Chapter 2
Measure-Valued Branching Processes

A measure-valued process describes the evolution of a population that evolves according to the law of chance. In this chapter we provide some basic characterizations and constructions for measure-valued branching processes. In particular, we establish a one-to-one correspondence between those processes and cumulant semigroups. Some results for nonlinear integral evolution equations are proved, which lead to an analytic construction of a class of measure-valued branching processes, the so-called Dawson–Watanabe superprocesses. We shall construct the superprocesses for admissible killing densities and general branching mechanisms that are not necessarily decomposable into local and non-local parts. A number of moment formulas for the superprocesses are also given.

2.1 Definitions and Basic Properties

Suppose that E is a Lusin topological space and $(Q_t)_{t\geq 0}$ is a conservative transition semigroup on $M(E)$. We say $(Q_t)_{t\geq 0}$ satisfies the *branching property* provided

$$Q_t(\mu_1 + \mu_2, \cdot) = Q_t(\mu_1, \cdot) * Q_t(\mu_2, \cdot), \qquad t \geq 0,\ \mu_1, \mu_2 \in M(E). \tag{2.1}$$

Given the transition semigroup $(Q_t)_{t\geq 0}$, for $t \geq 0$ and $f \in B(E)^+$ let

$$V_t f(x) = -\log \int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\delta_x, \mathrm{d}\nu), \qquad x \in E. \tag{2.2}$$

We say $(Q_t)_{t\geq 0}$ satisfies the *regular branching property* if for every $t \geq 0$ and $f \in B(E)^+$ the function $V_t f$ belongs to $B(E)^+$ and

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\mu, \mathrm{d}\nu) = \exp\{-\mu(V_t f)\}, \qquad \mu \in M(E). \tag{2.3}$$

Z. Li, *Measure-Valued Branching Markov Processes*,
Probability and Its Applications, DOI 10.1007/978-3-642-15004-3_2,

Clearly, $(Q_t)_{t\geq 0}$ has the branching property (2.1) if it satisfies the regular branching property (2.3).

Theorem 2.1 *If $(Q_t)_{t\geq 0}$ satisfies the branching property (2.1), then for any probability measures N_1 and N_2 on $M(E)$ we have*

$$(N_1 * N_2)Q_t = (N_1 Q_t) * (N_2 Q_t), \qquad t \geq 0. \tag{2.4}$$

Proof. For any $t \geq 0$ and $f \in B(E)^+$,

$$\begin{aligned}
&\int_{M(E)} \mathrm{e}^{-\nu(f)} (N_1 * N_2)Q_t(\mathrm{d}\nu) \\
&\quad = \int_{M(E)} (N_1 * N_2)(\mathrm{d}\mu) \int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\mu, \mathrm{d}\nu) \\
&\quad = \int_{M(E)^2} N_1(\mathrm{d}\mu_1) N_2(\mathrm{d}\mu_2) \int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\mu_1 + \mu_2, \mathrm{d}\nu) \\
&\quad = \int_{M(E)^2} N_1(\mathrm{d}\mu_1) N_2(\mathrm{d}\mu_2) \int_{M(E)^2} \mathrm{e}^{-\nu_1(f)-\nu_2(f)} Q_t(\mu_1, \mathrm{d}\nu_1) Q_t(\mu_2, \mathrm{d}\nu_2) \\
&\quad = \int_{M(E)} \mathrm{e}^{-\nu_1(f)} (N_1 Q_t)(\mathrm{d}\nu_1) \int_{M(E)} \mathrm{e}^{-\nu_2(f)} (N_2 Q_t)(\mathrm{d}\nu_2).
\end{aligned}$$

Then (2.4) follows by the uniqueness of the Laplace functional. □

Proposition 2.2 *If $(Q_t)_{t\geq 0}$ satisfies the branching property (2.1) and K is an infinitely divisible probability measure on $M(E)$, then KQ_t is an infinitely divisible probability measure on $M(E)$ for any $t \geq 0$.*

Proof. For $n \geq 1$ let K_n be the n-th root of K. By applying (2.4) inductively we have $(K_n Q_t)^{*n} = (K_n^{*n})Q_t = KQ_t$. Then KQ_t is infinitely divisible. □

Suppose that T is an interval on the real line and $(\mathscr{F}_t)_{t\in T}$ is a filtration. A Markov process $\{(X_t, \mathscr{F}_t) : t \in T\}$ in $M(E)$ with transition semigroup $(Q_t)_{t\geq 0}$ satisfying the branching property (2.1) is called a *measure-valued branching process* (MB-process). In particular, we call $\{(X_t, \mathscr{F}_t) : t \in T\}$ a *regular MB-process* if $(Q_t)_{t\geq 0}$ satisfies the regular branching property defined by (2.2) and (2.3).

Theorem 2.3 *Suppose that $\{(X_t, \mathscr{F}_t) : t \in T\}$ and $\{(Y_t, \mathscr{G}_t) : t \in T\}$ are two independent MB-processes with transition semigroup $(Q_t)_{t\geq 0}$. Let $Z_t = X_t + Y_t$ and $\mathscr{H}_t = \sigma(\mathscr{F}_t \cup \mathscr{G}_t)$. Then $\{(Z_t, \mathscr{H}_t) : t \in T\}$ is also an MB-process with transition semigroup $(Q_t)_{t\geq 0}$.*

Proof. Let $r \leq t \in T$ and suppose $F \in \mathrm{b}\mathscr{F}_r$ and $G \in \mathrm{b}\mathscr{G}_r$. For any $f \in B(E)^+$ we use the independence of $\{(X_t, \mathscr{F}_t) : t \in T\}$ and $\{(Y_t, \mathscr{G}_t) : t \in T\}$ and the branching property (2.1) to see that

$$\begin{aligned}
&\mathbf{E}\Big[FG\exp\{-Z_t(f)\}\Big] \\
&\quad = \mathbf{E}\Big[F\exp\{-X_t(f)\}\Big]\mathbf{E}\Big[G\exp\{-Y_t(f)\}\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{E}\Big[F \int_{M(E)} \mathrm{e}^{-\nu(f)} Q_{t-r}(X_r, \mathrm{d}\nu)\Big] \mathbf{E}\Big[G \int_{M(E)} \mathrm{e}^{-\nu(f)} Q_{t-r}(Y_r, \mathrm{d}\nu)\Big] \\
&= \mathbf{E}\Big[FG \int_{M(E)} \mathrm{e}^{-\nu(f)} Q_{t-r}(Z_r, \mathrm{d}\nu)\Big].
\end{aligned}
$$

Then Proposition A.1 implies

$$
\mathbf{E}\Big[H \exp\{-Z_t(f)\}\Big] = \mathbf{E}\Big[H \int_{M(E)} \mathrm{e}^{-\nu(f)} Q_{t-r}(Z_r, \mathrm{d}\nu)\Big]
$$

for any $H \in \mathrm{b}\mathscr{H}_r$. That gives the desired result. $\square$

Recall that $\mathscr{I}(E)$ denotes the convex cone of functionals on $B(E)^+$ with the representation (1.20). Let $(V_t)_{t\geq 0}$ be a family of operators on $B(E)^+$ and let $v_t(x, f) = V_t f(x)$. We call $(V_t)_{t\geq 0}$ a *cumulant semigroup* provided:

(1) $v_t(x, \cdot) \in \mathscr{I}(E)$ for all $t \geq 0$ and $x \in E$;
(2) $V_r V_t = V_{r+t}$ for every $r, t \geq 0$.

By Theorem 1.36, if $(V_t)_{t\geq 0}$ is a cumulant semigroup, each operator V_t has the canonical representation

$$
V_t f(x) = \lambda_t(x, f) + \int_{M(E)^\circ} \big(1 - \mathrm{e}^{-\nu(f)}\big) L_t(x, \mathrm{d}\nu), \quad f \in B(E)^+, \tag{2.5}
$$

where $\lambda_t(x, \mathrm{d}y)$ is a bounded kernel on E and $(1 \wedge \nu(1))L_t(x, \mathrm{d}\nu)$ is a bounded kernel from E to $M(E)^\circ$.

Theorem 2.4 *The relation* (2.3) *establishes a one-to-one correspondence between cumulant semigroups* $(V_t)_{t\geq 0}$ *on* $B(E)^+$ *and transition semigroups* $(Q_t)_{t\geq 0}$ *on* $M(E)$ *satisfying the regular branching property.*

Proof. Suppose that $(V_t)_{t\geq 0}$ is a cumulant semigroup. By Theorem 1.35 we see that (2.3) defines an infinitely divisible probability measure $Q_t(\mu, \cdot)$ on $M(E)$. From $V_r V_t = V_{r+t}$ we have $Q_r Q_t = Q_{r+t}$. That is, $(Q_t)_{t\geq 0}$ is a transition semigroup on $M(E)$. Conversely, suppose that $(Q_t)_{t\geq 0}$ is a transition semigroup on $M(E)$ satisfying the regular branching property. Then $Q_t(\mu, \cdot)$ is an infinitely divisible probability measure on $M(E)$. This is true in particular for $\mu = \delta_x$, and so $V_t f(x)$ has the representation (2.5) by Theorems 1.35 and 1.36. The semigroup property of $(V_t)_{t\geq 0}$ follows from that of $(Q_t)_{t\geq 0}$. $\square$

Example 2.1 Let $M_a(E)$ and $M_d(E)$ denote respectively the subset of $M(E)$ of purely atomic measures and that of diffuse measures. Then each $\mu \in M(E)$ has the unique decomposition $\mu = \mu_a + \mu_d$ for $\mu_a \in M_a(E)$ and $\mu_d \in M_d(E)$. The mappings $\mu \mapsto \mu_a$ and $\mu \mapsto \mu_d$ are measurable; see Kallenberg (1975, pp.10–11). Take two distinct real constants c_a and c_d and let $Q_t(\mu, \cdot)$ be the unit mass concentrated at $\mathrm{e}^{c_a t}\mu_a + \mathrm{e}^{c_d t}\mu_d$. Then $(Q_t)_{t\geq 0}$ satisfies the branching property (2.1), but it is not regular in the sense of (2.3).

Theorem 2.5 *Suppose that E is a compact metric space. If $(V_t)_{t\geq 0}$ is a cumulant semigroup on E preserving $C(E)^{++}$ and $V_tf(x) \to f(x)$ pointwise as $t \to 0$ for every $f \in C(E)^{++}$, then (2.3) defines a Feller semigroup $(Q_t)_{t\geq 0}$ on $M(E)$. Conversely, if $(Q_t)_{t\geq 0}$ is a Feller semigroup having the branching property (2.1), then it satisfies the regular branching property (2.3) with cumulant semigroup $(V_t)_{t\geq 0}$ preserving $C(E)^{++}$ and $V_tf(x) \to f(x)$ pointwise as $t \to 0$ for every $f \in C(E)^{++}$.*

Proof. If $(V_t)_{t\geq 0}$ is a cumulant semigroup on E that preserves $C(E)^{++}$ and $V_tf(x) \to f(x)$ pointwise as $t \to 0$ for every $f \in C(E)^{++}$, it is simple to see that (2.3) defines a Feller semigroup on $M(E)$. For the converse, suppose that $(Q_t)_{t\geq 0}$ is a Feller semigroup on $M(E)$ having the branching property (2.1). Given $f \in B(E)^+$ we define $V_tf(x)$ by (2.2). For any $f \in C(E)^{++}$ we clearly have $V_tf \in C(E)^+$. If $\mu = \sum_{i=1}^n (p_i/q_i)\delta_{x_i}$ for $x_i \in E$ and integers p_i and $q_i \geq 1$, we have (2.3) by easy calculations based on (2.1). By an approximating argument, the equality holds for all $\mu \in M(E)$ and $f \in C(E)^{++}$. The extension from $f \in C(E)^{++}$ to $f \in B(E)^+$ is immediate by Proposition 1.3. Then $(Q_t)_{t\geq 0}$ satisfies the regular branching property. If there exists $f \in C(E)^{++}$ so that $V_tf \notin C(E)^{++}$, the compactness of E assures the existence of a point $x_0 \in E$ satisfying $V_tf(x_0) = 0$, so the function $\mu \mapsto \exp\{-\mu(V_tf)\}$ does not belong to $C_0(M(E))$, yielding a contradiction. Then $(V_t)_{t\geq 0}$ preserves $C(E)^{++}$. Since $Q_tF(\mu) \to F(\mu)$ pointwise as $t \to 0$ for every $F \in C_0(M(E))$, we have $V_tf(x) \to f(x)$ pointwise as $t \to 0$ for every $f \in C(E)^{++}$. □

In the rest of the book, we will only consider regular MB-processes and will omit the adjective "regular". Given the transition semigroup $(Q_t)_{t\geq 0}$ of an MB-process in $M(E)$, we use $(Q_t^\circ)_{t\geq 0}$ to denote its restriction to $M(E)^\circ$.

