r,m,F).$$

Here and in what follows we fix the basis of fundamental physical units that is standard in mechanics, namely, length, mass and time, which, following Maxwell, we denote by $\{L,M,T\}$. (In thermodynamics the symbol T is used to denote absolute temperature, but unless otherwise stated, we shall meanwhile use this notation for the unit of time.)

Let us find the dimension vector of the quantities P, r, m, F in the basis $\{L,M,T\}$. We write them as the columns of the following table:

	P	r	m	F
L	0	1	0	1
M	0	0	1	1
T	1	0	0	-2

Since, as we have shown, the dimension function always has degree form, multiplication of such functions corresponds to addition of the degree exponents, in other words, they correspond to linear operations on the dimension vectors of the corresponding physical quantities.

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Hence, using standard linear algebra, one can find a system of independent quantities from the matrix formed by their dimension vectors; also, by decomposing the dimension vector of some quantity into the dimension vectors of the selected independent quantities, one can find a formula for the dimension of this quantity in the system of independent quantities of the concrete problem.

Thus, in our case the quantities r, m, F, are independent because the matrix formed by the vectors $[r]$, $[m]$, $[F]$ is non-singular. Finding the expansion $[P] = \frac{1}{2}[r] + \frac{1}{2}[m,] - \frac{1}{2}[F]$ on the basis of formula (1.6) of Chapter I we immediately see that

$$P = \left(\frac{mr}{F}\right)^{1/2} \cdot f(1,1,1).$$

Thus, to within the positive factor $c = f(1,1,1)$ (which can be found by a single laboratory experiment) we have found the dependence of P on r, m, F. Of course, knowing Newton's law $F = m \cdot a$, in the present instance we could easily have found the final formula, where $c = 2\pi$. However, everything that we have used is just a general indication of the existence of the dependence $P = f(r, m, F)$.

2.2 The gravitional constant. Kepler's third law and the degree exponent in Newton's law of universal gravitation.

After Newton we find the degree exponent α in the law of universal gravitation

$$F = G\frac{m_1 m_2}{r^\alpha}.$$

We use the previous problem and Kepler's third law (which was known to Newton) which, for circular orbits, implies that the square of the periods of rotation of the planets (with respect to a central body of mass M) are proportional to the cubes of the radii of their orbits. In view of the result of the previous problem and the law of universal gravitation (with the exponent α not yet found) we have

$$\left(\frac{P_1}{P_2}\right)^2 = \left(\frac{m_1 r_1}{F_1}\right)^{1/2} / \left(\frac{m_2 r_2}{F_2}\right)^{1/2} = \left(\frac{m_1 r_1}{\frac{m_1 M}{r^\alpha}}\right)^{1/2} / \left(\frac{m_2 r_2}{\frac{m_2 M}{r^\alpha}}\right)^{1/2} = \left(\frac{r_1}{r_2}\right)^{\alpha+1}.$$

But by Kepler's law $\left(\frac{P_1}{P_2}\right)^2 = \left(\frac{r_1}{r_2}\right)^3$. Hence $\alpha = 2$.

2.3 Period of oscillation of a heavy pendulum (inclusion of g).

After the detailed explanations involved in the solution of the first problem we can now allow ourselves a more compact account, pausing only at certain new circumstances.

We shall find the period of oscillation of a pendulum. More precisely, a load of mass m is fixed at the end of a weightless suspension of length l inclined from the equilibrium position at some initial angle φ_0, is let go and under the action of the force of gravity starts to perform a periodic oscillation. We shall find the period P of these oscillations.

To write $P = f(l, m, \varphi_0)$ would be wrong because a pendulum has different periods of oscillation on the Earth and the Moon in view of the difference in the forces of gravity at the surfaces of these two bodies. The force of gravity at the surface of a body, for example, the Earth, is characterized by the quantity g which is the acceleration of free fall at the surface of this body. Therefore, instead of the impossible relation $P = f(l, m, \varphi_0)$, one must assume that $P = f(l, m, g, \varphi_0)$.

We write the dimension vectors of all these quantities in the basis $\{L, M, T\}$:

$$\begin{array}{cccccc} & P & l & m & g & \varphi_0 \\ L & 0 & 1 & 0 & 1 & 0 \\ M & 0 & 0 & 1 & 0 & 0 \\ T & 1 & 0 & 0 & -2 & 0 \end{array}$$

Clearly the vectors $[l]$, $[m]$, $[g]$ are independent and $[P] = \frac{1}{2}[l] - \frac{1}{2}[g]$.

In view of the Π-Theorem in the form of relation 1.6 of Chapter I, it follows that

$$P = \left(\frac{l}{g}\right)^{\frac{1}{2}} \cdot f(1, 1, 1, \varphi_0).$$

We have found that $P = c(\varphi_0) \cdot \sqrt{\frac{l}{g}}$, where the dimensionless factor $c(\varphi_0)$ depends only on the dimensionless angle φ_0 of the initial inclination (measured in radians).

The precise value of $c(\varphi_0)$ can also be found, although this time this is no longer all that easy. It can be done by solving the equation of motion of an oscillating heavy pendulum and invoking the elliptic integral

$$F(k, \varphi) := \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

Namely, $c(\varphi_0) = 4K(\sin(\frac{1}{2}\varphi_0))$, where $K(k) := F(k, \frac{1}{2}\pi)$.

