

INTRODUCTORY BACKGROUND

The variational methods of image segmentation discussed in this book minimize functionals. In this chapter, we review some formulas we use repeatedly in the definition and minimization of these functionals: Euler-Lagrange equations, gradient descent minimization, level set representation. We also review optical flow basic expressions used in motion based segmentation.

2.1 Euler-Lagrange equations

Most of the image segmentation methods we study in this book are variational methods which use closed regular plane curves to define an image domain partition. The objective functionals they minimize have these curves as arguments. They are, typically, the sum of integrals of one of two types, namely, integrals along a regular closed plane curve and integrals over the region enclosed by such a curve. The Euler-Lagrange equations corresponding to these variable domain integrals can be derived using standard calculus of variations and, in particular, the Euler-Lagrange equations corresponding to definite integrals [1].

2.1.1 Definite integrals

Let x_1 and x_2 be fixed real numbers and consider an integral of the form

$$\mathcal{E}(y) = \int_{x_1}^{x_2} g(x, y, y') dx \quad (2.1)$$

where $y = y(x)$ is a twice differentiable real function, $y' = \frac{dy}{dx}$, and g is a function twice differentiable with respect to any of its arguments, x, y , and y' . Assuming there is a function y satisfying the end point conditions $y(x_1) = y_1$ and $y(x_2) = y_2$, which minimizes (2.1), then y satisfies the Euler-Lagrange differential equation

$$\frac{\partial g}{\partial y} - \frac{d}{dx} \left(\frac{\partial g}{\partial y'} \right) = 0 \quad (2.2)$$

A free end point endpoint at $x = x_1$ is expressed by the condition

$$\left. \frac{\partial g}{\partial y'} \right|_{x_1} = 0 \quad (2.3)$$

A similar expression holds for a free end point at $x = x_2$:

$$\left. \frac{\partial g}{\partial y'} \right|_{x_2} = 0 \quad (2.4)$$

For an integral involving several dependent variables $y(x), \dots, z(x)$, of the form

$$E(y, \dots, z) = \int_{x_1}^{x_2} g(x, y, \dots, z, y', \dots, z') dx \quad (2.5)$$

where $y' = \frac{dy}{dx}, \dots, z' = \frac{dz}{dx}$, there is an Euler-Lagrange equation similar to (2.2) for each dependent variable:

$$\begin{aligned} \frac{\partial g}{\partial y} - \frac{d}{dx} \left(\frac{\partial g}{\partial y'} \right) &= 0 \\ &\vdots \\ \frac{\partial g}{\partial z} - \frac{d}{dx} \left(\frac{\partial g}{\partial z'} \right) &= 0 \end{aligned} \quad (2.6)$$

In subsequent chapters, we will encounter integrals involving scalar functions of two independent variables. For an integral of the form

$$\mathcal{E}(w) = \int_{\mathbf{R}} g(x, y, w, w_x, w_y) dx dy \quad (2.7)$$

where \mathbf{R} is some fixed region of \mathbb{R}^2 , $w = w(x, y)$ assumes some prescribed values at all points on the boundary $\partial \mathbf{R}$ of \mathbf{R} , w_x and w_y are the partial derivatives of w , and g is twice differentiable with respect to each of its arguments, the Euler-Lagrange equation is

$$\frac{\partial g}{\partial w} - \frac{\partial}{\partial x} \left(\frac{\partial g}{\partial w_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial g}{\partial w_y} \right) = 0 \quad (2.8)$$

When w is not specified on $\partial \mathbf{R}$, we have

$$\frac{\partial g}{\partial w_x} \frac{dy}{ds} - \frac{\partial g}{\partial w_y} \frac{dx}{ds} = 0 \quad \text{on } \partial \mathbf{R} \quad (2.9)$$

where s is arc length and $(x(s), y(s))$ the corresponding parametric representation of $\partial \mathbf{R}$.

2.1.2 Variable domain of integration

Let $\gamma: s \in [0, 1] \rightarrow (x(s), y(s)) \in \mathbb{R}^2$ be a simple closed plane curve parametrized by arc length, and \mathbf{R}_γ its interior (the region it encloses). The segmentation functionals we will encounter in the subsequent chapters typically contain a term of the form

$$\int_{\mathbf{R}_\gamma} f(x, y) dx dy \quad (2.10)$$

where f is a scalar function, i.e., independent of γ . They also typically contain the term

$$\int_\gamma ds \quad (2.11)$$

which is the length of γ . Consider the following functional:

$$\mathcal{E}(\gamma) = \int_{\mathbf{R}_\gamma} f(x, y) dx dy + \lambda \int_\gamma ds \quad (2.12)$$

where f is a scalar function and λ is a positive constant. To determine the Euler-Lagrange equation corresponding to the minimization of (2.12) with respect to γ (we assume that the problem is to minimize \mathcal{E} , but the following discussions apply to maximization as well), the first integral in the functional is first transformed into a simple integral as follows. Let

