Integration by Parts Formula with Respect to Jump Times for Stochastic Differential Equations

Vlad Bally and Emmanuelle Clément

Abstract We establish an integration by parts formula based on jump times in an abstract framework in order to study the regularity of the law for processes solution of stochastic differential equations with jumps.

Keywords Integration by parts formula · Poisson Point Measures · Stochastic Equations

MSC (2010): Primary: 60H07, Secondary: 60G55, 60G57

1 Introduction

We consider the one-dimensional equation

$$X_t = x + \int_0^t \int_E c(u, a, X_{u-}) \mathrm{d}N(u, a) + \int_0^t g(u, X_u) \mathrm{d}u \tag{1}$$

where N is a Poisson point measure of intensity measure μ on some abstract measurable space E. We assume that c and g are infinitely differentiable with respect to t and x, have bounded derivatives of any order, and have linear growth with respect to x. Moreover we assume that the derivatives of c are bounded by a function $\overline{c}$ such that $\int_E \overline{c}(a) \mathrm{d}\mu(a) < \infty$. Under these hypotheses, the equation has a unique solution and the stochastic integral with respect to the Poisson point measure is a Stieltjes integral.

Our aim is to give sufficient conditions in order to prove that the law of X_t is absolutely continuous with respect to the Lebesgue measure and has a smooth density. If $E = \mathbb{R}^m$ and if the measure μ admits a smooth density h, then one may develop a Malliavin calculus based on the amplitudes of the jumps in order

V. Bally and E. Clément (✉)
Laboratoire d'Analyse et de Mathématiques Appliquées, UMR 8050, Université Paris-Est, 5 Bld Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée Cedex 2, France
e-mail: vlad.bally@univ-mlv.fr, emmanuelle.clement@univ-mlv.fr

D. Crisan (ed.), *Stochastic Analysis 2010*, DOI 10.1007/978-3-642-15358-7_2,

to solve this problem. This has been done first in Bismut [4] and then in Bichteler, Gravereaux, and Jacod [3]. But if μ is a singular measure, this approach fails and one has to use the noise given by the jump times of the Poisson point measure in order to settle a differential calculus analogous to the Malliavin calculus. This is a much more delicate problem and several approaches have been proposed. A first step is to prove that the law of X_t is absolutely continuous with respect to the Lebesgue measure, without taking care of the regularity. A first result in this sense was obtained by Carlen and Pardoux [5] and was followed by a lot of other papers (see [1,7,11,13]). The second step is to obtain the regularity of the density. Recently two results of this type have been obtained by Ishikawa and Kunita [10] and by Kulik [12]. In both cases, one deals with an equation of the form

$$\mathrm{d}X_t = g(t, X_t)\mathrm{d}t + f(t, X_{t-})\mathrm{d}U_t \tag{2}$$

where U is a Lévy process. The above equation is multi-dimensional (let us mention that the method presented in our paper may be used in the multi-dimensional case as well, but then some technical problems related to the control of the Malliavin covariance matrix have to be solved – and for simplicity we preferred to leave out this kind of difficulties in this paper). Ishikawa and Kunita [10] used the finite difference approach given by Picard [14] in order to obtain sufficient conditions for the regularity of the density of the solution of an equation of type (1) (in a somehow more particular form, close to linear equations). The result in that paper produces a large class of examples in which we get a smooth density even for an intensity measure which is singular with respect to the Lebesgue measure. The second approach is due to Kulik [12]. He settled a Malliavin type calculus based on perturbations of the time structure in order to give sufficient conditions for the smoothness of the density. In his paper, the coefficient f is equal to one so the non-degeneracy comes from the drift term g only. As before, he obtains the regularity of the density even if the intensity measure μ is singular. He also proves that under some appropriate conditions, the density is not smooth for a small t so that one has to wait before the regularization effect of the noise produces a regular density.

The result proved in our paper is the following. We consider the function

$$\alpha(t,a,x) = g(x) - g(x + c(t,a,x)) + (g\partial_x c + \partial_t c)(t,a,x).$$

Except the regularity and boundedness conditions on g and c we consider the following non-degeneracy assumption. There exists a measurable function $\underline{\alpha}$ such that $|\alpha(t,a,x)| \geq \underline{\alpha}(a) > 0$ for every $(t,a,x) \in \mathbb{R}_+ \times E \times \mathbb{R}$. We assume that there exists a sequence of subsets $E_n \uparrow E$ such that $\mu(E_n) < \infty$ and

$$\underline{\lim}_{n\to\infty} \frac{1}{\mu(E_n)} \ln\left(\int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu(a)\right) = \theta < \infty.$$

We need this hypothesis in order to control the error due to the fact that we localize our differential calculus on a non-degeneracy set. If $\theta = 0$, then for every $t > 0$, the law of X_t has a C^∞ density with respect to the Lebesgue measure. Suppose now that $\theta > 0$ and let $q \in \mathbb{N}$. Then, for $t > 16\theta(q+2)(q+1)^2$ the law of X_t has a density of class $\mathcal{C}^q$. Notice that for small t we are not able to prove that a density exists and we have to wait for a sufficiently large t in order to obtain a regularization effect.

In the paper of Kulik [12], one takes $c(t,a,x) = a$ so $\alpha(t,a,x) = g(x) - g(x + c(t,a,x))$. Then the non-degeneracy condition concerns just the drift coefficient g. And in the paper of Ishikawa and Kunita, the basic example (which corresponds to the geometric Lévy process) is $c(t,a,x) = xa(\mathrm{e}^a - 1)$ and g constant. So $\alpha(t,a,x) = a(\mathrm{e}^a - 1) \sim a^2$ as $a \to 0$. The drift coefficient does not contribute to the non-degeneracy condition (which is analogous to the uniform ellipticity condition).

The paper is organized as follows. In Sect. 2, we give an integration by parts formula of Malliavin type. This is analogous to the integration by parts formulas given in [2] and [1]. But there are two specific points: first of all the integration by parts formula take into account the border terms (in the above-mentioned papers the border terms cancel because one makes use of some weights which are null on the border; but in the paper of Kulik [12] such border terms appear as well). The second point is that we use here a "one shot" integration by parts formula: in the classical gaussian Malliavin calculus, one employs all the noise which is available – so one derives an infinite dimensional differential calculus based on "all the increments" of the Brownian motion. The analogous approach in the case of Poisson point measures is to use all the noise which comes from the random structure (jumps). And this is the point of view of almost all the papers on this topic. But in our paper, we use just "one jump time" which is chosen in a cleaver way (according to the non-degeneracy condition). In Sect. 3, we apply the general integration by parts formula to stochastic equations with jumps. The basic noise is given by the jump times.

2 Integration by Parts Formula

2.1 Notations-Derivative Operators

The abstract framework is quite similar to the one developed in Bally and Clément [2], but we introduce here some modifications in order to take into account the border terms appearing in the integration by parts formula. We consider a sequence of random variables $(V_i)_{i \in \mathbb{N}^*}$ on a probability space $(\Omega, \mathcal{F}, P)$, a sub σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ and a random variable J, $\mathcal{G}$ measurable, with values in $\mathbb{N}$. Our aim is to establish a differential calculus based on the variables (V_i), conditionally on $\mathcal{G}$. In order to derive an integration by parts formula, we need some assumptions on the random variables (V_i). The main hypothesis is that conditionally on $\mathcal{G}$, the law of V_i admits a locally smooth density with respect to the Lebesgue measure.