2.4 Outflow of volume and mass in a waterfall

On a broad shelf having the form of a step on the upper platform, water falling under the action of gravity forms a waterfall. The depth of the water on the upper platform is known and is equal to h. It is required to find the specific volumetric outflow V (per unit of time on a unit of width of the step) of the water. If we look at the mechanism of the phenomenon in the right way, then we see that $V = f(g,h)$.

Since this phenomenon is determined by gravitation, along with the dimensional constant g (free-fall acceleration), we could, as a precaution, introduce the density ϱ of the fluid, that is, we suppose that

$$V = f(\varrho,g,h).$$

We now carry out the standard procedure of finding the dimension vectors:

$$\begin{array}{ccccc} & V & \varrho & g & h \\ L & 2 & -3 & 1 & 1 \\ M & 0 & 1 & 0 & 0 \\ T & -1 & 0 & -2 & 0 \end{array}$$

Clearly the vectors $[\varrho],[g],[h]$ are independent and $[V] = \frac{1}{2}[g] + \frac{3}{2}[h]$.

In view of the Π-Theorem we now obtain that

$$V = g^{\frac{1}{2}} h^{\frac{3}{2}} \cdot f(1,1,1).$$

Thus, $V = c \cdot g^{\frac{1}{2}} h^{\frac{3}{2}}$, where c is a constant to be determined, for example, in a laboratory experiment. Here the specific outflow Q of the mass is clearly equal to ϱV. One could also have arrived at the same formula by applying the method of dimensions to the relation $Q = f(\varrho,g,h)$.

2.5 Drag force for the motion of a ball in a non-viscous medium

A ball of radius r moves with velocity v in a non-viscous medium of density ϱ. It is required to find the drag force acting on the ball. (One could, of course, assume that there is a flow moving with velocity v past a ball at rest, which is a typical situation in wind-tunnel tests.)

We write down the general formula $F = f(\varrho,v,r)$ and analyse it in terms of dimensions:

$$\begin{array}{ccccc} & F & \varrho & v & r \\ L & 1 & -3 & 1 & 1 \\ M & 1 & 1 & 0 & 0 \\ T & -2 & 0 & -1 & 0 \end{array}$$

Clearly the vectors $[\varrho],[v],[r]$ are independent and $[F] = [\varrho] + 2[v] + 2[r]$. In view of the Π-Theorem we now obtain

$$F = \varrho v^2 r^2 \cdot f(1,1,1). \tag{2.1}$$

Thus, $F = c \cdot \varrho v^2 r^2$, where c is a dimensionless constant coefficient.

2.6 Drag force for the motion of a ball in a viscous medium

Before we turn to the formulation of this problem we recall the notion of viscosity of a medium and find the dimension of viscosity.

If one places a sheet of paper on the surface of thick honey, then in order to move the sheet along the surface one needs to apply certain forces. In first approximation the force F applied to the sheet stuck on the surface of the honey will be proportional to the area S of the sheet, the speed v of its motion and inversely proportional to the distance h from the surface to the bottom where the honey is also stuck and stays motionless in spite of the motion at the top (like a river).

Thus, $F = \eta \cdot Sv/h$. The coefficient η in this formula depends on the medium (honey, water, air, and so on) and is called the *coefficient of viscosity* of the medium or simply the *viscosity*.

The ratio $\nu = \eta/\varrho$, where, as always, ϱ is the density of the medium, is frequently encountered in problems of hydrodynamics and is called the *kinematic viscosity* of the medium.

We now find the dimensions of these quantities in the standard $\{L,M,T\}$ basis. Since $[\eta] = [FhS^{-1}v^{-1}]$, the dimension function corresponding to the viscosity in this basis has the form $\varphi_\eta = L^{-1}M^1T^{-1}$, and the dimension vector is $[\eta] = (-1,1,-1)$. For the kinematic viscosity $[\nu] = [\eta/\varrho]$, therefore $\varphi_\nu = L^2M^0T^{-1}$ and $[\nu] = (2,0,-1)$.

We now try to solve the previous problem on the drag force arising in the motion of the same ball, but now in a viscous medium. The initial dependence now looks like this: $F = f(\eta,\varrho,v,r)$. We analyse it in terms of dimensions:

$$\begin{array}{cccccc} & F & \eta & \varrho & v & r \\ L & 1 & -1 & -3 & 1 & 1 \\ M & 1 & 1 & 1 & 0 & 0 \\ T & -2 & -1 & 0 & -1 & 0 \end{array}$$

Clearly the vectors $[\varrho],[v],[r]$ are independent; $[F] = [\varrho] + 2[v] + 2[r]$ and $[\eta] = [\varrho] + [v] + [r]$. In view of the Π-Theorem we have

$$F = \varrho v^2 r^2 \cdot f(\mathrm{Re}^{-1}, 1, 1, 1), \tag{2.2}$$

where the function $f(\mathrm{Re}^{-1}, 1, 1, 1)$ remains unknown. This last function depends on the dimensionless parameter

$$\mathrm{Re} = \varrho vr/\eta = vr/\nu, \tag{2.3}$$

which plays a key role in questions of hydrodynamics.

This dimensionless quantity Re (the indication of the ratio of the force of inertia and the viscosity) is called the *Reynolds number* after the English physicist and engineer Osborn Reynolds, who first drew attention to it in his papers on turbulence in 1883. It turns out that as the Reynolds number increases, for example, as the speed of the flow increases or as the viscosity of the medium decreases, the character of the flow undergoes structural transformations (called bifurcations) evolving from a calm stable laminar flow to turbulence and chaos.