Proposition 2.6 *Suppose that $(Q_t)_{t\geq 0}$ is defined by (2.3) with $(V_t)_{t\geq 0}$ given by (2.5). If $N = I(\eta, H)$ is an infinitely divisible probability measure on $M(E)$, then $NQ_t = I(\eta_t, H_t)$ is infinitely divisible for every $t \geq 0$, where*

$$\eta_t = \int_E \eta(\mathrm{d}y)\lambda_t(y, \cdot) \ \text{ and } \ H_t = \int_E \eta(\mathrm{d}y)L_t(y, \cdot) + HQ_t^\circ. \tag{2.6}$$

Proof. We first note that NQ_t is infinitely divisible by Proposition 2.2. For $t \geq 0$ and $f \in B(E)^+$ we have

$$\begin{aligned} -\log \int_{M(E)} &\mathrm{e}^{-\nu(f)} NQ_t(\mathrm{d}\nu) \\ &= \eta(V_tf) + \int_{M(E)^\circ} \big(1 - \mathrm{e}^{-\nu(V_tf)}\big) H(\mathrm{d}\nu) \\ &= \int_E \eta(\mathrm{d}y)\lambda_t(y, f) + \int_E \eta(\mathrm{d}y) \int_{M(E)^\circ} \big(1 - \mathrm{e}^{-\nu(f)}\big) L_t(y, \mathrm{d}\nu) \\ &\qquad + \int_{M(E)^\circ} \big(1 - \mathrm{e}^{-\nu(f)}\big) HQ_t^\circ(\mathrm{d}\nu). \end{aligned}$$

Then $NQ_t = I(\eta_t, H_t)$ with (η_t, H_t) given by (2.6). □

Corollary 2.7 *Suppose that* $(Q_t)_{t\geq 0}$ *is defined by* (2.3) *with* $(V_t)_{t\geq 0}$ *given by* (2.5). *Then for any* $t \geq r \geq 0$ *and* $x \in E$ *we have*

$$\lambda_{r+t}(x,\cdot) = \int_E \lambda_r(x,\mathrm{d}y)\lambda_t(y,\cdot) \tag{2.7}$$

and

$$L_{r+t}(x,\cdot) = \int_E \lambda_r(x,\mathrm{d}y)L_t(y,\cdot) + \int_{M(E)^\circ} L_r(x,\mathrm{d}\mu)Q_t^\circ(\mu,\cdot). \tag{2.8}$$

Proof. This follows by applying Proposition 2.6 to the infinitely divisible probability measure $Q_r(\delta_x,\cdot)$ on $M(E)$. □

Suppose that $(Q_t)_{t\geq 0}$ is the transition semigroup of an MB-process defined by (2.3). In general, the corresponding cumulant semigroup $(V_t)_{t\geq 0}$ has the representation (2.5). Let E° be the set of points $x \in E$ such that $\lambda_t(x,E) = 0$ for all $t > 0$. Then $x \in E^\circ$ if and only if

$$V_t f(x) = \int_{M(E)^\circ} \big(1 - \mathrm{e}^{-\nu(f)}\big) L_t(x,\mathrm{d}\nu), \quad t > 0, f \in B(E)^+. \tag{2.9}$$

In view of (2.8), we have the following:

Proposition 2.8 *For any* $x \in E^\circ$ *the family of* σ*-finite measures* $\{L_t(x,\cdot) : t > 0\}$ *on* $M(E)^\circ$ *constitute an entrance law for the restricted semigroup* $(Q_t^\circ)_{t\geq 0}$.

2.2 Integral Evolution Equations

Let E be a Lusin topological space. Suppose that $\xi = (\Omega, \mathscr{F}, \mathscr{F}_t, \xi_t, \mathbf{P}_x)$ is a Borel right process in E with transition semigroup $(P_t)_{t\geq 0}$. Let $\{K(t) : t \geq 0\}$ be a continuous additive functional of ξ which is *admissible* in the sense that each $\omega \mapsto K_t(\omega)$ is measurable with respect to the σ-algebra $\mathscr{F}^0 := \sigma(\{\xi_t : t \geq 0\})$ and

$$k(t) := \sup_{x\in E} \mathbf{P}_x\big[K(t)\big] \to 0, \qquad t \to 0. \tag{2.10}$$

For any $\beta \in B(E)$ we write

$$K_t(\beta) = \int_0^t \beta(\xi_s)K(\mathrm{d}s), \qquad t \geq 0.$$

Let $\mathrm{b}\mathscr{E}(K)$ denote the set of functions $\beta \in B(E)$ so that $t \mapsto \mathrm{e}^{-K_t(\beta)}$ is a locally bounded stochastic process. Recall that $\|\cdot\|$ denotes the supremum norm of functions on E.

Proposition 2.9 *Let $f \in B(E)$ and $b, \beta \in \mathrm{b}\mathscr{E}(K)$. If the two locally bounded functions $h, u \in \mathscr{B}([0,\infty) \times E)$ satisfy*

$$u(t,x) = \mathbf{P}_x\big[\mathrm{e}^{-K_t(\beta)} f(\xi_t)\big] + \mathbf{P}_x\bigg[\int_0^t \mathrm{e}^{-K_s(\beta)} h(t-s,\xi_s) K(\mathrm{d}s)\bigg], \quad (2.11)$$

they also satisfy

$$\begin{aligned} u(t,x) = \mathbf{P}_x\big[\mathrm{e}^{-K_t(b)} f(\xi_t)\big] &+ \mathbf{P}_x\bigg\{\int_0^t \mathrm{e}^{-K_s(b)} h(t-s,\xi_s) K(\mathrm{d}s)\bigg\} \\ &- \mathbf{P}_x\bigg\{\int_0^t \mathrm{e}^{-K_s(b)} [\beta(\xi_s) - b(\xi_s)] u(t-s,\xi_s) K(\mathrm{d}s)\bigg\}. \end{aligned} \quad (2.12)$$

Proof. Let $K_t^r(\beta) = K_t(\beta) - K_r(\beta)$ for $t \geq r \geq 0$. Since $s \mapsto \mathscr{F}_s$ is a right continuous filtration, the process

$$s \mapsto \mathbf{P}_x[\mathrm{e}^{-K_t^s(\beta)} f(\xi_t)|\mathscr{F}_s] = \mathrm{e}^{K_s(\beta)} \mathbf{P}_x[\mathrm{e}^{-K_t(\beta)} f(\xi_t)|\mathscr{F}_s]$$

is a.s. right continuous. Let $g = \beta - b \in \mathrm{b}\mathscr{E}(K)$. By the Markov property of ξ,

$$\begin{aligned}
&\mathbf{P}_x\bigg\{\int_0^t g(\xi_s)\mathrm{e}^{-K_s(b)} \mathbf{P}_{\xi_s}\big[\mathrm{e}^{-K_{t-s}(\beta)} f(\xi_{t-s})\big] K(\mathrm{d}s)\bigg\} \\
&= \mathbf{P}_x\bigg\{\int_0^t g(\xi_s)\mathrm{e}^{-K_s(b)} \mathbf{P}_x\big[\mathrm{e}^{-K_t^s(\beta)} f(\xi_t)|\mathscr{F}_s\big] K(\mathrm{d}s)\bigg\} \\
&= \lim_{n\to\infty} \mathbf{P}_x\bigg\{\sum_{i=1}^n \int_{(i-1)t/n}^{it/n} g(\xi_s)\mathrm{e}^{-K_s(b)} \mathbf{P}_x\big[\mathrm{e}^{-K_t^{it/n}(\beta)} f(\xi_t)|\mathscr{F}_{it/n}\big] K(\mathrm{d}s)\bigg\} \\
&= \lim_{n\to\infty} \sum_{i=1}^n \mathbf{P}_x\bigg\{\mathbf{P}_x\bigg[\int_{(i-1)t/n}^{it/n} g(\xi_s)\mathrm{e}^{-K_s(b)} \mathrm{e}^{-K_t^{it/n}(\beta)} f(\xi_t) K(\mathrm{d}s)\bigg|\mathscr{F}_{it/n}\bigg]\bigg\} \\
&= \lim_{n\to\infty} \sum_{i=1}^n \mathbf{P}_x\bigg\{\int_{(i-1)t/n}^{it/n} g(\xi_s)\mathrm{e}^{-K_s(b)} \mathrm{e}^{-K_t^{it/n}(\beta)} f(\xi_t) K(\mathrm{d}s)\bigg\} \\
&= \mathbf{P}_x\bigg\{\int_0^t g(\xi_s)\mathrm{e}^{-K_s(b)} \mathrm{e}^{-K_t^s(\beta)} f(\xi_t) K(\mathrm{d}s)\bigg\} \\
&= \mathbf{P}_x\bigg\{\int_0^t g(\xi_s)\mathrm{e}^{-K_t(b)} \mathrm{e}^{-K_t^s(g)} f(\xi_t) K(\mathrm{d}s)\bigg\} \\
&= \mathbf{P}_x\Big\{f(\xi_t)\mathrm{e}^{-K_t(b)}\big(1 - \mathrm{e}^{-K_t(g)}\big)\Big\}.
\end{aligned}$$

By similar calculations we have

$$\begin{aligned}
&\mathbf{P}_x\bigg\{\int_0^t g(\xi_s)\mathrm{e}^{-K_s(b)} \mathbf{P}_{\xi_s}\bigg[\int_0^{t-s} \mathrm{e}^{-K_r(\beta)} h(t-s-r,\xi_r) K(\mathrm{d}r)\bigg] K(\mathrm{d}s)\bigg\} \\
&= \mathbf{P}_x\bigg\{\int_0^t g(\xi_s)\mathrm{e}^{-K_s(b)} K(\mathrm{d}s) \int_0^{t-s} \mathrm{e}^{-K_{s+r}^s(\beta)} h(t-s-r,\xi_{s+r}) K(s+\mathrm{d}r)\bigg\} \\
&= \mathbf{P}_x\bigg\{\int_0^t g(\xi_s)\mathrm{e}^{-K_s(b)} K(\mathrm{d}s) \int_s^t \mathrm{e}^{-K_r^s(\beta)} h(t-r,\xi_r) K(\mathrm{d}r)\bigg\} \\
&= \mathbf{P}_x\bigg\{\int_0^t h(t-r,\xi_r)\mathrm{e}^{-K_r(b)} K(\mathrm{d}r) \int_0^r g(\xi_s)\mathrm{e}^{-K_r^s(g)} K(\mathrm{d}s)\bigg\} \\
&= \mathbf{P}_x\bigg\{\int_0^t h(t-r,\xi_r)\mathrm{e}^{-K_r(b)}\big(1 - \mathrm{e}^{-K_r(g)}\big) K(\mathrm{d}r)\bigg\}.
\end{aligned}$$

Then we add up both sides of the two equations and use (2.11) to get (2.12). □

In the sequel, we assume $\beta \in \mathrm{b}\mathscr{E}(K)$ and $f \mapsto \phi(\cdot, f)$ is an operator from $B(E)^+$ into $B(E)$ which is bounded on $B_a(E)^+$ for every $a \geq 0$. For $f \in B(E)^+$ we consider the integral evolution equation

$$v_t(x) = \mathbf{P}_x\big[\mathrm{e}^{-K_t(\beta)} f(\xi_t)\big] - \mathbf{P}_x\bigg[\int_0^t \mathrm{e}^{-K_s(\beta)} \phi(\xi_s, v_{t-s}) K(\mathrm{d}s)\bigg]. \quad (2.13)$$

For the convenience of statement of the results, we formulate the following conditions:

Condition 2.10 *There is a constant $L \geq 0$ so that $-\phi(x, f) \leq L\|f\|$ for $x \in E$ and $f \in B(E)^+$.*

Condition 2.11 *For every $a \geq 0$ there is a constant $L_a \geq 0$ so that*

$$\sup_{x\in E} |\phi(x, f) - \phi(x, g)| \leq L_a \|f - g\|, \qquad f, g \in B_a(E)^+.$$

