Chapter 2. Normed Rings

Norms $\|\cdot\|$ of associative rings are generalizations of absolute values $|\cdot|$ of integral domains, where the inequality $\|xy\| \le \|x\|\cdot\|y\|$ replaces the standard multiplication rule $|xy| = |x|\cdot|y|$. Starting from a complete normed commutative ring A, we study the ring $A\{x\}$ of all formal power series with coefficients in A converging to zero. This is again a complete normed ring (Lemma 2.2.1). We prove an analog of the Weierstrass division theorem (Lemma 2.2.4) and the Weierstrass preparation theorem for $A\{x\}$ (Corollary 2.2.5). If A is a field K and the norm is an absolute value, then $K\{x\}$ is a principal ideal domain, hence a factorial ring (Proposition 2.3.1). Moreover, $\mathrm{Quot}(K\{x\})$ is a Hilbertian field (Theorem 2.3.3). It follows that $\mathrm{Quot}(K\{x\})$ is not a Henselian field (Corollary 2.3.4). In particular, $\mathrm{Quot}(K\{x\})$ is not separably closed in $K((x))$. In contrast, the field $K((x))_0$ of all formal power series over K that converge at some element of K is algebraically closed in $K((x))$ (Proposition 2.4.5).

2.1 Normed Rings

In Section 4.4 we construct patching data over fields $K(x)$, where K is a complete ultrametric valued field. The 'analytic' fields P_i will be the quotient fields of certain rings of convergent power series in several variables over K. At a certain point in a proof by induction we consider a ring of convergent power series in one variable over a complete ultrametric valued ring. So, we start by recalling the definition and properties of the latter rings.

Let A be a commutative ring with 1. An **ultrametric absolute value** of A is a function $|\ |\colon A \to \mathbb{R}$ satisfying the following conditions:
(1a) $|a| \ge 0$, and $|a| = 0$ if and only if $a = 0$.
(1b) There exists $a \in A$ such that $0 < |a| < 1$.
(1c) $|ab| = |a| \cdot |b|$.
(1d) $|a + b| \le \max(|a|, |b|)$.

By (1a) and (1c), A is an integral domain. By (1c), the absolute value of A extends to an absolute value on the quotient field of A (by $|\frac{a}{b}| = \frac{|a|}{|b|}$). It follows also that $|1| = 1$, $|-a| = |a|$, and
(1d$'$) if $|a| < |b|$, then $|a + b| = |b|$.

Denote the ordered additive group of the real numbers by $\mathbb{R}^+$. The function $v\colon \mathrm{Quot}(A) \to \mathbb{R}^+ \cup \{\infty\}$ defined by $v(a) = -\log|a|$ satisfies the following conditions:
(2a) $v(a) = \infty$ if and only if $a = 0$.
(2b) There exists $a \in \mathrm{Quot}(A)$ such that $0 < v(a) < \infty$.
(2c) $v(ab) = v(a) + v(b)$.

M. Jarden, *Algebraic Patching*, Springer Monographs in Mathematics,
DOI 10.1007/978-3-642-15128-6_2, © Springer-Verlag Berlin Heidelberg 2011

(2d) $v(a+b) \geq \min\{v(a), v(b)\}$ (and $v(a+b) = v(b)$ if $v(b) < v(a)$).
In other words, v is a **real valuation** of $\mathrm{Quot}(A)$. Conversely, every real valuation v: $\mathrm{Quot}(A) \to \mathbb{R}^+ \cup \{\infty\}$ gives rise to a nontrivial ultrametric absolute value $|\cdot|$ of $\mathrm{Quot}(A)$: $|a| = \varepsilon^{v(a)}$, where ε is a fixed real number between 0 and 1.

An attempt to extend an absolute value from A to a larger ring A' may result in relaxing Condition (1c), replacing the equality by an inequality. This leads to the more general notion of a 'norm'.

Definition 2.1.1: Normed rings. Let R be an associative ring with 1. A **norm** on R is a function $\|\ \|$: $R \to \mathbb{R}$ that satisfies the following conditions for all $a, b \in R$:
(3a) $\|a\| \geq 0$, and $\|a\| = 0$ if and only if $a = 0$; further $\|1\| = \|-1\| = 1$.
(3b) There is an $x \in R$ with $0 < \|x\| < 1$.
(3c) $\|ab\| \leq \|a\| \cdot \|b\|$.
(3d) $\|a+b\| \leq \max(\|a\|, \|b\|)$.

The norm $\|\ \|$ naturally defines a topology on R whose basis is the collection of all sets $U(a_0, r) = \{a \in R \mid \|a - a_0\| < r\}$ with $a_0 \in R$ and $r > 0$. Both addition and multiplication are continuous under that topology. Thus, R is a **topological ring**. □

Definition 2.1.2: Complete rings. Let R be a normed ring. A sequence $a_1, a_2, a_3, \ldots$ of elements of R is **Cauchy** if for each $\varepsilon > 0$ there exists m_0 such that $\|a_n - a_m\| < \varepsilon$ for all $m, n \geq m_0$. We say that R is **complete** if every Cauchy sequence converges. □

Lemma 2.1.3: Let R be a normed ring and let $a, b \in R$. Then:
(a) $\|-a\| = \|a\|$.
(b) If $\|a\| < \|b\|$, then $\|a+b\| = \|b\|$.
(c) A sequence $a_1, a_2, a_3, \ldots$ of elements of R is Cauchy if for each $\varepsilon > 0$ there exists m_0 such that $\|a_{m+1} - a_m\| < \varepsilon$ for all $m \geq m_0$.
(d) The map $x \to \|x\|$ from R to $\mathbb{R}$ is continuous.
(e) If R is complete, then a series $\sum_{n=0}^{\infty} a_n$ of elements of R converges if and only if $a_n \to 0$.
(f) If R is complete and $\|a\| < 1$, then $1-a \in R^\times$. Moreover, $(1-a)^{-1} = 1+b$ with $\|b\| < 1$.

Proof of (a): Observe that $\|-a\| \leq \|-1\| \cdot \|a\| \leq \|a\|$. Replacing a by $-a$, we get $\|a\| \leq \|-a\|$, hence the claimed equality.

Proof of (b): Assume $\|a+b\| < \|b\|$. Then, by (a), $\|b\| = \|(-a)+(a+b)\| \leq \max(\|-a\|, \|a+b\|) < \|b\|$, which is a contradiction.

Proof of (c): With m_0 as above let $n > m \geq m_0$. Then

$$\|a_n - a_m\| \leq \max(\|a_n - a_{n-1}\|, \ldots, \|a_{m+1} - a_m\|) < \varepsilon.$$

Proof of (d): By (3d), $\|x\| = \|(x-y)+y\| \leq \max(\|x-y\|, \|y\|) \leq \|x-y\| + \|y\|$. Hence, $\|x\| - \|y\| \leq \|x-y\|$. Symmetrically, $\|y\| - \|x\| \leq \|y-x\| =$

$\|x-y\|$. Therefore, $|\,\|x\|-\|y\|\,| \le \|x-y\|$. Consequently, the map $x \mapsto \|x\|$ is continuous.

Proof of (e): Let $s_n = \sum_{i=0}^n a_i$. Then $s_{n+1} - s_n = a_{n+1}$. Thus, by (c), $s_1, s_2, s_3, \ldots$ is a Cauchy sequence if and only if $a_n \to 0$. Hence, the series $\sum_{n=0}^\infty a_n$ converges if and only if $a_n \to 0$.

Proof of (f): The sequence a^n tends to 0. Hence, by (e), $\sum_{n=0}^\infty a^n$ converges. The identities $(1-a)\sum_{i=0}^n a^i = 1 - a^{n+1}$ and $\sum_{i=0}^n a^i(1-a) = 1 - a^{n+1}$ imply that $\sum_{n=0}^\infty a^n$ is both the right and the left inverse of $1-a$. Moreover, $\sum_{n=0}^\infty a^n = 1 + b$ with $b = \sum_{n=1}^\infty a^n$ and $\|b\| \le \max_{n\ge 1} \|a\|^n < 1$. □

Example 2.1.4:

(a) Every field K with an ultrametric absolute value is a normed ring. For example, for each prime number p, $\mathbb{Q}$ has a p-adic absolute value $|\cdot|_p$ which is defined by $|x|_p = p^{-m}$ if $x = \frac{a}{b}p^m$ with $a, b, m \in \mathbb{Z}$ and $p \nmid a, b$.

(b) The ring $\mathbb{Z}_p$ of p-adic integers and the field $\mathbb{Q}_p$ of p-adic numbers are complete with respect to the p-adic absolute value.

(c) Let K_0 be a field and let $0 < \varepsilon < 1$. The ring $K_0[[t]]$ (resp. field $K_0((t))$) of formal power series $\sum_{i=0}^\infty a_i t^i$ (resp. $\sum_{i=m}^\infty a_i t^i$ with $m \in \mathbb{Z}$) with coefficients in K_0 is complete with respect to the absolute value $|\sum_{i=m}^\infty a_i t^i| = \varepsilon^{\min(i \mid a_i \ne 0)}$.

(d) Let $\|\cdot\|$ be a norm of a commutative ring A. For each positive integer n we extend the norm to the associative (and usually not commutative) ring $M_n(A)$ of all $n \times n$ matrices with entries in A by

$$\|(a_{ij})_{1\le i,j\le n}\| = \max(\|a_{ij}\|_{1\le i,j\le n}).$$

If $b = (b_{jk})_{1\le j,k\le n}$ is another matrix and $c = ab$, then $c_{ik} = \sum_{j=1}^n a_{ij}b_{jk}$ and $\|c_{ik}\| \le \max(\|a_{ij}\| \cdot \|b_{jk}\|) \le \|a\| \cdot \|b\|$. Hence, $\|c\| \le \|a\|\|b\|$. This verifies Condition (3c). The verification of (3a), (3b), and (3d) is straightforward. Note that when $n \ge 2$, even if the initial norm of A is an absolute value, the extended norm satisfies only the weak condition (3c) and not the stronger condition (1c), so it is not an absolute value.