It is very instructive to pause at this juncture and wonder why the results (2.1) and (2.2) of the last two problems appear to be essentially the same. The wonder disappears if one considers more closely the variable quantity $f(\mathrm{Re}^{-1}, 1, 1, 1)$. Under the assumptions that, in modern terminology, are equivalent to the relative smallness of the Reynolds number, Stokes as long ago as 1851 found that $F = 6\pi\eta vr$. This does not contradict formula (2.2) but merely states that for small Reynolds numbers the function $f(\mathrm{Re}^{-1}, 1, 1, 1)$ behaves asymptotically like $6\pi\mathrm{Re}^{-1}$. In fact, substituting this value in formula (2.2) and recalling the definition (2.3) we obtain Stokes's formula.

2.7 Exercises

1. Since orchestras exist, it is natural to suggest that the speed of sound is weakly dependent (or not dependent) on the wave length ?

(Recall the nature of a sound wave, introduce the modulus of elasticity E of the medium and, starting from the dependence $v = f(\varrho, E, \lambda)$, prove that $v = c \cdot (E/\varrho)^{1/2}$.)

2. What is the law of the change of speed of propagation of a shock wave resulting from a very strong explosion in the atmosphere ?

(Introduce the energy E_0 of the explosion. The pressure in front of the shock wave can be ignored; the elasticity of the air no longer plays a role. Start by finding the law $r = f(\varrho, E_0, t)$ of propagation of the shock wave.)

3. Obtain the formula $v = c \cdot (\lambda g)^{1/2}$ for the speed of propagation of a wave in a deep reservoir under the action of the force of gravity. (Here c is a numerical coefficient, g is the acceleration of free fall and λ is the wave length.)

4. The speed of propagation of a wave in shallow water does not depend on the wave length. Accepting this observation as a fact show that it is proportional to the square root of the depth of the reservoir.

5. The formula used for determining the quantity of liquid flowing along a cylindrical tube (for example, along an artery) has the form

$$v = \frac{\pi \varrho P r^4}{8\eta l},$$

where v is the speed of the flow, ϱ is the density of the liquid, P is the difference in pressure at the ends of the tube, r is the radius of cross-section of the tube, η is the viscosity of the liquid and l is the length of the tube. Derive this formula (to within a numerical factor) by verifying the agreement of the dimensions on both sides of the formula.

6. a) In a desert inhabited by animals it is required to overcome the large distances between the sources of water. How does the maximal time that the animal can run depend on the size L of the animal? (Assume that evaporation only occurs from the surface, the size of which is proportional to L^2.)

b) How does the speed of running (on the level and uphill) depend on the size of the animal? (Assume that the power developed and the corresponding intensity of heat loss (say, through evaporation) are proportional to each other, and the resistive force against horizontal motion (for example, air resistance) is proportional to the square of the speed and the area of the frontal surface.)

c) How does the distance that an animal can run depend on its size? (Compare with the answers to the previous two questions.)

d) How does the height of the jump of an animal depend on its size? (The critical load that can be borne by a column that is not too high is proportional to the area of cross-section of the column. Assume that the answer to the question depends only on the strength of the bones and the "capability" of the muscles (corresponding to the strengths of the bones).

Here we are dealing throughout with animals of size on the human scale, such as camels, horses, dogs, hares, kangaroos, jerboas, in their customary habitats. In this connection see the books by Arnold and Schmidt cited below.

7. After Lord Rayleigh, find the period of small oscillations of drops of liquid under the action of their surface tension, assuming that everything happens outside a gravitational field (in the cosmos).

(Answer: $c \cdot (\varrho r^3/s)^{1/2}$, where ϱ is the density of the liquid, r is the radius of the drop and s is the surface tension, $[s] = (0, 1, -2)$.)

8. Find the period of rotation of a double star. We have in mind that two bodies with masses m_1 and m_2 rotate in circular orbits about their common centre of mass. The system occurs in empty space and is maintained by the forces of mutual attraction between these bodies. (If you are puzzled, recall the gravitational constant and its dimension.)

9. “Discover” Wien’s displacement law $\varepsilon(\nu,T) = \nu^3 F(\nu/T)$ and also the Rayleigh-Jeans law $\varepsilon(\nu,T) = \nu^2 T\ G(\nu/T)$ for the distribution of the intensity of black-body radiation as a function of the frequency and the absolute temperature.

[Wien’s fundamental law (not the displacement law given above) has the form $\varepsilon(\nu,T) = \nu^3 \exp(-a\nu/T)$ and is valid for $\nu/T \gg 1$, while the Rayleigh-Jeans law $\varepsilon(\nu,T) = 8\pi\nu^2 kT/c^3$ is valid for relatively small values of ν/T.

Both these laws (the specific intensity of radiation in the frequency interval from ν to $\nu + d\nu$) are united by Planck’s formula (1900) launching the ground-breaking epoch of quantum theory:

$$\varepsilon(\nu,T) = \frac{8\pi}{c^3}\nu^2 \frac{h\nu}{e^{h\nu/kT} - 1}.$$

Here c is the velocity of light, h is Planck’s constant, k is the Boltzmann constant ($k = R/N$, where R is the universal gas constant and N is Avogadro’s number). Wien’s law and the Rayleigh-Jeans law are obtained from Planck’s formula for $h\nu \gg kT$ and $kT \gg h\nu$, respectively.]