H0. (a) Conditionally on $\mathcal{G}$, the random variables $(V_i)_{1\le i\le J}$ are independent and for each $i \in \{1,\dots,J\}$ the law of V_i is absolutely continuous with respect to the Lebesgue measure. We note p_i the conditional density.

(b) For all $i \in \{1,\dots,J\}$, there exist some $\mathcal{G}$ measurable random variables a_i and b_i such that $-\infty < a_i < b_i < +\infty$, $(a_i,b_i) \subset \{p_i > 0\}$. We also assume that p_i admits a continuous bounded derivative on (a_i,b_i) and that $\ln p_i$ is bounded on (a_i,b_i).

We define now the class of functions on which this differential calculus will apply. We consider in this paper functions $f : \Omega \times \mathbb{R}^{\mathbb{N}^*} \to \mathbb{R}$ which can be written as

$$f(\omega, v) = \sum_{m=1}^{\infty} f^m(\omega, v_1,\dots,v_m) 1_{\{J(\omega)=m\}} \tag{3}$$

where $f^m : \Omega \times \mathbb{R}^m \to \mathbb{R}$ are $\mathcal{G} \times \mathcal{B}(\mathbb{R}^m)-$measurable functions.

In the following, we fix $L \in \mathbb{N}$ and we will perform integration by parts L times. But we will use another set of variables for each integration by parts. So for $1 \le l \le L$, we fix a set of indices $I_l \subset \{1,\dots,J\}$ such that if $l \neq l'$, $I_l \cap I_{l'} = \emptyset$. In order to do l integration by parts, we will use successively the variables $V_i, i \in I_l$, then the variables $V_i, i \in I_{l-1}$ and end with $V_i, i \in I_1$. Moreover, given l we fix a partition $(\Lambda_{l,i})_{i\in I_l}$ of Ω such that the sets $\Lambda_{l,i} \in \mathcal{G}, i \in I_l$. If $\omega \in \Lambda_{l,i}$, we will use only the variable V_i in our integration by parts.

With these notations, we define our basic spaces. We consider in this paper random variables $F = f(\omega, V)$ where $V = (V_i)_i$ and f is given by (3). To simplify the notation we write $F = f^J(\omega, V_1,\dots,V_J)$ so that conditionally on $\mathcal{G}$ we have $J = m$ and $F = f^m(\omega, V_1,\dots,V_m)$. We denote by $\mathcal{S}^0$ the space of random variables $F = f^J(\omega, V_1,\dots,V_J)$ where f^J is a continuous function on $O_J = \prod_{i=1}^J (a_i,b_i)$ such that there exists a $\mathcal{G}$ measurable random variable C satisfying

$$\sup_{v\in O_J} |f^J(\omega, v)| \le C(\omega) < +\infty \quad \text{a.e.} \tag{4}$$

We also assume that f^J has left limits (respectively right limits) in a_i (respectively in b_i). Let us be more precise.

With the notations

$$V_{(i)} = (V_1,\dots,V_{i-1},V_{i+1},\dots,V_J), \quad (V_{(i)}, v_i) = (V_1,\dots,V_{i-1},v_i,V_{i+1},\dots,V_J),$$

for $v_i \in (a_i,b_i)$ our assumption is that the following limits exist and are finite:

$$\lim_{\varepsilon\to 0} f^J(\omega, V_{(i)}, a_i+\varepsilon) := F(a_i^+), \quad \lim_{\varepsilon\to 0} f^J(\omega, V_{(i)}, b_i-\varepsilon) := F(b_i^-). \tag{5}$$

Now for $k \ge 1$, $\mathcal{S}^k(I_l)$ denotes the space of random variables $F = f^J(\omega, V_1,\dots, V_J) \in \mathcal{S}^0$, such that f^J admits partial derivatives up to order k with respect to the variables $v_i, i \in I_l$ and these partial derivatives belong to $\mathcal{S}^0$.

We are now able to define our differential operators.

- *The derivative operators.* We define $D_l : \mathcal{S}^1(I_l) \to \mathcal{S}^0(I_l)$: by

$$D_l F := 1_{O_J}(V) \sum_{i \in I_l} 1_{\Lambda_{l,i}}(\omega) \partial_{v_i} f(\omega, V),$$

where $O_J = \prod_{i=1}^{J} (a_i, b_i)$.

- *The divergence operators.* We note

$$p_{(l)} = \sum_{i \in I_l} 1_{\Lambda_{l,i}} p_i, \tag{6}$$

and we define $\delta_l : \mathcal{S}^1(I_l) \to \mathcal{S}^0(I_l)$ by

$$\delta_l(F) = D_l F + F D_l \ln p_{(l)} = 1_{O_J}(V) \sum_{i \in I_l} 1_{\Lambda_{l,i}} (\partial_{v_i} F + F \partial_{v_i} \ln p_i)$$

We can easily see that if $F, U \in \mathcal{S}^1(I_l)$ we have

$$\delta_l(FU) = F\delta_l(U) + U D_l F. \tag{7}$$

- *The border terms.* Let $U \in \mathcal{S}^0(I_l)$. We define (using the notation (5))

$$[U]_l = \sum_{i \in I_l} 1_{\Lambda_{l,i}} 1_{O_{J,i}}(V_{(i)})((Up_i)(b_i^-) - (Up_i)(a_i^+))$$

with $O_{J,i} = \prod_{1 \le j \le J, j \ne i} (a_j, b_j)$

2.2 Duality and Basic Integration by Parts Formula

In our framework, the duality between δ_l and D_l is given by the following proposition. In the sequel, we denote by $E_{\mathcal{G}}$ the conditional expectation with respect to the sigma-algebra $\mathcal{G}$.

Proposition 1. *Assuming H0 then* $\forall F, U \in \mathcal{S}^1(I_l)$ *we have*

$$E_{\mathcal{G}}(U D_l F) = -E_{\mathcal{G}}(F \delta_l(U)) + E_{\mathcal{G}}[FU]_l. \tag{8}$$

For simplicity, we assume in this proposition that the random variables F and U take values in $\mathbb{R}$ but such a result can easily be extended to $\mathbb{R}^d$ value random variables.

Proof. We have $E_{\mathcal{G}}(UD_l F) = \sum_{i\in I_l} 1_{\Lambda_{l,i}} E_{\mathcal{G}} 1_{O_J}(V)(\partial_{v_i} f^J(\omega,V)u^J(\omega,V))$. From H0 we obtain

$$E_{\mathcal{G}} 1_{O_J}(V)(\partial_{v_i} f^J(\omega,V)u^J(\omega,V)) = E_{\mathcal{G}} 1_{O_{J,i}}(V_{(i)}) \int_{a_i}^{b_i} \partial_{v_i}(f^J)u^J p_i(v_i)\mathrm{d}v_i.$$

By using the classical integration by parts formula, we have

$$\int_{a_i}^{b_i} \partial_{v_i}(f^J)u^J p_i(v_i)\mathrm{d}v_i = [f^J u^J p_i]_{a_i}^{b_i} - \int_{a_i}^{b_i} f^J \partial_{v_i}(u^J p_i)\mathrm{d}v_i.$$

Observing that $\partial_{v_i}(u^J p_i) = (\partial_{v_i}(u^J) + u^J\partial_{v_i}(\ln p_i))p_i$, we have

$$\begin{aligned} E_{\mathcal{G}}(1_{O_J}(V)\partial_{v_i} f^J u^J) &= E_{\mathcal{G}} 1_{O_{J,i}}[(V_{(i)}) f^J u^J p_i]_{a_i}^{b_i} \\ &\quad - E_{\mathcal{G}} 1_{O_J}(V) F(\partial_{v_i}(U) + U\partial_{v_i}(\ln p_i)) \end{aligned}$$

and the proposition is proved. □

We can now state a first integration by parts formula.