$$P(x, y) = -\frac{1}{2} \int_0^y f(x, z) dz \quad (2.13)$$

and

$$Q(x, y) = \frac{1}{2} \int_0^x f(z, y) dz \quad (2.14)$$

Then, using the Green's theorem [2], we have

$$\int_{\mathbf{R}_\gamma} f(x, y) dx dy = \int_{\mathbf{R}_\gamma} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_\gamma P dx + Q dy = \int_0^l (Px' + Qy') ds \quad (2.15)$$

where $x' = \frac{dx}{ds}$ and $y' = \frac{dy}{ds}$. Applying (2.6) to the last integral in (2.15), i.e., using $g(s, x, y, x', y') = P(x(s), y(s))x'(s) + Q(x(s), y(s))y'(s)$, yields

$$\begin{aligned} \frac{\partial g}{\partial x} - \frac{d}{ds} \left(\frac{\partial g}{\partial x'} \right) &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) y' = fy' \\ \frac{\partial g}{\partial y} - \frac{d}{ds} \left(\frac{\partial g}{\partial y'} \right) &= \left(-\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} \right) x' = -fx' \end{aligned} \quad (2.16)$$

Therefore, orienting γ so that the outward normal is

$$\mathbf{n} = (y', -x'), \quad (2.17)$$

the component of the Euler-Lagrange equation corresponding to (2.12) due to its first term is

$$f\mathbf{n} \quad (2.18)$$

The second integral of (2.12) is rewritten as

$$\int_0^l \left(x'^2 + y'^2 \right)^{\frac{1}{2}} ds \quad (2.19)$$

Application of (2.6), i.e., using $g(s, x, y, x', y') = \left((x'(s))^2 + (y'(s))^2 \right)^{\frac{1}{2}}$, gives

$$\begin{aligned} \frac{\partial g}{\partial x} - \frac{d}{ds} \left(\frac{\partial g}{\partial x'} \right) &= -\frac{d}{ds} \left(\frac{x'}{\left((x')^2 + (y')^2 \right)^{\frac{1}{2}}} \right) = \kappa y' \\ \frac{\partial g}{\partial y} - \frac{d}{ds} \left(\frac{\partial g}{\partial y'} \right) &= -\frac{d}{ds} \left(\frac{y'}{\left((x')^2 + (y')^2 \right)^{\frac{1}{2}}} \right) = -\kappa x' \end{aligned} \quad (2.20)$$

where κ is the curvature of γ given by, independently of the parametrization [3],

$$\kappa = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{\frac{3}{2}}} = \operatorname{div} \left(\frac{y'}{\left((x')^2 + (y')^2 \right)^{\frac{1}{2}}}, \frac{-x'}{\left((x')^2 + (y')^2 \right)^{\frac{1}{2}}} \right) = \operatorname{div} \left(\frac{\mathbf{n}}{\|\mathbf{n}\|} \right) \quad (2.21)$$

with $x'' = \frac{d^2x}{ds^2}$, $y'' = \frac{d^2y}{ds^2}$, and div is the divergence operator. Therefore, the contribution of the second integral of (2.12) to the Euler-Lagrange equation is

$$\kappa \mathbf{n} \quad (2.22)$$

This gives the Euler-Lagrange equation corresponding to $\mathcal{E}(\gamma)$:

$$(f + \lambda \kappa) \mathbf{n} = \mathbf{0} \quad (2.23)$$

In the computer vision literature, the left-hand side of (2.23) is referred to as the *functional derivative* of \mathcal{E} with respect to γ , the sum of the functional derivatives of its two component terms. It is noted $\frac{d\mathcal{E}}{d\gamma}$, or $\frac{\partial \mathcal{E}}{\partial \gamma}$ when \mathcal{E} is also dependent on other variables, as it is the case in the methods discussed in this book.

In a subsequent chapter, we will encounter the following integral, called a *geodesic functional*:

$$\int_{\gamma} h ds \quad (2.24)$$

where $h = h(x(s), y(s))$ is a positive scalar function (the length integral in (2.11) is a special case, with $h = 1$). Using previous arguments, its functional derivative, with respect to γ , is

$$\frac{d}{d\gamma} \left(\int_{\gamma} h ds \right) = (\langle \nabla h, \mathbf{n} \rangle + h\kappa) \mathbf{n} \quad (2.25)$$

and the corresponding Euler-Lagrange equation is

$$(\langle \nabla h, \mathbf{n} \rangle + h\kappa) \mathbf{n} = \mathbf{0} \quad (2.26)$$

where $\langle \cdot \rangle$ denotes the scalar product, and $\nabla h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right) = (h_x, h_y)$ is the spatial gradient of h .

The Euler-Lagrange equations corresponding to the region integral (2.10) and to the geodesic functional (2.24) can be generalized to volume and surface integrals, respectively. Although not used in this book, such integrals appear, for instance, in spatio-temporal image segmentation [4]. Let S be a closed regular surface [3] in \mathbb{R}^3 . The Euler-Lagrange equation corresponding to $\int_S h dS$ is

$$(\langle \nabla h, \mathbf{n} \rangle + 2h\kappa) \mathbf{n} = \mathbf{0}, \quad (2.27)$$

where \mathbf{n} is the external unit normal to S and κ its mean curvature function [3]. The factor 2 appearing in this equation is due to the definition of the mean curvature. The equation corresponding to the volume integral $\int_{\mathbf{R}_S} f dV$, where \mathbf{R}_S is the interior of S , is

$$f \mathbf{n} = \mathbf{0} \quad (2.28)$$

A detailed proof of these formulas can be found in [4].