Let ν_T be the frequency at which the function $\varepsilon(\nu,T) = \nu^3 F(\nu/T)$ attains its maximum for a fixed value of the temperature T. Verify (after Wien) that we have the remarkable displacement law $\nu_T/T = \text{const}$. Find this constant using Planck’s formula.

10. Taking the gravitational constant G, the speed of light c and Planck’s constant h as the basic units, find the universal Planck units of length $L^* = (hG/c^3)^{1/2}$, time $T^* = (hG/c^5)^{1/2}$ and mass $M^* = (hc/G)^{1/2}$.

(The values $G = 6.67 \cdot 10^{-11}\mathrm{H} \cdot \mathrm{m}^2/\mathrm{kg}^2$, $c = 2.997925 \cdot 10^8 \mathrm{m/s}$ and $h = 6.625 \cdot 10^{-34}\mathrm{J{\cdot}s}$, other physical constants, as well as other information on units of measurement can be found in the books [4a], [4b], [4c].)

Many problems, analysed examples, instructive discussions and warnings relating to the analysis of dimensionality and principles of similarity can be found in the books [1], [2], [3], [4].

2.8 Concluding remarks

The little that has been said about the analysis of dimension and its applications already enables us to make the following observations.

The effectiveness of the use of the method mainly depends on a proper understanding of the nature of the phenomenon to which it is being applied. (By the way, in an early stage of analysis only people at the level of Newton, the brothers Bernoulli and Euler knew how to apply the analysis of infinitesimals without getting embroiled in paradoxes, which was required for extra intuition).[1]

Dimension analysis is particularly useful when the laws of the phenomenon have not yet been described. Namely, in this situation it sometimes reveals connections (albeit very general), which are useful for an understanding of the mechanism of the phenomenon and the choice of the direction of further investigations and refinements. We shall then demonstrate this by the example of Kolmogorov's approach to the description of the (still mysterious) fundamental phenomenon of turbulence.

The main postulate of dimension theory relates to the linear theory of similarity transformations, the theory of measurements, the notion of a rigid body and the homogeneity of a space, among other things. In Lobachevskii's hyperbolic geometry there are no similar figures at all, as is well known. Even so, locally this geometry admits a Euclidean approximation. Hence, as in all laws, the postulate of dimension theory is itself applicable in certain scales, depending on the problem. These scales were rarely known in advance and were most often discovered when incongruities arose.

The method shows that the larger the number of dimensionally independent quantities are, the simpler and more concrete the functional dependence of the quantities under study becomes. On the other hand, the more physical relations are discovered the less remains of the dimensionally independent quantities. (For example, distance can now be measured in light years.) So we see that the more we know, the less general dimension analysis gives us. Counterbalancing this, the penetration into essentially new areas is usually accompanied by the appearance of new dimensionally independent quantities (the algebraic aspect of dimensions and many other matters can be found, for example, in the book [14].)

Disregarding Problems 9 and 10 we restrict ourselves here to the discussion of phenomena described within the framework of the quantities of classical mechanics. This will suffice to begin with. But true enjoyment can only be obtained by reading the discussions of scholars, thinkers and, in general, professionals capable of a large-scale multischeme and unique view of the

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[1] I quote the justified misgivings of V.I. Arnol'd concerning the possible overestimation of the Π-theorem: "Such an approach is extremely dangerous because it opens up the possibility of irresponsible speculation (under the name of dimension theory) in those places where the corresponding laws of similarity should be verified experimentally, since they do not at all follow from the dimensions of the quantities describing the phenomenon under study, and they are deep subtle facts". Rather the same relates to a clumsy use of multiplication tables, statistics or catastrophe theory.

Using these new publications of the present book I add that in his recent book "Mathematical understanding of nature" (MCCME, Moscow, 2009) discussing such a theory of adiabatic invariants V.I. (on p.117) observes that " The theory of adiabatic invariants is a strange example of a physical theory seemingly contradicting the purport of easily verifiable mathematical facts. In spite of such an undesirable property of this "theory" it provided remarkable physical discoveries to those who were not afraid to use its conclusions (even though they were mathematically unjustified)". In a word: "Think it out for yourself, solve it yourself, take it or leave it".

world or the subject matter. And this is in connection with various areas. It is like a symphony and it captivates!

[If you resign yourself to the "obscurantism" of dimension analysis but what has been set out still does not seem rather crazy, then it amuses me to quote the following excerpt from a well-known physicist (whom I shall not name so as not to accidentally subject his good name to attacks by less free-thinking people).

"Physicists begin the study of a phenomenon by introducing suitable units of measurement. It is unreasonable to measure the radius of an atom in metres or the speed of an electron in kilometres per hour; one needs to find appropriate units. There are already important immediate consequences of one such choice of units. Thus, from the charge e of an electron and its mass m one cannot form a quantity having the dimensions of length. This means that in classical mechanics the atom is impossible — an electron cannot move in a stationary orbit. The situation changed with the appearance of Planck's constant $\hbar$ ($\hbar = h/2\pi$). As is clear from the definition, $\hbar = 1{,}054 \cdot 10^{-34}$ J·s has dimensions of energy times time.[2] We can now form the quantity of the dimension of length: $a_0 = \hbar/me^2$.

If in this relation we substitute the values of the constants occurring therein, then we should get a quantity of the order of the dimensions of the atom; one obtains $0{,}5 \cdot 10^{-10}$m. Thus from a simple dimensional estimate one has found the size of the atom.