Proposition 2. *Let H0 hold true and let $F \in \mathcal{S}^2(I_l)$, $G \in \mathcal{S}^1(I_l)$ and $\Phi : \mathbb{R} \to \mathbb{R}$ be a bounded function with bounded derivative. We assume that $F = f^J(\omega,V)$ satisfies the condition*

$$\min_{i\in I_l} \inf_{v\in O_J} |\partial_{v_i} f^J(\omega,v)| \geq \gamma(\omega), \tag{9}$$

where γ is $\mathcal{G}$ measurable and we define on $\{\gamma > 0\}$

$$(D_l F)^{-1} = 1_{O_J}(V) \sum_{i\in I_l} 1_{\Lambda_{l,i}} \frac{1}{\partial_{v_i} f(\omega,V)},$$

then

$$\begin{aligned} 1_{\{\gamma>0\}} E_{\mathcal{G}}(\Phi^{(1)}(F)G) &= -1_{\{\gamma>0\}} E_{\mathcal{G}}\left(\Phi(F) H_l(F,G)\right) \\ &\quad + 1_{\{\gamma>0\}} E_{\mathcal{G}}[\Phi(F)G(D_l F)^{-1}]_l \end{aligned} \tag{10}$$

with

$$H_l(F,G) = \delta_l(G(D_l F)^{-1}) = G\delta_l((D_l F)^{-1}) + D_l G(D_l F)^{-1}. \tag{11}$$

Proof. We observe that

$$D_l\Phi(F) = 1_{O_J}(V) \sum_{i\in I_l} 1_{\Lambda_{l,i}} \partial_{v_i}\Phi(F) = 1_{O_J}(V)\Phi^{(1)}(F) \sum_{i\in I_l} 1_{\Lambda_{l,i}} \partial_{v_i} F,$$

so that

$$D_l \Phi(F).D_l F = \Phi^{(1)}(F)(D_l F)^2,$$

and then $1_{\{\gamma>0\}}\Phi^{(1)}(F) = 1_{\{\gamma>0\}}D_l \Phi(F).(D_l F)^{-1}$. Now since $F \in \mathcal{S}^2(I_l)$, we deduce that $(D_l F)^{-1} \in \mathcal{S}^1(I_l)$ on $\{\gamma > 0\}$ and applying Proposition 1 with $U = G(D_l F)^{-1}$ we obtain the result. □

2.3 Iterations of the Integration by Parts Formula

We will iterate the integration by parts formula given in Proposition 2. We recall that if we iterate l times the integration by parts formula, we will integrate by parts successively with respect to the variables $(V_i)_{i\in I_k}$ for $1 \le k \le l$. In order to give some estimates of the weights appearing in these formulas, we introduce the following norm on $\mathcal{S}^l(\cup_{k=1}^l I_k)$, for $1 \le l \le L$.

$$|F|_l = |F|_\infty + \sum_{k=1}^{l} \sum_{1\le l_1<\ldots<l_k\le l} |D_{l_1}\ldots D_{l_k} F|_\infty, \tag{12}$$

where $|.|_\infty$ is defined on $\mathcal{S}^0$ by

$$|F|_\infty = \sup_{v\in O_J} |f^J(\omega, v)|.$$

For $l = 0$, we set $|F|_0 = |F|_\infty$. We remark that we have for $1 \le l_1 < \ldots < l_k \le l$

$$|D_{l_1}\ldots D_{l_k} F|_\infty = \sum_{i_1\in I_{l_1},\ldots,i_k\in I_{l_k}} \left(\prod_{j=1}^{k} 1_{\Lambda_{l_j,i_j}}\right)|\partial_{v_{i_1}}\ldots\partial_{v_{i_k}} F|_\infty,$$

and since for each l $(\Lambda_{l,i})_{i\in I_l}$ is a partition of Ω, for ω fixed, the preceding sum has only one term not equal to zero. This family of norms satisfies for $F \in \mathcal{S}^{l+1}(\cup_{k=1}^{l+1} I_k)$:

$$|F|_{l+1} = |D_{l+1}F|_l + |F|_l \quad \text{so} \quad |D_{l+1}F|_l \le |F|_{l+1}. \tag{13}$$

Moreover, it is easy to check that if $F, G \in \mathcal{S}^l(\cup_{k=1}^l I_k)$

$$|FG|_l \le C_l |F|_l |G|_l, \tag{14}$$

where C_l is a constant depending on l. Finally for any function $\phi \in \mathcal{C}^l(\mathbb{R}, \mathbb{R})$ we have

$$|\phi(F)|_l \leq C_l \sum_{k=0}^{l} |\phi^{(k)}(F)|_\infty |F|_l^k \leq C_l \max_{0\leq k\leq l} |\phi^{(k)}(F)|_\infty (1+|F|_l^l). \tag{15}$$

With these notations, we can iterate the integration by parts formula.

Theorem 1. *Let H0 hold true and let $\Phi : \mathbb{R} \mapsto \mathbb{R}$ be a bounded function with bounded derivatives up to order L. Let $F = f^J(w,V) \in \mathcal{S}^1(\cup_{l=1}^L I_l)$ such that*

$$\inf_{i\in\{1,\ldots,J\}} \inf_{v\in O_J} |\partial_{v_i} f^J(\omega,v)| \geq \gamma(\omega), \quad \gamma \in [0,1] \quad \mathcal{G} \text{ measurable} \tag{16}$$

then we have for $l \in \{1,\ldots,L\}$, $G \in \mathcal{S}^l(\cup_{k=1}^l I_k)$ and $F \in \mathcal{S}^{l+1}(\cup_{k=1}^l I_k)$

$$1_{\{\gamma>0\}} |E_{\mathcal{G}} \Phi^{(l)}(F) G| \leq C_l ||\Phi||_\infty 1_{\{\gamma>0\}} E_{\mathcal{G}} \left(|G|_l (1+|p|_0)^l \Pi_l(F) \right) \tag{17}$$

where $||\Phi||_\infty = \sup_x |\Phi(x)|$, $|p|_0 = \max_{l=1,\ldots,L} |p_{(l)}|_\infty$, C_l is a constant depending on l and $\Pi_l(F)$ is defined on $\{\gamma > 0\}$ by

$$\Pi_l(F) = \prod_{k=1}^{l} (1+|(D_k F)^{-1}|_{k-1})(1+|\delta_k((D_k F)^{-1})|_{k-1}). \tag{18}$$

Moreover, we have the bound

$$\Pi_l(F) \leq C_l \frac{(1+|\ln p|_1)^l}{\gamma^{l(l+2)}} \prod_{k=1}^{l} (1+|F|_k^{k-1} + |D_k F|_k^{k-1})^2, \tag{19}$$

where $|\ln p|_1 = \max_{i=1,\ldots,J} |(\ln p_i)'|_\infty$.