Proposition 2.12 *Let $r \geq 0$ and $f \in B(E)^+$. Then $(t, x) \mapsto v_t(x)$ satisfies (2.13) for $t \geq 0$ if and only if it satisfies the equation for $0 \leq t \leq r$ and $(t, x) \mapsto v_{r+t}(x)$ satisfies*

$$v_{r+t}(x) = \mathbf{P}_x\big[\mathrm{e}^{-K_t(\beta)} v_r(\xi_t)\big] - \mathbf{P}_x\bigg[\int_0^t \mathrm{e}^{-K_s(\beta)} \phi(\xi_s, v_{r+t-s}) K(\mathrm{d}s)\bigg]. \quad (2.14)$$

Proof. Suppose that $(t, x) \mapsto v_t(x)$ satisfies (2.13) for $0 \leq t \leq r$ and $(t, x) \mapsto v_{r+t}(x)$ satisfies (2.14) for $t \geq 0$. Then we have

$$\begin{aligned}
v_{r+t}(x) &= \mathbf{P}_x\big[\mathrm{e}^{-K_t(\beta)} v_r(\xi_t)\big] - \mathbf{P}_x\bigg[\int_0^t \mathrm{e}^{-K_s(\beta)} \phi(\xi_s, v_{r+t-s}) K(\mathrm{d}s)\bigg] \\
&= \mathbf{P}_x\Big\{\mathrm{e}^{-K_t(\beta)} \mathbf{P}_{\xi_t}\big[\mathrm{e}^{-K_r(\beta)} f(\xi_r)\big]\Big\} \\
&\quad - \mathbf{P}_x\bigg\{\mathrm{e}^{-K_t(\beta)} \mathbf{P}_{\xi_t}\bigg[\int_0^r \mathrm{e}^{-K_s(\beta)} \phi(\xi_s, v_{r-s}) K(\mathrm{d}s)\bigg]\bigg\} \\
&\quad - \mathbf{P}_x\bigg[\int_0^t \mathrm{e}^{-K_s(\beta)} \phi(\xi_s, v_{r+t-s}) K(\mathrm{d}s)\bigg] \\
&= \mathbf{P}_x\big[\mathrm{e}^{-K_{r+t}(\beta)} f(\xi_{r+t})\big] - \mathbf{P}_x\bigg[\int_0^r \mathrm{e}^{-K_{t+s}(\beta)} \phi(\xi_{t+s}, v_{r-s}) K(t + \mathrm{d}s)\bigg] \\
&\quad - \mathbf{P}_x\bigg[\int_0^t \mathrm{e}^{-K_s(\beta)} \phi(\xi_s, v_{r+t-s}) K(\mathrm{d}s)\bigg] \\
&= \mathbf{P}_x\big[\mathrm{e}^{-K_{r+t}(\beta)} f(\xi_{r+t})\big] - \mathbf{P}_x\bigg[\int_t^{r+t} \mathrm{e}^{-K_s(\beta)} \phi(\xi_s, v_{r+t-s}) K(\mathrm{d}s)\bigg] \\
&\quad - \mathbf{P}_x\bigg[\int_0^t \mathrm{e}^{-K_s(\beta)} \phi(\xi_s, v_{r+t-s}) K(\mathrm{d}s)\bigg] \\
&= \mathbf{P}_x\big[\mathrm{e}^{-K_{r+t}(\beta)} f(\xi_{r+t})\big] - \mathbf{P}_x\bigg[\int_0^{r+t} \mathrm{e}^{-K_s(\beta)} \phi(\xi_s, v_{r+t-s}) K(\mathrm{d}s)\bigg].
\end{aligned}$$

Therefore $(t, x) \mapsto v_t(x)$ satisfies (2.13) for $t \geq 0$. For the converse, suppose that (2.13) holds for $t \geq 0$. The equation certainly holds for $0 \leq t \leq r$. By calculations similar to the above we see $(t, x) \mapsto v_{r+t}(x)$ satisfies (2.14). □

Corollary 2.13 *If for every $f \in B(E)^+$ there is a unique locally bounded positive solution $(t,x) \mapsto v_t(x,f)$ to (2.13), then the operators $V_t : f \mapsto v_t(\cdot,f)$ on $B(E)^+$ constitute a semigroup.*

Proof. Fix $r \geq 0$ and define $u_t = v_t$ for $0 \leq t \leq r$ and $u_{r+t} = v_t(\cdot, v_r)$ for $t \geq 0$. By Proposition 2.12 we see $(t,x) \mapsto u_t(x)$ solves (2.13) for $t \geq 0$. Then the uniqueness of the solution implies $u_{r+t} = v_{r+t}$ for all $t \geq 0$. That gives the semigroup property of $(V_t)_{t\geq 0}$. □

Proposition 2.14 *Suppose that Condition 2.10 holds. Then there is an increasing function $t \mapsto C(t)$ on $[0,\infty)$ so that for any locally bounded positive solution $(t,x) \mapsto v_t(x,f)$ to (2.13) we have*

$$\sup_{0\leq s\leq t} \|v_s(\cdot,f)\| \leq C(t)\|f\|, \qquad t \geq 0. \tag{2.15}$$

Proof. Let $t \mapsto l(t)$ be an increasing function so that $\mathrm{e}^{-K_t(\beta)} \leq l(t)$ for all $t \geq 0$. By (2.13) and Condition 2.10 we have

$$\|v_t(\cdot,f)\| \leq l(t)\|f\| + Ll(t) \sup_{x\in E} \mathbf{P}_x \bigg[\int_0^t \|v_{t-s}(\cdot,f)\| K(\mathrm{d}s) \bigg].$$

It follows that

$$\sup_{0\leq s\leq t} \|v_s(\cdot,f)\| \leq l(t)\|f\| + Lk(t)l(t) \sup_{0\leq s\leq t} \|v_s(\cdot,f)\|.$$

Let $\delta > 0$ be sufficiently small so that $Lk(\delta)l(\delta) < 1$. For $0 \leq t \leq \delta$ the above inequality implies

$$\sup_{0\leq s\leq t} \|v_s(\cdot,f)\| \leq l(t)\big[1 - Lk(t)l(t)\big]^{-1}\|f\|.$$

Then the desired result follows by Proposition 2.12 and a successive application of the above estimate. □

Proposition 2.15 *If Condition 2.11 holds, there is at most one locally bounded positive solution $(t,x) \mapsto v_t(x,f)$ to (2.13).*

Proof. Suppose that $(t,x) \mapsto u_t(x)$ and $(t,x) \mapsto v_t(x)$ are two locally bounded positive solutions of (2.13). Let $h_t(x) = u_t(x) - v_t(x)$ and let $l(t)$ be as in the proof of Proposition 2.14. For fixed $T > 0$ we can use Proposition 2.14 to find a constant $a \geq 0$ so that $\|u_t\| \leq a$ and $\|v_t\| \leq a$ for all $0 \leq t \leq T$. By (2.13) and Condition 2.11 we have

$$\begin{aligned}\|h_t\| &\leq l(t)\mathbf{P}_x \bigg[\int_0^t |\phi(\xi_s, u_{t-s}) - \phi(\xi_s, v_{t-s})| K(\mathrm{d}s) \bigg] \\ &\leq L_a l(t) \sup_{x\in E} \mathbf{P}_x \bigg[\int_0^t \|h_{t-s}\| K(\mathrm{d}s) \bigg].\end{aligned}$$

Then it is easy to get

$$\sup_{0\le s\le t}\|h_s\| \le L_a k(t)l(t) \sup_{0\le s\le t}\|h_s\|, \quad 0\le t\le T.$$

Take $0<\delta\le T$ so that $L_a k(\delta)l(\delta)<1$. The above inequality implies $\|h_t\|=0$ and hence $u_t=v_t$ for $0\le t\le \delta$. Then an application of Proposition 2.12 gives the uniqueness of the solution to (2.13). □

Proposition 2.16 *Let $\{\phi_n\}$ be a sequence of operators from $B(E)^+$ into $B(E)$ satisfying Conditions 2.10 and 2.11 with the constants L and L_a independent of $n\ge 1$. Suppose that $\lim_{n\to\infty}\phi_n(x,f)=\phi(x,f)$ uniformly on $E\times B_a(E)^+$ for every $a\ge 0$ and for $f_n\in B(E)^+$ there is a unique locally bounded positive solution $t\mapsto v_n(t)=v_n(t,x)$ to the equation*

$$v_n(t,x)=\mathbf{P}_x\big[\mathrm{e}^{-K_t(\beta)}f_n(\xi_t)\big]-\mathbf{P}_x\bigg[\int_0^t \mathrm{e}^{-K_s(\beta)}\phi_n(\xi_s,v_n(t-s))K(\mathrm{d}s)\bigg]. \quad (2.16)$$

If $\lim_{n\to\infty}f_n=f$ in the supremum norm, then the limit $\lim_{n\to\infty}v_n(t,x)=v_t(x)$ exists and is uniform on $[0,T]\times E$ for every $T\ge 0$. Moreover, $(t,x)\mapsto v_t(x)$ is a solution of (2.13).

Proof. Choose a sufficiently large constant $a\ge 0$ so that $\{f_n\}\subset B_a(E)^+$. By Proposition 2.14 there is an increasing $t\mapsto C(t)$ on $[0,\infty)$ so that

$$\sup_{0\le s\le t}\|v_n(s)\|\le C(t)\|f_n\|\le aC(t), \qquad t\ge 0.$$

Fix $T>0$ and let $c=aC(T)$. For $\varepsilon>0$ let $N=N(\varepsilon,c)$ be an integer so that $\|f_n-f\|\le\varepsilon$ and $\|\phi_n(\cdot,h)-\phi(\cdot,h)\|\le\varepsilon$ for $n\ge N$ and $h\in B_c(E)^+$. Let $l(t)$ be as in the proof of Proposition 2.14 and let

$$H_t(n_1,n_2)=\sup_{0\le s\le t}\|v_{n_2}(s)-v_{n_1}(s)\|.$$

By (2.16) and Condition 2.11 we have

$$H_t(n_1,n_2)\le 2l(t)[1+k(t)]\varepsilon+L_c k(t)l(t)H_t(n_1,n_2)$$

for $0\le t\le T$ and $n_1,n_2\ge N$. Take $0<\delta\le T$ so that $L_c k(\delta)l(\delta)<1$. The above inequality implies

$$H_t(n_1,n_2)\le 2l(t)[1+k(t)][1-L_c k(t)l(t)]^{-1}\varepsilon$$

for $0\le t\le\delta$. Then $v_n(t,x)$ converges uniformly on $[0,\delta]\times E$. By repeating the above arguments and applying Proposition 2.12 we see the limit $\lim_{n\to\infty}v_n(t,x)=v_t(x)$ exists and is uniform on $[0,T]\times E$. Then letting $n\to\infty$ in (2.16) we obtain (2.13). □

2.3 Dawson–Watanabe Superprocesses

In this section we give the construction of a general class of Dawson–Watanabe superprocesses. For this purpose we need to discuss the existence of solutions of some nonlinear integral evolution equations which define cumulant semigroups. Let E be a Lusin topological space. Suppose that ξ is a Borel right process in E with transition semigroup $(P_t)_{t\geq 0}$ and $\{K(t) : t \geq 0\}$ is a continuous admissible additive functional of ξ.

Lemma 2.17 *Suppose that $b \in B(E)$ and $\gamma(x, \mathrm{d}y)$ is a bounded kernel on E. Then for each $f \in B(E)$ there is a unique locally bounded solution $(t, x) \mapsto \pi_t f(x)$ to the linear evolution equation*

$$\begin{aligned}\pi_t f(x) = \mathbf{P}_x f(\xi_t) + \mathbf{P}_x\bigg\{\int_0^t \gamma(\xi_s, \pi_{t-s} f) K(\mathrm{d}s)\bigg\}\\ - \mathbf{P}_x\bigg\{\int_0^t b(\xi_s)\pi_{t-s} f(\xi_s) K(\mathrm{d}s)\bigg\},\end{aligned} \tag{2.17}$$

which defines a locally bounded semigroup $(\pi_t)_{t\geq 0}$ of kernels on E.

