Introduction

The central problem of contemporary Galois theory is to describe the absolute Galois group $\mathrm{Gal}(K)$ of a given field K. A less ambitious problem, also known as the "inverse Galois problem over K" is to list all finite quotients of $\mathrm{Gal}(K)$; that is, to find which finite groups occur as Galois groups over K.

1. Background.

1.1 The inverse problem of Galois theory over $\mathbb{Q}$. The case where K is the field $\mathbb{Q}$ of rational numbers has been the most prominent one. It is the consequence of the theory of cyclotomic extensions, developed in the 19th century, that every finite Abelian group can be realized as a Galois group over $\mathbb{Q}$. Using class field theory, Shafarevich was able in the 20th century to realize every finite solvable group over $\mathbb{Q}$. For a non-Abelian simple group, one usually tries to realize G over a finitely generated purely transcendental extension $E = \mathbb{Q}(x_1, \ldots, x_n)$ of $\mathbb{Q}$. Once this has been successfully done, it is possible to apply Hilbert's irreducibility theorem to specialize the Galois extension of E to a Galois extension of $\mathbb{Q}$ with an isomorphic Galois group. This procedure was initiated by Hilbert in [Hil1892], where he realized each one of the symmetric groups S_n and the alternating groups A_n over $\mathbb{Q}$. Further work along these lines was done by Matzat, Thompson, Völklein, and others in the fourth quarter of the 20th century. Starting from the Riemann existence theorem which gives a realization of each finite group G over $\mathbb{C}(x)$ with a detailed description of ramification, they used several criteria, most notably rigidity, to descend those realizations to Galois extensions over $\mathbb{Q}$. In this way they succeeded to realize all sporadic simple groups (except M_{23}) over $\mathbb{Q}$ and several families of simple non-Abelian finite groups. For more details, the reader is referred to [MaM99] and [Voe96].

1.2 Complex analytic methods. While a solution to the inverse Galois problem over $\mathbb{Q}$ seems to be still out of reach, interest in Galois theory has been extended to other base fields, especially to fields of local flavor. Starting again from the Riemann existence theorem, Fried and Völklein constructed for each finite group G and "ramification data" for G a complex analytic space H. Then they used a higher dimensional analog to the Riemann existence theorem due to Grauert and Remmert to prove that H is indeed an absolutely irreducible algebraic variety defined over $\mathbb{C}$. Moreover, if K is a subfield of $\mathbb{C}$ over which H is defined and $H(K) \neq \emptyset$, then G is realizable over $K(x)$ with additional information about branch points and inertia groups. If K is a **PAC** subfield of $\mathbb{C}$ (i.e. every absolutely irreducible variety defined over K has a K-rational point), then H can be chosen to be defined over K and then, by definition, $H(K) \neq \emptyset$. This shows that the inverse Galois problem over $K(x)$ has a positive solution. In the case when K is in addition Hilbertian, Fried and Völklein exploited the fact that $\mathrm{Gal}(K)$ is projective and proved that every finite split embedding problem over K is solvable. In particular, if K is countable, it follows that $\mathrm{Gal}(K)$ is isomorphic to the free profinite group $\hat{F}_\omega$ of rank $\aleph_0$, giving a satisfactory solution to the structure problem of Galois theory over K, and solving one of the open problems of Field Arithmetic [FrJ86, Problem 24.41] in characteristic 0. The reader may consult the original papers [FrV91] and [FrV92] as well as Völklein's book [Voe96] on that subject.

1.3 Formal Patching. The application of complex analytic methods restricts the results of the former subsection to fields of characteristic 0. In the general case one uses one of several methods of "patching".

The first method of this kind is called "Formal Patching". It uses Grothendieck's formal schemes and was developed by Harbater in [Hrb87] in order to prove that if R is a complete local domain which is not a field and $K = \mathrm{Quot}(R)$, then each finite group occurs as a Galois group over $K(x)$. It follows for example that the inverse Galois problem has a positive solution over $\mathbb{Q}_p(x)$ or over $K_0((x_1, \ldots, x_n))(x)$, where K_0 is a field and $n \geq 1$. Among further applications of Formal Patching by Harbater and his collaborators let us mention the solution of the generalized Abhyankar's problem over curves in positive characteristic [Hrb94a] (see also Remark 9.2.2). Moreover, they proved that if F is a function field of one variable over a separably closed field, then $\mathrm{Gal}(F)$ is a free profinite group of rank $\mathrm{card}(F)$ [Hrb95].

1.4 Rigid Patching. Following an idea of Serre, Liu translated in 1990 Harbater's method to the language of rigid analytic geometry and reproved that for each complete field K under a nonarchimedean absolute value, every finite group is realizable over $K(x)$ [Liu95]. Instead of "Formal Patching" one speaks here about "Rigid Patching". An account of rigid analytic geometry can be found in [FrP04].

At the end of 1990, Pop proved that if a field K is either PAC or Henselian, then each finite group has a K-regular realization over $K(x)$. As explained in a letter from Roquette to Geyer from December 1990, this follows from the result of Harbater-Liu, because in both cases, K is existentially closed in $K((t))$. This common property of PAC fields and Henselian fields was at that time somewhat surprising, because by Frey-Prestel, a field K cannot be both PAC and Henselian except if K is separably closed [FrJ08, Cor. 11.5.5].

Pop formalizes that property in [Pop96]. He calls a field K **large** if K is existentially closed in $K((t))$. Alternatively, K is large (or **ample** as we prefer to call it) if every absolutely irreducible K-curve C with a K-rational simple point has infinitely many K-rational points [Pop96, Prop. 1.1]. In particular, $K((t))$ itself is ample, because it is Henselian.