Proof. We proceed by induction. For $l = 1$, we have from Proposition 2 since $G \in \mathcal{S}^1(I_1)$ and $F \in \mathcal{S}^2(I_1)$

$$\begin{aligned} 1_{\{\gamma>0\}} E_{\mathcal{G}}(\Phi^{(1)}(F)G) = &-1_{\{\gamma>0\}} E_{\mathcal{G}}(\Phi(F) H_1(F,G)) \\ &+1_{\{\gamma>0\}} E_{\mathcal{G}}[\Phi(F) G (D_1 F)^{-1}]_1. \end{aligned} \tag{20}$$

We have on $\{\gamma > 0\}$

$$\begin{aligned} |H_1(F,G)| &\leq |G||\delta_1((D_1F)^{-1})| + |D_1 G||(D_1 F)^{-1}|, \\ &\leq (|G|_\infty + |D_1 G|_\infty)(1+|(D_1F)^{-1}|_\infty)(1+|\delta_1((D_1F)^{-1})|_\infty), \\ &= |G|_1 (1+|(D_1F)^{-1}|_0)(1+|\delta_1((D_1F)^{-1})|_0). \end{aligned}$$

Turning to the border term $[\Phi(F)G(D_1F)^{-1}]_1$, we check that

$$|[\Phi(F)G(D_1F)^{-1}]_1| \leq 2||\Phi||_\infty |G|_\infty \sum_{i\in I_1} 1_{\Lambda_{1,i}} |\frac{1}{\partial_{v_i} F}|_\infty \sum_{i\in I_1} 1_{\Lambda_{1,i}} |p_i|_\infty,$$
$$\leq 2||\Phi||_\infty |G|_0 |(D_1F)^{-1}|_0 |p|_0.$$

This proves the result for $l = 1$.

Now assume that Theorem 1 is true for $l \geq 1$ and let us prove it for $l + 1$. By assumption, we have $G \in \mathcal{S}^{l+1}(\cup_{k=1}^{l+1} I_k) \subset \mathcal{S}^1(I_{l+1})$ and $F \in \mathcal{S}^{l+2}(\cup_{k=1}^{l+1} I_k) \subset \mathcal{S}^2(I_{l+1})$. Consequently, we can apply Proposition 2 on I_{l+1}. This gives

$$1_{\{\gamma>0\}} E_{\mathcal{G}}(\Phi^{(l+1)}(F)G) = -1_{\{\gamma>0\}} E_{\mathcal{G}}\left(\Phi^{(l)}(F)H_{l+1}(F,G)\right)$$
$$+1_{\{\gamma>0\}} E_{\mathcal{G}}[\Phi^{(l)}(F)G(D_{l+1}F)^{-1}]_{l+1}, \quad (21)$$

with

$$H_{l+1}(F,G) = G\delta_{l+1}((D_{l+1}F)^{-1}) + D_{l+1}G(D_{l+1}F)^{-1},$$

$$[\Phi^{(l)}(F)G(D_{l+1}F)^{-1}]_{l+1} = \sum_{i\in I_{l+1}} 1_{\Lambda_{l+1,i}} 1_{O_{J,i}}(V_{(i)}) \left(\left(\Phi^{(l)}(F)G\frac{1}{\partial_{v_i} F} p_i\right)(b_i^-)\right.$$
$$\left. - \left(\Phi^{(l)}(F)G\frac{1}{\partial_{v_i} F} p_i\right)(a_i^+)\right).$$

We easily see that $H_{l+1}(F,G) \in \mathcal{S}^l(\cup_{k=1}^{l} I_k)$, and so using the induction hypothesis we obtain

$$1_{\{\gamma>0\}} |E_{\mathcal{G}}\Phi^{(l)}(F)H_{l+1}(F,G)|$$
$$\leq C_l ||\Phi||_\infty 1_{\{\gamma>0\}} E_{\mathcal{G}} |H_{l+1}(F,G)|_l (1 + |p|_0)^l \Pi_l(F),$$

and we just have to bound $|H_{l+1}(F,G)|_l$ on $\{\gamma > 0\}$. But using successively (14) and (13)

$$|H_{l+1}(F,G)|_l \leq C_l (|G|_l |\delta_{l+1}((D_{l+1}F)^{-1})|_l + |D_{l+1}G|_l |(D_{l+1}F)^{-1})|_l,$$
$$\leq C_l |G|_{l+1} (1 + |(D_{l+1}F)^{-1})|_l)(1 + |\delta_{l+1}((D_{l+1}F)^{-1})|_l).$$

This finally gives

$$|E_{\mathcal{G}}\Phi^{(l)}(F)H_{l+1}(F,G)| \leq C_l ||\Phi||_\infty E_{\mathcal{G}} |G|_{l+1} (1 + |p|_0)^l \Pi_{l+1}(F). \quad (22)$$

So we just have to prove a similar inequality for $E_{\mathcal{G}}[\Phi^{(l)}(F)G(D_{l+1}F)^{-1}]_{l+1}$. This reduces to consider

$$\sum_{i\in I_{l+1}} 1_{\Lambda_{l+1,i}} p_i(b_i^-) E_{\mathcal{G}} 1_{O_{J,i}}(V_{(i)}) \left(\Phi^{(l)}(F)G\frac{1}{\partial_{v_i} F}\right)(b_i^-) \quad (23)$$

since the other term can be treated similarly. Consequently, we just have to bound

$$|E_{\mathcal{G}} 1_{O_{J,i}}(V_{(i)}) \left(\Phi^{(l)}(F) G \frac{1}{\partial_{v_i} F} \right)(b_i^-)|.$$

Since all variables satisfy (4), we obtain from Lebesgue Theorem, using the notation (5)

$$\begin{aligned} & E_{\mathcal{G}} 1_{O_{J,i}}(V_{(i)}) \left(\Phi^{(l)}(F) G \frac{1}{\partial_{v_i} F} \right)(b_i^-) \\ & \quad = \lim_{\varepsilon \to 0} E_{\mathcal{G}} 1_{O_{J,i}}(V_{(i)}) \Phi^{(l)}(f^J(V_{(i)}, b_i - \varepsilon)) \left(g^J \frac{1}{\partial_{v_i} f^J} \right)(V_{(i)}, b_i - \varepsilon). \end{aligned}$$

To shorten the notation, we write simply $F(b_i - \varepsilon) = f^J(V_{(i)}, b_i - \varepsilon)$.

Now one can prove that if $U \in \mathcal{S}^{l'}(\cup_{k=1}^{l+1} I_k)$ for $1 \leq l' \leq l$ then $\forall i \in I_{l+1}$, $U(b_i - \varepsilon) \in \mathcal{S}^{l'}(\cup_{k=1}^{l} I_k)$ and $|U(b_i - \varepsilon)|_{l'} \leq |U|_{l'}$. We deduce then that $\forall i \in I_{l+1}$ $F(b_i - \varepsilon) \in \mathcal{S}^{l+1}(\cup_{k=1}^{l} I_k)$ and that $(G \frac{1}{\partial_{v_i} F})(b_i - \varepsilon) \in \mathcal{S}^{l}(\cup_{k=1}^{l} I_k)$ and from induction hypothesis

$$\begin{aligned} & |E_{\mathcal{G}} \Phi^{(l)}(F(b_i - \varepsilon)) 1_{O_{J,i}} (G \frac{1}{\partial_{v_i} F})(b_i - \varepsilon)| \\ & \quad \leq C_l ||\Phi||_\infty E_{\mathcal{G}} \{ |G(b_i - \varepsilon)|_l | \frac{1}{\partial_{v_i} F(b_i - \varepsilon)} |_l (1 + |p|_0)^l \Pi_l(F(b_i - \varepsilon)) \}, \\ & \quad \leq C_l ||\Phi||_\infty E_{\mathcal{G}} |G|_l | \frac{1}{\partial_{v_i} F} |_l (1 + |p|_0)^l \Pi_l(F). \end{aligned}$$