It is easy to see that $e^2/\hbar$ has the dimension of velocity, it is roughly 100 times smaller than the speed of light. If one divides this quantity by the speed of light c, then one obtains the dimensionless quantity $\alpha = e^2/\hbar c = 1/137$, characterizing the interaction of the electron with an electric field. This quantity is called the *fine structure constant*.

We have given estimates for the hydrogen atom. It is easy to obtain them for an atom with nuclear charge Ze. The motion of an electron in an atom is determined by its interaction with the nucleus, which is proportional to the product of the charge on the nucleus and the charge on the electron. Therefore for a nucleus with charge Ze, in the formulae for α and a_0 we must replace e^2 by Ze^2. In heavy elements with $Z \sim 100$ the velocity of the electrons is close to the speed of light."]

Finally we make some practical observations.

Dimension analysis is a good means of double checking:

a) if the dimensions of the left- and right-hand sides of an equation are not equal, then one must look for the error;

b) if under a sign that is not a degree function (for example, under a logarithm or exponential sign) there is a quantity that is not dimensionless, then one must look for the error (or one must look for a transformation getting rid of this situation);

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[2] *Author's comment*: The dimension of h can be worked out from Planck's formula in Problem 9.

c) only quantities of the same dimension can be added.

(If $v = at$ is the velocity and $s = \frac{1}{2}at^2$ is the distance passed under uniform acceleration, then formally it is, of course, true that $v + s = at + \frac{1}{2}at^2$. However, from a physical point of view this equality reduces to two: $v = at$ and $s = \frac{1}{2}at^2$. Bridgeman, in whose cited book we gave this example, indicates a complete analogy with the equality of vectors , which gives rise to equalities of coordinates with the same name.)

Chapter 3
Further applications: hydrodynamics and turbulence

3.1 Equations of hydrodynamics (general information)

The basic classical characteristics of a moving continuous medium (liquid, gas) are, as is well known, velocity $v = v(x,t)$, pressure $p = p(x,t)$ and density $\rho = \rho(x,t)$ of the medium as a function of the point x of the region of flow and time t.

The analogue of Newton's equation $ma = F$ for the motion of an ideal continuous medium is Euler's equation

$$\rho \frac{dv}{dt} = -\nabla p. \tag{3.1}$$

If the medium is viscous, then on the right-hand side one must add to the force relating to the fall in pressure the force of internal friction; this gives us the equation

$$\rho \frac{dv}{dt} = -\nabla p + \eta \triangle v, \tag{3.2}$$

where η is the viscosity of the medium.

The above equation was introduced in 1827 by Navier in a certain special case and was then further generalized in turn by Poisson (1831), St. Venant (1843) and Stokes (1845). Since v is a vector field, this vector equation is equivalent to a system of equations for the coordinates of the field v. This system is called the Navier–Stokes system of equations, often called the NS-system.

For $\eta = 0$ we revert to Euler's equation relating to an ideal (non-viscous) fluid, for which one can ignore the loss of energy due to internal friction.

In addition to Euler's equation there is the so-called continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0, \tag{3.3}$$

expressing in differential form the law of conservation of mass (the change in the quantity of matter in any region of flow is the same as its flow through the boundary of this region).

For a homogeneous incompressible liquid $\rho \equiv \text{const}$, $\operatorname{div}(v) = 0$, the continuity equation holds automatically and the Navier–Stokes equation (3.2) takes the form

$$\frac{dv}{dt} = -\nabla \left(\frac{p}{\rho}\right) + \nu \triangle v, \tag{3.4}$$

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where $\nu = \eta/\rho$ is the kinematic viscosity of the medium.

If the flow proceeds in the presence of mass forces (for example, in a gravitational field), then on the right-hand side of the NS-equation (3.2) the density f of these forces is added. Further, if we expand the total derivative on the left-hand side of the NS-equation bearing in mind that $\dot{x} = v$, then we obtain another way of writing the NS-equation:

$$\partial_t v + (v\nabla)v = \nu \triangle v + \frac{1}{\rho}(f - \nabla p). \tag{3.5}$$

If the flow is stationary, that is, the velocity field v is independent of time, then the last equation takes the form

$$(v\nabla)v = \nu \triangle v + \frac{1}{\rho}(f - \nabla p). \tag{3.6}$$

Here we do not intend to get further immersed in the enormous range of works relating to the Navier–Stokes equation. We merely recall that the following problem is included in the list of problems of the century (and a worthwhile prize has been set aside for its solution): if the initial and boundary conditions of the three-dimensional NS–equation (3.2) are smooth, then will the smoothness of the solution be preserved for ever or can a singularity spring up after a finite time ? (In the two-dimensional case no singularities arise.)

From the physical point of view there are probably other problems that present great interest, for example, the question how one can obtain from the NS–equation (3.2) a satisfactory description of turbulent flows and how the transition from turbulence to chaos occurs.

Thus, the equations of the dynamics of a continuous medium exist (for actual media they are supplemented by thermodynamic equations of state). In a number of cases these equations of the dynamics can be solved explicitly. In other cases it is possible to carry out the calculations of concrete flows on a computer. But, on the whole, a lot of further investigation is required, including, possibly, ideas new in principle.

Turning back to the very beginning, we now imagine that we do not yet know about the Navier–Stokes equations but are nevertheless interested in the flow of a continuous medium. For example, suppose that the flow of a homogeneous incompressible liquid having a velocity u at infinity runs into an object having a certain characteristic dimension l. We are interested in the stationary regime of the flow, that is, the resulting vector velocity field $v = v(r)$ as a function of the radius vector r of a point in space with respect to some fixed system of Cartesian coordinates. Let ϱ be the density of the liquid, η its viscosity and $\nu = \eta/\rho$ its kinematic viscosity.