Image segmentation functionals can involve integrals with \mathbf{R}_{γ} -dependent parameters. A typical such integral (called a data term) is

$$\int_{\mathbf{R}_{\gamma}} (I - \mu)^2 dx dy \quad (2.29)$$

where I is the image and μ is its mean over \mathbf{R}_{γ} . The minimization of these functionals can use the Euler-Lagrange equations obtained by assuming that the region parameters are fixed (i.e., independent of γ), leading to iterations of a greedy two-step algorithm, one step to minimize with respect to the parameters, the other to minimize with respect to the curve assuming that the parameters are fixed. However, other frameworks, such as the shape calculus, which uses shape gradients [5, 6, 7], can take the integrand dependence on γ into account to derive the necessary conditions for a minimum, leading to iterations of gradient descent (fastest descent). We will see examples of both schemes in subsequent chapters.

2.2 Descent methods for unconstrained optimization

2.2.1 Real functions

Consider the problem of finding an unconstrained local minimum of a C^1 function $f : \mathbf{x} \in \mathbb{R}^N \rightarrow f(\mathbf{x}) \in \mathbb{R}$ (assuming such a minimum exists). Let $\mathbf{x} : t \geq 0 \rightarrow \mathbf{x}(t) \in \mathbb{R}^N$ be C^1 , and $g(t) = f(\mathbf{x}(t))$. We have

$$\frac{dg}{dt} = \langle \nabla f, \frac{d\mathbf{x}}{dt} \rangle \quad (2.30)$$

Therefore, if \mathbf{x} varies according to

$$\begin{aligned} \frac{d\mathbf{x}}{dt}(t) &= -\alpha(t)d(\mathbf{x}(t)) \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned} \quad (2.31)$$

where $\alpha(t) \in \mathbb{R}^+$ and $\langle \nabla f, d \rangle > 0$, function g will vary according to

$$\frac{dg}{dt} = -\alpha \langle \nabla f, d \rangle \quad (2.32)$$

Because $\nabla f = 0$ is a necessary condition for a local minimum of f and $-\alpha \langle \nabla f, d \rangle$ is negative, \mathbf{x} varying by (2.31) will converge to a local minimum of f , assuming such a minimum exists on the trajectory of \mathbf{x} . Methods of unconstrained minimization based on (2.31) are called *descent* methods. Most often $d = \nabla f$ is used (*gradient*, or *fastest*, descent). The scaling function α is often predetermined. For instance, $\alpha(t) = \text{constant}$, or $\alpha(t) = 1/t$ [8]. In general, descent methods are discretized and implemented as follows.

1. $k = 0$; $\mathbf{x}^0 = \mathbf{x}_0$
2. Repeat until a test of convergence is verified

$$\begin{aligned} d^k &= d(\mathbf{x}^k) \\ \alpha_k &= \arg \min_{\alpha \geq 0} f(\mathbf{x}^k - \alpha d^k) \\ \mathbf{x}^{k+1} &= \mathbf{x}^k - \alpha_k d^k \\ k &\leftarrow k + 1 \end{aligned}$$

Similar vectorial formulas apply to vectorial functions $F = (f_1, \dots, f_n)^t$ by treating each component function f_i as described.

2.2.2 Integral functionals

Consider the problem of minimizing (2.1), i.e., $\mathcal{E}(y) = \int_{x_1}^{x_2} g(x, y, y') dx$. Let y vary in time, i.e., y is embedded in a one-parameter family of functions indexed by (algorithmic) time t , $y = y(x, t)$. Consider the functional

$$\mathcal{E}(y, t) = \int_{x_1}^{x_2} g(x, y(x, t), y'(x, t)) dx \quad (2.33)$$

where $y' = \partial y / \partial x$. Let us calculate the derivative of \mathcal{E} with respect to time:

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial t} &= \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial g}{\partial y'} \frac{\partial y'}{\partial t} \right) dx \\ &= \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial g}{\partial y'} \frac{\partial}{\partial t} \left(\frac{\partial y}{\partial x} \right) \right) dx \\ &= \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial g}{\partial y'} \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial t} \right) \right) dx \end{aligned}$$

Integration by parts of the second term of the integrand gives

$$\frac{\partial \mathcal{E}}{\partial t} = \left[\frac{\partial g}{\partial y'} \frac{\partial y}{\partial t} \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial y} - \frac{\partial}{\partial x} \left(\frac{\partial g}{\partial y'} \right) \right) \frac{\partial y}{\partial t} dx \quad (2.34)$$

Assuming that $\frac{\partial y}{\partial t}(x_1, t) = \frac{\partial y}{\partial t}(x_2, t) \forall t$, which is the case for each of the two component functions of the closed curves we consider in this book, we finally have

$$\frac{\partial \mathcal{E}}{\partial t} = \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial y} - \frac{\partial}{\partial x} \left(\frac{\partial g}{\partial y'} \right) \right) \frac{\partial y}{\partial t} dx \quad (2.35)$$