Putting this in (23) we obtain

$$\begin{aligned} & \left| E_{\mathcal{G}} \sum_{i \in I_{l+1}} 1_{\Lambda_{l+1,i}} 1_{O_{J,i}} \left(\Phi^{(l)}(F) G \frac{1}{\partial_{v_i} F} p_i \right)(b_i^-) \right| \\ & \quad \leq C_l ||\Phi||_\infty E_{\mathcal{G}} \left\{ |G|_l (1 + |p|_0)^l \Pi_l(F) \sum_{i \in I_{l+1}} 1_{\Lambda_{l+1,i}} |p_i|_\infty | \frac{1}{\partial_{v_i} F} |_l \right\}, \\ & \quad \leq C_l ||\Phi||_\infty E_{\mathcal{G}} \{ |G|_l (1 + |p|_0)^{l+1} \Pi_l(F) |(D_{l+1} F)^{-1}|_l \}. \end{aligned} \tag{24}$$

Finally plugging (22) and (24) in (21)

$$\begin{aligned} |E_{\mathcal{G}}(\Phi^{(l+1)}(F) G)| & \leq C_l ||\Phi||_\infty \Big(E_{\mathcal{G}} |G|_{l+1} (1 + |p|_0)^l \Pi_{l+1}(F) \\ & \qquad + E_{\mathcal{G}} |G|_l (1 + |p|_0)^{l+1} \Pi_l(F) |(D_{l+1} F)^{-1}|_l \Big), \\ & \leq C_l ||\Phi||_\infty E_{\mathcal{G}} |G|_{l+1} (1 + |p|_0)^{l+1} \Pi_{l+1}(F), \end{aligned}$$

and inequality (17) is proved for $l+1$. This achieves the first part of the proof of Theorem 1.

It remains to prove (19). We assume that $\omega \in \{\gamma > 0\}$.

Let $1 \leq k \leq l$. We first notice that combining (13) and (14), we obtain

$$|\delta_k(F)|_{k-1} \leq |F|_k \left(1 + \left|D_k \ln p_{(k)}\right|_\infty\right),$$

since $p_{(k)}$ only depends on the variables $V_i, i \in I_k$. So we deduce the bound

$$\left|\delta_k((D_k F)^{-1})\right|_{k-1} \leq \left|(D_k F)^{-1}\right|_k (1 + |\ln p|_1). \tag{25}$$

Now we have

$$|(D_k F)^{-1}|_{k-1} = \sum_{i \in I_k} 1_{\Lambda_{k,i}} \left|\frac{1}{\partial_{v_i} F}\right|_{k-1}$$

From (15) with $\phi(x) = 1/x$

$$\left|\frac{1}{\partial_{v_i} F}\right|_{k-1} \leq C_k \frac{(1 + |F|_k^{k-1})}{\gamma^k},$$

and consequently

$$|(D_k F)^{-1}|_{k-1} \leq C_k \frac{(1 + |F|_k^{k-1})}{\gamma^k}. \tag{26}$$

Moreover, we have, using successively (13) and (26),

$$\begin{aligned}
|(D_k F)^{-1}|_k &= |(D_k F)^{-1}|_{k-1} + |D_k (D_k F)^{-1}|_{k-1}, \\
&\leq C_k \left(\frac{(1+|F|_k^{k-1})}{\gamma^k} + \frac{(1+|D_k F|_k^{k-1})}{\gamma^{k+1}}\right), \\
&\leq C_k \frac{(1+|F|_k^{k-1} + |D_k F|_k^{k-1})}{\gamma^{k+1}}.
\end{aligned}$$

Putting this in (25)

$$\left|\delta_k((D_k F)^{-1})\right|_{k-1} \leq C_k \frac{(1 + |F|_k^{k-1} + |D_k F|_k^{k-1})}{\gamma^{k+1}} (1 + |\ln p|_1). \tag{27}$$

Finally from (26) and (27), we deduce

$$\Pi_l(F) \leq C_l \frac{(1 + |\ln p|_1)^l}{\gamma^{l(l+2)}} \prod_{k=1}^{l} (1 + |F|_k^{k-1} + |D_k F|_k^{k-1})^2,$$

and Theorem 1 is proved. $\square$

3 Stochastic Equations with Jumps

3.1 Notations and Hypotheses

We consider a Poisson point process p with measurable state space $(E, \mathcal{B}(E))$. We refer to Ikeda and Watanabe [9] for the notation. We denote by N the counting measure associated to p so $N_t(A) := N((0,t) \times A) = \#\{s < t; p_s \in A\}$. The intensity measure is $\mathrm{d}t \times \mathrm{d}\mu(a)$ where μ is a sigma-finite measure on $(E, \mathcal{B}(E))$ and we fix a non-decreasing sequence (E_n) of subsets of E such that $E = \cup_n E_n$, $\mu(E_n) < \infty$ and $\mu(E_{n+1}) \leq \mu(E_n) + K$ for all n and for a constant $K > 0$.

We consider the one-dimensional stochastic equation

$$X_t = x + \int_0^t \int_E c(s, a, X_{s-})\mathrm{d}N(s,a) + \int_0^t g(s, X_s)\mathrm{d}s. \tag{28}$$

Our aim is to give sufficient conditions on the coefficients c and g in order to prove that the law of X_t is absolutely continuous with respect to the Lebesgue measure and has a smooth density. We make the following assumptions on the coefficients c and g.

H1. We assume that the functions c and g are infinitely differentiable with respect to the variables (t,x) and that there exist a bounded function $\overline{c}$ and a constant $\overline{g}$, such that

$$\forall (t,a,x) \quad |c(t,a,x)| \leq \overline{c}(a)(1+|x|), \qquad \sup_{l+l' \geq 1} |\partial_t^{l'} \partial_x^l c(t,a,x)| \leq \overline{c}(a);$$
$$\forall (t,x) \quad |g(t,x)| \leq \overline{g}(1+|x|), \qquad \sup_{l+l' \geq 1} |\partial_t^{l'} \partial_x^l g(t,x)| \leq \overline{g};$$

We assume moreover that $\int_E \overline{c}(a)\mathrm{d}\mu(a) < \infty$.

Under H1, (28) admits a unique solution.

H2. We assume that there exists a measurable function $\hat{c} : E \mapsto \mathbb{R}_+$ such that $\int_E \hat{c}(a)\mathrm{d}\mu(a) < \infty$ and

$$\forall (t,a,x) \quad |\partial_x c(t,a,x)(1 + \partial_x c(t,a,x))^{-1}| \leq \hat{c}(a).$$

To simplify the notation, we take $\hat{c} = \overline{c}$. Under H2, the tangent flow associated to (28) is invertible. At last we give a non-degeneracy condition which will imply (16). We denote by α the function

$$\alpha(t,a,x) = g(t,x) - g(t, x + c(t,a,x)) + (g\partial_x c + \partial_t c)(t,a,x). \tag{29}$$

H3. We assume that there exists a measurable function $\underline{\alpha} : E \mapsto \mathbb{R}_+$ such that

$$\forall (t,a,x) \quad |\alpha(t,a,x)| \geq \underline{\alpha}(a) > 0,$$

$$\forall n \int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu(a) < \infty \quad \text{and} \quad \liminf_n \frac{1}{\mu(E_n)} \ln \left(\int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu(a) \right) = \theta < \infty.$$

We give in the following some examples where $E = (0, 1]$ and $\underline{\alpha}(a) = a$.