If A is complete, then so is $M_n(A)$. Indeed, let $a_i = (a_{i,rs})_{1\le r,s\le n}$ be a Cauchy sequence in $M_n(A)$. Since $\|a_{i,rs} - a_{j,rs}\| \le \|a_i - a_j\|$, each of the sequences $a_{1,rs}, a_{2,rs}, a_{3,rs}, \ldots$ is Cauchy, hence converges to an element b_{rs} of A. Set $b = (b_{rs})_{1\le r,s\le n}$. Then $a_i \to b$. Consequently, $M_n(A)$ is complete.

(e) Let $\mathfrak{a}$ be a proper ideal of a Noetherian domain A. By a theorem of Krull, $\bigcap_{n=0}^\infty \mathfrak{a}^n = 0$ [AtM69, p. 110, Cor. 10.18]. We define an $\mathfrak{a}$-adic norm on A by choosing an ε between 0 and 1 and setting $\|a\| = \varepsilon^{\max(n \mid a \in \mathfrak{a}^n)}$. If $\|a\| = \varepsilon^m$ and $\|b\| = \varepsilon^n$, and say $m \le n$, then $\mathfrak{a}^n \subseteq \mathfrak{a}^m$, so $a + b \in \mathfrak{a}^m$, hence $\|a+b\| \le \varepsilon^m = \max(\|a\|, \|b\|)$. Also, $ab \in \mathfrak{a}^{m+n}$, so $\|ab\| \le \|a\| \cdot \|b\|$. □

Like absolute valued rings, every normed ring has a completion:

LEMMA 2.1.5: *Every normed ring $(R, \| \ \|)$ can be embedded into a complete normed ring $(\hat{R}, \| \ \|)$ such that R is dense in $\hat{R}$ and the following universal condition holds:*

(4) *Each continuous homomorphism f of R into a complete ring S uniquely extends to a continuous homomorphism $\hat{f}\colon \hat{R} \to S$.*

The normed ring $(\hat{R}, \| \ \|)$ is called the **completion** *of $(R, \| \ \|)$.*

Proof: We consider the set A of all Cauchy sequences $\mathbf{a} = (a_n)_{n=1}^{\infty}$ with $a_n \in R$. For each $\mathbf{a} \in A$, the values $\|a_n\|$ of its components are bounded. Hence, A is closed under componentwise addition and multiplication and contains all constant sequences. Thus, A is a ring. Let $\mathfrak{n}$ be the ideal of all sequences that converge to 0. We set $\hat{R} = A/\mathfrak{n}$ and identify each $x \in R$ with the coset $(x)_{n=1}^{\infty} + \mathfrak{n}$.

If $\mathbf{a} \in A \smallsetminus \mathfrak{n}$, then $\|a_n\|$ eventually becomes constant. Indeed, there exists $\beta > 0$ such that $\|a_n\| \geq \beta$ for all sufficiently large n. Choose n_0 such that $\|a_n - a_m\| < \beta$ for all $n, m \geq n_0$. Then, $\|a_n - a_{n_0}\| < \beta \leq \|a_{n_0}\|$, so $\|a_n\| = \|(a_n - a_{n_0}) + a_{n_0}\| = \|a_{n_0}\|$. We define $\|\mathbf{a}\|$ to be the eventual absolute value of a_n and note that $\|\mathbf{a}\| \neq 0$. If $\mathbf{b} \in \mathfrak{n}$, we set $\|\mathbf{b}\| = 0$ and observe that $\|\mathbf{a} + \mathbf{b}\| = \|\mathbf{a}\|$. It follows that $\|\mathbf{a} + \mathfrak{n}\| = \|\mathbf{a}\|$ is a well defined function on $\hat{R}$ which extends the norm of R.

One checks that $\| \ \|$ is a norm on $\hat{R}$ and that R is dense in $\hat{R}$. Indeed, if $\mathbf{a} = (a_n)_{n=1}^{\infty} \in A$, then $a_n + \mathfrak{n} \to \mathbf{a} + \mathfrak{n}$. To prove that $\hat{R}$ is complete under $\| \ \|$ we consider a Cauchy sequence $(a_k)_{k=1}^{\infty}$ of elements of $\hat{R}$. For each k we choose an element $b_k \in R$ such that $\|b_k - a_k\| < \frac{1}{k}$. Then $(b_k)_{k=1}^{\infty}$ is a Cauchy sequence of R and the sequence $(\mathbf{a}_k)_{k=1}^{\infty}$ converges to the element $(b_k)_{k=1}^{\infty} + \mathfrak{n}$ of $\hat{R}$.

Finally, let S be a complete normed ring and $f\colon R \to S$ a continuous homomorphism. Then, for each $\mathbf{a} = (a_n)_{n=1}^{\infty} \in A$, the sequence $(f(a_n))_{n=1}^{\infty}$ of S is Cauchy, hence it converges to an element s. Define $\hat{f}(\mathbf{a} + \mathfrak{n}) = s$ and check that $\hat{f}$ has the desired properties. $\square$

Example 2.1.6: Let A be a commutative ring. We consider the ring $R = A[x_1, \ldots, x_n]$ of polynomials over A in the variables $x_1, \ldots, x_n$ and the ideal $\mathfrak{a}$ of R generated by $x_1, \ldots, x_n$. The completion of R with respect to $\mathfrak{a}$ is the ring $\hat{R} = A[[x_1, \ldots, x_n]]$ of all formal power series $f(x_1, \ldots, x_n) = \sum_{i=0}^{\infty} f_i(x_1, \ldots, x_n)$, where $f_i \in A[x_1, \ldots, x_n]$ is a homogeneous polynomial of degree i. Moreover, $\hat{R} = A[[x_1, \ldots, x_{n-1}]][[x_n]]$ and $\hat{R}$ is complete with respect to the ideal $\hat{\mathfrak{a}}$ generated by $x_1, \ldots, x_n$ [Lan93, Chap. IV, Sec. 9]. If R is a Noetherian integral domain, then so is $\hat{R}$ [Lan93, p. 210, Cor. 9.6]. If $A = K$ is a field, then $\hat{R}$ is a unique factorization domain [Mat94, Thm. 20.3].

If A is an integral domain, then the function $v\colon \hat{R} \to \mathbb{Z} \cup \{\infty\}$ defined for f as in the preceding paragraph by $v(f) = \min_{i \geq 0}(f_i \neq 0)$ satisfies Condition (2), so it extends to a discrete valuation of $\hat{K} = \mathrm{Quot}(\hat{R})$. However, by Weissauer, $\hat{K}$ is Hilbertian if $n \geq 2$. [FrJ08, Example 15.5.2]. Hence, $\hat{K}$

is Henselian with respect to no valuation [FrJ08, Lemma 15.5.4]. Since v is discrete, $\hat{K}$ is not complete with respect to v. $\square$

2.2 Rings of Convergent Power Series

Let A be a complete normed commutative ring and x a variable. Consider the following subset of $A[[x]]$:

$$A\{x\} = \Big\{ \sum_{n=0}^{\infty} a_n x^n \mid a_n \in A, \lim_{n\to\infty} \|a_n\| = 0 \Big\}.$$

For each $f = \sum_{n=0}^{\infty} a_n x^n \in A\{x\}$ we define $\|f\| = \max(\|a_n\|)_{n=0,1,2,\ldots}$. This definition makes sense because $a_n \to 0$, hence $\|a_n\|$ is bounded.

We prove the Weierstrass division and the Weierstrass preparation theorems for $A\{x\}$ in analogy to the corresponding theorems for the ring of formal power series in one variable over a local ring.

LEMMA 2.2.1:
(a) *$A\{x\}$ is a subring of $A[[x]]$ containing A.*
(b) *The function $\| \ \|$: $A\{x\} \to \mathbb{R}$ is a norm.*
(c) *The ring $A\{x\}$ is complete under that norm.*
(d) *Let B be a complete normed ring extension of A. Then each $b \in B$ with $\|b\| \le 1$ defines an* **evaluation homomorphism** *$A\{x\} \to B$ given by*

$$f = \sum_{n=0}^{\infty} a_n x^n \mapsto f(b) = \sum_{n=0}^{\infty} a_n b^n.$$

Proof of (a): We prove only that $A\{x\}$ is closed under multiplication. To that end let $f = \sum_{i=0}^{\infty} a_i x^i$ and $g = \sum_{j=0}^{\infty} b_j x^j$ be elements of $A\{x\}$. Consider $\varepsilon > 0$ and let n_0 be a positive number such that $\|a_i\| < \varepsilon$ if $i \ge \frac{n_0}{2}$ and $\|b_j\| < \varepsilon$ if $j \ge \frac{n_0}{2}$. Now let $n \ge n_0$ and $i + j = n$. Then $i \ge \frac{n_0}{2}$ or $j \ge \frac{n_0}{2}$. It follows that $\|\sum_{i+j=n} a_i b_j\| \le \max(\|a_i\| \cdot \|b_j\|)_{i+j=n} \le \varepsilon \cdot \max(\|f\|, \|g\|)$. Thus, $fg = \sum_{n=0}^{\infty} \sum_{i+j=n} a_i b_j x^n$ belongs to $A\{x\}$, as claimed.

Proof of (b): Standard checking.