Therefore, varying y according to

$$\frac{\partial y}{\partial t} = - \left(\frac{\partial g}{\partial y} - \frac{\partial}{\partial x} \left(\frac{\partial g}{\partial y'} \right) \right), \quad (2.36)$$

i.e.,

$$\frac{\partial y}{\partial t} = - \frac{\partial \mathcal{E}}{\partial y}, \quad (2.37)$$

implies

$$\frac{\partial \mathcal{E}}{\partial t} = - \int_{x_1}^{x_2} \left(\frac{\partial g}{\partial y} - \frac{\partial}{\partial x} \left(\frac{\partial g}{\partial y'} \right) \right)^2 \leq 0 \quad (2.38)$$

As a result, \mathcal{E} decreases in time. As with real functions, starting from $y(0) = y_0$, y will converge to a local minimum of \mathcal{E} , assuming such a minimum exists. Equation (2.36) is the fastest descent equation to minimize functional (2.33). Functionals of several dependent variables are treated in a similar way.

As an example, to minimize (2.12), we adopt the descent equation

$$\frac{\partial \gamma}{\partial t} = -(f + \lambda \kappa) \mathbf{n} \quad (2.39)$$

In the computer vision literature, the partial differential equation (2.39) is seen as the evolution equation of curve γ along its normal at speed $-(f + \lambda \kappa)$, and γ is called an *active curve*. Direct implementation of this equation, as with the algorithm for real functions above, will iteratively displace each of the points of the curve.

However, a curve can split and join during evolution, and such changes in the curve topology are quite difficult, if at all possible, to effect by an explicit representation of the curve as a set of mobile marker points. An efficient and numerically stable implementation of such descent equations is via *level sets* which we outline next.

2.3 Level sets

Let γ be a simple closed curve in the image domain Ω . In the problems we address, such curves are made to move so as to converge to delimit the desired segmentation regions. Consider a curve which evolves according to a velocity vector in the direction of its normal at each point (Figure 2.1 a),

$$\mathbf{V} = V\mathbf{n} \quad (2.40)$$

The velocities we will encounter in this book have speeds of one of three distinct types.

Type 1. V is a function of the curvature of the evolving curve.

Type 2. V is of the form $\langle \mathbf{F}, \mathbf{n} \rangle$ where \mathbf{F} is a vector field dependent on position and possibly time (via the underlying image function, for instance) but not on the curve. Such terms are called advection speeds in [9].

Type 3. V is a scalar function which depends on position and time but is not of the other two types.

The closed curves delineating a segmentation region can split or merge during evolution. These changes in a curve topology are at best extensively cumbersome to effect using an explicit representation of a curve as a set of marker points and an implementation of the evolution by a descent algorithm such as given in Section 2.2 for the minimization of real functions. Level sets offer an efficient and numerically stable alternative implementation.

Let Γ be the set of smooth (C^2) plane curves $\gamma : s \in [0, 1] \rightarrow \gamma(s) \in \Omega$ which are closed, simple, and regular [3]. Let an active curve be represented by a one-parameter (algorithmic time) family of curves in Γ , i.e, a function $\gamma : s, t \in [0, 1] \times \mathbb{R}^+ \rightarrow \gamma(s, t) = (x(s, t), y(s, t), t) \in \Omega \times \mathbb{R}^+$ such that $\forall t$ curve $\gamma_t : s \rightarrow (x(s, t), y(s, t))$ is in Γ . With the level set implementation, an active curve γ is represented implicitly as the zero level set of a function $\phi : \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow \mathbb{R}$:

$$\forall s, t \quad \phi(\gamma(s, t)) = \phi(x(s, t), y(s, t), t) = 0 \quad (2.41)$$

The total derivative of (2.41) with respect to time gives, assuming ϕ is sufficiently smooth,

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial\phi}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial\phi}{\partial t} = \langle \nabla\phi, \frac{\partial\gamma}{\partial t} \rangle + \frac{\partial\phi}{\partial t} = 0 \quad (2.42)$$