3.2 Main Results and Examples

Following the methodology introduced in Bally and Clément [2], our aim is to bound the Fourier transform of X_t, $\hat{p}_{X_t}(\xi)$, in terms of $1/|\xi|$, recalling that if $\int_{\mathbb{R}} |\xi|^q |\hat{p}_{X_t}(\xi)| \mathrm{d}\xi < \infty$, for $q > 0$, then the law of X_t is absolutely continuous and its density is $\mathcal{C}^{[q]}$. This is done in the next proposition. The proof of this proposition relies on an approximation of X_t which will be given in the next section.

Proposition 3. *Assuming H1, H2 and H3 we have for all $n, L \in \mathbb{N}^*$*

$$|\hat{p}_{X_t}(\xi)| \leq C_{t,L} \left(\mathrm{e}^{-\mu(E_n)t/(2L)} + \frac{1}{|\xi|^L} A_{n,L} \right),$$

with $A_{n,L} = \mu(E_n)^L (\int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu(a))^{L(L+2)}$.

From this proposition, we deduce our main result.

Theorem 2. *We assume that H1, H2 and H3 hold. Let $q \in \mathbb{N}$, then for $t > 16\theta(q+2)(q+1)^2$, the law of X_t is absolutely continuous with respect to the Lebesgue measure and its density is of class $\mathcal{C}^q$. In particular if $\theta = 0$, the law of X_t is absolutely continuous with respect to the Lebesgue measure and its density is of class $\mathcal{C}^\infty$ for every $t > 0$.*

Proof. From Proposition 3, we have

$$|\hat{p}_{X_t}(\xi)| \leq C_{t,L} \left(\mathrm{e}^{-\mu(E_n)t/2L} + \frac{1}{|\xi|^L} A_{n,L} \right).$$

Now $\forall k, k_0 > 0$, if $t/2L > k\theta$, we deduce from H3 that for $n \geq n_L$

$$t/2L > \frac{k}{\mu(E_n)} \ln \left(\int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu(a) \right) + \frac{k \ln \mu(E_n)}{k_0 \mu(E_n)}$$

since the second term on the right-hand side tends to zero. This implies

$$\mathrm{e}^{\mu(E_n)t/2L} > \left(\int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu(a) \right)^k \mu(E_n)^{k/k_0}.$$

Choosing $k = L(L+2)$ and $k/k_0 = L$, we obtain that for $n \geq n_L$ and $t/2L > L(L+2)\theta$

$$\mathrm{e}^{\mu(E_n)t/2L} > A_{n,L}.$$

and then

$$|\hat{p}_{X_t}(\xi)| \leq C_{t,L}\left(\mathrm{e}^{-\mu(E_n)t/2L} + \frac{1}{|\xi|^L}\mathrm{e}^{\mu(E_n)t/2L}\right),$$
$$\leq C_{t,L}\left(\frac{1}{B_n(t)} + \frac{B_n(t)}{|\xi|^L}\right),$$

with $B_n(t) = \mathrm{e}^{\mu(E_n)t/2L}$. Now recalling that $\mu(E_n) < \mu(E_{n+1}) \leq K + \mu(E_n)$, we have $B_n(t) < B_{n+1}(t) \leq K_t B_n(t)$. Moreover, since $B_n(t)$ goes to infinity with n we have

$$1_{\{|\xi|^{L/2} \geq B_{n_L}(t)\}} = \sum_{n \geq n_L} 1_{\{B_n(t) \leq |\xi|^{L/2} < B_{n+1}(t)\}}.$$

But if $B_n(t) \leq |\xi|^{L/2} < B_{n+1}(t)$, $|\hat{p}_{X_t}(\xi)| \leq C_{t,L}/|\xi|^{L/2}$ and so

$$\int |\xi|^q |\hat{p}_{X_t}(\xi)| \mathrm{d}\xi = \int_{|\xi|^{L/2} < B_{n_L}(t)} |\xi|^q |\hat{p}_{X_t}(\xi)| \mathrm{d}\xi + \int_{|\xi|^{L/2} \geq B_{n_L}(t)} |\xi|^q |\hat{p}_{X_t}(\xi)| \mathrm{d}\xi,$$
$$\leq C_{t,L,n_L} + \int_{|\xi|^{L/2} \geq B_{n_L}(t)} |\xi|^{q-L/2} \mathrm{d}\xi.$$

For $q \in \mathbb{N}$, choosing L such that $L/2 - q > 1$, we obtain $\int |\xi|^q |\hat{p}_{X_t}(\xi)| \mathrm{d}\xi < \infty$ for $t/2L > L(L+2)\theta$ and consequently the law of X_t admits a density $\mathcal{C}^q$ for $t > 2L^2(L+2)\theta$ and $L > 2(q+1)$, that is $t > 16\theta(q+1)^2(q+2)$ and Theorem 2 is proved. □

We end this section with two examples

Example 1. We take $E = (0,1]$, $\mu_\lambda = \sum_{k \geq 1} \frac{1}{k^\lambda} \delta_{1/k}$ with $0 < \lambda < 1$ and $E_n = [1/n, 1]$. We have $\cup_n E_n = E$, $\mu(E_n) = \sum_{k=1}^n \frac{1}{k^\lambda}$ and $\mu_\lambda(E_{n+1}) \leq \mu_\lambda(E_n) + 1$. We consider the process (X_t) solution of (28) with $c(t,a,x) = a$ and $g(t,x) = g(x)$ assuming that the derivatives of g are bounded and that $|g'(x)| \geq \underline{g} > 0$. We have $\int_E a \mathrm{d}\mu_\lambda(a) = \sum_{k \geq 1} \frac{1}{k^{\lambda+1}} < \infty$ so H1 and H2 hold. Moreover, $\alpha(t,a,x) = g(x) - g(x+a)$ so $\underline{\alpha}(a) = \underline{g}a$. Now $\int_{E_n} \frac{1}{a} \mathrm{d}\mu_\lambda(a) = \sum_{k=1}^n k^{1-\lambda}$, which is equivalent as n, go to infinity to $n^{2-\lambda}/(2-\lambda)$. Now we have

$$\frac{1}{\mu_\lambda(E_n)} \ln\left(\int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu_\lambda(a)\right) = \frac{\ln(\underline{g} \sum_{k=1}^n k^{1-\lambda})}{\sum_{k=1}^n \frac{1}{k^\lambda}} \sim_{n\to\infty} C \frac{\ln(n^{2-\lambda})}{n^{1-\lambda}} \to 0,$$

and then H3 is satisfied with $\theta = 0$. We conclude from Theorem 2 that $\forall t > 0$, X_t admits a density $\mathcal{C}^\infty$.

In the case $\lambda = 1$, we have $\mu_1(E_n) = \sum_{k=1}^n \frac{1}{k} \sim_{n\to\infty} \ln n$ then

$$\frac{1}{\mu_1(E_n)} \ln\left(\int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu_1(a)\right) = \frac{\ln(\underline{g} \sum_{k=1}^n 1)}{\sum_{k=1}^n \frac{1}{k}} \sim_{n\to\infty} 1,$$

and this gives H3 with $\theta = 1$. So the density of X_t is regular as soon as t is large enough. In fact it is proved in Kulik [12] that under some appropriate conditions the density of X_t is not continuous for small t.