Using rigid patching, Pop proved that every finite split embedding problem over $K((t))$ has a $K((t))$-regular solution over $K((t))(x)$ [Pop96, Lemma 1.4]. In particular, this reproves the result of Harbater and Liu. Again, if K is ample, then the same statement holds over K [Pop96, Main Theorem A].

In particular, if K is PAC, then every finite split embedding problem over $K(x)$ is solvable. If, in addition, K is Hilbertian, then every finite split embedding problem over K is solvable. Since the absolute Galois group of a PAC field is projective, each finite embedding problem over K can be reduced to a finite split embedding problem [FrJ08, Thm. 11.6.2]. It follows that every finite embedding problem over K is solvable. If further, K is countable, then by Iwasawa [FrJ08, Thm. 24.8.1], $\mathrm{Gal}(K) \cong \hat{F}_\omega$ [Pop96, Thm. 1]. This completes the proof of [FrJ86, Problem 24.41] in the general case.

If K is algebraically closed, then by Tsen Theorem (Proposition 9.4.6), $\mathrm{Gal}(K(x))$ is projective. Since K is ample, every finite split embedding problem over $K(x)$ is solvable. If in addition, K is countable, then as in the preceding paragraph, $\mathrm{Gal}(K(x)) \cong \hat{F}_\omega$. In particular, $\mathrm{Gal}(\tilde{\mathbb{F}}_p(x)) \cong \hat{F}_\omega$. Note that in the latter case, $\tilde{\mathbb{F}}_p(x)$ is obtained from $\mathbb{F}_p(x)$ by adjoining all roots of unity. Thus, the latter result appears as an analog to the still open problem of Shafarevich that asks whether $\mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}})$ is free.

Generalizing the analog of Shafarevich's conjecture to the case where K is an algebraically closed field of an arbitrary infinite cardinality m, Harbater [Hrb95] and Pop [Pop95] independently proved that if E is a function field of one variable over K, then every finite split embedding problem over E has m solutions. Adding the projectivity of $\mathrm{Gal}(E)$ a generalization of Iwasawa's theorem due to Melnikov-Chatzidakis, that implies that $\mathrm{Gal}(E) \cong \hat{F}_m$.

We note that [Pop96] was printed from a manuscript that was ready in 1993. In a subsequent manuscript [Pop93], Pop applies rigid patching once more to prove that if E is a function field of one variable over an ample field K (called "a field with a universal local-global principle" in that paper), then every finite split embedding problem over E has a "regular solution".

Harbater improved Pop's result and proved that if K is an ample field of cardinality m, E is a function field of one variable over K, and $\mathcal{E}$ is a finite split embedding problem, than the number of solutions of $\mathcal{E}$ is m [Hrb09, Thm. 3.4]. Theorem A below improves Harbater's result even further.

Another application of Rigid Patching by Pop was to reduce the general Abhyankar's conjecture to the special one over the affine line proved by Raynaud [Pop95]. A detailed account of the patching methods mentioned so far can be found in [Hrb03].

Both Formal Patching and Rigid Patching draw inspiration from "cut-and-paste" methods in topology and analysis, in which spaces are constructed on metric open sets and glued on overlaps. In the case of Formal Patching, one considers "formal opens" which are defined by rings of formal power series and patches them together in order to get a formal total space. By applying the "formal GAGA", more precisely Gorthendieck's existence theorem [Gro61, Thm. 2.1.1], one concludes that the formal total space originates actually from the "usual" algebraic geometry. In the context of Rigid Patching, one considers "affinoids" which are defined by Tate algebras and patches them together to get a total rigid analytic space. Applying the "rigid GAGA" one concludes that the total rigid analytic space originates from algebraic geometry. Both of these constructions are actually parallel to the complex analytic construction, where one concludes by using Serre's complex analytic

GAGA and the Grauert-Remmert theorem. The key technical point in proving GAGA type results is always some form of Cartan's lemma on matrix factorization [Hrb03, discussion proceeding Theorem 2.2.5].

2. ALGEBRAIC PATCHING. Inspired by a talk of v. d. Put, Haran and Völklein realized that the formal/rigid GAGA, on which formal/rigid patching relies in an essential way, can be actually replaced by a more specialized and simpler algebraization technique. In a few words, Haran and Völklein work in [HaV96] directly with the Tate algebras defining the affinoids to be patched, and using Cartan's Lemma, show that the final patching results originates from Galois theory.

The paper [HaV96], develops all the theory needed from scratch, without any prerequisites, and solves the inverse Galois problem over $K(x)$, where K is a complete discrete valued field. It also proves that $\mathrm{Gal}(K(x)) \cong \hat{F}_\omega$ if K is an algebraically closed countable field. The method Haran and Völklein developed got the name "Algebraic Patching". That method is further developed in [HaJ98a], [HaJ98b], and [HaJ00a]. In those papers, most of the results about absolute Galois groups of fields previously achieved by Formal Patching and Rigid Patching are proved by Algebraic Patching.

The basic idea behind each of the patching methods is that every finite group G is generated by finite cyclic groups. It is not very difficult to realize each of these groups over a given rational function field $K(x)$. The question is how to construct a Galois extension F of $K(x)$ with Galois group G out of these extensions. For example, their compositum will almost never give the desired field F.

2.1 Patching data. Algebraic patching takes an axiomatic approach to the problem of realizing finite groups, like the one used in [Pop94, Subsection (1.1)] for Rigid Patching. Starting from a field E and a finite group G, we choose finitely many cyclic subgroups G_i, $i \in I$, that generate G. For each $i \in I$ we construct a Galois extension F_i of E with Galois group G_i. Similar to the formal/rigid patching, algebraic patching proceeds in three steps: lifting, inducing, and algebraization. The first two steps are easier and of rather formal nature, whereas the the third one is more difficult and includes a GAGA type assertion.