Proof. Let $b^+ = 0 \vee b$ and $b^- = 0 \vee (-b)$. By Proposition A.41 there is a unique locally bounded solution $(t, x) \mapsto \pi_t f(x)$ to the equation

$$\begin{aligned}\pi_t f(x) = \mathbf{P}_x\big[\mathrm{e}^{-K_t(b^+)} f(\xi_t)\big] + \mathbf{P}_x\bigg\{\int_0^t \mathrm{e}^{-K_s(b^+)}\gamma(\xi_s, \pi_{t-s} f) K(\mathrm{d}s)\bigg\}\\ + \mathbf{P}_x\bigg\{\int_0^t \mathrm{e}^{-K_s(b^+)} b^-(\xi_s)\pi_{t-s} f(\xi_s) K(\mathrm{d}s)\bigg\},\end{aligned}$$

which defines a locally bounded semigroup $(\pi_t)_{t\geq 0}$ of kernels on E. By Proposition 2.9 the above equation is equivalent to (2.17). □

Suppose that $\eta(x, \mathrm{d}y)$ is a bounded kernel on E and $\nu(1)H(x, \mathrm{d}\nu)$ is a bounded kernel from E to $M(E)^\circ$. We consider a function $b \in B(E)$ and an operator $f \mapsto \psi(\cdot, f)$ on $B(E)^+$ with the representation

$$\psi(x, f) = \eta(x, f) + \int_{M(E)^\circ} \big(1 - \mathrm{e}^{-\nu(f)}\big) H(x, \mathrm{d}\nu). \tag{2.18}$$

From $\eta(x, \mathrm{d}y)$ and $H(x, \mathrm{d}\nu)$ we can define the bounded kernel $\gamma(x, \mathrm{d}y)$ on E by

$$\gamma(x, \mathrm{d}y) = \eta(x, \mathrm{d}y) + \int_{M(E)^\circ} \nu(\mathrm{d}y) H(x, \mathrm{d}\nu). \tag{2.19}$$

Let $\beta \geq 0$ be a constant so that $b(x) \leq \beta$ for all $x \in E$. For fixed $f \in B(E)^+$ set $u_0(t, x) = 0$ and define $u_n(t, x) = u_n(t, x, f)$ inductively by

$$u_{n+1}(t, x) = \mathbf{P}_x[\mathrm{e}^{-K_t(\beta)} f(\xi_t)] + \mathbf{P}_x\bigg\{\int_0^t \mathrm{e}^{-K_s(\beta)}\psi(\xi_s, u_n(t-s)) K(\mathrm{d}s)\bigg\}$$

$$+\mathbf{P}_x\bigg\{\int_0^t \mathrm{e}^{-K_s(\beta)}[\beta - b(\xi_s)]u_n(t-s,\xi_s)K(\mathrm{d}s)\bigg\}. \tag{2.20}$$

Proposition 2.18 *For every $f \in B(E)^+$ there is a unique locally bounded positive solution $(t,x) \mapsto u_t(x,f)$ to the evolution equation*

$$u_t(x) = \mathbf{P}_x[f(\xi_t)] + \mathbf{P}_x\bigg\{\int_0^t [\psi(\xi_s, u_{t-s}) - b(\xi_s)u_{t-s}(\xi_s)]K(\mathrm{d}s)\bigg\}. \tag{2.21}$$

Moreover, we have $\pi_t f(x) \geq u_t(x,f) =\uparrow \lim_{n\to\infty} u_n(t,x,f)$ for all $t \geq 0$ and $x \in E$, where $(\pi_t)_{t\geq 0}$ is the semigroup defined by (2.17).

Proof. The operator $f \mapsto \psi(\cdot, f) - bf$ clearly satisfies Condition 2.11 with $L_a = \|b\| + \|\gamma(\cdot,1)\|$ for all $a \geq 0$. By Proposition 2.15 there is at most one locally bounded positive solution to (2.21). We next claim that

$$0 \leq u_{n-1}(t,x,f) \leq u_n(t,x,f) \leq \pi_t f(x), \qquad t \geq 0, x \in E \tag{2.22}$$

for every $n \geq 1$. By Proposition 2.9 we can also define $(t,x) \mapsto \pi_t f(x)$ by the evolution equation

$$\begin{aligned}\pi_t f(x) = \mathbf{P}_x[\mathrm{e}^{-K_t(\beta)}f(\xi_t)] &+ \mathbf{P}_x\bigg\{\int_0^t \mathrm{e}^{-K_s(\beta)}\gamma(\xi_s, \pi_{t-s}f)K(\mathrm{d}s)\bigg\}\\ &+ \mathbf{P}_x\bigg\{\int_0^t \mathrm{e}^{-K_s(\beta)}[\beta - b(\xi_s)]\pi_{t-s}f(\xi_s)K(\mathrm{d}s)\bigg\}.\end{aligned} \tag{2.23}$$

Then for $n = 1$ the inequalities in (2.22) are trivial. Suppose they are true for some $n \geq 1$. By the monotonicity of the operator $f \mapsto \psi(\cdot, f) + (\beta - b)f$ we have

$$0 \leq u_n(t,x,f) \leq u_{n+1}(t,x,f) \leq v(t,x,f),$$

where

$$\begin{aligned}v(t,x,f) = \mathbf{P}_x[\mathrm{e}^{-K_t(\beta)}f(\xi_t)] &+ \mathbf{P}_x\bigg\{\int_0^t \mathrm{e}^{-K_s(\beta)}\psi(\xi_s, \pi_{t-s}f)K(\mathrm{d}s)\bigg\}\\ &+ \mathbf{P}_x\bigg\{\int_0^t \mathrm{e}^{-K_s(\beta)}[\beta - b(\xi_s)]\pi_{t-s}f(\xi_s)K(\mathrm{d}s)\bigg\}.\end{aligned} \tag{2.24}$$

In view of (2.23) and (2.24) we have $v(t,x,f) \leq \pi_t f(x)$. Then (2.22) holds for all $n \geq 1$. Let $u_t(x,f) =\uparrow \lim_{n\to\infty} u_n(t,x,f)$. From (2.20) we see that $(t,x) \mapsto u_t(x,f)$ is a locally bounded positive solution of

$$\begin{aligned}u_t(x) = \mathbf{P}_x[\mathrm{e}^{-K_t(\beta)}f(\xi_t)] &+ \mathbf{P}_x\bigg\{\int_0^t \mathrm{e}^{-K_s(\beta)}\psi(\xi_s, u_{t-s})K(\mathrm{d}s)\bigg\}\\ &+ \mathbf{P}_x\bigg\{\int_0^t \mathrm{e}^{-K_s(\beta)}[\beta - b(\xi_s)]u_{t-s}(\xi_s)K(\mathrm{d}s)\bigg\},\end{aligned}$$

which is equivalent to (2.21) by Proposition 2.9. □

Proposition 2.19 *In the case where* $K(\mathrm{d}s) = \mathrm{d}s$ *is the Lebesgue measure, we have* $u_t(x, f) = \uparrow\lim_{n\to\infty} u_n(t, x, f)$ *uniformly on* $[0, T] \times E \times B_a(E)^+$ *for every* $T \geq 0$ *and* $a \geq 0$.

Proof. Let $D_n(t) = \sup_{0\leq s\leq t} \|u_n(s) - u_{n-1}(s)\|$. From (2.20) it is easy to get

$$\begin{aligned}
D_n(t) &\leq (\beta + \|b\| + \|\gamma(\cdot, 1)\|) \int_0^t D_{n-1}(s_1)\mathrm{d}s_1 \\
&\leq (\beta + \|b\| + \|\gamma(\cdot, 1)\|)^2 \int_0^t \mathrm{d}s_1 \int_0^{s_1} D_{n-2}(s_2)ds_2 \\
&\leq \cdots \\
&\leq (\beta + \|b\| + \|\gamma(\cdot, 1)\|)^{n-1} \int_0^t \mathrm{d}s_1 \int_0^{s_1} \cdots \int_0^{s_{n-2}} \|f\|\mathrm{d}s_{n-1} \\
&\leq \frac{1}{(n-1)!}(\beta + \|b\| + \|\gamma(\cdot, 1)\|)^{n-1}t^{n-1}\|f\|,
\end{aligned}$$

and hence

$$D(t) := \sum_{n=1}^{\infty} D_n(t) \leq \|f\| \exp\{(\beta + \|b\| + \|\gamma(\cdot, 1)\|)t\} < \infty.$$

Then $\lim_{n\to\infty} u_n(t, x, f) = u_t(x, f)$ uniformly on $[0, T] \times E \times B_a(E)^+$. □

Now we consider a more general operator $f \mapsto \phi(\cdot, f)$ as follows. Let $b \in B(E)$ and $c \in B(E)^+$. Let $\eta(x, \mathrm{d}y)$ be a bounded kernel on E and $H(x, \mathrm{d}\nu)$ a σ-finite kernel from E to $M(E)^\circ$. Suppose that

$$\sup_{x\in E} \int_{M(E)^\circ} \Big[\nu(1) \wedge \nu(1)^2 + \nu_x(1)\Big] H(x, \mathrm{d}\nu) < \infty, \tag{2.25}$$

where $\nu_x(\mathrm{d}y)$ denotes the restriction of $\nu(\mathrm{d}y)$ to $E \setminus \{x\}$. For $x \in E$ and $f \in B(E)^+$ write

$$\begin{aligned}
\phi(x, f) = b(x)f(x) + c(x)f(x)^2 - \int_E f(y)\eta(x, \mathrm{d}y) \\
+ \int_{M(E)^\circ} \big[\mathrm{e}^{-\nu(f)} - 1 + \nu(\{x\})f(x)\big] H(x, \mathrm{d}\nu).
\end{aligned} \tag{2.26}$$

By Taylor's expansion it is easy to see that

$$\mathrm{e}^{-\nu(f)} - 1 + \nu(\{x\})f(x) = -\nu_x(f) + \frac{1}{2}\mathrm{e}^{-\theta}\nu(f)^2,$$

where $0 \leq \theta \leq \nu(f)$. Observe also that

$$\big|\mathrm{e}^{-\nu(f)} - 1 + \nu(\{x\})f(x)\big| \leq \nu(f) + \nu(\{x\})f(x).$$

Then the second integral on the right-hand side of (2.26) is bounded on $E\times B_a(E)^+$ for every $a\ge 0$. Moreover, we can rewrite (2.26) into

$$\begin{aligned}\phi(x,f) = b(x)f(x) + c(x)f(x)^2 - \int_E f(y)\gamma(x,\mathrm{d}y) \\ + \int_{M(E)^\circ}\big[\mathrm{e}^{-\nu(f)} - 1 + \nu(f)\big]H(x,\mathrm{d}\nu),\end{aligned}\tag{2.27}$$

where

$$\gamma(x,\mathrm{d}y) = \eta(x,\mathrm{d}y) + \int_{M(E)^\circ}\nu_x(\mathrm{d}y)H(x,\mathrm{d}\nu).\tag{2.28}$$

For each integer $n\ge 1$ define

$$\begin{aligned}\phi_n(x,f) = b(x)f(x) + 2nc(x)f(x) + \int_{M(E)^\circ}\nu(f)h_n(\nu)H(x,\mathrm{d}\nu) \\ - \int_E f(y)\gamma(x,\mathrm{d}y) - 2n^2c(x)(1-\mathrm{e}^{-f(x)/n}) \\ - \int_{M(E)^\circ}\big(1-\mathrm{e}^{-\nu(f)}\big)h_n(\nu)H(x,\mathrm{d}\nu),\end{aligned}\tag{2.29}$$

where $h_n(\nu) = 1\wedge[n\nu(1)]$. It is easy to see $\phi_n(x,f)\to\phi(x,f)$ increasingly as $n\to\infty$. For $n\ge 1$ and $f\in B(E)^+$ we consider the equation

$$v(t,x) = \mathbf{P}_x[f(\xi_t)] - \mathbf{P}_x\bigg\{\int_0^t \phi_n(\xi_s, v(t-s))K(\mathrm{d}s)\bigg\}.\tag{2.30}$$

This is clearly a special case of (2.21). By Proposition 2.18 there is a unique locally bounded positive solution $(t,x)\mapsto v_n(t,x,f)$ to (2.30).

Proposition 2.20 *Suppose that ϕ and γ are defined respectively by (2.27) and (2.28). Let $(\pi_t)_{t\ge 0}$ be defined by (2.17). Then for every $f\in B(E)^+$ there is a unique locally bounded positive solution $(t,x)\mapsto v_t(x,f)$ to*

$$v_t(x) = \mathbf{P}_x f(\xi_t) - \mathbf{P}_x\bigg[\int_0^t \phi(\xi_s, v_{t-s})K(\mathrm{d}s)\bigg],\quad t\ge 0, x\in E.\tag{2.31}$$

Moreover, we have $\pi_t f(x)\ge v_t(x,f) = \downarrow\lim_{n\to\infty} v_n(t,x,f)$ for $t\ge 0$ and $x\in E$.