Assuming that $v = f(r, \eta, \rho, l, u)$, we shall try to draw on a dimensional analysis:

$$\begin{array}{ccccccc} & v & r & \eta & \rho & l & u \\ L & 1 & 1 & -1 & -3 & 1 & 1 \\ M & 0 & 0 & 1 & 1 & 0 & 0 \\ T & 1 & 0 & -1 & 0 & 0 & -1 \end{array}$$

Hence, in view of the Π-theorem, we obtain the following relation between dimensionless quantities:

$$\frac{v}{u} = f(\frac{r}{l}, 1, \frac{\rho u l}{\eta}, 1, 1), \tag{3.7}$$

in which we find the Reynolds number $\mathrm{Re} := \frac{\rho u l}{\eta} = \frac{ul}{\nu}$.

We would have obtained the same result by supposing that $v = f(r, \nu, \rho, l, u)$, or if we had started with $v = f(r, \nu, l, u)$, in which the density is hidden in the kinematic viscosity ν.

Thus one can change the values of ρ, u, l, η; however, if at the same time we do not change their combination expressed by the Reynolds number Re, then the character of the flow remains the same to within the scale of the measurements of length and time (or length and velocity). Remarkable!

3.2 Loss of stability of the flow and comments on bifurcations in dynamical systems

The character of the flow for different values of the Reynolds number is, in general, different. As the Reynolds number increases there occur topological restructurings (bifurcations) of the flow. Its character changes from stable laminar flow to turbulence and chaos: for $\mathrm{Re} \asymp 10^0$ the flow is laminar; then for $Re \asymp 10^1$ the first critical value Re_1 appears and also the first bifurcation (first restructuring of the topology of the flow), and so on. The sequence $\mathrm{Re}_1 < \mathrm{Re}_2 < ... < \mathrm{Re}_n < ...$ of critical values rapidly converges, which, however, is a fairly universal phenomenon.

(This universality, which was discovered for the first time by M.J. Feigenbaum in 1978, bears his name *Feigenbaum universality*; it states that the following limit exists:

$$\lim_{n\to\infty} \frac{R_{n+1} - R_n}{R_n - R_{n-1}} = \delta^{-1},$$

where $\{R_n, n \in \mathbb{N}\}$ is the sequence of critical values at which the restructuring of the dynamical system (called bifurcation of period doubling) occurs, and $\delta = 4.6692...$.) Thus the sequence of numbers Re_n has a limit Re_∞.

If the parameter Re of the problem is increased beyond the value Re_∞, then there occurs the regime, which in hydrodynamics is called turbulence. At very large values of the Reynolds number the motion becomes quite chaotic (as though it were an indeterminate random process).

At present the fundamental question remains open: can one arrive at a description of turbulence starting from the classical Navier–Stokes equations? Here we recall the words of Richard Feynman who, in another context, said (if I am not mistaken) that possibly Schrödinger's equation already contains the formula for life, but that does not rule out biology, which studies the living cell without waiting meanwhile to find out whether the existence of life is justified via Schrödinger's equation.

Having shown himself to be a true natural scientist A.N. Kolmogorov proposed (in 1941) a model for the development of turbulence, which, although it was subsequently refined, has remained basic and deserves special consideration. We have just chosen Kolmogorov's model to demonstrate the non-trivial application of dimensional analysis in the study of a phenomenon of which we have no fundamental description at our disposal.

Of course, the discovery of strange attractors (E.N. Lorenz, 1963) was a serious new general achievement of the theory of dynamical systems. This made it possible to concretize the ideology of the emergence of chaotic manifestations of a determinate system as its sensitivity to small changes in the initial conditions, and also to give a general-dynamical explanation of a phenomenon of turbulence (D. Ruelle & F. Takens, 1971) (See also the commentary in the paper [5]).

3.3 Turbulence (initial ideas)

Introducing the collection [9] of articles on turbulence, Academician O.M. Belotserkovskiĭ recalled that when he studied at the physics faculty in Moscow University, lectures on electricity were given to him by Professor S.G. Kalashnikov, who at the very first lecture related the following. Once in an examination he (Kalashnikov) asked a student "What is electricity?" The student began to fidget and fuss and replied "Oh dear, I knew it yesterday but now I have forgotten". To this Kalashnikov observed "There was only one man who knew it, and even he forgot it!"

The situation with turbulence is roughly the same, although people of various disciplines have speculated about this, of course primarily physicists, mathematicians and astronomers.

A fast river flowing past the pier of a bridge forms a vortex, which as it percolates forms patterns drawn by Leonardo da Vinci, who penetrated everything with his thought and eye. Vortices can also be observed in the air in clouds of dust behind a whirling machine or, much more pleasingly simply in clouds capriciously deforming themselves in front of one's eyes. Cosmic vortices form galaxies. Water from a tap stops flowing peacefully when the tap is switched on too violently. A small aircraft is not allowed to take off behind a large liner. And through the porthole of an aeroplane one observes with interest the seemingly minute (as seen from above) ocean ships behind which there trails a clearly distinguishable turbulent wake.

In science the term *turbulence* was established by the end of the nineteenth century, after Maxwell, Lord Kelvin and Reynolds, although already Da Vinci made good use of it.