To be more precise, algebraic patching assumes the existence of an extension field P_i of E, (which we view as an "analytic field"), $i \in I$, and a field Q containing all P_i's. The data obtained should satisfies the following conditions:

(1a) $F_i \subseteq P_i'$, where $P_i' = \bigcap_{j \neq i} P_j$, $i \in I$;
(1b) $\bigcap_{i \in I} P_i = E$; and
(1c) Let $n = |G|$. Then for every $B \in \mathrm{GL}_n(Q)$ and each $i \in I$ there exist $B_1 \in \mathrm{GL}_n(P_i)$ and $B_2 \in \mathrm{GL}_n(P_i')$ such that $B = B_1 B_2$ (Cartan's decomposition).

We call $\mathcal{E} = (E, F_i, P_i, Q; G_i, G)_{i \in I}$ a **patching data**. The lifting step takes F_i to $Q_i = P_i F_i$ and we observe that Q_i is a Galois extension of P_i

with $\mathrm{Gal}(Q_i/P_i) = G_i$. Then we consider the induced vector spaces $N_i = \mathrm{Ind}_{G_i}^G Q_i$, $i \in I$, and $N = \mathrm{Ind}_1^G Q$, and use (1c) to construct a basis for N/Q which is also a basis for N_i/Q_i for each $i \in I$. Once this is done, we prove that a certain proper E-translate F of $\bigcap_{i\in I} N_i$ into Q (called the **compound** of $\mathcal{E}$) is a Galois extension of E with $\mathrm{Gal}(F/E) \cong G$ (Lemma 1.1.7).

Next suppose E is a finite Galois extension of a field E_0 with a Galois group Γ and Γ **acts properly** on $\mathcal{E}$. This means that Γ acts on the group G, on the set I, and on the field Q in a compatible way. Thus:

(2a) The action of Γ on Q extends the action of Γ on E.
(2b) $F_i^\gamma = F_{i^\gamma}$, $P_i^\gamma = P_{i^\gamma}$, and $G_i^\gamma = G_{i^\gamma}$, for all $i \in I$ and $\gamma \in \Gamma$.
(2c) $(a^\tau)^\gamma = (a^\gamma)^{\tau^\gamma}$ for all $i \in I$, $a \in F_i$, $\tau \in G_i$, and $\gamma \in \Gamma$.

By Proposition 1.2.2, the compound F is a Galois extension of E_0 that solves the **finite split embedding problem** $\Gamma \ltimes G \to \mathrm{Gal}(E/E_0)$; that is, there is an isomorphism $\mathrm{Gal}(F/E_0) \cong \Gamma \ltimes G$ identifying the projection of $\Gamma \ltimes G$ onto Γ with the restriction map $\mathrm{Gal}(F/E_0) \to \mathrm{Gal}(E/E_0)$. Moreover, the action of Γ on F as a subgroup of $\mathrm{Gal}(F/E_0)$ coincides with the action of Γ on Q restricted to F. In this case we also have, by Lemma 1.1.7, that $P_iF = Q_i$ for each $i \in I$.

2.2 Complete fields under ultra-metric absolute values. We are able to put together a patching data for fields of the form $E = \hat{K}(x)$, where $\hat{K}$ is a complete field with respect to an ultra-metric absolute value $|\ |$. Parallel to [Pop94], we start with a finite set I. For each $i \in I$ we choose an element $c_i \in \hat{K}$ such that $c_i \neq c_j$ if $i \neq j$ and an element $r \in \hat{K}^\times$ satisfying $|r| \leq |c_i - c_j|$ for $i \neq j$. Then we set $w_i = \frac{r}{x - c_i}$ and consider the ring $R = R_I = \hat{K}\{w_i\}_{i\in I}$ of all power series $f = a_0 + \sum_{i\in I}\sum_{n=1}^\infty a_{in}w_i^n$, where $a_0, a_{in} \in \hat{K}$ and $|a_{in}|$ tends to 0 as $n \to \infty$ (Section 3.2). It turns out that R is a complete ring with respect to the norm $\|f\| = \max(|a_0|, |a_{in}|)_{i,n}$ (Lemma 3.2.1). Also, R is a principal ideal domain, hence a unique factorization domain. Moreover, for each $i \in I$, every ideal of R is generated by an element of $\hat{K}[w_i]$ (Proposition 3.2.9). We let $Q = P_I = \mathrm{Quot}(R_I)$. Similarly, we construct the fields $P_i = \mathrm{Quot}(\hat{K}\{w_j\}_{j\neq i})$ and $P_i' = \mathrm{Quot}(\hat{K}\{w_i\})$. By Corollary 3.3.2, $P_i' = \bigcap_{j\neq i} P_j$ and $\bigcap_{i\in I} P_i = E$. Thus, the "analytic" fields P_i satisfy Condition (1b) and the second part of Condition (1a). By definition, each element of R is a sum of an element of $R_{I\smallsetminus\{i\}}$ and an element of $R_{\{i\}}$. This implies that the P_i's also satisfy Condition (1c) (Corollary 3.4.4).

Given a finite group G, we choose I such that for each $i \in I$ there is a cyclic subgroup G_i whose order is a power of a prime number and $G = \langle G_i\rangle_{i\in I}$. It is classical that E has a cyclic extension F_i in $\hat{K}((x))$ with Galois group G_i [FrJ08, Section 16]. Here we construct F_i/E with control on its ramification. In particular, each prime divisor of $E/\hat{K}$ that ramifies in F_i is totally ramified (Lemma 4.2.5). Now we apply Proposition 2.4.5 saying that every power series $z \in \hat{K}[[x]]$ which is algebraic over E converges at some $c \in \hat{K}^\times$. This allows us to shift F_i into P_i'. Thus, the first half of Condition (1a) is also satisfied.