Proof. Since $\phi_n(x,f)$ is increasing in $n\ge 1$ and $f\in B(E)^+$, by Proposition 2.18 we see $v_n(t,x,f)$ is decreasing in $n\ge 1$. Let $v_t(x,f) = \lim_{n\to\infty} v_n(t,x,f)\le \pi_t(x,f)$. In view of (2.29) and (2.30), we conclude by dominated convergence that $(t,x)\mapsto v_t(x,f)$ is a locally bounded positive solution of (2.31). For $a\ge 0$ and $f,g\in B_a(E)^+$ we can use (2.27) to see

$$|\phi(x,f)-\phi(x,g)| \le (\|b\| + 2a\|c\|)\|f-g\| + \gamma(x,1)\|f-g\|$$

$$+\int_{M(E)^\circ} |\nu(f-g) + \mathrm{e}^{-\nu(f)} - \mathrm{e}^{-\nu(g)}| H(x, \mathrm{d}\nu).$$

By the mean-value theorem we have

$$\nu(f-g) + \mathrm{e}^{-\nu(f)} - \mathrm{e}^{-\nu(g)} = \nu(f-g)(1 - \mathrm{e}^{-\theta}),$$

where $\nu(f \wedge g) \leq \theta \leq \nu(f \vee g) \leq a\nu(1)$. It follows that

$$|\nu(f-g) + \mathrm{e}^{-\nu(f)} - \mathrm{e}^{-\nu(g)}| \leq \|f-g\|(\nu(1) \wedge a\nu(1)^2).$$

Then $f \mapsto \phi(\cdot, f)$ satisfies Condition 2.11 for some constant $L_a \geq 0$ and the uniqueness of the solution of (2.31) follows by Proposition 2.15. □

Theorem 2.21 *Let ϕ be given by (2.26) or (2.27). For every $f \in B(E)^+$ let $(t, x) \mapsto V_t f(x)$ denote the unique locally bounded positive solution of (2.31). Then the operators $(V_t)_{t\geq 0}$ constitute a cumulant semigroup.*

Proof. By (2.20) and Theorem 1.37 one checks inductively $u_n(t, x, \cdot) \in \mathscr{I}(E)$ for each $n \geq 1$. Now Corollary 1.34 and Propositions 2.18 and 2.20 imply first $u_t(x, \cdot) \in \mathscr{I}(E)$ for the solution of (2.21), and then $v_t(x, \cdot) \in \mathscr{I}(E)$ for the solution of (2.31). The semigroup property of $(V_t)_{t\geq 0}$ follows from Corollary 2.13. □

Let ϕ be given by (2.26) or (2.27) and let $(V_t)_{t\geq 0}$ be the cumulant semigroup defined by (2.31). Then we can define a Markov transition semigroup $(Q_t)_{t\geq 0}$ on $M(E)$ by

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\mu, \mathrm{d}\nu) = \exp\{-\mu(V_t f)\}, \qquad f \in B(E)^+. \tag{2.32}$$

If X is a Markov process in $M(E)$ with transition semigroup $(Q_t)_{t\geq 0}$, we call it a *Dawson–Watanabe superprocess* with parameters (ξ, K, ϕ), or simply a (ξ, K, ϕ)-*superprocess*, where ξ is the *spatial motion*, K is the *killing functional* or *killing density*, and ϕ is the *branching mechanism*. If $K(\mathrm{d}s) = \mathrm{d}s$ is the Lebesgue measure, we call X a (ξ, ϕ)-*superprocess*. In this case, we can rewrite (2.31) into

$$v_t(x) = P_t f(x) - \int_0^t \mathrm{d}s \int_E \phi(y, v_s) P_{t-s}(x, \mathrm{d}y), \quad x \in E, t \geq 0. \tag{2.33}$$

We say the branching mechanism is *spatially constant* if $f \mapsto \phi(\cdot, f)$ maps constant functions to constant functions. In Chapter 4 we shall give some intuitive interpretations of the superprocesses in terms of limit theorems of branching particle systems.

Theorem 2.22 *A realization $\{X_t : t \geq 0\}$ of the (ξ, K, ϕ)-superprocess is right continuous in probability.*

Proof. Let $f \in C(E)^+$. Since ξ is right continuous, the map $t \mapsto P_t f(x)$ is right continuous for every $x \in E$, so (2.31) implies $\lim_{t\to 0} V_t f(x) = f(x)$. From (2.32) we get

$$\lim_{t\to 0}\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\mu, \mathrm{d}\nu) = \exp\{-\mu(f)\}.$$

Then we have $\lim_{t\to 0} Q_t(\mu,\cdot) = \delta_\mu$ weakly. For any $\varepsilon > 0$ let $B(\mu,\varepsilon)^c = \{\nu \in M(E) : \rho(\nu,\mu) > \varepsilon\}$, where ρ is the metric on $M(E)$ defined by (1.3). Then we infer $\lim_{t\to 0} Q_t(\mu, B(\mu,\varepsilon)^c) = 0$. Using the Markov property of X and dominated convergence we get

$$\lim_{t\to r+} \mathbf{P}\{\rho(X_t, X_r) > \varepsilon\} = \lim_{t\to r+} \mathbf{P}\{Q_{t-r}(X_r, B(X_r,\varepsilon)^c)\} = 0$$

for every $r \geq 0$. Therefore $t \mapsto X_t$ is right continuous in probability. □

For the (ξ,ϕ)-superprocess we can give an alternate characterization of the cumulant semigroup. Given a function $b \in B(E)$, we define a locally bounded semigroup of Borel kernels $(P_t^b)_{t\geq 0}$ on E by the following *Feynman–Kac formula*:

$$P_t^b f(x) = \mathbf{P}_x\Big[\mathrm{e}^{-\int_0^t b(\xi_s)\mathrm{d}s} f(\xi_t)\Big], \qquad x \in E, f \in B(E). \tag{2.34}$$

Then (2.17) can be rewritten into

$$\pi_t f(x) = P_t f(x) + \int_0^t P_{t-s}(\gamma - b)\pi_s f(x)\mathrm{d}s, \quad t \geq 0, x \in E, \tag{2.35}$$

which is equivalent to

$$\pi_t f(x) = P_t^b f(x) + \int_0^t P_{t-s}^b \gamma \pi_s f(x)\mathrm{d}s, \qquad t \geq 0, x \in E. \tag{2.36}$$

From Proposition A.41 we have

$$\pi_t f(x) = P_t^b f(x) + \sum_{n=1}^\infty \int_0^t \mathrm{d}s_1 \cdots \int_0^{s_{n-1}} P_{t-s_1}^b \gamma P_{s_1-s_2}^b \cdots \gamma P_{s_n}^b f(x)\mathrm{d}s_n. \tag{2.37}$$

Let $b^+ = 0 \vee b$ and $b^- = 0 \vee (-b)$. By Proposition A.49 we have $\|\pi_t\| \leq \mathrm{e}^{c_0 t}$ for all $t \geq 0$, where $c_0 = \|b^-\| + \|\gamma(\cdot,1)\|$.

Theorem 2.23 *Suppose that ϕ and γ are defined respectively by* (2.27) *and* (2.28)*. Let $(\pi_t)_{t\geq 0}$ be defined by* (2.35)*. Then* (2.33) *is equivalent to the evolution equation*

$$v_t(x) = \pi_t f(x) - \int_0^t \mathrm{d}s \int_E \phi_0(y, v_s)\pi_{t-s}(x,\mathrm{d}y), \tag{2.38}$$

where

$$\phi_0(y,f) = c(y)f(y)^2 + \int_{M(E)^\circ} \big[\mathrm{e}^{-\nu(f)} - 1 + \nu(f)\big] H(y,\mathrm{d}\nu). \tag{2.39}$$

Proof. We first show (2.33) implies (2.38). By applying Proposition 2.9 to (2.33) we have

$$v_t(x) = P_t^b f(x) - \int_0^t P_{t-s}^b\big[\phi(v_s) - bv_s\big](x)\mathrm{d}s.$$

This combined with (2.36) implies

$$v_t(x) = \pi_t f(x) - \int_0^t P_{t-s}^b \phi_0(v_s)(x)\mathrm{d}s + \int_0^t P_{t-s}^b \gamma(v_s - \pi_s f)(x)\mathrm{d}s.$$

Then we use the above relation inductively to see

$$\begin{aligned} v_t(x) = \pi_t f(x) &- \int_0^t P_{t-s_1}^b \phi_0(v_{s_1})(x)\mathrm{d}s_1 + w_n(t,x) \\ &- \sum_{i=2}^n \int_0^t \mathrm{d}s_1 \cdots \int_0^{s_{i-1}} P_{t-s_1}^b \gamma P_{s_1-s_2}^b \\ &\qquad\qquad \cdots \gamma P_{s_{i-1}-s_i}^b g_{s_i}(x)\mathrm{d}s_i, \end{aligned} \tag{2.40}$$

where $g_{s_i}(x) = \phi_0(x, v_{s_i})$ and

$$w_n(t,x) = \int_0^t \mathrm{d}s_1 \cdots \int_0^{s_n} P_{t-s_1}^b \gamma \cdots P_{s_n-s_{n+1}}^b \gamma(v_{s_{n+1}} - \pi_{s_{n+1}} f)(x)\mathrm{d}s_{n+1}.$$

Since $0 \le v_{s_{n+1}}(x) \le \pi_{s_{n+1}} f(x) \le \|f\|\mathrm{e}^{c_0 s_{n+1}}$, we have

$$\begin{aligned} \|w_n(t,\cdot)\| &\le \|f\|\|\gamma(\cdot,1)\|^{n+1}\mathrm{e}^{c_0 t} \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \cdots \int_0^{s_n} \mathrm{d}s_{n+1} \\ &\le \|f\|\|\gamma(\cdot,1)\|^{n+1}\mathrm{e}^{c_0 t} \frac{t^{n+1}}{(n+1)!}. \end{aligned}$$

Then letting $n \to \infty$ in (2.40) and using (2.37) we obtain (2.38). The uniqueness of the solution to (2.38) follows from Gronwall's inequality by standard arguments. Then the two equations are equivalent. □

2.4 Examples of Superprocesses

The (ξ, K, ϕ)- and (ξ, ϕ)-superprocesses we have constructed are quite wide. From these one can derive the existence of various special classes of superprocesses. Some special cases of the parameters are discussed in the following examples.

Example 2.2 Let $|\cdot|$ and $(\cdot,\cdot)$ denote respectively the Euclidean norm and inner product of $\mathbb{R}^d$. For each $1 \le i \le d$ suppose that $\lambda \mapsto \phi_i(\lambda)$ is a function on $\mathbb{R}_+^d$ with the representation

$$\phi_i(\lambda) = b_i\lambda_i + c_i\lambda_i^2 + (\eta_i, \lambda) + \int_{\mathbb{R}_+^d\setminus\{0\}} \big(\mathrm{e}^{-(\lambda,u)} - 1 + \lambda_i u_i\big) H_i(\mathrm{d}u),$$

where $c_i \geq 0$ and b_i are constants, $\eta_i \in \mathbb{R}_+^d$ is a vector, and $H_i(\mathrm{d}u)$ is a σ-finite measure on $\mathbb{R}_+^d \setminus \{0\}$ so that

$$\int_{\mathbb{R}_+^d\setminus\{0\}} \Big(|u| \wedge |u|^2 + \sum_{j\neq i} u_j\Big) H_i(\mathrm{d}u) < \infty.$$

By Proposition 2.20 and Theorem 2.21 for any $\lambda \in \mathbb{R}_+^d$ there is a unique locally bounded vector-valued solution $t \mapsto v(t,\lambda) \in \mathbb{R}_+^d$ to the evolution equation system

$$v_i(t,\lambda) = \lambda_i - \int_0^t \phi_i(v(s,\lambda))\mathrm{d}s, \qquad t \geq 0, i = 1,\dots,d, \tag{2.41}$$

and there is a transition semigroup $(Q_t)_{t\geq 0}$ on $\mathbb{R}_+^d$ defined by

$$\int_{\mathbb{R}_+^d} \mathrm{e}^{-(\lambda,y)} Q_t(x,\mathrm{d}y) = \mathrm{e}^{-(x,v(t,\lambda))}, \qquad \lambda, x \in \mathbb{R}_+^d. \tag{2.42}$$

From (2.41) we see that $t \mapsto v_i(t,\lambda)$ is continuously differentiable. Then we can rewrite the equation into the equivalent differential form

$$\frac{\mathrm{d}v_i}{\mathrm{d}t}(t,\lambda) = -\phi_i(v(t,\lambda)), \quad v_i(0,\lambda) = \lambda_i, \qquad i = 1,\dots,d.$$

A Markov process in $\mathbb{R}_+^d$ with transition semigroup $(Q_t)_{t\geq 0}$ given by (2.42) is called a *continuous-state branching process* (CB-process).