Proof of (c): Let $f_i = \sum_{n=0}^{\infty} a_{in} x^n$, $i = 1, 2, 3, \ldots$, be a Cauchy sequence in $A\{x\}$. For each $\varepsilon > 0$ there exists i_0 such that $\|a_{in} - a_{jn}\| \le \|f_i - f_j\| < \varepsilon$ for all $i, j \ge i_0$ and for all n. Thus, for each n, the sequence $a_{1n}, a_{2n}, a_{3n}, \ldots$ is Cauchy, hence converges to an element $a_n \in A$. If we let j tend to infinity in the latter inequality, we get that $\|a_{in} - a_n\| < \varepsilon$ for all $i \ge i_0$ and all n. Set $f = \sum_{i=0}^{\infty} a_n x^n$. Then $a_n \to 0$ and $\|f_i - f\| = \max(\|a_{in} - a_n\|)_{n=0,1,2,\ldots} < \varepsilon$ if $i \ge i_0$. Consequently, the f_i's converge in $A\{x\}$.

Proof of (d): Note that $\|a_n b^n\| \le \|a_n\| \to 0$, so $\sum_{n=0}^{\infty} a_n b^n$ is an element of B. $\square$

Definition 2.2.2: Let $f = \sum_{n=0}^{\infty} a_n x^n$ be a nonzero element of $A\{x\}$. We define the **pseudo degree** of f to be the integer $d = \max\{n \geq 0 \mid \|a_n\| = \|f\|\}$ and set $\mathrm{pseudo.deg}(f) = d$. The element a_d is the **pseudo leading coefficient** of f. Thus, $\|a_d\| = \|f\|$ and $\|a_n\| < \|f\|$ for each $n > d$. If $f \in A[x]$ is a polynomial, then $\mathrm{pseudo.deg}(f) \leq \deg(f)$. If a_d is invertible in A and satisfies $\|ca_d\| = \|c\| \cdot \|a_d\|$ for all $c \in A$, we call f **regular**. In particular, if A is a field and $\| \ \|$ is an ultrametric absolute value, then each $0 \neq f \in A\{x\}$ is regular. The next lemma implies that in this case $\| \ \|$ is an absolute value of $A\{x\}$. □

Lemma 2.2.3 (Gauss' Lemma): *Let $f, g \in A\{x\}$. Suppose f is regular of pseudo degree d and $f, g \neq 0$. Then $\|fg\| = \|f\| \cdot \|g\|$ and* $\mathrm{pseudo.deg}(fg) = \mathrm{pseudo.deg}(f) + \mathrm{pseudo.deg}(g)$.

Proof: Let $f = \sum_{i=0}^{\infty} a_i x^i$ and $g = \sum_{j=0}^{\infty} b_j x^j$. Let a_d (resp. b_e) be the pseudo leading coefficient of f (resp. g). Then $fg = \sum_{n=0}^{\infty} c_n x^n$ with $c_n = \sum_{i+j=n} a_i b_j$.

If $i + j = d + e$ and $(i, j) \neq (d, e)$, then either $i > d$ or $j > e$. In each case, $\|a_i b_j\| \leq \|a_i\| \|b_j\| < \|f\| \cdot \|g\|$. By our assumption on a_d, we have $\|a_d b_e\| = \|a_d\| \cdot \|b_e\| = \|f\| \cdot \|g\|$. By Lemma 2.1.3(b), this implies $\|c_{d+e}\| = \|f\| \cdot \|g\|$.

If $i+j > d+e$, then either $i > d$ and $\|a_i\| < \|f\|$ or $j > e$ and $\|b_j\| < \|g\|$. In each case $\|a_i b_j\| \leq \|a_i\| \cdot \|b_j\| < \|f\| \cdot \|g\|$. Hence, $\|c_n\| < \|c_{d+e}\|$ for each $n > d + e$. Therefore, c_{d+e} is the pseudo leading coefficient of fg, and the lemma is proved. □

Proposition 2.2.4 (Weierstrass division theorem): *Let $f \in A\{x\}$ and let $g \in A\{x\}$ be regular of pseudo degree d. Then there are unique $q \in A\{x\}$ and $r \in A[x]$ such that $f = qg + r$ and* $\deg(r) < d$. *Moreover,*

$$(1) \qquad \|qg\| = \|q\| \cdot \|g\| \leq \|f\| \qquad \textit{and} \qquad \|r\| \leq \|f\|$$

Proof: We break the proof into several parts.

Part A: *Proof of (1).* First we assume that there exist $q \in A\{x\}$ and $r \in A[x]$ such that $f = qg + r$ with $\deg(r) < d$. If $q = 0$, then (1) is clear. Otherwise, $q \neq 0$ and we let $e = \mathrm{pseudo.deg}(q)$. By Lemma 2.2.3, $\|qg\| = \|q\| \cdot \|g\|$ and $\mathrm{pseudo.deg}(qg) = e + d > \deg(r)$. Hence, the coefficient c_{d+e} of x^{d+e} in qg is also the coefficient of x^{d+e} in f. It follows that $\|qg\| = \|c_{d+e}\| \leq \|f\|$. Consequently, $\|r\| = \|f - qg\| \leq \|f\|$.

Part B: *Uniqueness.* Suppose $f = qg + r = q'g + r'$, where $q, q' \in A\{x\}$ and $r, r' \in A[x]$ are of degrees less than d. Then $0 = (q - q')g + (r - r')$. By Part A, applied to 0 rather than to f, $\|q - q'\| \cdot \|g\| = \|r - r'\| = 0$. Hence, $q = q'$ and $r = r'$.

Part C: *Existence if g is a polynomial of degree d.* Write $f = \sum_{n=0}^{\infty} b_n x^n$ with $b_n \in A$ converging to 0. For each $m \geq 0$ let $f_m = \sum_{n=0}^{m} b_n x^n \in$

$A[x]$. Then the $f_1, f_2, f_3, \ldots$ converge to f, in particular they form a Cauchy sequence. Since g is regular of pseudo degree d, its leading coefficient is invertible. Euclid's algorithm for polynomials over A produces $q_m, r_m \in A[x]$ with $f_m = q_m g + r_m$ and $\deg(r_m) < \deg(g)$. Thus, for all k, m we have $f_m - f_k = (q_m - q_k)g + (r_m - r_k)$. By Part A, $\|q_m - q_k\| \cdot \|g\|, \|r_m - r_k\| \leq \|f_m - f_k\|$. Thus, $\{q_m\}_{m=0}^{\infty}$ and $\{r_m\}_{m=0}^{\infty}$ are Cauchy sequences in $A\{x\}$. Since $A\{x\}$ is complete (Lemma 2.2.1), the q_m's converge to some $q \in A\{x\}$. Since A is complete, the r_m's converge to an $r \in A[x]$ of degree less than d. It follows that $f = qg + r$

Part D: *Existence for arbitrary g.* Let $g = \sum_{n=0}^{\infty} a_n x^n$ and set $g_0 = \sum_{n=0}^{d} a_n x^n \in A[x]$. Then $\|g - g_0\| < \|g\|$. By Part C, there are $q_0 \in A\{x\}$ and $r_0 \in A[x]$ such that $f = q_0 g_0 + r_0$ and $\deg(r_0) < d$. By Part A, $\|q_0\| \leq \frac{\|f\|}{\|g\|}$ and $\|r_0\| \leq \|f\|$. Thus, $f = q_0 g + r_0 + f_1$, where $f_1 = -q_0(g - g_0)$, and $\|f_1\| \leq \frac{\|g-g_0\|}{\|g\|} \cdot \|f\|$.

Set $f_0 = f$. By induction we get, for each $k \geq 0$, elements $f_k, q_k \in A\{x\}$ and $r_k \in A[x]$ such that $\deg(r_k) < d$ and

$$f_k = q_k g + r_k + f_{k+1}, \quad \|q_k\| \leq \frac{\|f_k\|}{\|g\|}, \quad \|r_k\| \leq \|f_k\|, \quad \text{and}$$
$$\|f_{k+1}\| \leq \frac{\|g - g_0\|}{\|g\|} \|f_k\|.$$

It follows that $\|f_k\| \leq \left(\frac{\|g-g_0\|}{\|g\|}\right)^k \|f\|$, so $\|f_k\| \to 0$. Hence, also $\|q_k\|, \|r_k\| \to 0$. Therefore, $q = \sum_{k=0}^{\infty} q_k \in A\{x\}$ and $r = \sum_{k=0}^{\infty} r_k \in A[x]$. By construction, $f = \sum_{n=0}^{k} q_n g + \sum_{n=0}^{k} r_n + f_{k+1}$ for each k. Taking k to infinity, we get $f = qg + r$ and $\deg(r) < d$. ☐

Corollary 2.2.5 (Weierstrass preparation theorem): *Let $f \in A\{x\}$ be regular of pseudo degree d. Then $f = qg$, where q is a unit of $A\{x\}$ and $g \in A[x]$ is a monic polynomial of degree d with $\|g\| = 1$. Moreover, q and g are uniquely determined by these conditions.*

Proof: By Proposition 2.2.4 there are $q' \in A\{x\}$ and $r' \in A[x]$ of degree $< d$ such that $x^d = q'f + r'$ and $\|r'\| \leq \|x^d\| = 1$. Set $g = x^d - r'$. Then g is monic of degree d, $g = q'f$, and $\|g\| = 1$. It remains to show that $q' \in A\{x\}^\times$.

Note that g is regular of pseudo degree d. By Proposition 2.2.4, there are $q \in A\{x\}$ and $r \in A[x]$ such that $f = qg + r$ and $\deg(r) < d$. Thus, $f = qq'f + r$. Since $f = 1 \cdot f + 0$, the uniqueness part of Proposition 2.2.4 implies that $qq' = 1$. Hence, $q' \in A\{x\}^\times$.