Finally, assume that $\hat{K}$ is a finite Galois extension of a field $\hat{K}_0$ with Galois group Γ that acts on G and $\hat{K}_0$ is complete with respect to the restriction of $|\ |$. Set $E_0 = \hat{K}_0(x)$. Then the proof of Proposition 4.4.2 shows how to choose the set I, the groups G_i, the fields F_i, and the fields P_i such that Γ acts properly on the patching data $\mathcal{E} = (E, F_i, P_i, Q; G_i, G)_{i\in I}$. It follows that the finite split embedding problem $\Gamma \ltimes G \to \mathrm{Gal}(\hat{K}/\hat{K}_0)$ (also called a **constant finite split embedding problem**) is solvable over E_0. Moreover, the solution field F has a $\hat{K}$-rational place φ unramified over E_0 and $\varphi(x) \in \hat{K}_0$. In particular, F is a regular extension of $\hat{K}$. Thus, F is a **regular solution** of the embedding problem.

2.3 Ample fields. If K_0 is an ample field and K is a finite Galois extension of K_0 with Galois group Γ, then K is also an ample field (Lemma 5.5.1). Let $\hat{K}_0 = K_0((t))$ and $\hat{K} = K((t))$. Then $\hat{K}/\hat{K}_0$ is a Galois extension of complete fields under the t-adic absolute value with $\Gamma = \mathrm{Gal}(\hat{K}/\hat{K}_0)$. Thus, if Γ acts on a finite group G, then $\hat{K}(x)$ has a finite extension $\hat{F}$, Galois over $\hat{K}_0(x)$ and regular over $\hat{K}$, that solves the constant finite split embedding problem $\Gamma \ltimes G \to \mathrm{Gal}(\hat{K}/\hat{K}_0)$. By definition, K_0 is existentially closed in $\hat{K}_0$. Since $\hat{K}_0/K_0$ is a regular extension, we may apply the Bertini-Noether theorem and descend from $\hat{F}$ to a field F, Galois over $K_0(x)$ and regular over K, that solves the finite split embedding problem $\Gamma \ltimes G \to \mathrm{Gal}(K/K_0)$ (Lemma 5.9.1) over $K_0(x)$. As noticed above, in the special case where K_0 is a countable Hilbertian PAC field, this reproves the isomorphism $\mathrm{Gal}(K_0) \cong \hat{F}_\omega$.

One of the equivalent conditions for a field K to be ample is that the set $V(K)$ of K-rational points of each absolutely irreducible variety V defined over K with a simple K-rational point is Zariski-dense in V (Lemma 5.3.1). It turns out that under the above conditions, $\mathrm{card}(V(K)) = \mathrm{card}(K)$ (Proposition 5.4.3). Moreover, if h is a nonconstant rational function of V, then $\mathrm{card}\{h(\mathbf{a}) \mid \mathbf{a} \in V(K)\} = \mathrm{card}(K)$ (Corollary 5.4.4).

In addition to PAC fields and Henselian fields we find that real closed fields are ample. So are the quotient fields of Henselian domains (Proposition 5.7.7). In particular, for every field K_0 and $n \geq 1$, the field of formal power series $K_0((X_1, \ldots, X_n))$ is ample. Similarly, $\mathrm{Quot}(\mathbb{Z}_p[[X_1, \ldots, X_n]])$ is ample for every prime number p and $n \geq 0$ (Remark 5.7.8). Finally, if the absolute Galois group of a field K is pro-p for some prime number p, then K is ample (Theorem 5.8.3).

2.4 Non-ample fields. Chapter 6 reveals the other side of the coin. It gives examples of nonample fields. By Corollary 5.3.3, every finite field is nonample. Elementary arguments that apply the Riemann-Roch formula show that every finitely generated transcendental extension F of a field K is nonample. Moreover, if F is a union of a directed family of function fields of one variable over K of bounded genus, then F is nonample (Theorem 6.1.8). The proof that every number field is nonample uses a deep result, namely Faltings' theorem that curves of genus at least 2 over number fields have only finitely many rational points (Proposition 6.2.5). To give examples of infinite algebraic extensions of $\mathbb{Q}$ that are nonample, we use even more advanced tools, namely the Mordell-Lang conjecture proved in characteristic 0 by Faltings and others. That theorem implies that if A is a nonzero Abelian variety defined over an ample field K of characteristic 0, then $\dim_{\mathbb{Q}}(A(K)\otimes\mathbb{Q}) = \infty$. Now we refer to an example of Kato and Rohrlich of an Abelian infinite extension K of $\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q})$ finitely generated and an elliptic curve E defined over $\mathbb{Q}$ such that $E(K)$ is finitely generated. Thus, K is nonample (Example 6.5.5). Using Faltings' result again and the concept of gonality of curves, we construct for every positive integer d a linearly disjoint sequence $K_1, K_2, K_3, \ldots$ of extensions of degree d of a given number field whose compositum $K = K_1K_2K_3\cdots$ is nonample (Proposition 6.8.8).