Example 2.3 By a *super-Brownian motion* we mean a superprocess with Brownian motion as underlying spatial motion. A particular super-Brownian motion is described as follows. Let ξ be a standard Brownian motion in $\mathbb{R}$. It is well-known that ξ has a continuous local time $\{2l(t,y) : t \geq 0, y \in \mathbb{R}\}$, that is,

$$\int_0^t 1_B(\xi_s)\mathrm{d}s = \int_B 2l(t,y)\mathrm{d}y, \qquad t \geq 0, B \in \mathscr{B}(\mathbb{R}); \tag{2.43}$$

see, e.g., Ikeda and Watanabe (1989, p.113). Let $\rho \in M(\mathbb{R})$ and define the continuous additive functional $t \mapsto K(t)$ by

$$K(t) = \int_{\mathbb{R}} 2l(t,y)\rho(\mathrm{d}y), \qquad t \geq 0.$$

Then we have

$$\mathbf{P}_x[K(t)] = \int_{\mathbb{R}} \rho(\mathrm{d}y) \int_0^t g_s(y-x)\mathrm{d}s \leq \frac{\sqrt{2t}}{\sqrt{\pi}}\rho(\mathbb{R}),$$

where

$$g_t(z) = \frac{1}{\sqrt{2\pi t}} \exp\{-z^2/2t\}, \qquad t > 0, z \in \mathbb{R}. \tag{2.44}$$

Thus $t \mapsto K(t)$ is admissible. In this case, we can rewrite (2.31) as

$$v_t(x) = P_t f(x) - \int_0^t \mathrm{d}s \int_{\mathbb{R}} \phi(y, v_{t-s}) g_s(y - x) \rho(\mathrm{d}y).$$

The corresponding (ξ, K, ϕ)-superprocess is called a *catalytic super-Brownian motion* with *catalyst measure* $\rho(\mathrm{d}y)$.

Example 2.4 Let $b \in B(E)$ and $c \in B(E)^+$. Let $(u \wedge u^2)m(x, \mathrm{d}u)$ be a bounded kernel from E to $(0, \infty)$. We define a Borel function $(x, z) \mapsto \phi(x, z)$ on $E \times [0, \infty)$ by

$$\phi(x, z) = b(x)z + c(x)z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu) m(x, \mathrm{d}u). \tag{2.45}$$

Then $(x, f) \mapsto \phi(x, f(x))$ can be represented in the form (2.26) or (2.27). In this case, we say the corresponding superprocess has a *local branching mechanism*. If there is $c \in B(E)^+$ so that $\phi(x, z) = c(x)z^2$ for all $x \in E$ and $z \geq 0$, we say the superprocess has a *binary local branching mechanism*.

Example 2.5 Let $(x, f) \mapsto \psi(x, f)$ be given by (2.18) and let $(x, z) \mapsto \phi(x, z)$ be given by (2.45). Then the operator $f \mapsto \phi(\cdot, f(\cdot)) - \psi(\cdot, f)$ can be represented in the form (2.26) or (2.27), so it defines a branching mechanism. A branching mechanism of this type is said to be *decomposable* with *local part* ϕ and *non-local part* ψ. A superprocess with such a branching mechanism is referred to as a (ξ, K, ϕ, ψ)*-superprocess*. In the special case of Lebesgue killing density, we call it a (ξ, ϕ, ψ)*-superprocess*. Of course, the expression $\phi(\cdot, f(\cdot)) - \psi(\cdot, f)$ of a decomposable branching mechanism is not unique.

Example 2.6 Let $\pi(x, \mathrm{d}y)$ be a probability kernel on E. Suppose that $\beta \in B(E)^+$ and $un(x, \mathrm{d}u)$ is a bounded kernel from E to $(0, \infty)$. Given the function

$$\zeta(x, z) = \beta(x)z + \int_0^\infty (1 - \mathrm{e}^{-zu}) n(x, \mathrm{d}u), \qquad x \in E, z \geq 0, \tag{2.46}$$

we can define a non-local branching mechanism by

$$\psi(x, f) = \zeta(x, \pi(x, f)), \qquad x \in E, f \in B(E)^+. \tag{2.47}$$

If $\zeta(x, y, z)$ is given by (2.46) with $x \in E$ replaced by $(x, y) \in E^2$, we can define another special non-local branching mechanism by

$$\psi(x, f) = \int_E \zeta(x, y, f(y)) \pi(x, \mathrm{d}y), \qquad x \in E, f \in B(E)^+. \tag{2.48}$$

Example 2.7 Let $1 < \alpha < 2$ be a constant and let π_0 be a diffuse probability measure on E. We can define a branching mechanism on E by

$$\phi(x,f) = \int_0^1 \big[\exp\{-uf(x) - u^2\pi_0(f)\} - 1 + uf(x)\big]\frac{\mathrm{d}u}{u^{1+\alpha}}.$$

In fact, it is easy to see

$$\phi(x,f) = \int_{M(E)^\circ} \big[\mathrm{e}^{-\nu(f)} - 1 + \nu(\{x\})f(x)\big]H(x,\mathrm{d}\nu),$$

where $H(x,\mathrm{d}\nu)$ is the image of $u^{-1-\alpha}\mathrm{d}u$ under the mapping $u \mapsto u\delta_x + u^2\pi_0$ of $(0,1]$ into $M(E)^\circ$. This branching mechanism cannot be decomposed into local and non-local parts.

2.5 Some Moment Formulas

In this section, we prove some moment formulas for Dawson–Watanabe superprocesses. Suppose that E is a Lusin topological space. Let ξ be a Borel right process in E with transition semigroup $(P_t)_{t\geq 0}$ and resolvent $(U^\alpha)_{\alpha>0}$. Let $t \mapsto K(t)$ be a continuous admissible additive functional of ξ and let ϕ be a branching mechanism given by (2.26) or (2.27). Recall that $c_0 = \|b^-\| + \|\gamma(\cdot,1)\|$.

Proposition 2.24 *Let $(V_t)_{t\geq 0}$ denote the cumulant semigroup of the (ξ,K,ϕ)-superprocess represented by (2.5). Then for $t \geq 0$, $x \in E$ and $f \in B(E)$ we have*

$$\pi_t f(x) = \lambda_t(x,f) + \int_{M(E)^\circ} \nu(f)L_t(x,\mathrm{d}\nu), \tag{2.49}$$

where $(\pi_t)_{t\geq 0}$ is defined by (2.17).

Proof. For any $n \geq 1$ and $f \in B(E)^+$ we have $nv_t(x,f/n) \leq \pi_t f(x)$ by Proposition 2.20. From (2.5) we see $nv_t(x,f/n)$ is increasing in $n \geq 1$. Then we use (2.27) and (2.31) to see $\pi_t f(x) = \lim_{n\to\infty} nv_t(x,f/n)$ is the unique solution of (2.17). By (2.5) we have

$$nv_t(x,f/n) = \lambda_t(x,f) + \int_{M(E)^\circ} n\big(1 - \mathrm{e}^{-\nu(f/n)}\big)L_t(x,\mathrm{d}\nu).$$

Then (2.49) follows by monotone convergence. The equality for $f \in B(E)$ follows by linearity. □

Corollary 2.25 *If $(V_t)_{t\geq 0}$ is the cumulant semigroup of the (ξ,K,ϕ)-superprocess represented by (2.5), then for any $f,g \in B(E)^+$ we have*

$$|V_tf(x) - V_tg(x)| \le \pi_t(x, |f-g|), \qquad t \ge 0, x \in E, \tag{2.50}$$

where $(\pi_t)_{t\ge 0}$ *is defined by* (2.17).

Proof. By the canonical representation (2.5) we have

$$\begin{aligned}|V_tf(x) - V_tg(x)| &\le \lambda_t(x, |f-g|) + \int_{M(E)^\circ} |\mathrm{e}^{-\nu(f)} - \mathrm{e}^{-\nu(f)}| L_t(x, \mathrm{d}\nu) \\ &\le \lambda_t(x, |f-g|) + \int_{M(E)^\circ} \nu(|f-g|) L_t(x, \mathrm{d}\nu).\end{aligned}$$

Then (2.50) follows from (2.49). □

Corollary 2.26 *Let* $(V_t)_{t\ge 0}$ *be the cumulant semigroup of the* (ξ, ϕ)*-superprocess represented canonically by* (2.5). *Then* (2.49) *holds with* $(\pi_t)_{t\ge 0}$ *defined by* (2.35). *In particular, if* ϕ *is the local branching mechanism given by* (2.45)*, the equality holds with* $\pi_t = P_t^b$ *for all* $t \ge 0$.

Proposition 2.27 *Let* $(Q_t)_{t\ge 0}$ *denote the transition semigroup of the* (ξ, K, ϕ)*-superprocess. Then for* $t \ge 0$*,* $\mu \in M(E)$ *and* $f \in B(E)$ *we have*

$$\int_{M(E)} \nu(f) Q_t(\mu, \mathrm{d}\nu) = \mu(\pi_t f), \tag{2.51}$$

where $(\pi_t)_{t\ge 0}$ *is defined by* (2.17).

Proof. This follows by differentiating both sides of (2.32) and applying Proposition 2.24. □

Corollary 2.28 *Let* $(Q_t)_{t\ge 0}$ *be the transition semigroup of the* (ξ, ϕ)*-superprocess. Then* (2.51) *holds with* $(\pi_t)_{t\ge 0}$ *defined by* (2.35). *In particular, if* ϕ *is the local branching mechanism given by* (2.45)*, the equality holds with* $\pi_t = P_t^b$ *for all* $t \ge 0$.

If $b(x) \ge \gamma(x, 1)$ for all $x \in E$, then (2.17) defines a Borel right transition semigroup $(\pi_t)_{t\ge 0}$ by Theorem A.43. In this case, we say the (ξ, K, ϕ)-superprocess is *subcritical*. In particular, if $(P_t)_{t\ge 0}$ is conservative and $b(x) = \gamma(x, 1)$ for all $x \in E$, then $(\pi_t)_{t\ge 0}$ is a conservative transition semigroup and we say the superprocess is *critical*. If $(P_t)_{t\ge 0}$ is conservative and $b(x) \le \gamma(x, 1)$ for all $x \in E$, then $\pi_t 1(x) \ge 1$ for all $t \ge 0$ and $x \in E$ and we say the (ξ, K, ϕ)-superprocess is *supercritical*. The meanings of the notions are made clear by Proposition 2.27.