Finally suppose $f = q_1 g_1$, where $q \in A\{x\}^\times$ and $g_1 \in A[x]$ is monic of degree d with $\|g_1\| = 1$. Then $g_1 = (q_1^{-1} q_2)g$ and $g_1 = 1 \cdot g + (g_1 - g)$, where $g_1 = g$ is a polynomial of degree at most $d - 1$. By the uniqueness part of Proposition 2.2.4, $q_1^{-1} q_2 = 1$, so $q_1 = q_2$ and $g_1 = g$. ☐

COROLLARY 2.2.6: *Let* $f = \sum_{n=0}^{\infty} a_n x^n$ *be a regular element of* $A\{x\}$ *such that* $\|a_0 b\| = \|a_0\| \cdot \|b\|$ *for each* $b \in A$. *Then* $f \in A\{x\}^\times$ *if and only if* pseudo.deg$(f) = 0$ *and* $a_0 \in A^\times$.

Proof: If there exists $g \in \sum_{n=0}^{\infty} b_n x^n$ in $A\{x\}$ such that $fg = 1$, then pseudo.deg(f) + pseudo.deg$(g) = 0$ (Lemma 2.2.3 applied to 1 rather than to f), so pseudo.deg$(f) = 0$. In addition, $a_0 b_0 = 1$, so $a_0 \in A^\times$.

Conversely, suppose pseudo.deg$(f) = 0$ and $a_0 \in A^\times$. Then f is regular. Hence, by Corollary 2.2.5, $f = q \cdot 1$ where $q \in A\{x\}^\times$.

Alternatively, $a_0^{-1} f = 1 - h$, where $h = -\sum_{n=1}^{\infty} a_0^{-1} a_n x^n$. By our assumption on a_0, we have $\|a_0^{-1}\| \cdot \|a_0\| = \|a_0^{-1} a_0\| = 1$, so $\|a_0^{-1}\| = \|a_0\|^{-1}$. Since pseudo.deg$(f) = 0$, we have $\|a_0\| < \|a_n\|$, so $\|a_0^{-1} a_n\| \leq \|a_0\|^{-1} \|a_n\| < 1$ for each $n \geq 1$. It follows that $\|h\| = \max(\|a_0^{-1} a_n\|)_{n=1,2,3,\ldots} < 1$. By Lemma 2.1.3(f), $a_0^{-1} f \in A\{x\}^\times$, so $f \in A\{x\}^\times$. $\square$

2.3 Properties of the Ring $K\{x\}$

We turn our attention in this section to the case where the ring A of the previous sections is a complete field K under an ultrametric absolute value $|\ |$ and $O = \{a \in K \mid |a| \leq 1\}$ its **valuation ring**. We fix K and O for the whole section and prove that $K\{x\}$ is a principal ideal domain and that $F = \mathrm{Quot}(K\{x\})$ is a Hilbertian field.

Note that in our case $|ab| = |a| \cdot |b|$ for all $a, b \in K$ and each nonzero element of K is invertible. Hence, each nonzero $f \in K\{x\}$ is regular. It follows from Lemma 2.2.3 that the norm of $K\{x\}$ is multiplicative, hence it is an absolute value which we denote by $|\ |$ rather than by $\|\ \|$.

PROPOSITION 2.3.1:

(a) $K\{x\}$ *is a principal ideal domain. Moreover, each ideal in* $K\{x\}$ *is generated by an element of* $O[x]$.

(b) $K\{x\}$ *a unique factorization domain.*

(c) *A nonzero element* $f \in K\{x\}$ *is invertible if and only if* pseudo.deg$(f) = 0$.

(d) pseudo.deg(fg) = pseudo.deg(f) + pseudo.deg(g) *for all* $f, g \in K\{x\}$ *with* $f, g \neq 0$.

(e) *Every prime element* f *of* $K\{x\}$ *can be written as* $f = ug$, *where* u *is invertible in* $K\{x\}$ *and* g *is an irreducible element of* $K[x]$.

(f) *If a* $g \in K[x]$ *is monic of degree* d, *irreducible in* $K[x]$, *and* $|g| = 1$, *then* g *is irreducible in* $K\{x\}$.

(g) *There are irreducible polynomials in* $K[x]$ *that are not irreducible in* $K\{x\}$.

(h) *There are reducible polynomials in* $K[x]$ *that are irreducible in* $K\{x\}$.

Proof of (a): By the Weierstrass preparation theorem (Corollary 2.2.5) (applied to K rather than to A) each nonzero ideal $\mathfrak{a}$ of $K\{x\}$ is generated by the ideal $\mathfrak{a} \cap K[x]$ of $K[x]$. Since $K[x]$ is a principal ideal domain, $\mathfrak{a} \cap K[x] = fK[x]$

for some nonzero $f \in K[x]$. Consequently, $\mathfrak{a} = fK\{x\}$ is a principal ideal. Moreover, dividing f by one of its coefficients with highest absolute value, we may assume that $f \in O[x]$.

Proof of (b): Since every principal ideal domain has a unique factorization, (b) is a consequence of (a).

Proof of (c): Apply Corollary 2.2.6.

Proof of (d): Apply Lemma 2.2.3.

Proof of (e): By (a), $f = u_1 f_1$ with $u_1 \in K\{x\}^\times$ and $f_1 \in K[x]$. Write $f_1 = g_1 \cdots g_n$ with irreducible polynomials $g_1, \ldots, g_n \in K[x]$. Then $f = u_1 g_1 \cdots g_n$. Since f is irreducible in $K\{x\}$, one of the g_i's, say g_n is irreducible in $K\{x\}$ and all the others, that is $g_1, \ldots, g_{n-1}$, are invertible in $K\{x\}$. Set $u = u_1 g_1 \cdots g_{n-1}$ and $g = g_n$. Then $f = ug$ is the desired presentation.

Proof of (f): The irreducibility of g in $K[x]$ implies that $d > 0$. Our assumptions imply that pseudo.deg$(g) = d$. Hence, by Corollary 2.2.6, $g \not\in K\{x\}^\times$.

Now assume $g = g_1 g_2$, where $g_1, g_2 \in K\{x\}$ are nonunits. By Corollary 2.2.5, we may assume that $g_1 \in K[x]$ is monic, say of degree d_1, and $|g_1| = 1$. Thus pseudo.deg$(g_1) = d_1$. By Euclid's algorithm, there are $q, r \in K[x]$ such that $g = qg_1 + r$ and $\deg(r) < d_1$. Applying the additional presentation $g = g_2 g_1 + 0$ and the uniqueness part of Proposition 2.2.4, we get that $g_2 = q \in K[x]$. Thus, either $g_1 \in K[x]^\times \subseteq K\{x\}^\times$ or $g_1 \in K[x]^\times \subseteq K\{x\}^\times$. In both cases we get a contradiction.

Proof of (g): Let a be an element of K with $|a| < 1$. Then $ax - 1$ is irreducible in $K[x]$. On the other hand, pseudo.deg$(ax - 1) = 0$, so, by (c), $ax - 1 \in K\{x\}^\times$. In particular, $ax - 1$ is not irreducible in $K\{x\}$.

Proof of (h): We choose a as in the proof of (f) and consider the reducible polynomial $f(x) = (ax-1)(x-1)$. By the proof of (f), $ax-1 \in K\{x\}^\times$. Next we note that pseudo.deg$(x - 1) = 1$, so by (d) and (c), $x - 1$ is irreducible in $K\{x\}$. Consequently, $f(x)$ is irreducible in $K\{x\}$. □

Let $E = K(x)$ be the field of rational functions over K in the variable x. Then $K[x] \subseteq K\{x\}$ and the restriction of $|\ |$ to $K[x]$ is an absolute value. By the multiplicativity of $|\ |$, it extends to an absolute value of E. Let $\hat{E}$ be the completion of E with respect to $|\ |$ [CaF67, p. 47]. For each $\sum_{n=0}^\infty a_n x^n \in K\{x\}$ we have, by definition, $a_n \to 0$, hence $\sum_{n=0}^\infty a_n x^n = \lim_{n\to\infty} \sum_{i=0}^n a_i x^i$. Thus, $K[x]$ is dense in $K\{x\}$. Since $K\{x\}$ is complete (Lemma 2.2.1(c)), this implies that $K\{x\}$ is the closure of $K[x]$ in $\hat{E}$.

Remark 2.3.2:

(a) $|x| = 1$.

(b) Let $\bar{K} \subseteq \bar{E}$ be the residue fields of $K \subseteq E$ with respect to $|\ |$. Denote the image in $\bar{E}$ of an element $u \in K(x)$ with $|u| \le 1$ by $\bar{u}$. Then $\bar{x}$ is transcendental over $\bar{K}$. Indeed, let h be a monic polynomial over $\bar{K}$. Choose a monic polynomial p with coefficients in the valuation ring of K such that $\bar{p} = h$. Since $|p(x)| = 1$, we have $h(\bar{x}) = \bar{p}(\bar{x}) \neq 0$. It follows that $\bar{K}(\bar{x})$ is the field of rational functions over $\bar{K}$ in the variable $\bar{x}$ and $\bar{K}(\bar{x}) \subseteq \bar{E}$. Moreover, $\bar{K}(\bar{x}) = \bar{E}$. Indeed, let $u = \frac{f(x)}{g(x)}$ with $f = \sum_{i=0}^{m} a_i x^i$, $g = \sum_{j=0}^{n} b_j x^j \neq 0$, and $a_i, b_j \in K$ such that $|u| \le 1$. Then $\max_i |a_i| \le \max_j |b_j|$. Choose $c \in K$ with $|c| = \max_j |b_j|$. Then replace a_i with $c^{-1}a_i$ and b_j with $c^{-1}b_j$, if necessary, to assume that $|a_i|, |b_j| \le 1$ for all i, j and there exists k with $|b_k| = 1$. Under these assumptions, $\bar{u} = \frac{\bar{f}(\bar{x})}{\bar{g}(\bar{x})} \in \bar{K}(\bar{x})$, as claimed.

(c) If $|\cdot|'$ is an absolute value of E which coincides with $|\ |$ on K and the residue x' of x with respect to $|\ |'$ is transcendental over $\bar{K}$, then $|\ |'$ coincides with $|\ |$.