2.5 Many solutions. While the solvability of constant finite split embedding problems for $\mathrm{Gal}(K)$ over $K(x)$ suffices to prove that $\mathrm{Gal}(K) \cong \hat{F}_\omega$ if K is a countable PAC Hilbertian field, it does not give us enough information about nonconstant embedding problems over $K(x)$ and ignores the uncountable case. Both problems are addressed in Chapter 7 over complete fields. The most effective way to create uncountably many solutions to a finite split embedding problem is to solve the embedding problem with information about the branch points. Proposition 7.3.1 considers a complete field $\hat{K}_0$ with respect to an ultrametric absolute value, a finite Galois extension E/E_0 over $E_0 = \hat{K}_0(x)$ such that $\mathrm{Gal}(E/E_0)$ acts on a finite group H (we have replaced E' appearing in Proposition 7.3.1 by E). We assume that E has a $\hat{K}$-rational place unramified over the algebraic closure $\hat{K}$ of $\hat{K}_0$ in E. Then the embedding problem $\mathrm{Gal}(E/E_0) \ltimes H \to \mathrm{Gal}(E/E_0)$ has a solution field $\hat{F}$ regular over $\hat{K}$. Moreover, if G_j, $j \in J$, are finitely many cyclic subgroups of H of prime power orders that generate H, then for each $j \in J$ the extension $\hat{F}/E_0$ has a branch point b_j with G_j as an inertia group. Moreover, if $\hat{K}_0$ is an extension of infinite transcendence degree of a field K_0, then we may choose the b_j's to be algebraically independent over K_0.

The next step is to solve a finite split embedding problem

$$\mathrm{Gal}(E/K_0(x)) \ltimes H \to \mathrm{Gal}(E/K_0(x))$$

(which we denote by $\mathcal{E}$) for an ample field K_0 in many ways. As in the case of constant split embedding problems, we go over to the field $\hat{K}_0 = K_0((t))$, let $\hat{E} = E\hat{K}_0$, and solve the finite split embedding problem $\mathrm{Gal}(\hat{E}/\hat{K}_0(x)) \ltimes H \to \mathrm{Gal}(\hat{E}/\hat{K}_0(x))$ (which we denote by $\hat{\mathcal{E}}$) as in Proposition 7.3.1 (with E replacing E'). Then we choose appropriate $u_1, \ldots, u_n \in \hat{K}_0$ and descend the embedding problem with its solution field to an embedding problem (which we denote by $\mathcal{E}_{\mathbf{u}}$) with a solution field $F_{\mathbf{u}}$ over $K_0(\mathbf{u})$, keeping the branch points and the corresponding inertia groups. The new decisive step is to reduce $F_{\mathbf{u}}$ to a solution field F of $\mathcal{E}$ with sufficient information on the reduced branch points $\bar{b}_j$. To this aim we apply good reduction to the function field of one variable $F_{\mathbf{u}}/K(\mathbf{u})$ such that the inertia group $\bar{I}_j$ over $\bar{b}_j$ contains G_j. We also use the information that b_j is transcendental over K_0 to choose the reduction such that the branch point $\bar{b}_j$ is unramified in E and in the compositum N of all solution fields of $\mathcal{E}$ obtained in a transfinite induction up to that point. This implies that $\bar{I}_j$ is contained in $\mathrm{Gal}(F/F \cap N)$. Since the G_j's were chosen to generate H (which we identify with $\mathrm{Gal}(F/E)$), the $\bar{I}_j$'s generate $\mathrm{Gal}(F/E)$. This implies that $F \cap N = E$. In this way our transfinite induction constructs a transfinite sequence $(F_\kappa)_{\kappa<\mathrm{card}(K_0)}$ of solutions to $\mathcal{E}$ that are linearly disjoint over E (see also Lemma 7.4.1).

2.6 Algebraically closed base fields. Section 9.1 surveys the classical results about fundamental groups of Riemann surfaces. Given a finite set S of prime divisors of $\mathbb{C}(x)/\mathbb{C}$, we denote the maximal extension of $\mathbb{C}(x)$ in its algebraic closure that ramifies at most over S by $\mathbb{C}(x)_S$. A consequence of the Riemann existence theorem then describes $\mathrm{Gal}(\mathbb{C}(x)_S/\mathbb{C}(x))$ by generators and relations, where each generator generates an inertia group over an element of S. We are then able to take the limit over all the sets S and deduce that $\mathrm{Gal}(\mathbb{C}(x))$ is a free profinite group of rank $\mathrm{card}(\mathbb{C})$. In particular, $\mathrm{Gal}(\mathbb{C}(x))$ is projective (Corollary 9.1.11).

That result can be carried over to a result over an arbitrary algebraically closed field C of characteristic 0. If $p = \mathrm{char}(C) > 0$, the same result holds provided one stays away from p (Proposition 9.2.1). However, it is not true any more in its general form (Proposition 9.9.4). In particular, in the notation of the preceding paragraph, $\mathrm{Gal}(C(x)_S/C(x))$ is not free. Consequently, we are not able to repeat the proof that works in characteristic 0 that $\mathrm{Gal}(C(x))$ is free in the general case.

Nevertheless the latter result is still true for each algebraically closed field. The first step is a proof that $\mathrm{Gal}(C(x))$ is projective. Since we try to be as self-contained as possible, we give a direct proof of the projectivity of $\mathrm{Gal}(C(x))$ based on simple results about homogeneous equations that we prove and basic results about Galois cohomology that we survey (Proposition 9.4.6). Combined with Proposition 8.6.3, this proves that $\mathrm{Gal}(C(x))$ is isomorphic to the free profinite group of rank $\mathrm{card}(C)$. In positive characteristic p, we generalize that result and prove that if the absolute Galois group of a field K is pro-p and F is a function field of one variable over K, then $\mathrm{Gal}(F)$ is free of rank $\mathrm{card}(K)$ (Theorem 9.4.8).