Proposition 2.29 *Let* $(Q_t)_{t\ge 0}$ *denote the transition semigroup of the* (ξ, K, ϕ)*-superprocess. Then for* $t \ge 0$*,* $\mu \in M(E)$ *and* $(f, g) \in B(E)^+ \times B(E)$ *we have*

$$\int_{M(E)} \nu(g) \mathrm{e}^{-\nu(f)} Q_t(\mu, \mathrm{d}\nu) = \exp\{-\mu(V_tf)\} \mu(V_t^g f), \tag{2.52}$$

where $(t, x) \mapsto V_t^g f(x)$ *is the unique locally bounded solution of*

$$V_t^g f(x) = \mathbf{P}_x g(\xi_t) - \mathbf{P}_x\bigg[\int_0^t \psi(\xi_s, V_{t-s}f, V_{t-s}^g f)K(\mathrm{d}s)\bigg] \tag{2.53}$$

and $(f,g) \mapsto \psi(\cdot, f, g)$ *is the operator from* $B(E)^+ \times B(E)$ *to* $B(E)$ *defined by*

$$\begin{aligned}\psi(x,f,g) = b(x)g(x) + 2c(x)f(x)g(x) - \int_E g(y)\gamma(x,\mathrm{d}y) \\ + \int_{M(E)^\circ} \nu(g)\big(1-\mathrm{e}^{-\nu(f)}\big)H(x,\mathrm{d}\nu).\end{aligned} \tag{2.54}$$

Proof. By Proposition 2.27 the left-hand side of (2.52) is finite. For $(f,g) \in B(E)^+ \times B(E)^+$ let $V_t^g f(x) = (\mathrm{d}/\mathrm{d}\theta)V_t(f+\theta g)(x)|_{\theta=0+}$. Then we get (2.52) and (2.53) by differentiating both sides of (2.32) and (2.31). For $(f,g) \in B(E)^+ \times B(E)$ the result follows by linearity. For any $r \geq 0$ it is not hard to show that (2.53) holds for all $t \geq 0$ if and only if it holds for $0 \leq t \leq r$ and

$$V_{r+t}^g f(x) = \mathbf{P}_x V_r^g f(\xi_t) - \mathbf{P}_x\bigg[\int_0^t \psi(\xi_s, V_{r+t-s}f, V_{r+t-s}^g f)K(\mathrm{d}s)\bigg]$$

holds for all $t \geq 0$. Based on this fact, the uniqueness of the solution to (2.53) follows by arguments similar to those in the proofs of Propositions 2.15 and 2.20. □

Corollary 2.30 *Let* $(f,g) \in B(E)^+ \times B(E)$ *and let* $(t,x) \mapsto V_t^g f(x)$ *be defined by* (2.53). *Then we have* $V_{r+t}^g f(x) = V_r^{V_t^g f} V_t f(x)$ *for all* $r,t \geq 0$ *and* $x \in E$.

Proof. For any $(f,g) \in B(E)^+ \times B(E)$ we can use Proposition 2.29 and the semigroup property of $(Q_t)_{t\geq 0}$ to see

$$\begin{aligned}&\int_{M(E)} \nu(g)\mathrm{e}^{-\nu(f)}Q_{r+t}(\mu,\mathrm{d}\nu) \\ &\quad= \int_{M(E)} Q_r(\mu,\mathrm{d}\eta)\int_{M(E)} \nu(g)\mathrm{e}^{-\nu(f)}Q_t(\eta,\mathrm{d}\nu).\end{aligned}$$

By applying (2.52) to both sides for $\mu = \delta_x$ we obtain the desired equality. □

Proposition 2.31 *Let* $(V_t)_{t\geq 0}$ *denote the cumulant semigroup of the* (ξ, K, ϕ)-*superprocess represented canonically by* (2.5). *Then for* $t \geq 0$, $x \in E$ *and* $(f,g) \in B(E)^+ \times B(E)$ *we have*

$$V_t^g f(x) = \lambda_t(x,g) + \int_{M(E)^\circ} \nu(g)\mathrm{e}^{-\nu(f)}L_t(x,\mathrm{d}\nu), \tag{2.55}$$

where the left-hand side is defined by (2.53).

Proof. Using the notation in the proof of Proposition 2.29, for $(f,g) \in B(E)^+ \times B(E)^+$ we get (2.55) by differentiating both sides of (2.5). Then the result for $g \in B(E)$ follows by linearity. □

A direct proof of the existence of the solution of (2.53) can be given in the special case $K(\mathrm{d}s) = \mathrm{d}s$. Let us rewrite the equation into

$$V_t^g f(x) = P_t g(x) - \int_0^t \mathrm{d}s \int_E \psi(y, V_s f, V_s^g f) P_{t-s}(x, \mathrm{d}y). \tag{2.56}$$

Proposition 2.32 *Let $(t, x) \mapsto V_t^g f(x)$ be defined by (2.56). Let $v_0(t, x) = 0$ and define $v_n(t, x) = v_n(t, x, f, g)$ inductively by*

$$v_{n+1}(x) = P_t g(x) - \int_0^t \mathrm{d}s \int_E \psi(y, V_s f, v_n) P_{t-s}(x, \mathrm{d}y). \tag{2.57}$$

Then for every $T \geq 0$ we have $v_n(x) \to V_t^g f(x)$ uniformly on $[0, T] \times E$.

Proof. Let $D_n(t) = \sup_{0 \leq s \leq t} \|v_n(s) - v_{n-1}(s)\|$. For any $T \geq 0$, since $t \mapsto V_t f$ is locally bounded on $[0, T]$, by (2.54) and (2.57) there is a constant $L \geq 0$ so that

$$D_n(t) \leq L \int_0^t D_{n-1}(s) \mathrm{d}s \leq \cdots \leq \frac{1}{(n-1)!} L^{n-1} t^{n-1} \|g\|, \quad 0 \leq t \leq T.$$

Then the result follows as in the proof of Proposition 2.19. □

Proposition 2.33 *Let $(X_t, \mathscr{G}_t, \mathbf{P})$ be a realization of the (ξ, K, ϕ)-superprocess such that $\mathbf{P}[X_0(1)] < \infty$ and $(\pi_t)_{t \geq 0}$ is the semigroup defined by (2.17). Let $\alpha \geq 0$ and let $f \in B(E)^+$ be α-super-mean-valued for $(\pi_t)_{t \geq 0}$. Then $t \mapsto \mathrm{e}^{-\alpha t} X_t(f)$ is a $(\mathscr{G}_t)$-supermartingale.*

Proof. Since $f \in B(E)^+$ is α-super-mean-valued for $(\pi_t)_{t \geq 0}$, by Proposition 2.27 for any $t \geq r \geq 0$ we have

$$\mathbf{P}\big[\mathrm{e}^{-\alpha t} X_t(f) \big| \mathscr{G}_r\big] = \mathrm{e}^{-\alpha t} X_r(\pi_{t-r} f) \leq \mathrm{e}^{-\alpha r} X_r(f).$$

Therefore $t \mapsto \mathrm{e}^{-\alpha t} X_t(f)$ is a $(\mathscr{G}_t)$-supermartingale. □

Corollary 2.34 *Suppose that $(X_t, \mathscr{G}_t, \mathbf{P})$ is a realization of the (ξ, ϕ)-superprocess such that $\mathbf{P}[X_0(1)] < \infty$ and $(\pi_t)_{t \geq 0}$ is defined by (2.35). Let $\alpha \geq 0$ and let $f \in B(E)^+$ be an α-super-mean-valued function for $(P_t)_{t \geq 0}$ satisfying $\varepsilon := \inf_{x \in E} f(x) > 0$. Then for $\beta \geq \alpha + c_0 \varepsilon^{-1} \|f\|$, the process $t \mapsto \mathrm{e}^{-2\beta t} X_t(f)$ is a $(\mathscr{G}_t)$-supermartingale.*

Proof. Since $f \in B(E)^+$ is α-super-mean-valued for $(P_t)_{t \geq 0}$, we have $P_t f(x) \leq \mathrm{e}^{\alpha t} f(x)$. Recall that $\|\pi_t f\| \leq \|f\| \mathrm{e}^{c_0 t}$ by Proposition A.49. Then we use (2.36) to see

$$\begin{aligned} \pi_t f(x) &\leq \mathrm{e}^{\|b^-\| t} P_t f(x) + \|\gamma(\cdot, 1)\| \|f\| \mathrm{e}^{c_0 t} \int_0^t P_{t-s} 1(x) \mathrm{d}s \\ &\leq \mathrm{e}^{(\|b^-\| + \alpha) t} f(x) + c_0 \varepsilon^{-1} \|f\| \mathrm{e}^{c_0 t} \int_0^t P_s f(x) \mathrm{d}s \end{aligned}$$

$$\leq \mathrm{e}^{\beta t} f(x) + \beta \mathrm{e}^{\beta t} \int_0^t \mathrm{e}^{\beta s} f(x) \mathrm{d}s \;\leq\; \mathrm{e}^{2\beta t} f(x).$$

Then the result follows by Proposition 2.33. □

Corollary 2.35 *Let ϕ be a local branching mechanism given by (2.45). Suppose that $(X_t, \mathscr{G}_t, \mathbf{P})$ is a realization of the (ξ, ϕ)-superprocess such that $\mathbf{P}[X_0(1)] < \infty$. Let $\alpha \geq 0$ and let $f \in B(E)^+$ be an α-super-mean-valued function for $(P_t)_{t\geq 0}$. Then for any $\alpha_1 \geq \alpha + \|b^-\|$, the process $t \mapsto \mathrm{e}^{-\alpha_1 t} X_t(f)$ is a $(\mathscr{G}_t)$-supermartingale.*

Proof. Since $f \in B(E)^+$ is α-super-mean-valued for $(P_t)_{t\geq 0}$, we have

$$P_t^b f(x) \leq \mathrm{e}^{\|b^-\| t} P_t f(x) \leq \mathrm{e}^{(\|b^-\| + \alpha)t} f(x) \leq \mathrm{e}^{\alpha_1 t} f(x).$$

Then we have the result by Proposition 2.33. □

Let $\mathbf{F}$ be the set of functions $f \in B(E)$ that are finely continuous relative to ξ. Fix $\beta > 0$ and let $(A, \mathscr{D}(A))$ be the weak generator of $(P_t)_{t\geq 0}$ defined by $\mathscr{D}(A) = U^\beta \mathbf{F}$ and $Af = \beta f - g$ for $f = U^\beta g \in \mathscr{D}(A)$.

Theorem 2.36 *Suppose that $(X_t, \mathscr{G}_t, \mathbf{P})$ is a progressive realization of the (ξ, ϕ)-superprocess such that $\mathbf{P}[X_0(1)] < \infty$. Then for any $f \in \mathscr{D}(A)$, the process*

$$M_t(f) := X_t(f) - X_0(f) - \int_0^t X_s(Af + \gamma f - bf) \mathrm{d}s, \qquad t \geq 0,$$

is a $(\mathscr{G}_t)$-martingale.

Proof. Let $(\pi_t)_{t\geq 0}$ be defined by (2.35). For any $t \geq r \geq 0$ we use Corollary 2.28 and the Markov property of $\{(X_t, \mathscr{G}_t) : t \geq 0\}$ to see that

$$\begin{aligned}
\mathbf{P}\big[M_t(f) \big| \mathscr{G}_r\big] &= \mathbf{P}\bigg[X_t(f) - X_0(f) - \int_0^t X_s(Af + \gamma f - bf) \mathrm{d}s \bigg| \mathscr{G}_r\bigg] \\
&= \mathbf{P}\bigg[X_t(f) - \int_0^{t-r} X_{r+s}(Af + \gamma f - bf) \mathrm{d}s \bigg| \mathscr{G}_r\bigg] \\
&\qquad - X_0(f) - \int_0^r X_s(Af + \gamma f - bf) \mathrm{d}s \\
&= X_r(\pi_{t-r} f) - \int_0^{t-r} X_r(\pi_s(A + \gamma - b) f) \mathrm{d}s \\
&\qquad - X_0(f) - \int_0^r X_s(Af + \gamma f - bf) \mathrm{d}s \\
&= X_r(f) - X_0(f) - \int_0^r X_s(Af + \gamma f - bf) \mathrm{d}s,
\end{aligned}$$

where we have also used Theorem A.55 for the last equality. That gives the martingale property of $\{M_t(f) : t \geq 0\}$. □

We next give some second-moment formulas. For simplicity we only consider the (ξ, ϕ)-superprocess. In this case, the semigroup $(\pi_t)_{t\geq 0}$ is defined by (2.35). We shall need the integral condition

$$\sup_{x\in E} \int_{M(E)^\circ} \nu(1)^2 H(x, \mathrm{d}\nu) < \infty. \tag{2.58}$$

Proposition 2.37 *Suppose that (2.58) holds. Let $(Q_t)_{t\geq 0}$ be the transition semigroup of the (ξ, ϕ)-superprocess. Then for $t > 0$, $x \in E$ and $f \in B(E)$ we have*

$$\int_{M(E)} \nu(f)^2 L_t(x, \mathrm{d}\nu) = \int_0^t \mathrm{d}s \int_E q(y, \pi_s f)\pi_{t-s}(x, \mathrm{d}y),$$

where $(\pi_t)_{t\geq 0}$ is defined by (2.35) and

$$q(y, f) = 2c(y)f(y)^2 + \int_{M(E)^\circ} \nu(f)^2 H(y, \mathrm{d}\nu). \tag{2.59}$$

Proof. We first assume $f \in B(E)^+$. By applying Proposition 1.38 to (2.5), for any $\theta > 0$ we can define the function $u_t'(x, \theta) := (\mathrm{d}/\mathrm{d}\theta)v_t(x, \theta f)$, which is given by

$$u_t'(x, \theta) = \lambda_t(x, f) + \int_{M(E)^\circ} \nu(f)\mathrm{e}^{-\theta\nu(f)} L_t(x, \mathrm{d}\nu). \tag{2.60}$$