Indeed, let $p(x) = \sum_{i=0}^{n} a_i x^i$ be a nonzero polynomial in $K[x]$. Choose a $c \in K^\times$ with $|c| = \max_i |a_i|$. Then $(c^{-1}p(x))' = \sum_{i=0}^{n} (c^{-1}a_i)'(x')^i \neq 0$ (the prime indicates the residue with respect to $|\ |'$), hence $|c^{-1}p(x)|' = 1$, so $|p(x)|' = |c| = |p(x)|$.

(d) It follows from (c) that if γ is an automorphism of E that leaves K invariant, preserves the absolute value of K, and $\overline{x^\gamma}$ is transcendental over $\bar{K}$, then γ preserves the absolute value of E.

In particular, γ is $|\ |$-continuous. Moreover, if $(x_1, x_2, x_3, \ldots)$ is a $|\ |$-Cauchy sequence in E, then so is $(x_1^\gamma, x_2^\gamma, x_3^\gamma, \ldots)$. Hence γ extends uniquely to a continuous automorphism of the $|\ |$-completion $\hat{E}$ of E.

(e) Now suppose K is a finite Galois extension of a complete field K_0 with respect to $|\ |$ and set $E_0 = K_0(x)$. Let $\gamma \in \mathrm{Gal}(K/K_0)$ and extend γ in the unique possible way to an element $\gamma \in \mathrm{Gal}(E/E_0)$. Then γ preserves $|\ |$ on K. Indeed, $|z|' = |z^\gamma|$ is an absolute valued of K. Since K_0 is complete with respect to $|\ |$, K_0 is Henselian, so $|\ |'$ is equivalent to $|\ |$. Thus, there exists $\varepsilon > 0$ with $|z^\gamma| = |z|^\varepsilon$ for each $z \in K$. In particular, $|z| = |z|^\varepsilon$ for each $z \in K_0$, so $\varepsilon = 1$, as claimed. In addition $x^\gamma = x$. By (d), γ preserves $|\ |$ also on E.

(f) Under the assumptions of (e) we let $\hat{E}_0$ and $\hat{E}$ be the $|\ |$-completions of E and E_0, respectively. Then $\hat{E}_0 E$ is a finite separable extension of $\hat{E}_0$ in $\hat{E}$. As such $\hat{E}_0 E$ is complete [CaF67, p. 57, Cor. 2] and contains E, so $\hat{E}_0 E = \hat{E}$. Thus, $\hat{E}/\hat{E}_0$ is a finite Galois extension.

By (d) and (e) each $\gamma \in \mathrm{Gal}(E/E_0)$ extends uniquely to a continuous automorphism γ of $\hat{E}$. Every $x \in \hat{E}_0$ is the limit of a sequence $(x_1, x_2, x_3, \ldots)$ of elements of E_0. Since $x_i^\gamma = x_i$ for each i, we have $x^\gamma = x$. It follows that res: $\mathrm{Gal}(\hat{E}/\hat{E}_0) \to \mathrm{Gal}(E/E_0)$ is an isomorphism.

(g) Finally suppose $y = \frac{ax+b}{cx+d}$ with $a, b, c, d \in K$ such that $|a|, |b|, |c|, |d| \le 1$ and $\bar{a}\bar{d} - \bar{b}\bar{c} \neq 0$. Then $\bar{a}\bar{x} + \bar{b}$ and $\bar{c}\bar{x} + \bar{d}$ are nonzero elements of $\bar{K}(\bar{x})$, so $\bar{y} = \frac{\bar{a}\bar{x}+\bar{b}}{\bar{c}\bar{x}+\bar{d}} \in \bar{K}(\bar{x})$. Moreover, $\bar{K}(\bar{x}) = \bar{K}(\bar{y})$, hence $\bar{y}$ is transcendental over $\bar{K}$. We conclude from (c) that the map $x \mapsto y$ extends to a K-automorphism of $K(x)$ that preserves the absolute value. It therefore extends to an isomorphism $\sum a_n x^n \to \sum a_n y^n$ of $K\{x\}$ onto $K\{y\}$. $\square$

In the following theorem we refer to an equivalence class of a valuation of a field F as a **prime** of F. For each prime $\mathfrak{p}$ we choose a valuation $v_{\mathfrak{p}}$ representing the prime and let $O_{\mathfrak{p}}$ be the corresponding valuation ring.

We say that an ultrametric absolute value $|\ |$ of a field K is **discrete**, if the group of all values $|a|$ with $a \in K^\times$ is isomorphic to $\mathbb{Z}$.

THEOREM 2.3.3: *Let K be a complete field with respect to a nontrivial ultrametric absolute value $|\ |$. Then $F = \mathrm{Quot}(K\{x\})$ is a Hilbertian field.*

Proof: Let $O = \{a \in K \mid |a| \le 1\}$ be the valuation ring of K with respect to $|\ |$ and let $D = O\{x\} = \{f \in K\{x\} \mid |f| \le 1\}$. Each $f \in K\{x\}$ can be written as af_1 with $a \in K$, $f_1 \in D$, and $|f_1| = 1$. Hence, $\mathrm{Quot}(D) = F$.

We construct a set S of prime divisors of F that satisfies the following conditions:

(1a) For each $\mathfrak{p} \in S$, $v_{\mathfrak{p}}$ is a real valuation (i.e. $v_{\mathfrak{p}}(F) \subseteq \mathbb{R}$).

(1b) The valuation ring $O_{\mathfrak{p}}$ of $v_{\mathfrak{p}}$ is the local ring of D at the prime ideal $\mathfrak{m}_{\mathfrak{p}} = \{f \in D \mid v_{\mathfrak{p}}(f) > 0\}$.

(1c) $D = \bigcap_{\mathfrak{p} \in S} O_{\mathfrak{p}}$.

(1d) For each $f \in F^\times$ the set $\{\mathfrak{p} \in S \mid v_{\mathfrak{p}}(f) \ne 0\}$ is finite.

(1e) The Krull dimension of D is at least 2.

Then D is a **generalized Krull domain of dimension exceeding** 1. A theorem of Weissauer [FrJ05, Thm. 15.4.6] will then imply that F is Hilbertian.

THE CONSTRUCTION OF S: The absolute value $|\ |$ of $K\{x\}$ extends to an absolute value of F. The latter determines a prime $\mathfrak{M}$ of F with a real valuation $v_{\mathfrak{M}}$ (Section 2.1). Each $u \in F$ with $|u| \le 1$ can be written as $u = a\frac{f_1}{g_1}$ with $a \in O$ and $f_1, g_1 \in D$, $|f_1| = |g_1| = 1$. Hence, $O_{\mathfrak{M}} = D_{\mathfrak{m}}$, where $\mathfrak{m} = \{f \in D \mid |f| < 1\}$.

By Proposition 2.3.1, each nonzero prime ideal of $K\{x\}$ is generated by a prime element $p \in K\{x\}$. Divide p by its pseudo leading coefficient, if necessary, to assume that $|p| = 1$. Then let v_p be the discrete valuation of F determined by p and let $\mathfrak{p}_p$ be its equivalence class. We prove that p is a prime element of D. This will prove that pD is a prime ideal of D and its local ring will coincide with the valuation ring of v_p.

Indeed, let f, g be nonzero elements of D such that p divides fg in D. Write $f = af_1$, $g = bg_1$ with nonzero $a, b \in O$, $f_1, g_1 \in D$, $|f_1| = |g_1| = 1$. Then p divides f_1g_1 in $K\{x\}$ and therefore it divides, say, f_1 in $K\{x\}$. Thus, there exists $q \in K\{x\}$ with $pq = f_1$. But then $|q| = 1$, so $q \in D$. Consequently, p divides f in D, as desired.

Let P be the set of all prime elements p as in the paragraph before the preceding one. Then $S = \{\mathfrak{p}_p \mid p \in P\} \cup \{\mathfrak{M}\}$ satisfies (1a) and (1b).

By Proposition 2.3.1(b), $K\{x\}$ is a unique factorization domain, hence $K\{x\} = \bigcap_{p \in P} O_p$, hence $\bigcap_{\mathfrak{p} \in S} O_{\mathfrak{p}} = \{f \in K\{x\} \mid |f| \le 1\} = D$. This settles (1c).

Next observe that for each $f \in F^\times$ there are only finitely many $p \in P$ such that $v_p(f) \neq 0$, so (1d) holds.

Finally note that if $f = \sum_{n=0}^\infty a_n x^n$ is in D, then $|a_n| \leq 1$ for all n and $|a_n| < 1$ for all large n. Hence $D/\mathfrak{m} \cong \bar{K}[\bar{x}]$, where $\bar{K}$ and $\bar{x}$ are as in Remark 2.3.2(b). Since $\bar{x}$ is transcendental over $\bar{K}$, $\mathfrak{m}$ is a nonzero prime ideal and $\mathfrak{m} + Ox$ is a prime ideal of D that properly contains $\mathfrak{m}$. This proves (1e) and concludes the proof of the theorem. □

Corollary 2.3.4: $\mathrm{Quot}(K\{x\})$ *is not a Henselian field.*

Proof: Since $K\{x\}$ is Hilbertian (Theorem 2.3.3), $K\{x\}$ can not be Henselian [FrJ08, Lemma 15.5.4]. □

2.4 Convergent Power Series

Let K be a complete field with respect to an ultrametric absolute value $|\ |$. We say that a formal power series $f = \sum_{n=m}^\infty a_n x^n$ in $K((x))$ **converges** at an element $c \in K$, if $f(c) = \sum_{n=m}^\infty a_n c^n$ converges, i.e. $a_n c^n \to 0$. In this case f converges at each $b \in K$ with $|b| \leq |c|$. For example, each $f \in K\{x\}$ converges at 1. We say that f **converges** if f converges at some $c \in K^\times$.