Example 2. We take the intensity measure μ_λ as in the previous example and we consider the process (X_t) solution of (28) with $g = 1$ and $c(t,a,x) = ax$. This gives $\overline{c}(a) = a$ and $\underline{\alpha}(a) = a$. So the conclusions are similar to example 1 in both cases $0 < \lambda < 1$ and $\lambda = 1$. But in this example we can compare our result to the one given by Ichikawa and Kunita [10]. They assume the condition

$$\liminf_{u\to 0} \frac{1}{u^h} \int_{|a|\le u} a^2 \mathrm{d}\mu(a) > 0, \quad (\star)$$

for some $h \in (0,2)$. Here we have

$$\int_{|a|\le u} a^2 \mathrm{d}\mu(a) = \sum_{k\ge 1/u} \frac{1}{k^{2+\lambda}} \sim_{u\to 0} \frac{u^{1+\lambda}}{1+\lambda}.$$

So if $0 < \lambda < 1$, $(\star)$ holds and their results apply. In the case $\lambda = 1$, $(\star)$ fails and they do not conclude. However, in our approach we conclude that the density of X_t is $\mathcal{C}^q$ for $t > 16(q+2)(q+1)^2$.

The next section is devoted to the proof of Proposition 3.

3.3 Approximation of X_t and Integration by Parts Formula

In order to bound the Fourier transform of the process X_t solution of (28), we will apply the differential calculus developed in Sect. 2. The first step consists in an approximation of X_t by a random variable X_t^N which can be viewed as an element of our basic space $\mathcal{S}^0$. We assume that the process (X_t^N) is solution of the discrete version of (28)

$$X_t^N = x + \int_0^t \int_{E_N} c(s,a,X_{s-}^N)\mathrm{d}N(s,a) + \int_0^t g(s,X_s^N)\mathrm{d}s. \tag{30}$$

Since $\mu(E_N) < \infty$, the number of jumps of the process X^N on the interval $(0,t)$ is finite and consequently we may consider the random variable X_t^N as a function of these jump times and apply the methodology proposed in Sect. 2. We denote by

(J_t^N) the Poisson process defined by $J_t^N = N((0,t), E_N) = \#\{s < t; p_s \in E_N\}$ and we note $(T_k^N)_{k\geq 1}$ its jump times. We also introduce the notation $\Delta_k^N = p_{T_k^N}$. With these notations, the process solution of (30) can be written

$$X_t^N = x + \sum_{k=1}^{J_t^N} c(T_k^N, \Delta_k^N, X_{T_k^N-}^N) + \int_0^t g(s, X_s^N)\mathrm{d}s. \tag{31}$$

We will not work with all the variables $(T_k^N)_k$ but only with the jump times (T_k^n) of the Poisson process J_t^n, where $n < N$. In the following we will keep n fixed and we will make N go to infinity. We note $(T_k^{N,n})_k$ the jump times of the Poisson process $J_t^{N,n} = N((0,t), E_N \backslash E_n)$ and $\Delta_k^{n,N} = p_{T_k^{n,N}}$. Now we fix $L \in \mathbb{N}^*$, the number of integration by parts and we note $t_l = tl/L$, $0 \leq l \leq L$. Assuming that $J_{t_l}^n - J_{t_{l-1}}^n = m_l$ for $1 \leq l \leq L$, we denote by $(T_{l,i}^n)_{1\leq i\leq m_l}$ the jump times of J_t^n belonging to the time interval (t_{l-1}, t_l). In the following we assume that $m_l \geq 1$, $\forall l$. For $i = 0$ we set $T_{l,0}^n = t_{l-1}$ and for $i = m_l + 1$, $T_{l,m_l+1}^n = t_l$. With these definitions we choose our basic variables $(V_i, i \in I_l)$ as

$$(V_i, i \in I_l) = (T_{l,2i+1}^n, 0 \leq i \leq [(m_l - 1)/2]). \tag{32}$$

The σ-algebra which contains the noise which is not involved in our differential calculus is

$$\mathcal{G} = \sigma\{(J_{t_l}^n)_{1\leq l\leq L}; (T_{l,2i}^n)_{1\leq 2i\leq m_l, 1\leq l\leq L}; (T_k^{N,n})_k; (\Delta_k^N)_k\}. \tag{33}$$

Using some well-known results on Poisson processes, we easily see that conditionally on $\mathcal{G}$ the variables (V_i) are independent and for $i \in I_l$ the law of V_i conditionally on $\mathcal{G}$ is uniform on $(T_{l,2i}^n, T_{l,2i+2}^n)$ and we have

$$p_i(v) = \frac{1}{T_{l,2i+2}^n - T_{l,2i}^n} 1_{(T_{l,2i}^n, T_{l,2i+2}^n)}(v), \quad i \in I_l, \tag{34}$$

Consequently, taking $a_i = T_{l,2i}^n$ and $b_i = T_{l,2i+2}^n$ we check that hypothesis H0 holds. It remains to define the localizing sets $(\Lambda_{l,i})_{i\in I_l}$.

We denote

$$h_l^n = \frac{t_l - t_{l-1}}{2m_l} = \frac{t}{2Lm_l}$$

and $n_l = [(m_l - 1)/2]$. We will work on the $\mathcal{G}$ measurable set

$$\Lambda_l^n = \cup_{i=0}^{n_l}\{T_{l,2i+2}^n - T_{l,2i}^n \geq h_l^n\}, \tag{35}$$

and we consider the following partition of this set:

$$\Lambda_{l,0} = \{T^n_{l,2} - T^n_{l,0} \geq h^n_l\},$$
$$\Lambda_{l,i} = \cap_{k=1}^{i}\{T^n_{l,2k} - T^n_{l,2k-2} < h^n_l\} \cap \{T^n_{l,2i+2} - T^n_{l,2i} \geq h^n_l\}, \quad i = 1, \ldots, n_l.$$

After $L - l$ iterations of the integration by parts we will work with the variables $V_i, i \in I_l$ so the corresponding derivative is

$$D_l F = \sum_{i \in I_l} 1_{\Lambda_{l,i}} \partial_{V_i} F = \sum_{i \in I_l} 1_{\Lambda_{l,i}} \partial_{T^n_{l,2i+1}} F.$$

If we are on Λ^n_l then we have at least one i such that $t_{l-1} \leq T^n_{l,2i} < T^n_{l,2i+1} < T^n_{l,2i+2} \leq t_l$ and $T^n_{l,2i+2} - T^n_{l,2i} \geq h^n_l$. Notice that in this case $1_{\Lambda_{l,i}} |p_i|_\infty \leq (h^n_l)^{-1}$ and roughly speaking this means that the variable $V_i = T^n_{l,2i+1}$ gives a sufficiently large quantity of noise. Moreover, in order to perform L integrations by parts we will work on

$$\Gamma^n_L = \cap_{l=1}^{L} \Lambda^n_l \tag{36}$$

and we will leave out the complementary of Γ^n_L. The following lemma says that on the set Γ^n_L we have enough noise and that the complementary of this set may be ignored.

Lemma 1. *Using the notation given in Theorem 1 one has*

(i) $|p|_0 := \max_{1 \leq l \leq L} \sum_{i \in I_l} 1_{\Lambda_{l,i}} |p_i|_\infty \leq \frac{2L}{t} J^n_t$,
(ii) $P((\Gamma^n_L)^c) \leq L \exp(-\mu(E_n)t/2L)$.