Then we differentiate both sides of (2.38) to obtain

$$\begin{aligned} u_t'(x, \theta) = \pi_t f(x) &- 2\int_0^t ds \int_E c(y)v_s(y, \theta f)u_s'(y, \theta)\pi_{t-s}(x, \mathrm{d}y) \\ &- \int_0^t ds \int_E h_s(y, \theta, f)\pi_{t-s}(x, \mathrm{d}y), \end{aligned} \tag{2.61}$$

where

$$h_s(y, \theta, f) = \int_{M(E)^\circ} \nu(u_s'(\cdot, \theta))\big(1 - \mathrm{e}^{-\nu(v_s(\cdot,\theta f))}\big)H(y, \mathrm{d}\nu).$$

For any $\theta > 0$ let $u_t''(x, \theta) = (\mathrm{d}^2/\mathrm{d}\theta^2)v_t(x, \theta f)$. By Proposition 1.38,

$$u_t''(x, \theta) = -\int_{M(E)^\circ} \nu(f)^2\mathrm{e}^{-\theta\nu(f)} L_t(x, \mathrm{d}\nu). \tag{2.62}$$

On the other hand, from (2.61) we have

$$\begin{aligned} u_t''(x, \theta) = -2\int_0^t ds \int_E c(y)\big[u_s'(y, \theta)^2 + v_s(y, \theta f)u_s''(y, \theta)\big]\pi_{t-s}(x, \mathrm{d}y) \\ - \int_0^t ds \int_E h_s'(y, \theta, f)\pi_{t-s}(x, \mathrm{d}y) \end{aligned}$$

where

$$h'_s(y,\theta,f) = \int_{M(E)^\circ} \nu(u'_s(\cdot,\theta))^2 \mathrm{e}^{-\nu(v_s(\cdot,\theta f))} H(y,\mathrm{d}\nu) + \int_{M(E)^\circ} \nu(u''_s(\cdot,\theta))\big(1-\mathrm{e}^{-\nu(v_s(\cdot,\theta f))}\big) H(y,\mathrm{d}\nu).$$

By dominated convergence we have

$$\lim_{\theta\to 0} u''_t(x,\theta) = -\int_0^t ds \int_E q(y,\pi_s f)\pi_{t-s}(x,\mathrm{d}y).$$

From this and (2.62) we get the desired equality for $f \in B(E)^+$. The extension to $f \in B(E)$ is elementary. □

Proposition 2.38 *Suppose that* (2.58) *holds. Let* $(Q_t)_{t\geq 0}$ *be the transition semigroup of the* (ξ,ϕ)*-superprocess. Then for* $t \geq 0$, $\mu \in M(E)$ *and* $f \in B(E)$ *we have*

$$\int_{M(E)} \nu(f)^2 Q_t(\mu,\mathrm{d}\nu) = \mu(\pi_t f)^2 + \int_0^t \mathrm{d}s \int_E q(y,\pi_s f)\mu\pi_{t-s}(\mathrm{d}y), \tag{2.63}$$

where $(\pi_t)_{t\geq 0}$ *is defined by* (2.35) *and* $q(y,f)$ *is defined by* (2.59).

Proof. Let $u'_t(x,\theta)$ and $u''_t(x,\theta)$ be defined as in the proof of Proposition 2.37. In view of (2.32), we have

$$\int_{M(E)} \nu(f)^2 \mathrm{e}^{-\theta\nu(f)} Q_t(\mu,\mathrm{d}\nu) = \big[\mu(u'_t(\cdot,\theta))^2 - \mu(u''_t(\cdot,\theta))\big] \exp\{-\mu(v_t(\cdot,\theta))\}.$$

By letting $\theta \to 0$ in the above equation we obtain (2.63), first for $f \in B(E)^+$ and then for $f \in B(E)$. □

Corollary 2.39 *Let* $(Q_t)_{t\geq 0}$ *be the transition semigroup of the* (ξ,ϕ)*-superprocess with local branching mechanism given by* (2.45) *and assume*

$$x \mapsto \phi''(x,0) := 2c(x) + \int_0^\infty u^2 m(x,\mathrm{d}u) \tag{2.64}$$

is bounded on E*. Then for* $t \geq 0$, $\mu \in M(E)$ *and* $f \in B(E)$ *we have*

$$\int_{M(E)} \nu(f)^2 Q_t(\mu,\mathrm{d}\nu) = \mu(P^b_t f)^2 + \int_0^t \mathrm{d}s \int_E \phi''(x,0) P^b_s f(x)^2 \mu P^b_{t-s}(\mathrm{d}x).$$

Example 2.8 Suppose that $X = (W,\mathscr{G},\mathscr{G}_t,X_t,\mathbf{Q}_\mu)$ is a (ξ,ϕ)-superprocess with binary local branching mechanism $\phi(x,z) = c(x)z^2/2$. Let $(V_t)_{t\geq 0}$ denote the cumulant semigroup of X. Fix $f \in B(E)^+$ and define

$$v_t^{(n)}(x) = (-1)^{n-1} \frac{\partial^n}{\partial \theta^n} V_t(\theta f)(x)\Big|_{\theta=0+}.$$

Then we have $v_t^{(1)}(x) = P_t f(x)$ and

$$v_t^{(n)}(x) = \sum_{k=1}^{n-1} \binom{n-1}{k} \int_0^t P_{t-s}(cv_s^{(k)} v_s^{(n-k)})(x)\mathrm{d}s$$

for $n = 2, 3, \ldots$. The moments of X are determined by $\mathbf{Q}_\mu[X_t(f)] = \mu(P_t f)$ and

$$\mathbf{Q}_\mu[X_t(f)^n] = \sum_{k=0}^{n-1} \binom{n-1}{k} \mu(v_t^{(n-k)}) \mathbf{Q}_\mu[X_t(f)^k].$$

2.6 Notes and Comments

The one-to-one correspondence stated in Theorem 2.4 was established in Watanabe (1968) under some stronger assumptions. Theorem 2.5 was also proved in Watanabe (1968). Jiřina (1964) studied the extinction problem of discrete-time branching processes taking values of finite measures on the positive half line. A class of superprocesses over compact metric spaces were constructed in Watanabe (1968), where it was shown those processes arise as high-density limits of branching particle systems. Silverstein (1969) constructed more general superprocesses with decomposable branching mechanisms; see also Dawson et al. (2002c) and Dynkin (1993a). Some inhomogeneous superprocesses with general branching mechanisms were constructed in Dynkin (1994), who assumed the existence of a càdlàg realization of the underlying spatial motion and a technical condition on the tail behavior of the kernel in the expression of the branching mechanism. The superprocesses constructed in Dynkin (1994) are not necessarily conservative. Dawson et al. (1998) proved that a general class of local branching (ξ, ϕ, K)-superprocesses with a fixed underlying spatial motion ξ depend on the parameters (ϕ, K) continuously and the superprocesses with Lebesgue killing density constitute a dense subset of the class. Leduc (2000) constructed some Hunt superprocesses under a second-moment condition.

Our assumptions on the branching mechanism guarantee that the corresponding superprocesses have finite first-moments in the sense of (2.51). Let $X = (W, \mathscr{G}, \mathscr{G}_t, X_t, \mathbf{Q}_\mu)$ be a realization of the (ξ, K, ϕ)-superprocess. For $t \geq 0$ and $\mu \in M(E)$ we can define the mean measure $I_{\mu,t}$ on E by

$$I_{\mu,t}(B) = \mathbf{Q}_\mu[X_t(B)], \qquad B \in \mathscr{B}(E).$$

The *Campbell measure* of the random measure X_t is the unique finite measure $R_{\mu,t}$ on $E \times M(E)$ such that

$$R_{\mu,t}(B \times A) = \mathbf{Q}_\mu[X_t(B)1_A(X_t)], \quad B \in \mathscr{B}(E), A \in \mathscr{B}(M(E)).$$

In view of (2.52) we have

$$\int_E \int_{M(E)} g(x)\mathrm{e}^{-\nu(f)} R_{\mu,t}(\mathrm{d}x, \mathrm{d}\nu) = \exp\{-\mu(V_t f)\}\mu(V_t^g f),$$

where $f \in B(E)^+$ and $g \in B(E)$. By the existence of regular conditional probabilities, there is a probability kernel $J_{\mu,t}(x, \mathrm{d}\nu)$ from E to $M(E)$ so that

$$R_{\mu,t}(\mathrm{d}x, \mathrm{d}\nu) = I_{\mu,t}(\mathrm{d}x)J_{\mu,t}(x, \mathrm{d}\nu), \quad x \in E, \nu \in M(E).$$

The probability measures $\{J_{\mu,t}(x, \cdot) : x \in E\}$ are called *Palm distributions* of X_t. If (η, Y) is a random variable on $E \times M(E)$ distributed according to the Campbell measure $R_{\mu,t}$, then η is chosen according to the random measure Y and $J_{\mu,t}(x, \cdot)$ is the conditional distribution of Y given $\eta = x$. See Dawson (1993) and Dawson and Perkins (1991) for some applications of the Campbell measure and the Palm distributions in the study of the superprocess.

Example 2.1 was given by Dynkin et al. (1994). Rhyzhov and Skorokhod (1970) and Watanabe (1969) constructed CB-processes under conditions on the branching mechanism weaker than those of Example 2.2. Moment formulas for superprocesses as in Example 2.8 were established in Dynkin (1989a) and Konno and Shiga (1988). A construction for super-Brownian motions was given in Ren (2001) under a weaker admissibility assumption on the killing additive functional. A super-stable process with infinite mean was constructed in Fleischmann and Sturm (2004) by a passage to the limit.

The catalyst measure $\rho(\mathrm{d}y)$ in Example 2.3 can be time dependent. In fact, it can be replaced by a measure-valued process $\{\rho_t : t \geq 0\}$. The study of superprocesses with measure-valued catalysts was initiated by Dawson and Fleischmann (1991, 1992). A binary local branching super-Brownian motion with super-Brownian catalyst was constructed in Dawson and Fleischmann (1997a). The property of persistence (no loss of expected mass in the long-time behavior) of the process with underlying dimensions $d \leq 3$ was proved in Dawson and Fleischmann (1997a, 1997b) and Etheridge and Fleischmann (1998). This phenomenon is in contrast to the super-Brownian motion with Lebesgue catalyst, where persistence only holds in high dimensions. A construction of catalytic super-Brownian motion via collision local times was given in Mörters and Vogt (2005). The long-time behavior of a branching random walk in a random catalytic medium was investigated in Greven et al. (1999). Engländer (2007) gave a survey of some recent topics in spatial branching processes in deterministic and random media.

There is another important class of measure-valued Markov processes, the so-called *Fleming–Viot superprocesses*. A Fleming–Viot superprocess takes values of probability measures and describes the evolution of a genetic system involving mutation, selection and recombination. The Saint-Flour lecture notes of Dawson (1993) provide a complete survey of the literature before 1992 on both Dawson–Watanabe and Fleming–Viot superprocesses. For a survey of the latter see also Ethier and

Kurtz (1993). It was conjectured in Ethier and Kurtz (1993) that a Fleming–Viot superprocess is reversible if and only if its mutation operator is of the uniform jump type. This was proved in Li et al. (1999); see also Handa (2002) and Schmuland and Sun (2002). A nice introduction of the theory of superprocesses was given by Etheridge (2000), where Brownian spatial motion was mainly considered. The connections between Dawson–Watanabe and Fleming–Viot superprocesses were investigated in Etheridge and March (1991), Perkins (1992) and Shiga (1990). The two classes of superprocesses model large population systems in which branching or splitting occurs. The dual phenomenon is coalescent or coagulation. Bertoin (2006) gave a comprehensive account of stochastic models involving fragmentation and coagulation. A kind of generalized Fleming–Viot superprocesses arising from coalescent processes were studied in Bertoin and Le Gall (2003, 2005, 2006). Feng (2010) provided an up-to-date account of Fleming–Viot superprocesses and Poisson–Dirichlet type distributions. Durrett (2008) and Ewens (2004) gave comprehensive coverage of mathematical population genetics.

http://www.springer.com/978-3-642-15003-6

Measure-Valued Branching Markov Processes
Li, Z.
2011, XI, 350 p., Hardcover
ISBN: 978-3-642-15003-6