We denote the set of all convergent power series in $K((x))$ by $K((x))_0$ and prove that $K((x))_0$ is a field that contains $K\{x\}$ and is algebraically closed in $K((x))$.

Lemma 2.4.1: *A power series* $f = \sum_{n=m}^\infty a_n x^n$ *in* $K((x))$ *converges if and only if there exists a positive real number* γ *such that* $|a_n| \leq \gamma^n$ *for each* $n \geq 0$.

Proof: First suppose f converges at $c \in K^\times$. Then $a_n c^n \to 0$, so there exists $n_0 \geq 1$ such that $|a_n c^n| \leq 1$ for each $n \geq n_0$. Choose

$$\gamma = \max\{|c|^{-1}, |a_k|^{1/k} \mid k = 0, \ldots, n_0 - 1\}.$$

Then $|a_n| \leq \gamma^n$ for each $n \geq 0$.

Conversely, suppose $\gamma > 0$ and $|a_n| \leq \gamma^n$ for all $n \geq 0$. Increase γ, if necessary, to assume that $\gamma > 1$. Then choose $c \in K^\times$ such that $|c| \leq \gamma^{-1.5}$ and observe that $|a_n c^n| \leq \gamma^{-0.5n}$ for each $n \geq 0$. Therefore, $a_n c^n \to 0$, hence f converges at c. □

Lemma 2.4.2: $K((x))_0$ *is a field that contains* $\mathrm{Quot}(K\{x\})$, *hence also* $K(x)$.

Proof: The only difficulty is to prove that if $f = 1 + \sum_{n=1}^\infty a_n x^n$ converges, then also $f^{-1} = 1 + \sum_{n=1}^\infty a'_n x^n$ converges.

Indeed, for $n \geq 1$, a'_n satisfies the recursive relation $a'_n = -a_n - \sum_{i=1}^{n-1} a_i a'_{n-i}$. By Lemma 2.4.1, there exists $\gamma > 1$ such that $|a_i| \leq \gamma^i$ for each $i \geq 1$. Set $a'_0 = 1$. Suppose, by induction, that $|a'_j| \leq \gamma^j$ for $j = 1, \ldots, n-1$. Then $|a'_n| \leq \max_i(|a_i| \cdot |a'_{n-i}|) \leq \gamma^n$. Hence, f^{-1} converges. □

Let v be the valuation of $K((x))$ defined by

$$v(\sum_{n=m}^{\infty} a_n x^n) = m \qquad \text{for } a_m, a_{m+1}, a_{m+2}, \ldots \in K \text{ with } a_m \neq 0.$$

It is discrete, complete, its valuation ring is $K[[x]]$, and $v(x) = 1$. The residue of an element $f = \sum_{n=0}^{\infty} a_n x^n$ of $K[[x]]$ at v is a_0, and we denote it by $\bar{f}$. We also consider the valuation ring $O = K[[x]] \cap K((x))_0$ of $K((x))_0$ and denote the restriction of v to $K((x))_0$ also by v. Since $K((x))_0$ contains $K(x)$, it is v-dense in $K((x))$. Finally, we also denote the unique extension of v to the algebraic closure of $K((x))$ by v.

Remark 2.4.3: $K((x))_0$ is not complete. Indeed, choose $a \in K$ such that $|a| > 1$. Then there exists no $\gamma > 0$ such that $|a^{n^2}| \leq \gamma^n$ for all $n \geq 1$. By Lemma 2.4.1, the power series $f = \sum_{n=0}^{\infty} a^{n^2} x^n$ does not belong to $K((x))_0$. Therefore, the valued field $(K((x))_0, v)$ is not complete. $\square$

LEMMA 2.4.4: *The field $K((x))_0$ is separably algebraically closed in $K((x))$.*

Proof: Let $y = \sum_{n=m}^{\infty} a_n x^n$, with $a_n \in K$, be an element of $K((x))$ which is separably algebraic of degree d over $K((x))_0$. We have to prove that $y \in K((x))_0$.

PART A: *A shift of y.* Assume that $d > 1$ and let $y_1, \ldots, y_d$, with $y = y_1$, be the (distinct) conjugates of y over $K((x))_0$. In particular $r = \max(v(y - y_i) \mid i = 2, \ldots, d)$ is an integer. Choose $s \geq r + 1$ and let

$$y_i' = \frac{1}{x^s}\big(y_i - \sum_{n=m}^{s} a_n x^n\big), \qquad i = 1, \ldots, d.$$

Then $y_1', \ldots, y_d'$ are the distinct conjugates of y_1' over $K((x))_0$. Also, $v(y_1') \geq 1$ and $y_i' = \frac{1}{x^s}(y_i - y) + y_1'$, so $v(y_i') \leq -1$, $i = 2, \ldots, d$. If y_1' belongs to $K((x))_0$, then so does y, and conversely. Therefore, we replace y_i by y_i', if necessary, to assume that

$$(1) \qquad v(y) \geq 1 \text{ and } v(y_i) \leq -1, \; i = 2, \ldots, d.$$

In particular $y = \sum_{n=0}^{\infty} a_n x^n$ with $a_0 = 0$. The elements $y_1, \ldots, y_d$ are the roots of an irreducible separable polynomial

$$h(Y) = p_d Y^d + p_{d-1} Y^{d-1} + \cdots + p_1 Y + p_0$$

with coefficients $p_i \in O$. Let $e = \min(v(p_0), \ldots, v(p_d))$. Divide the p_i, if necessary, by x^e, to assume that $v(p_i) \geq 0$ for each i between 0 and d and that $v(p_j) = 0$ for at least one j between 0 and d.

Part B: *We prove that* $v(p_0), v(p_d) > 0,\ \ v(p_k) > v(p_1)$ *if* $2 \le k \le d-1$ *and* $v(p_1) = 0$. Indeed, since $v(y) > 0$ and $h(y) = 0$, we have $v(p_0) > 0$. Since $v(y_2) < 0$ and $h(y_2) = 0$, we have $v(p_d) > 0$. Next observe that

$$\frac{p_1}{p_d} = \pm y_2 \cdots y_d \pm \sum_{i=2}^{d} \frac{y_1 \cdots y_d}{y_i}.$$

If $2 \le i \le d$, then $v(y_i) < v(y_1)$, so $v(y_2 \cdots y_d) < v(\frac{y_1}{y_i}) + v(y_2 \cdots y_d) = v(\frac{y_1 \cdots y_d}{y_i})$. Hence,

$$v\Big(\frac{p_1}{p_d}\Big) = v(y_2 \cdots y_d). \tag{2}$$

For k between 1 and $d-2$ we have

$$\frac{p_{d-k}}{p_d} = \pm \sum_{\sigma} \prod_{i=1}^{k} y_{\sigma(i)}, \tag{3}$$

where σ ranges over all monotonically increasing maps from $\{1, \ldots, k\}$ to $\{1, \ldots, d\}$. If $\sigma(1) \ne 1$, then $\{y_{\sigma(1)}, \ldots, y_{\sigma(k)}\}$ is properly contained in $\{y_2, \ldots, y_d\}$. Hence, $v(\prod_{i=1}^{k} y_{\sigma(i)}) > v(y_2 \cdots y_d)$. If $\sigma(1) = 1$, then

$$v\Big(\prod_{i=1}^{k} y_{\sigma(i)}\Big) > v\Big(\prod_{i=2}^{k} y_{\sigma(i)}\Big) > v(y_2 \cdots y_d).$$

Hence, by (2) and (3), $v(\frac{p_{d-k}}{p_d}) > v(\frac{p_1}{p_d})$, so $v(p_{d-k}) > v(p_1)$. Since $v(p_j) = 0$ for some j between 0 and d, since $v(p_i) \ge 0$ for every i between 0 and d, and since $v(p_0), v(p_d) > 0$, we conclude that $v(p_1) = 0$ and $v(p_i) > 0$ for all $i \ne 1$. Therefore,

$$p_k = \sum_{n=0}^{\infty} b_{kn} x^n, \qquad k = 0, \ldots, d \tag{4}$$

with $b_{kn} \in K$ such that $b_{1,0} \ne 0$ and $b_{k,0} = 0$ for each $k \ne 1$. In particular, $|b_{1,0}| \ne 0$ but unfortunately, $|b_{1,0}|$ may be smaller than 1.

Part C: *Making* $|b_{1,0}|$ *large.* We choose $c \in K$ such that $|c^{d-1} b_{1,0}| \ge 1$ and let $z = cy$. Then z is a zero of the polynomial $g(Z) = p_d Z^d + c p_{d-1} Z^{d-1} + \cdots + c^{d-1} p_1 Z + c^d p_0$ with coefficients in O. Relation (4) remains valid except that the zero term of the coefficient of Z in g becomes $c^{d-1} b_{1,0}$. By the choice of c, its absolute value is at least 1. So, without loss, we may assume that

$$|b_{1,0}| \ge 1. \tag{5}$$

Part D: *An estimate for* $|a_n|$. By Lemma 2.4.1, there exists $\gamma > 0$ such that $|b_{kn}| \leq \gamma^n$ for all $0 \leq k \leq d$ and $n \geq 1$. By induction we prove that $|a_n| \leq \gamma^n$ for each $n \geq 0$. This will prove that $y \in O$ and will conclude the proof of the lemma.