3.4 The Kolmogorov model

3.4.1 The multiscale property of turbulent motions

As above, we consider the flow round a body, that is, a flow running into a body of characteristic dimension l. If the velocity of the flow is large or the viscosity of the medium is small, then for very large values of the Reynolds number a region of turbulence is developed in some volume behind the body. In that region the flow is extremely unstable, chaotic and has the character of pulsations of different scales propagated in the region of turbulence. The change in the velocity field $v = v(x,t)$ is reminiscent here of a stationary random process when not the velocity field is stationary, but only certain averaged probabilistic characteristics (expressed, for example, by histograms of the probability distribution of some or other quantities connected with the flow).

Kolmogorov observed that for very large values of the Reynolds number, in the region of turbulence the picture of the flow, was locally homogeneous and isotropic, although still complicated.

Turbulence can (or should) be regarded as a manifestation of the interaction of the motions of different scales. On the motion of large scales pulsations of smaller scales are imposed and they are transferred by the motions of the larger scales (train passengers moving in the restaurant car participate in the motions of scales of the distance between towns, but can be looked at in the scales of the restaurant car as well).

Let us explain the idea of the multiscale property. A cell lives its own life. Associations of cells interacting with each other form a certain tissue. A group of tissues form an organ. A group of organs form an organism. An organism sits in a machine and goes to work. On the streets of a town a transport flow forms. All these flows together with the towns and countrysides are carried in space by a rotating Earth, and so on. Even closer to our theme could be the example relating to the multiscale life of the ocean or the atmosphere.

When we talk about the movement of a passenger, then it is clear that we have in view the characteristics of his motion within the scales of the restaurant car. We do not single out an individual passenger when we talk about speed in the scales of the motion of the entire train.

Turbulence is primarily the relative motions in the locality of the fluid and not the absolute transfer of motion in which it participates as an element of a larger-scale motion.

If this is the case, then in turbulence we have in front of us an entire spectrum of motions of different scales and we want, for example, to indicate some characteristic parameters of the motions of different scales: distribution of the energy of turbulent flow with respect to the scales of motion, the relative velocities of motions of different scales, the velocity of dispersion of particles in a turbulent flow, and so on.

3.4.2 Developed turbulence in the inertia interval

We now turn to a more concrete discussion.

Consider a flow, for example, (as above) round a body of characterisic dimension l for large values of the Reynolds number ($\mathrm{Re} \gg 1$). For $\mathrm{Re} \gg 1$ the turbulence that arises is usually called *developed turbulence*. We shall suppose, after Kolmogorov, that the developed turbulence in motions of scales $\lambda \ll l$ and far from solid walls (that is, at a distance much larger than the size of λ) is isotropic and homogeneous.

The condition $\mathrm{Re} \gg 1$ can be treated as smallness of viscosity. Viscosity only manifests itself in small-scale motions of some scale λ_0 since internal friction only occurs between close particles of the liquid. When one goes over to large masses of the liquid, the viscosity is insignificant since in this case the dissipation of energy is negligibly small by comparison with the kinetic energy of the inertial motion of the large mass.

The interval of scales $\lambda_0 \ll \lambda \ll l$ is called the *inertia interval*. In motions of these scales the viscosity can be ignored. The quantity λ_0 is called the *inner scale of the turbulent motion* and l is called the *outer scale of the turbulent motion*.

3.4.3 Specific energy

The steady turbulent regime is maintained by the expenditure of external energy dispersing in the liquid due to its viscosity. Let ε be the specific power of dissipation, more precisely, the amount of energy dissipated by a unit mass of the liquid in unit time. In accordance with this definition the quantity ε has the following dimension vector in the standard basis $\{L, M, T\}$: $[\varepsilon] = (2, 0, -3)$. [The force $F = ma$ has dimension (1, 1, -2). Energy, work and potential energy $F \cdot h$ have dimension (2, 1, -2). Hence $[\varepsilon] = (2, 0, -3)$.]

The kinetic energy of a flow running at a velocity u is decreased as a result of the dissipation of energy due to the internal friction in the viscous liquid.

The change Δu in the average velocity of the basic motion occurs in a spread of order l (up to the encounter with the body and beyond)

The quantity ε must be determined by this loss of kinetic energy of the basic motion, that is, it must be a function $\varepsilon = f(\rho, \Delta u, l)$. We then conclude on the basis of the Π-theorem that this quantity is of the order

$$\varepsilon \sim \frac{(\Delta u)^3}{l}. \tag{3.8}$$

Similarly we obtain the following equation for the fall in pressure:

$$\Delta p \sim (\Delta u)^2 \rho. \tag{3.9}$$

3.4.4 Reynolds number of motions of a given scale

Since we are interested in motions of different scales, we can associate with each scale λ the Reynolds number corresponding to it

$$Re_\lambda := \frac{v_\lambda \cdot \lambda}{\nu}. \tag{3.10}$$

In terms of these the inner scale of turbulence, that is, the quantity λ_0, must be determined by the condition that the Reynolds number have the order $\mathrm{Re}_{\lambda_0} \sim 1$, since a greater value of the Reynolds number would be equivalent to a small viscosity.