Because $\frac{\partial\gamma}{\partial t} = V\mathbf{n}$, we have

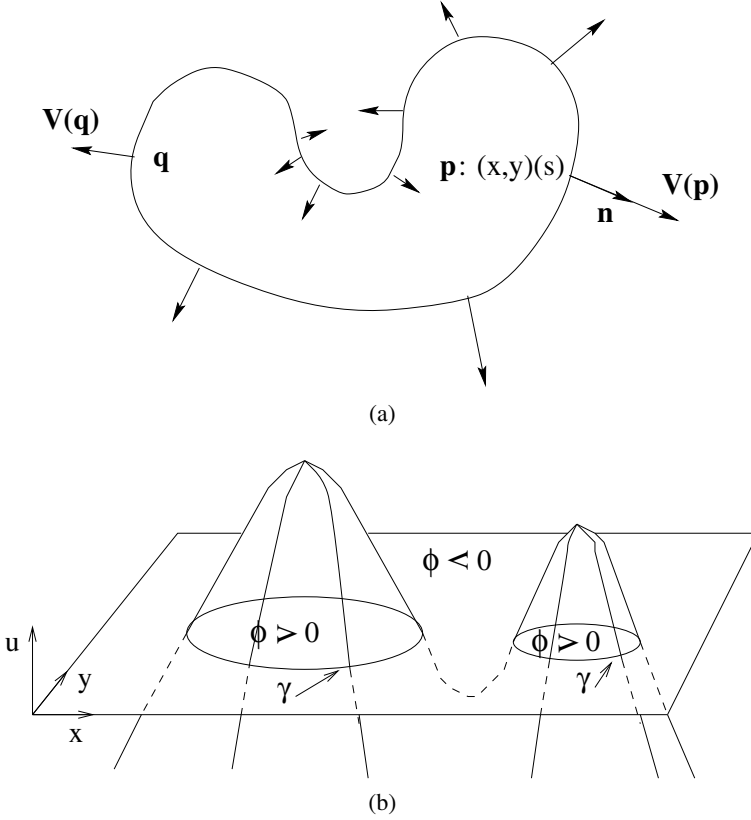


Fig. 2.1. (a) Active curve γ is moving at all times and at each point according to a velocity vector which is along its normal; (b) active curve γ is represented implicitly by the zero level curve of the graph of function ϕ . In this figure, γ is split into two regular curves while ϕ remains a function. In general, such a possibility cannot be implemented by an explicit representation of γ as a set points.

$$\frac{\partial \phi}{\partial t} = -V \langle \nabla \phi, \mathbf{n} \rangle \quad (2.43)$$

We also have

$$\forall s \in [0, 1] \quad \frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial s} = \langle \nabla \phi, \frac{\partial \gamma}{\partial s} \rangle = 0 \quad (2.44)$$

Therefore, because $\frac{\partial \gamma}{\partial s}$ is the tangent to the curve at s , $\nabla \phi$ is normal to the curve. With the convention that \mathbf{n} is oriented outward and ϕ is positive inside its zero level set, we have

$$\mathbf{n} = -\frac{\nabla\phi}{\|\nabla\phi\|} \quad (2.45)$$

Substitution of (2.45) in (2.43) gives the temporal evolution of ϕ :

$$\frac{\partial\phi}{\partial t} = V\|\nabla\phi\| \quad (2.46)$$

For type 1 speeds, the curvature is given in terms of the level set function by:

$$\kappa = \operatorname{div}\left(\frac{\mathbf{n}}{\|\mathbf{n}\|}\right) = -\operatorname{div}\left(\frac{\nabla\phi}{\|\nabla\phi\|}\right) = -\frac{\phi_{xx}\phi_y^2 - 2\phi_x\phi_y\phi_{xy} + \phi_{yy}\phi_x^2}{(\phi_x^2 + \phi_y^2)^{\frac{3}{2}}} \quad (2.47)$$

A priori, the level set evolution is specified for points on the level set function zero level. Therefore, one must define extension velocities [9] to evolve the level set function elsewhere. For instance, the extension velocity at a point is the velocity at the point closest to it on the evolving curve. Extension velocities can also be defined so that the level set function is at all times the distance function from the evolving curve. Both of these definitions, often implemented via narrow banding [9], require the initial curves intersect the regions they segment. This is important when a region has unconnected components. An alternative robust to initialization, which we use in all the methods described in this book, extends the expression of the velocity on the evolving curve to the image domain when it can be evaluated at each point, as it is often the case [4, 10].

At all times t , active curve γ subject to velocity (2.40) can be recovered as the zero level set of level set function ϕ evolving according to (2.46). Regardless of variations in the topology of the active curve, ϕ remains a function (Figure 2.1 b). Another advantage of the level set implementation is that region membership is explicitly maintained and readily available in the level set representation because the sign of ϕ determines which points are inside curve γ , and which are outside. This information is computationally very expensive to determine with an explicit representation of active curves.

The book of Sethian [9] is about efficient and numerically stable discretization of evolution equations such as those corresponding to the level set segmentation functionals in this book. It also contains several examples of applications in various domains. Velocities of the types 1, 2, and 3 are discretized differently as summarized below [9]. Because the velocity of an active curve is, in general, a compound of velocities of the three types, the discretization of (2.46) can be written as follows:

$$\phi_{ij}^{k+1} = \phi_{ij}^k + \Delta t \begin{cases} +V_{ij}^k \left((D_{ij}^{0x})^2 + (D_{ij}^{0y})^2 \right)^{\frac{1}{2}} & \text{for type 1 terms} \\ \left(\max(F_{1ij}^k, 0)D_{ij}^{-x} + \min(F_{1ij}^k, 0)D_{ij}^{+x} \right. \\ \quad \left. + \max(F_{2ij}^k, 0)D_{ij}^{-y} + \min(F_{2ij}^k, 0)D_{ij}^{+y} \right) & \text{for type 2 terms} \\ \left(\max(V_{ij}^k, 0)\nabla^+ + \min(V_{ij}^k, 0)\nabla^- \right) & \text{for type 3 terms} \end{cases} \quad (2.48)$$