Indeed, $|a_0| = 0 < 1 = \gamma^0$. Now assume that $|a_m| \leq \gamma^m$ for each $0 \leq m \leq n-1$. For each k between 0 and d we have that $p_k y^k = \sum_{n=0}^{\infty} c_{kn} x^n$, where

$$c_{kn} = \sum_{\sigma \in S_{kn}} b_{k,\sigma(0)} \prod_{j=1}^{k} a_{\sigma(j)},$$

and

$$S_{kn} = \{\sigma\colon \{0,\dots,k\} \to \{0,\dots,n\} \mid \sum_{j=0}^{k} \sigma(j) = n\}.$$

It follows that

$$c_{0n} = b_{0n} \text{ and } c_{1n} = b_{1,0}a_n + b_{11}a_{n-1} + \cdots + b_{1,n-1}a_1. \tag{6}$$

For $k \geq 2$ we have $b_{k,0} = 0$. Hence, if a term $b_{k,\sigma(0)} \prod_{j=1}^{k} a_{\sigma(j)}$ in c_{kn} contains a_n, then $\sigma(0) = 0$, so $b_{k,\sigma(0)} = 0$. Thus,

$$\begin{aligned} c_{kn} = \text{sum of products of the form } b_{k,\sigma(0)} \prod_{j=1}^{k} a_{\sigma(j)}, \\ \text{with } \sigma(j) < n,\ j = 1,\dots,k. \end{aligned} \tag{7}$$

From the relation $\sum_{k=0}^{d} p_k y^k = h(y) = 0$ we conclude that $\sum_{k=0}^{d} c_{kn} = 0$ for all n. Hence, by (6),

$$b_{1,0}a_n = -b_{0n} - b_{11}a_{n-1} - \cdots - b_{1,n-1}a_1 - c_{2n} - \cdots - c_{dn}.$$

Therefore, by (7),

$$\begin{aligned} b_{1,0}a_n = \text{sum of products of the form } -b_{k,\sigma(0)} \prod_{j=1}^{k} a_{\sigma(j)}, \\ \text{with } \sigma \in S_{kn},\ 0 \leq k \leq d, \text{ and } \sigma(j) < n,\ j = 1,\dots,k. \end{aligned} \tag{8}$$

Note that $b_{k,0} = 0$ for each $k \neq 1$ (by (4)), while $b_{1,0}$ does not occur on the right hand side of (8). Hence, for a summand in the right hand side of (8) indexed by σ we have

$$|b_{k,\sigma(0)} \prod_{j=1}^{k} a_{\sigma(j)}| \leq \gamma^{\sum_{j=0}^{k} \sigma(j)} = \gamma^n.$$

We conclude from $|b_{1,0}| \geq 1$ that $|a_n| \leq \gamma^n$, as contended. $\square$

Proposition 2.4.5: *The field $K((x))_0$ is algebraically closed in $K((x))$. Thus, each $f \in K((x))$ which is algebraic over $K(x)$ converges at some $c \in K^\times$. Moreover, there exists a positive integer m such that f converges at each $b \in K^\times$ with $|b| \leq \frac{1}{m}$.*

Proof: In view of Lemma 2.4.4, we have to prove the proposition only for $\mathrm{char}(K) > 0$. Let $f = \sum_{n=m}^{\infty} a_n x^n \in K((x))$ be algebraic over $K((x))_0$. Then $K((x))_0(f)$ is a purely inseparable extension of a separable algebraic extension of $K((x))_0$. By Lemma 2.4.4, the latter coincides with $K((x))_0$. Hence, $K((x))_0(f)$ is a purely inseparable extension of $K((x))_0$.

Thus, there exists a power q of $\mathrm{char}(K)$ such that $\sum_{n=m}^{\infty} a_n^q x^{nq} = f^q \in K((x))_0$. By Lemma 2.4.1, there exists $\gamma > 0$ such that $|a_n^q| \leq \gamma^{nq}$ for all $n \geq 1$. It follows that $|a_n| \leq \gamma^n$ for all $n \geq 1$. By Lemma 2.4.1, $f \in K((x))_0$, so there exists $c \in K^\times$ such that f converges at c. If $\frac{1}{m} \leq |c|$, then f converges at each $b \in K^\times$ with $|b| \leq \frac{1}{m}$. □

Corollary 2.4.6: *The valued field $(K((x))_0, v)$ is Henselian.*

Proof: Consider the valuation ring $O = K[[x]] \cap K((x))_0$ of $K((x))_0$ at v. Let $f \in O[X]$ be a monic polynomial and $a \in O$ such that $v(f(a)) > 0$ and $v(f'(a)) \neq 0$. Since $(K((x)), v)$ is Henselian, there exists $z \in K[[x]]$ such that $f(z) = 0$ and $v(z - a) > 0$. By Proposition 2.4.5, $z \in K((x))_0$, hence $z \in O$. It follows that $(K((x))_0, v)$ is Henselian. □

2.5 The Regularity of $K((x))/K((x))_0$

Let K be a complete field with respect to an ultrametric absolute value $|\ |$. We extend $|\ |$ in the unique possible way to $\tilde{K}$. We also consider the discrete valuation v of $K(x)/K$ defined by $v(a) = 0$ for each $a \in K^\times$ and $v(x) = 1$. Then $K((x))$ is the completion of $K(x)$ at v. Let $K((x))_0$ be the subfield of $K((x))$ of all convergent power series.

Proposition 2.4.5 states that $K((x))_0$ is algebraically closed in $K((x))$. In this section we prove that $K((x))$ is even a regular extension of $K((x))_0$. To do this, we only have to assume that $p = \mathrm{char}(K) > 0$ and prove that $K((x))/K((x))_0$ is a separable extension. In other words, we have to prove that $K((x))$ is linearly disjoint from $K((x))_0^{1/p}$ over $K((x))_0$. We do that in several steps.

Lemma 2.5.1: *The fields $K((x))$ and $K((x^{1/p}))_0$ are linearly disjoint over $K((x))_0$.*

Proof: First note that $1, x^{1/p}, \ldots, x^{p-1/p}$ is a basis for $K(x^{1/p})$ over $K(x)$. Then $1, x^{1/p}, \ldots, x^{p-1/p}$ have distinct v-values modulo $\mathbb{Z} = v(K((x)))$, so they are linearly independent over $K((x))$.

Next we observe that $1, x^{1/p}, \ldots, x^{p-1/p}$ also generate $K((x^{1/p}))$ over $K((x))$. Indeed, each $f \in K((x^{1/p}))$ may be multiplied by an appropriate power of x to be presented as

$$f = \sum_{n=0}^{\infty} a_n x^{n/p}, \tag{1}$$

with $a_0, a_1, a_2, \ldots \in K$. We write each n as $n = kp + l$ with integers $k \geq 0$ and $0 \leq l \leq p-1$ and rewrite f as

$$f = \sum_{l=0}^{p-1} \Big(\sum_{k=0}^{\infty} a_{kp+l} x^k \Big) x^{l/p}. \tag{2}$$

If $f \in K((x^{1/p}))_0$, then there exists $b \in K^\times$ such that $\sum_{n=0}^{\infty} a_n b^{n/p}$ converges in K, hence $a_n b^{n/p} \to 0$ as $n \to \infty$, so $a_{kp+l} b^k b^{l/p} \to 0$ as $k \to \infty$ for each l. Therefore, for each l, we have $a_{kp+l} b^k \to 0$ as $k \to \infty$, hence $\sum_{k=0}^{\infty} a_{kp+l} x^k$ converges, so belongs to $K((x))_0$.

It follows that $1, x^{1/p}, \ldots, x^{p-1/p}$ form a basis for $K((x^{1/p}))_0/K((x))_0$ as well as for $K((x^{1/p}))/K((x))$. Consequently, $K((x))$ is linearly disjoint from $K((x^{1/p}))_0$ over $K((x))_0$. $\square$

We set $K[[x]]_0 = K[[x]] \cap K((x))_0$.

Lemma 2.5.2: *Let $u_1, \ldots, u_m \in \tilde{K}[[x]]_0$ and $f_1, \ldots, f_m \in K[[x]]$. Set $u_{i0} = u_i(0)$ for $i = 1, \ldots, m$ and*

$$f = \sum_{i=1}^{m} f_i u_i. \tag{3}$$

Suppose $u_{10}, \ldots, u_{m0}$ are linearly independent over K, $f \in \tilde{K}[[x]]_0$, and $f(0) = 0$. Then $f_1, \ldots, f_m \in K[[x]]_0$.

Proof: We break up the proof into several parts.

Part A: *Comparison of norms.* We consider the K-vector space $V = \sum_{i=1}^{m} K u_{i0}$ and define a function $\mu\colon V \to \mathbb{R}$ by

$$\mu\Big(\sum_{i=1}^{m} a_i u_{i0}\Big) = \max(|a_1|, \ldots, |a_m|). \tag{4}$$

It satisfies the following rules:
(5a) $\mu(v) > 0$ for each nonzero $v \in V$.
(5b) $\mu(v + v') \leq \max(\mu(v), \mu(v'))$ for all $v, v' \in V$.
(5c) $\mu(av) = |a|\mu(v)$ for all $a \in K$ and $v \in V$.

Thus, v is a **norm** of V. On the other hand, $|\ |$ extends to an absolute value of $\tilde{K}$ and its restriction to V is another norm of V. Since K is complete under $|\ |$, there exists a positive real number s such that
(6) $\mu(v) \leq s|v|$ for all $v \in V$
[CaF67, p. 52, Lemma].

Part B: *Power series.* For each i we write $u_i = u_{i0} + u_i'$ where $u_i' \in \tilde{K}[[x]]_0$ and $u_i'(0) = 0$. Then

(7a) $$f = \sum_{n=1}^{\infty} a_n x^n \quad \text{with } a_1, a_2, \ldots \in \tilde{K},$$

(7b) $$u_i' = \sum_{n=1}^{\infty} b_{in} x^n \quad \text{with } b_{i1}, b_{i2}, \ldots \in \tilde{K}, \text{ and}$$

(7c) $$f_i = \sum_{n=0}^{\infty} a_{in} x^n \quad \text{with } a_{i0}, a_{i1}, a_{i2}, \ldots \in K.$$

If a power series converges at a certain element of $\tilde{K}^\times$, it converges at each element with a smaller absolute value. Since to each element of $\tilde{K}^\times$ there exists an element of $K^\times$ with a smaller absolute value, there exists $d \in K^\times$ such that $\sum_{n=1}^{\infty} a_n d^n$ and $\sum_{n=1}^{\infty} b_{in} d^n$, $i = 1, \ldots, m$, converge. In particular, the numbers $|a_n d^n|$ and $|b_{in} d^n|$ are bounded. It follows from the identities $|a_n c^n| = |a_n d^n| \cdot \left|\frac{c}{d}\right|^n$ and $|b_{in} c^n| = |b_{in} d^n| \cdot \left|\frac{c}{d}\right|^n$ that there exists $c \in K^\times$ such that

(8) $$\max_{n \ge 1} |a_n c^n| \le s^{-1} \quad \text{and} \quad \max_{n \ge 1} |b_{in} c^n| \le s^{-1}$$

for $i = 1, \ldots, m$.