3.4.5 The Kolmogorov–Obukhov law

We now find the average velocities v_λ of motions of the scale λ (or, which is the same, the change of the average velocity of a turbulent flow in a spread of distances of order λ). In the inertia interval, when $\lambda_0 \ll \lambda \ll l$, we can assume that $v_\lambda = f(\rho, \varepsilon, \lambda)$. Then, on the basis of the Π-theorem we conclude that

$$v_\lambda \sim (\varepsilon\lambda)^{1/3}. \tag{3.11}$$

This relation is called the *Kolmogorov–Obukhov law*. (A.M. Obukhov, who was student of Kolmogorov during the 1940s subsequently became an academician and director of the Institute of Physics of the Atmosphere in Moscow. Another student of Kolmogorov at the same time was A.S. Monin, who also became an academician and was director of the Institute of Oceanography. About this Kolmogorov joking said that one of his students is in charge of the ocean and another is in charge of the atmosphere.)

3.4.6 *Inner scale of turbulence*

We now find the inner scale λ_0 of turbulent flow. As we know, we need to find it from the condition that $\mathrm{Re}_{\lambda_0} \sim 1$.

For the Reynolds number Re for the basic motion of scale l, in accordance with the general definition of Reynolds number and in accordance with formula (3.10), we have $\mathrm{Re} \sim (\Delta u \cdot l)/\nu$. Taking into account relations (3.8) and (3.11) we get

$$\mathrm{Re}_\lambda \sim \frac{v_\lambda \cdot \lambda}{\nu} \sim \frac{(\varepsilon\lambda)^{1/3}\lambda}{\nu} \sim \frac{\Delta u \cdot (\lambda)^{4/3}}{\nu l^{1/3}} = \mathrm{Re}\left(\frac{\lambda}{l}\right)^{4/3}.$$

Setting $\mathrm{Re}_\lambda \sim 1$ we find that

$$\lambda_0 \sim \frac{l}{\mathrm{Re}^{3/4}}. \tag{3.12}$$

For the corresponding velocity we have

$$v_{\lambda_0} \sim (\varepsilon\lambda_0)^{1/3} \sim \frac{\Delta u}{l^{1/3}} \cdot \frac{l^{1/3}}{\mathrm{Re}^{1/4}} = \frac{\Delta u}{\mathrm{Re}^{1/4}}. \tag{3.13}$$

3.4.7 *Energy spectrum of turbulent pulsations*

We associate with the scale of length λ as a wave length the number $k := 1/\lambda$. Let $E(k)dk$ be the kinetic energy in the pulsations (motions) with wave number k in the interval dk referring to unit mass of the liquid.

We shall find the density $E(k)$ of this distribution. Since $E(k)dk$ has the dimension of energy relative to unit mass and $[dk] = (-1,0,0)$, we find that $[E(k)] = (3,0,-2)$. Combining ε and k and using Kolmogorov's dimensional arguments we obtain

$$E(k) \sim \varepsilon^{2/3}k^{-5/3}. \tag{3.14}$$

Assuming that v_λ determines the order of magnitude of the kinetic energy of the motions of all scales not exceeding λ we can again obtain the Kolmogorov–Obukhov law

$$v_\lambda^2 \sim \int_{k=1/\lambda}^{\infty} E(k)dk \sim \varepsilon^{2/3}k^{-2/3} \sim (\varepsilon\lambda)^{2/3}$$

and $v_\lambda \sim (\varepsilon\lambda)^{1/3}$.

3.4.8 Turbulent mixing and dispersion of particles

Two particles situated in a turbulent flow at a mutual distance λ apart become separated to a distance $\lambda(t)$ over an interval of time t. We shall find the speed $\lambda'(t)$ of separation of the particles. As in the derivation of the Kolmogorov–Obukhov law, we assume that $\lambda' = f(\rho, \varepsilon, \lambda)$, and in complete accordance with formula (3.11) we obtain

$$\frac{d\lambda}{dt} \sim (\varepsilon\lambda)^{1/3}. \tag{3.15}$$

As is clear, the speed of separation increases as λ increases. This is explained by the fact that in the process under consideration only motions of scale less than λ participate. The larger-scale motions transfer the particles but do not lead to their separation.

Part II

Multidimensional Geometry and Functions of a Very Large Number of Variables

Introduction

Almost all the bulk of a multidimensional body is concentrated at its boundary. For example, if one removes from a 1000-dimensional watermelon of radius one metre the peel of thickness 1 centimetre, then there remains less than one thousandth of the entire watermelon.

This phenomenon of localization or concentration of the measure has numerous unexpected manifestations. For example, any more-or-less regular function on a multidimensional sphere is almost constant in the sense that if one takes randomly and independently a pair of points of the sphere and calculates the values of the function at these points, then with high probability they will turn out to be almost the same.

From the point of view of a mathematician used to dealing with functions of one, two or several (but not a great number of) variables this may appear implausible. But in fact this ensures the stability of the basic parameters within our habitat (temperature, pressure, and so on), it lies in the foundations of statistical physics, is studied in probability theory under the name of the law of large numbers and has many applications (for example, in the transmission of information along a communication channel in the presence of noise).

The phenomenon of the concentration of measure explains in some respect both the statistical stability of the values of thermodynamic quantities which gave rise to the Boltzmann ergodic hypothesis and the remarkable ergodic theorems that arose with the aim of justifying it.

The principle of concentration is set forth in Chapter 2, which can be read independently of Chapter 1, where we give examples of areas in which functions of a large number of variables appear in a natural way.

In Chapter 1 (which, of course, can also be read independently) we dwell in detail on a less popular example — the transmission of information along a communication channel. We introduce and discuss the sampling theorem — Kotel'nikov's formula – the basis of modern digital representation of a signal. In Chapter 3 we supplement these discussions with Shannon's theorem on the speed of transmission along a communication channel in the presence of noise.

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