where i, j are indices on the discretization grid of Ω , k is the iteration index, F_1, F_2 are the coordinates of \mathbf{F} appearing in the general expression of terms of type 2. Finite difference x -derivative operators D^{+x} (forward scheme), D^{-x} (backward scheme), and D^{0x} (central scheme), are applied to ϕ at i, j and iteration k , i.e., $D_{ij}^{+x}, D_{ij}^{-x}, D_{ij}^{0x}$ in (2.48) stand for $D^{+x}(\phi^k)_{ij}, D^{-x}(\phi^k)_{ij}, D^{0x}(\phi^k)_{ij}$ and are given by

$$\begin{aligned} D_{ij}^{+x} &= \phi_{i+1,j}^k - \phi_{ij}^k \\ D_{ij}^{-x} &= \phi_{ij}^k - \phi_{i-1,j}^k \\ D_{ij}^{0x} &= \frac{1}{2}(\phi_{i+1,j}^k - \phi_{i-1,j}^k) \end{aligned}$$

Similar comments and expressions apply to the y -derivative operators D^{+y}, D^{-y} , and D^{0y} . Finally, ∇^+ and ∇^- are defined by

$$\begin{aligned} \nabla^+ &= \left(\max(D_{ij}^{-x}, 0)^2 + \min(D_{ij}^{+x}, 0)^2 \right. \\ &\quad \left. + \max(D_{ij}^{-y}, 0)^2 + \min(D_{ij}^{+y}, 0)^2 \right)^{\frac{1}{2}} \\ \nabla^- &= \left(\max(D_{ij}^{+x}, 0)^2 + \min(D_{ij}^{-x}, 0)^2 \right. \\ &\quad \left. + \max(D_{ij}^{+y}, 0)^2 + \min(D_{ij}^{-y}, 0)^2 \right)^{\frac{1}{2}} \end{aligned} \quad (2.49)$$

2.4 Optical flow

Let I be an image sequence, considered a differentiable function,

$$I : (x, y, t) \in \Omega \times]0, T[\mapsto I(x, y, t) \in \mathbb{R}^+, \quad (2.50)$$

where x, y are image coordinates, Ω is an open subset of the real plane to represent the image domain, and T is the time of duration of the image sequence.

2.4.1 The gradient equation

Let \mathbf{P} be a point on an imaged environmental surface moving relative to the viewing system. The viewing system is symbolized as in Figure 2.3. The image \mathbf{p} of \mathbf{P} on the viewing system projection plane can be viewed as moving along a trajectory in the space-time (actual time) domain $x - y - t$. Let $c(t) = (x(t), y(t), t)$ be the Cartesian parametric representation of this trajectory. Finally, let the restriction of I to c be $h = I \circ c$, where \circ designates composition, i.e., $h(t) = I(x(t), y(t), t)$. The hypothesis that h is constant in time, i.e., that I does not vary along the motion trajectory c of \mathbf{p} ,¹ leads to the *gradient equation* of Horn and Schunck [11]:

¹ This is strictly true for points on a Lambertian surface in translation relative to the viewing system, under constant, uniform lighting.

$$\frac{dh}{dt} = \frac{\partial I}{\partial x} \frac{dx}{dt} + \frac{\partial I}{\partial y} \frac{dy}{dt} + \frac{\partial I}{\partial t} \frac{dt}{dt} = 0 \quad (2.51)$$

If we designate the partial derivatives of I by I_x, I_y, I_t , and $\frac{dx}{dt}, \frac{dy}{dt}$ by u, v , we have

$$I_x u + I_y v + I_t = 0 \quad (2.52)$$

Vector (u, v) is the *optical velocity* of \mathbf{p} , the velocity of the projection of \mathbf{P} onto the image plane. The field of optical velocities over the image domain is the *optical flow* [12]. The gradient equation is also referred to as the *optical flow constraint*. It can be written in vector form:

$$\langle \nabla I, W \rangle + I_t = 0 \quad (2.53)$$

where $\nabla I = (I_x, I_y)$ is the spatial gradient of I and $W = (u, v)$. The projection W^\perp of W on ∇I is given by

$$W^\perp = \left\langle \frac{\nabla I}{\|\nabla I\|}, W \right\rangle \frac{\nabla I}{\|\nabla I\|} \quad (2.54)$$

or, using (2.53):

$$W^\perp = \left(-\frac{I_t}{\|\nabla I\|} \right) \frac{\nabla I}{\|\nabla I\|} \quad (2.55)$$

This projection can be estimated from the first order spatio-temporal variations of I (Figure 2.2). Therefore, the gradient equation gives the component of the optical velocity vector in the direction of the image gradient, i.e., the direction normal to the isophote, and only this component. This is a manifestation of the *aperture problem*: the movement of a straight edge seen through an aperture is ambiguous because only the component of motion in the direction perpendicular to the edge is determined (Figure 2.2).