2.7 Semi-free groups. Chapter 10 develops consequences of Proposition 7.3.1 from the point of view of profinite groups. We say that a profinite group G of infinite rank m is **semi-free** if every finite split embedding problem for G with a nontrivial kernel has m independent solutions. This condition is transferred to every open subgroup of G (Lemma 10.4.1), to every closed normal subgroup N such that G/N is finitely generated (Lemma 10.4.2) or Abelian (Theorem 10.5.4), and to every closed subgroup M of G that is **contained in a diamond** (Theorem 10.5.3).

As we saw above, one of the major steps to prove that the absolute Galois group of a field is free is to show that this group is projective. If K is PAC, then this is guaranteed by an old theorem of Ax [FrJ08, Thm. 11.6.2]. Going over to a function field E of one variable over K raises the cohomological dimension by 1, in particular, $\mathrm{Gal}(E)$ is usually not projective. However, we prove a local-global principle for the Brauer group $\mathrm{Br}(E)$ of E: the restriction map $\mathrm{Br}(E) \to \prod_{\mathfrak{p}} \mathrm{Br}(E_{\mathfrak{p}})$ is injective and the image of each element of $\mathrm{Br}(E)$ in $\mathrm{Br}(E_{\mathfrak{p}})$ is 0 for all but finitely many $\mathfrak{p}$ (Lemma 11.5.4). Here $\mathfrak{p}$ ranges over all prime divisors of E/K and $E_{\mathfrak{p}}$ is the Henselian closure of E at $\mathfrak{p}$. This implies an embedding $\mathrm{Br}(F) \to \prod \mathrm{Br}(F_{\mathfrak{p}})$ for each regular extension F of K of transcendence degree 1 (Proposition 11.5.5). In particular, if $x \in F$ is transcendental over K and F contains an nth root of each monic irreducible polynomial in $K[x]$ for each n with $\mathrm{char}(K) \nmid n$, then the valuation groups of $F_{\mathfrak{p}}$ are divisible away from $\mathrm{char}(K)$. This implies that $\mathrm{Br}(F_{\mathfrak{p}}) = 0$ for each $\mathfrak{p}$ (Proposition 11.1.3). It follows that $\mathrm{Br}(F) = 0$. Since the same holds for each finite extension of F, $\mathrm{Gal}(F)$ is projective (Proposition 11.6.6).

On the other hand, we use the previous results to prove that if E is a function field of one variable over an ample field K, then $\mathrm{Gal}(E)$ is semi-free. In particular, this is the case when K is PAC. We then choose nth roots $\sqrt[n]{f}$ in a compatible way for each monic irreducible polynomial $f \in K[x]$ and every positive integer n with $\mathrm{char}(K) \nmid n$. We let $F = K(\sqrt[n]{f})_{f,n}$ and prove that F lies in a diamond over $K(x)$. This implies that $\mathrm{Gal}(F)$ is semi-free of rank $m = \mathrm{char}(K)$ and projective. It follows that $\mathrm{Gal}(F) \cong \hat{F}_m$ (Theorem 11.7.6). We call F a **special K-radical extension of** $K(x)$. In the special case where K contains all roots of unity, we get that $\mathrm{Gal}(K(x)_{\mathrm{ab}}) \cong \hat{F}_m$. This is an analog to a well known conjecture of Shafarevich that $\mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}}) \cong \hat{F}_\omega$.

2.8 Hilbertian ample Krull fields. In the last chapter we consider an ample Hilbertian field K. Although every finite split embedding problem over K is solvable (Theorem 5.10.2), $\mathrm{Gal}(K)$ need not be semi-free of rank $\mathrm{card}(K)$ (Example 10.6.7). However, if K is the quotient field of a complete local Noetherian domain of height at least 2, then K is ample, Hilbertian, and $\mathrm{Gal}(K)$ is semi-free (Theorem 12.4.3). One of the main ingredients of the proof is a quantitative Chebotarev type theorem for K. We prove that K is a **Krull field**. Thus, K has a set $\mathcal{V}$ of discrete valuations such that $\{v \in \mathcal{V} \mid v(a) \neq 0\}$ is finite for each $a \in K^\times$ and for each finite Galois extension L of K there are $\mathrm{card}(K)$ elements $v \in \mathcal{V}$ that completely split in L.

The leading examples for fields K as in the preceding paragraph are $K_0((X_1, \ldots, X_n))$, for any field K_0 and $n \geq 2$, $\mathrm{Quot}(\mathbb{Z}_p[[X_1, \ldots, X_n]])$, where p is prime and $n \geq 1$, and $\mathrm{Quot}(\mathbb{Z}[[X_1, \ldots, X_n]])$, where $n \geq 1$ (Example 12.4.4). In the special case where $K = C((X_1, X_2))$ and C is algebraically closed of cardinality m, it follows that $\mathrm{Gal}(K_{\mathrm{ab}}) \cong \hat{F}_m$ (Theorem 12.4.6).

3. Main Results.

3.1 List of main results. Each result is followed by a short reference. We expand on it in the notes at the end of the chapters where the theorems are proved.

The two main results of the book are perhaps the following theorems.

THEOREM A: *Let K be an ample field of cardinality m and E a function field of one variable over K. Then $\mathrm{Gal}(E)$ is semi-free of rank m (Theorem 11.7.1).*

See notes to Chapters 7 and 11 for the history of Theorem A.

THEOREM B: *Let K be a Hilbertian ample Krull field of cardinality m. Then $\mathrm{Gal}(K)$ is semi-free of rank m (Theorem 12.4.1).*

Pop proves this result in [Pop10], where the notion of a Krull field is introduced.

Among the consequences of Theorems A and B we mention the following:

THEOREM C: *Every Hilbertian PAC field is ω-free (Theorem 5.10.3).*

This theorem is proved by Fried-Völklein [FrV92] when $\mathrm{char}(K) = 0$ and by Pop [Pop96] in general. Haran-Jarden reprove the result using algebraic patching in [HaJ98a].