Part C: *Claim:* $|a_{in} c^n| \le 1$ *for* $i = 1, \ldots, m$ *and* $n = 0, 1, 2, \ldots$. To prove the claim we substitute the presentations (7) of f, u_i', f_i in the relation (3) and get:

(9) $$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \sum_{j=1}^{m} a_{jn} u_{j0} x^n + \sum_{n=1}^{\infty} \sum_{j=1}^{m} \sum_{k=0}^{n-1} a_{jk} b_{j,n-k} x^n.$$

In particular, for $n = 0$ we get $0 = \sum_{j=1}^{m} a_{j0} u_{j0}$. Since $u_{10}, \ldots, u_{m0}$ are linearly independent over K and $a_{10}, \ldots, a_{m0} \in K$, we get $a_{10} = \cdots = a_{m0} = 0$, so our claim holds in this case.

Proceeding by induction, we assume $|a_{ik} c^k| \le 1$ for $i = 1, \ldots, m$ and $k = 0, \ldots, n-1$. By (5) and (6),

$$|a_{in}| \le \max(|a_{1n}|, \ldots, |a_{mn}|) = \mu\Big(\sum_{j=1}^{m} a_{jn} u_{j0}\Big) \le s\Big|\sum_{j=1}^{m} a_{jn} u_{j0}\Big|,$$

hence

(10) $$|a_{in} c^n| \le s\Big|\sum_{j=1}^{m} a_{jn} u_{j0} c^n\Big|.$$

Next we compare the coefficients of x^n on both sides of (9),

$$a_n = \sum_{j=1}^{m} a_{jn} u_{j0} + \sum_{j=1}^{m} \sum_{k=0}^{n-1} a_{jk} b_{j,n-k},$$

change sides and multiply the resulting equation by c^n:

$$\sum_{j=1}^{m} a_{jn} u_{j0} c^n = a_n c^n - \sum_{j=1}^{m} \sum_{k=0}^{n-1} a_{jk} c^k \cdot b_{j,n-k} c^{n-k}.$$

By the induction hypothesis and by (8),

$$(11)\quad |\sum_{j=1}^{m} a_{jn} u_{j0} c^n| \le \max\big(|a_n c^n|, \max_{1\le j\le m} \max_{0\le k\le n-1} |a_{jk} c^k| \cdot |b_{j,n-k} c^{n-k}|\big)$$
$$\le \max(s^{-1}, 1 \cdot s^{-1}) = s^{-1}$$

It follows from (10) and (11) that $|a_{in} c^n| \le 1$. This concludes the proof of the claim.

PART D: *End of the proof.* We choose $a \in K^\times$ such that $|a| < |c|$. Then $|a_{in} a^n| = \left|a_{in} c^n \left(\frac{a}{c}\right)^n\right| \le \left|\frac{a}{c}\right|^n$. Since the right hand side tends to 0 as $n \to \infty$, so does the left hand side. We conclude that f_i converges at a. $\square$

LEMMA 2.5.3: *The fields $K((x))$ and $K^{1/p}((x))_0$ are linearly disjoint over $K((x))_0$.*

Proof: We have to prove that every finite extension F' of $K((x))_0$ in $K^{1/p}((x))_0$ is linearly disjoint from $K((x))$ over $K((x))_0$.

If $F' = K((x))_0$, there is nothing to prove, so we assume F' is a proper extension of $K((x))$. Each element $f' \in F'$ has the form $f' = \sum_{i=k}^{\infty} b_i x^i$ with $b_i \in K^{1/p}$ and $\sum_{i=k}^{\infty} b_i c^i$ converges for some $c \in (K^{1/p})^\times$. Thus, $(f')^p = \sum_{i=k}^{\infty} b_i^p x^{ip} \in K((x))$ and $\sum_{i=k}^{\infty} b_i^p (c^p)^i$ converges, so $(f')^p \in K((x))_0$. We may therefore write $F' = F(f)$, where F is a finite extension of $K((x))_0$ in F' and $[F' : F] = p$.

By induction on the degree, F is linearly disjoint from $K((x))$ over $K((x))_0$. Let $m = [F : K((x))_0]$.

Moreover, $K((x))$ is the completion of $K(x)$, so also of $K((x))_0$. Hence, $\hat{F} = K((x))F$ is the completion of F under v. By the linear disjointness, $[\hat{F} : K((x))] = m$.

The residue field of $K((x))$ and of $K((x))_0$ is K and the residue field of $\hat{F}$ is equal to the residue field $\bar{F}$ of F. Both $K((x))$ and $K^{1/p}((x))$ have the same valuation group under v, namely $\mathbb{Z}$. Therefore, also $v(\hat{F}^\times) = \mathbb{Z}$, so $e(\hat{F}/K((x))) = 1$. Since $K((x))$ is complete and discrete, $[\hat{F} : K((x))] = e(\hat{F} : K((x)))[\bar{F} : K] = [\bar{F} : K]$ [CaF65, p. 19, Prop. 3].

Now we choose a basis $u_{10}, \ldots, u_{m0}$ for $\bar{F}/K$ and lift each u_{i0} to an element u_i of $F \cap \tilde{K}[[x]]_0$. Then, $u_1, \ldots, u_m$ are linearly independent over $K((x))_0$ and over $K((x))$, hence they form a basis for $F/K((x))_0$ and for $\hat{F}/K((x))$.

As before, $\widehat{F'} = K((x))F'$ is the completion of F'. Again, both F' and $\widehat{F'}$ have the same residue field $\overline{F'}$ and $[\widehat{F'} : \hat{F}] = [\overline{F'} : \bar{F}]$. Note that $\overline{F'} \subseteq K^{1/p}$ and $[\overline{F'} : \bar{F}] \le [F' : F] = p$. Therefore, $\overline{F'} = \bar{F}$ or $[\overline{F'} : \bar{F}] = p$.

In the first case $f \in \hat{F}$, so by the paragraph before the preceding one, there exist $f_1, \ldots, f_m \in K((x))$ such that $f = \sum_{i=1}^m f_i u_i$. Multiplying both sides by a large power of x, we may assume that $f_1, \ldots, f_m \in K[[x]]$ and $f(0) = 0$. By Lemma 2.5.2, $f_1, \ldots, f_m \in K((x))_0$, hence $f \in F$. This contradiction to the choice of f implies that $[\overline{F'} : \bar{F}] = p$. Hence, $[K((x))F' : K((x))F] = [\widehat{F'} : \hat{F}] = p = [F' : F]$. This implies that $\hat{F}$ and F' are linearly disjoint over F. By the tower property of linear disjointness, $K((x))$ and F' are linearly disjoint over $K((x))_0$, as claimed. □

PROPOSITION 2.5.4: *Let K be a complete field under an ultrametric absolute value $|\ |$ and denote the field of all convergent power series in x with coefficients in K by $K((x))_0$. Then $K((x))$ is a regular extension of $K((x))_0$.*

Proof: In view of Proposition 2.4.5, it suffices to assume that $p = \mathrm{char}(K) > 0$ and to prove that $K((x))$ is linearly disjoint from $K((x))_0^{1/p}$ over $K((x))_0$.

Indeed, by Lemma 2.5.3, $K((x))$ is linearly disjoint from $K^{1/p}((x))_0$ over $K((x))_0$. Next observe that $K^{1/p}$ is also complete under $|\ |$. Hence, by Lemma 2.5.1, applied to $K^{1/p}$ rather than to K, $K^{1/p}((x))$ is linearly disjoint from $K^{1/p}((x^{1/p}))_0$ over $K^{1/p}((x))_0$.

$$\begin{tikzcd}[arrows=dash]
K((x)) \arrow[r] \arrow[d] & K^{1/p}((x)) \arrow[r] \arrow[d] & K((x))^{1/p} = K^{1/p}((x^{1/p})) \arrow[d] \\
K((x))_0 \arrow[r] & K^{1/p}((x))_0 \arrow[r] & K((x))_0^{1/p} = K^{1/p}((x^{1/p}))_0
\end{tikzcd}$$

Finally we observe that $K((x))_0^{1/p} = K^{1/p}((x^{1/p}))_0$ to conclude that $K((x))$ is linearly disjoint from $K((x))_0^{1/p}$ over $K((x))_0$. □

Notes

The rings of convergent power series in one variable introduced in Section 2.2 are the rings of holomorphic functions on the closed unit disk that appear in [FrP04, Example 2.2]. Weierstrass Divison Theorem (Proposition 2.2.4) appears in [FrP, Thm. 3.1.1]. Our presentation follows the unpublished manuscript [Har05].

Proposition 2.4.5 appears as [Art67, p. 48, Thm. 14]. The proof given by Artin uses the method of Newton polynomials.

The property of $K\{x\}$ of being a principle ideal domain appears in [FrP, Thm. 2.2.9].

The proof that $K((x))/K((x))_0$ is a separable extension (Proposition 2.5.4) is due to Kuhlmann and Roquette [KuR96].

http://www.springer.com/978-3-642-15127-9

Algebraic Patching
Jarden, M.
2011, XXIV, 292 p., Hardcover
ISBN: 978-3-642-15127-9