2.4.2 The Horn and Schunck formulation

Horn and Schunck minimize the following functional to estimate optical flow:

$$E(u, v) = \int_{\Omega} (I_x u + I_y v + I_t)^2 dx dy + \lambda \int_{\Omega} (\|\nabla u\|^2 + \|\nabla v\|^2) dx dy \quad (2.56)$$

where λ is a positive constant to weigh the relative contribution of the two terms of the functional. The first integral, the data term, measures the conformity of the motion field to the image sequence first-order spatial and temporal variations. The second integral is a regularization term which measures the smoothness of the motion field. The Euler-Lagrange equations corresponding to (2.56) are two coupled partial differential equations:

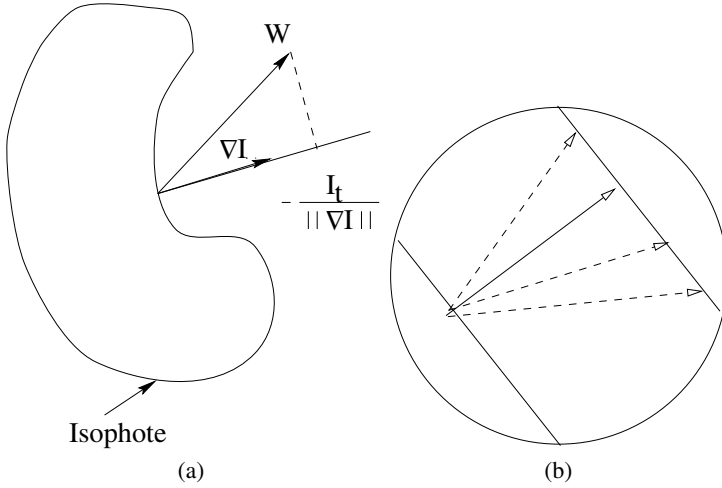


Fig. 2.2. (a) The projection of the optical flow on the image gradient can be estimated from the image first-order spatio-temporal data; whenever $\nabla I \neq 0$ it is equal to $-\frac{I_t}{\|\nabla I\|}$; (b) the aperture problem: the movement of the straight edge seen through an aperture (the circular window in this figure) is ambiguous because only the component of motion in the direction perpendicular to the edge is determined (the solid arrow).

$$\begin{aligned} I_x(I_x u + I_y v + I_t) - \lambda \nabla^2 u &= 0 \\ I_y(I_x u + I_y v + I_t) - \lambda \nabla^2 v &= 0, \end{aligned} \quad (2.57)$$

to which the (Neumann) boundary conditions are added:

$$\begin{aligned} \frac{\partial u}{\partial \mathbf{n}} &= 0 \\ \frac{\partial v}{\partial \mathbf{n}} &= 0 \end{aligned} \quad (2.58)$$

where ∇^2 designates the Laplacian and $\frac{\partial}{\partial \mathbf{n}}$ indicates differentiation in the direction of the normal \mathbf{n} to the image domain boundary $\partial\Omega$.

There are very efficient numerical implementations to solve (2.57) [13]. Refer to the appendix for more details.

The main problem with the Horn and Shunck method comes from the occurrence of the Laplacian in (2.57), which causes isotropic smoothing and, therefore, blurred

motion boundaries. The method of [14] formulates the problem so that smoothing of motion is inhibited across motion boundaries. This formulation is described next.

2.4.3 The Aubert, Kornprobst, and Deriche formulation

The study in [14, 15] investigates the following generalization of the Horn and Schunck functional:

$$E(u, v) = \int_{\Omega} (I_x u + I_y v + I_t)^2 dx dy + \lambda \int_{\Omega} (\rho(\|\nabla u\|) + \rho(\|\nabla v\|)) dx dy \quad (2.59)$$

where ρ is a function of class C^2 . With $\rho(z) = z^2$, (2.59) reduces to the Horn and Schunck functional (2.56).

The Euler-Lagrange equations corresponding to (2.59) are

$$\begin{aligned} I_x(I_x u + I_y v + I_t) &= \frac{\lambda}{2} \operatorname{div} \left(\rho'(\|\nabla u\|) \frac{\nabla u}{\|\nabla u\|} \right) \\ I_y(I_x u + I_y v + I_t) &= \frac{\lambda}{2} \operatorname{div} \left(\rho'(\|\nabla v\|) \frac{\nabla v}{\|\nabla v\|} \right), \end{aligned} \quad (2.60)$$

with the boundary conditions

$$\begin{aligned} \frac{\rho'(\|\nabla u\|)}{\|\nabla u\|} \frac{\partial u}{\partial \mathbf{n}} &= 0 \\ \frac{\rho'(\|\nabla v\|)}{\|\nabla v\|} \frac{\partial v}{\partial \mathbf{n}} &= 0 \end{aligned} \quad (2.61)$$

The Aubert-Deriche-Kornprobst function

$$\rho(s) = 2\sqrt{1+s^2} - 2 \quad (2.62)$$

follows conditions which require that it allows smoothing of motion along motion boundaries and inhibits it across [14, 15].

An efficient implementation of the formulation is described in [14, 15, 16]. Refer to the appendix for more details.

Next, we will examine the relationship between optical flow and the three-dimensional structure and motion of rigid objects. We will need this relationship in Chapter 8 on image segmentation based on the movement of real objects.