THEOREM D: *Let $\mathbb{Q}_{\mathrm{tr}}$ be the field of totally real algebraic numbers. Then $\mathrm{Gal}(\mathbb{Q}_{\mathrm{tr}}(\sqrt{-1})) \cong \hat{F}_\omega$ (Example 5.10.7).*

This example is a special case of a more general example: Let S be a finite set of primes of $\mathbb{Q}$ and let $\mathbb{Q}_S$ be the field of totally S-adic numbers. Then $\mathrm{Gal}(\mathbb{Q}_S(\mu_\infty)) \cong \hat{F}_\omega$ [Pop96]. Notice that $\mathbb{Q}_{\mathrm{tr}}(\sqrt{-1}) = \mathbb{Q}_{\mathrm{tr}}(\mu_\infty)$ is even PAC, whereas the former fields are not.

THEOREM E: *Let K be a field of characteristic p and cardinality m and let E be a function field of one variable over K. Suppose that $\mathrm{Gal}(K)$ is a pro-p group. Then $\mathrm{Gal}(E) \cong \hat{F}_m$ (Theorem 9.4.8).*

In the case where K is algebraically closed the theorem was proved independently by Harbater [Hrb95] and Pop [Pop95]. Haran-Jarden use algebraic patching in [HaJ98b] to reprove the theorem. The proof of the general case follows along the same lines.

THEOREM F: *Let K be a PAC field of cardinality m, x a variable, and F a special K-radical extension of $K(x)$. Then F is Hilbertian and $\mathrm{Gal}(F) \cong \hat{F}_m$. In particular, if K contains all roots of unity, then $\mathrm{Gal}(K(x)_{\mathrm{ab}}) \cong \hat{F}_m$ (Theorem 11.7.6).*

The result is proved by Jarden-Pop in [JaP09] and gives evidence for a conjecture of Bogomolov-Positselski.

THEOREM G: *Let R be a Noetherian domain, $\mathfrak{m}$ a prime ideal of R of height at least 2, and $\mathfrak{a}$ an ideal of R in $\mathfrak{m}$ such that R is complete in the $\mathfrak{a}$-adic topology. Then $K = \mathrm{Quot}(R)$ is Hilbertian, ample, and Krull. Moreover, $\mathrm{Gal}(K)$ is semi-free of rank $\mathrm{card}(K)$ (Theorem 12.4.3).*

The proof of Theorem G is based, among others, on Theorem B. It appears in [Pop10]. The following special cases of Theorem G are proved by Paran in [Par10] by other methods.

THEOREM H: *Let R be a complete local Noetherian domain of height at least 2 and of cardinality m. Set $K = \mathrm{Quot}(R)$. Then $\mathrm{Gal}(K)$ is semi-free of rank m (Theorem 12.4.5).*

THEOREM I: *In each of the following cases $\mathrm{Gal}(K)$ is semi-free of rank $\mathrm{card}(K)$:*
(a) $K = K_0((X_1,\ldots,X_n))$, where K_0 is an arbitrary field and $n \geq 2$;
(b) $K = \mathrm{Quot}(\mathbb{Z}_p[[X_1,\ldots,X_n]])$, where p is a prime number and $n \geq 1$;
(c) $K = \mathrm{Quot}(\mathbb{Z}[[X_1,\ldots,X_n]])$, where $n \geq 1$;
(Example 12.4.4).

Two interesting results that we prove use ingredients whose proofs are unfortunately beyond the scope of this book:

THEOREM J (Fehm-Petersen): *Let A be a nonzero Abelian variety defined over an ample field K of characteristic 0. Then, the rational rank of $A(K)$ is infinite (Theorem 6.5.2).*

THEOREM K: *Let K be a separably closed field of cardinality m. Then $\mathrm{Gal}(K((X_1,X_2))_{\mathrm{ab}}) \cong \hat{F}_m$ (Theorem 12.4.6).*

The case where K is algebraically closed is due to Harbater [Hrb09]. We have observed that Harbater's proof applies to the general case. A generalization appears in [Pop10]. □

3.2 Sources. The proofs of Theorems A and B are self-contained up to basic results of Field Arithmetic that we quote from [FrJ08]. A few of the applications, C – K, rely on extra information that we properly quote.

The first three chapters of the book give a quick self-contained introduction into Algebraic Patching. Over an ample field K, it leads to the result that every constant split embedding problem over $K(x)$ is solvable (Theorem 5.9.2). Using basic results of Field Arithmetic taken from [FrJ08], that part of the book culminates with the proof of Theorem C.

Theorem A generalizes Theorem 5.9.2, and its proof requires much more effort to achieve linear disjointness of the solution fields through a careful choice of the branch points with additional information about their inertia groups. Again, the proof is self-contained.

The first application of Theorem A appears in the proof of Theorem F. Here the first task is to prove that $\mathrm{Gal}(K(x)_{\mathrm{ab}})$ is projective. The proof uses some basic Galois cohomology that we survey in Section 9.3.

The proofs of Theorems G, H, and I use several results from commutative algebra. The proof of Theorem J uses, among others, the Mordel-Lang Conjecture proved by Faltings. The proof of Theorem K applies a result whose proof applies étale cohomology.