2.4.4 Optical flow of rigid body motion

We symbolize physical space by the Euclidean space \mathbb{R}^3 and the viewing system by an orthonormal direct coordinate system $\mathcal{S} = (\mathbf{O}; \mathbf{I}, \mathbf{J}, \mathbf{K})$ and central projection

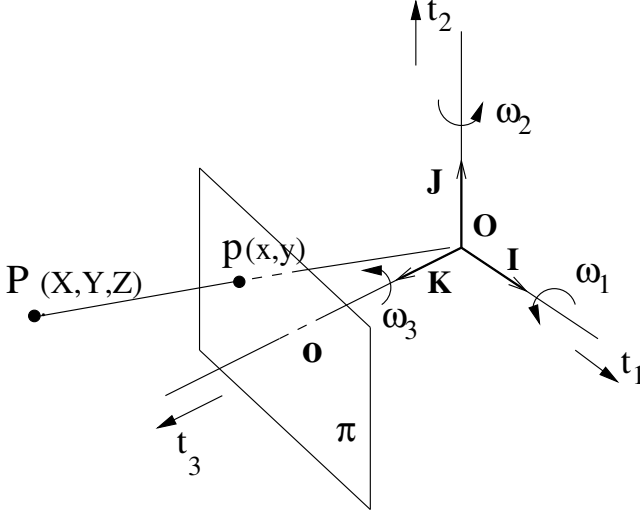


Fig. 2.3. The viewing system is symbolized by a direct orthonormal coordinate system $\mathcal{S} = (\mathbf{O}; \mathbf{I}, \mathbf{J}, \mathbf{K})$ and central projection through \mathbf{O} on plane π (the image plane) parallel to plane $P_{\mathbf{IJ}}$ and at focal distance f from \mathbf{O} .

through \mathbf{O} on plane π (the image plane) parallel to plane $P_{\mathbf{IJ}}$ and at distance f from \mathbf{O} (the focal length). This is illustrated in Figure 2.3.

Let $t \rightarrow \mathcal{B}(t)$ be a rigid body moving in space and \mathbf{P} a point of \mathcal{B} . Let \mathbf{p} be the image of \mathbf{P} on π . If $\mathbf{P} = X\mathbf{I} + Y\mathbf{J} + Z\mathbf{K}$ and $\mathbf{p} = x\mathbf{I} + y\mathbf{J}$, we have the following projection relations:

$$x = f \frac{X}{Z} \quad y = f \frac{Y}{Z} \quad (2.63)$$

If the kinematic screw of the (rigid) motion of \mathcal{B} relative to \mathcal{S} is $(\boldsymbol{\omega}, \mathbf{T})$ then the velocity of \mathbf{P} is $\mathbf{P}' = \mathbf{T} + \boldsymbol{\omega} \times \mathbf{OP}$. Substitution of this expression in the derivative with respect to time of each equation in (2.63) gives the following expression of optical velocity [17]:

$$\begin{aligned} u &= \frac{1}{Z}(ft_1 - xt_3) - \frac{xy}{f}\omega_1 + \frac{f^2 + x^2}{f}\omega_2 - y\omega_3 \\ v &= \frac{1}{Z}(ft_2 - yt_3) - \frac{f^2 + y^2}{f}\omega_1 + \frac{xy}{f}\omega_2 + x\omega_3 \end{aligned} \quad (2.64)$$

where $\mathbf{T} = (t_1, t_2, t_3)$ and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$. Substitution of (2.64) in the gradient equation (2.52) gives the *optical flow 3D rigid body constraint*:

$$\langle \mathbf{s}, \boldsymbol{\tau} \rangle + \langle \mathbf{q}, \boldsymbol{\omega} \rangle + I_t = 0 \quad (2.65)$$

where vectors $\boldsymbol{\tau}$, \mathbf{s} , and \mathbf{q} are given by

$$\tau = \frac{\mathbf{T}}{Z}, \quad \mathbf{s} = \begin{pmatrix} fI_x \\ fI_y \\ -xI_x - yI_y \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} -fI_y - \frac{y}{f}(xI_x + yI_y) \\ fI_x + \frac{x}{f}(xI_x + yI_y) \\ -yI_x + xI_y \end{pmatrix} \quad (2.66)$$

Pulling $\frac{1}{z}$ out of each equation in (2.64), equating the resulting expressions, and making a change of variables, leads to the following depth-free homogeneous linear equation [18, 19]:

$$\langle \mathbf{d}, \mathbf{e} \rangle = 0 \quad (2.67)$$

where $\mathbf{d} = (x^2, y^2, f^2, 2xy, 2xf, 2yf, -fv, fu, -uy + vx)$, and \mathbf{e} is the vector of *essential parameters*:

$$\begin{aligned} e_1 &= -\omega_3 t_3 - \omega_2 t_2, & e_2 &= -\omega_3 t_3 - \omega_1 t_1, & e_3 &= -\omega_2 t_2 - \omega_1 t_1 \\ e_4 &= \frac{\omega_2 t_1 + \omega_1 t_2}{2}, & e_5 &= \frac{\omega_1 t_3 + \omega_3 t_1}{2}, & e_6 &= \frac{\omega_2 t_3 + \omega_3 t_2}{2} \\ e_7 &= t_1, & e_8 &= t_2, & e_9 &= t_3 \end{aligned} \quad (2.68)$$

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