3.3 Advantages and disadvantages of Algebraic Patching. Our method of Algebraic Patching has the advantage of being quickly accessible. The cost of this convenience is its inability to deal with fundamental groups of curves over algebraically closed fields K. Indeed, every Galois extension F_i of $K(x)$

involved in patching data (see 2.1), contributes at least one extra branch point to the solution field of the given embedding problem. In addition, our patching is carried out only over rational fields $K(x)$ (geometrically, over the line) and not over algebraic function fields of one variable (geometrically, over curves). Thus, that method seems not to be suitable to handle questions like the general Abhyankar's conjecture that was reduced to the special Abhyankar's conjecture by both the Formal Patching and the Rigid Patching methods. Nevertheless, we have been able to apply several methods of descent from function fields of one variable to rational function fields and prove all of the above mentioned results about absolute Galois groups.

4. Field Patching. David Harbater and Julia Hartmann have recently developed a new kind of patching called "field patching". In its simplest form, that method considers fields $F \subseteq F_1, F_2 \subseteq F_0$ such that $F = F_1 \cap F_2$ and for each matrix $A_0 \in \mathrm{GL}_n(F_0)$ there exist matrices $A_1 \in \mathrm{GL}_n(F_1)$ and $A_2 \in \mathrm{GL}_n(F_2)$ such that $A_0 = A_1 A_2$. Thus, these fields satisfy the second part of Condition (1a) and Conditions (1b) and (1c) in the special case where $I = \{1,2\}$, E is replaced by F, and P_1, P_2 are replaced by F_1, F_2. They prove that if for $i = 0,1,2$, V_i is a vector space of dimension n over F_i and $F_0 V_i = V_0$ for $i = 1,2$, then $V = V_1 \cap V_2$ is a vector space of dimension n over F [HaH10, Prop. 2.1]. This corresponds to Lemma 1.1.7, where V_i is replaced by the algebra N_i (again for $I = \{1,2\}$) and V is replaced by the field F' (which is a Galois extension of E of degree equal to $\dim_{P_i} N_i$).

Harbater and Hartmann verify the axioms of Field Patching over a complete discrete valuation ring T with uniformizer t and quotient field K. They consider a smooth projective curve $\hat{X}$ over T, let X be its closed fiber and F the function field of X. Then they consider proper subsets U_1, U_2 of X such that $X = U_1 \cup U_2$, and set $U_0 = U_1 \cap U_2$. For $i = 0,1,2$ they let R_i be the ring of rational functions of $\hat{X}$ that are regular on U_i. Now they consider the t-adic completion $\hat{R}_i$ of R_i and let $F_i = \mathrm{Quot}(\hat{R}_i)$. They prove that these objects satisfy the axioms of the preceding paragraph, hence also their conclusion [HaH10, Thm. 4.10]. In contrast to Formal Patching, the proofs rely only on elementary arguments such as the Riemann-Roch theorem for function fields of one variable.

In the special case where $\hat{X}$ is the projective line over T, $R_1 = T[x]$, $R_2 = T[x^{-1}]$, and $R_0 = T[x, x^{-1}]$, the corresponding t-adic completions are $\hat{R}_1 = T\{x\}$, $\hat{R}_2 = T\{x^{-1}\}$, and $\hat{R}_0 = T\{x, x^{-1}\}$. These rings are respectively contained in the rings $R'_1 = K\{x\}$, $R'_2 = K\{x^{-1}\}$, and $R'_0 = K\{x, x^{-1}\}$ that appear in [HaV96] and also, in a more general form in Chapters 2 and 3. Moreover, $\mathrm{Quot}(\hat{R}_i) = \mathrm{Quot}(R'_i)$ for $i = 0,1,2$ and $F = K(x)$.

While Algebraic Patching works only over rational function fields of one variable and aims at Galois Theory and in particular toward applications to absolute Galois groups, Field Patching works over arbitrary function fields of one variable (over $\mathrm{Quot}(T)$) and has been also applied to other areas of algebra, most notably to differential algebra, quadratic forms, and Brauer groups.

We mention here only one result: In the notation of the first paragraph of 4, let G be a connected linear algebraic group over F whose function field over F is rational and let H be an F-variety. Suppose $G(F')$ acts transitively on $H(F')$ for each extension F' of F and that $H(F_i) \neq \emptyset$ for $i = 1, 2$. Then $H(F) \neq \emptyset$ [HHK09].

For each field K let $u(K)$ be the maximal number of variables of a quadratic form over K with no nontrivial zero. As an application, [HHK09] proves that if K is a complete discrete valued field whose residue field is a C_d-field, then $u(F) \leq 2^{d+2}$ for each function field F of one variable over K. In particular, if F is a function field of one variable over $\mathbb{Q}_p$, a finite extension of $\mathrm{Quot}(\mathbb{Z}_p[[X]])$, or a finite extension of $\mathrm{Quot}(\mathbb{F}_p[[X, Y]])$, with $p \neq 2$, then $u(F) = 8$. Thus, every quadratic form in 9 variables over F has a nontrivial zero but there is a quadratic form in 8 variables over F that fails to have a nontrivial zero. See also an earlier proof of that result by Parimala and Suresh [PaS07] that use other methods and a recent generalization of the result by David Leep to function fields of several variables over $\mathbb{Q}_p$.

Acknowledgement: I would like to thank Aharon Razon for thorough reading of the whole manuscript and for many suggestions that helped me clean the book from many mistakes and typos. Elad Paran's, Lior Bary-Soroker's, Arno Fehm's, and Wulf-Dieter Geyer's many comments are also warmly appreciated. Special recognition goes to Peter Roquette for his simplification and considerable shortening of the proof of Lemma 8.2.4. Finally, my thanks go to David Harbater, Florian Pop, and Dan Haran for their constant support and essential advise.

Mevasseret Zion, Spring 2010 Moshe Jarden

Springer
http://www.springer.com/978-3-642-15127-9

Algebraic Patching
Jarden, M.
2011, XXIV, 292 p., Hardcover
ISBN: 978-3-642-15127-9