Proof. As mentioned before $1_{\Lambda_{l,i}} |p_i|_\infty \leq (h^n_l)^{-1} = 2Lm_l/t \leq \frac{2L}{t} J^n_t$ and so we have (i). In order to prove (ii) we have to estimate $P((\Lambda^n_l)^c)$ for $1 \leq l \leq L$. We denote $s_l = \frac{1}{2}(t_l + t_{l-1})$ and we will prove that $\{J^n_{t_l} - J^n_{s_l} \geq 1\} \subset \Lambda^n_l$. Suppose first that $m_l = J^n_{t_l} - J^n_{t_{l-1}}$ is even. Then $2n_l + 2 = m_l$. If $T^n_{l,2i+2} - T^n_{l,2i} < h^n_l$ for every $i = 0, \ldots, n_l$ then

$$T^n_{l,m_l} - t_{l-1} = \sum_{i=0}^{n_l} (T^n_{l,2i+2} - T^n_{l,2i}) \leq (n_l + 1) \times \frac{t}{2Lm_l} \leq \frac{t}{4L} \leq s_l - t_{l-1}$$

so there are no jumps in (s_l, t_l). Suppose now that m_l is odd so $2n_l + 2 = m_l + 1$ and $T^n_{l,2n_l+2} = t_l$. If we have $T^n_{l,2i+2} - T^n_{l,2i} < h^n_l$ for every $i = 0, \ldots, n_l$, then we deduce

$$\sum_{i=0}^{n_l} (T^n_{l,2i+2} - T^n_{l,2i}) < (n_l + 1) \times \frac{t}{2Lm_l} < \frac{m_l + 1}{m_l} \frac{t}{4L} \leq \frac{t}{2L},$$

and there are no jumps in (s_l, t_l). So we have proved that $\{J^n_{t_l} - J^n_{s_l} \geq 1\} \subset \Lambda^n_l$ and since $P(J^n_{t_l} - J^n_{s_l} = 0) = \exp(-\mu(E_n)t/2L)$ the inequality (ii) follows. □

Now we will apply Theorem 1, with $F^N = X^N_t$, $G = 1$ and $\Phi_\xi(x) = e^{i\xi x}$. So we have to check that $F^N \in \mathcal{S}^{L+1}(\cup_{l=1}^{L} I_l)$ and that condition (16) holds.

Moreover, we have to bound $|F^N|_l^{l-1}$ and $|D_l F^N|_l^{l-1}$, for $1 \le l \le L$. This needs some preliminary lemma.

Lemma 2. *Let $v = (v_i)_{i \ge 0}$ be a positive non-increasing sequence with $v_0 = 0$ and $(a_i)_{i \ge 1}$ a sequence of E. We define $J_t(v)$ by $J_t(v) = v_i$ if $v_i \le t < v_{i+1}$ and we consider the process solution of*

$$X_t = x + \sum_{k=1}^{J_t} c(v_k, a_k, X_{v_k-}) + \int_0^t g(s, X_s)\mathrm{d}s. \tag{37}$$

We assume that H1 holds. Then X_t admits some derivatives with respect to v_i and if we note $U_i(t) = \partial_{v_i} X_t$ and $W_i(t) = \partial^2_{v_i} X_t$, the processes $(U_i(t))_{t \ge v_i}$ and $(W_i(t))_{t \ge v_i}$ solve, respectively,

$$U_i(t) = \alpha(v_i, a_i, X_{v_i}-) + \sum_{k=i+1}^{J_t} \partial_x c(v_k, a_k, X_{v_k}-)U_i(v_k-) + \int_{v_i}^t \partial_x g(s, X_s)U_i(s)\mathrm{d}s, \tag{38}$$

$$W_i(t) = \beta_i(t) + \sum_{k=i+1}^{J_t} \partial_x c(v_k, a_k, X_{v_k-})W_i(v_k-) + \int_{v_i}^t \partial_x g(s, X_s)W_i(s)\mathrm{d}s, \tag{39}$$

with

$$\begin{aligned}
\alpha(t, a, x) &= g(t, x) - g(t, x + c(t, a, x)) + g(t, x)\partial_x c(t, a, x) + \partial_t c(t, a, x),\\
\beta_i(t) &= \partial_t \alpha(v_i, a_i, X_{v_i}-) + \partial_x \alpha(v_i, a_i, X_{v_i}-)g(v_i, X_{v_i}-) - \partial_x g(v_i, X_{v_i})U_i(v_i)\\
&\quad + \sum_{k=i+1}^{J_t} \partial_x^2 c(v_k, a_k, X_{v_k}-)(U_i(v_k-))^2 + \int_{v_i}^t \partial_x^2 g(s, X_s)(U_i(s))^2\mathrm{d}s.
\end{aligned}$$

Proof. If $s < v_i$, we have $\partial_{v_i} X_s = 0$. Now we have

$$X_{v_i-} = x + \sum_{k=1}^{v_{i-1}} c(v_k, a_k, X_{v_k-}) + \int_0^{v_i} g(s, X_s)\mathrm{d}s,$$

and consequently

$$\partial_{v_i} X_{v_i-} = g(v_i, X_{v_i}-).$$

For $t > v_i$, we observe that

$$X_t = X_{v_i-} + \sum_{k=v_i}^{J_t} c(v_k, a_k, X_{v_k-}) + \int_{v_i}^t g(s, X_s)\mathrm{d}s,$$

this gives

$$
\begin{aligned}
\partial_{v_i} X_t = {} & g(v_i, X_{v_i-}) + g(v_i, X_{v_i-})\partial_x c(v_i, a_i, X_{v_i-}) + \partial_t c(v_i, a_i, X_{v_i-}) \\
& - g(v_i, X_{v_i}) + \sum_{k=i+1}^{J_t} \partial_x c(v_k, a_k, X_{v_k-})\partial_{v_i} X_{v_k-} \\
& + \int_{v_i}^{t} \partial_x g(s, X_s)\partial_{v_i} X_s \mathrm{d}s.
\end{aligned}
$$

Remarking that $X_{v_i} = X_{v_i-} + c(v_i, a_i, X_{v_i-})$, we obtain (38). The proof of (39) is similar and we omit it. □

We give next a bound for X_t and its derivatives with respect to the variables (v_i).

Lemma 3. *Let (X_t) be the process solution of (37). We assume that H1 holds and we note*

$$
n_t(\overline{c}) = \sum_{k=1}^{J_t} \overline{c}(a_k).
$$

Then we have:

$$
\sup_{s \le t} |X_t| \le C_t(1 + n_t(\overline{c}))\mathrm{e}^{n_t(\overline{c})}.
$$

Moreover $\forall l \ge 1$, there exist some constants $C_{t,l}$ and C_l such that $\forall (v_{k_i})_{i=1,\dots,l}$ with $t > v_{k_l}$, we have

$$
\begin{aligned}
& \sup_{v_{k_l} \le s \le t} |\partial_{v_{k_1}} \dots \partial_{v_{k_{l-1}}} U_{k_l}(s)| + \sup_{v_{k_l} \le s \le t} |\partial_{v_{k_1}} \dots \partial_{v_{k_{l-1}}} W_{k_l}(s)| \\
& \quad \le C_{t,l}(1 + n_t(\overline{c}))^{C_l} \mathrm{e}^{C_l n_t(\overline{c})}.
\end{aligned}
$$

We observe that the previous bound does not depend on the variables (v_i).

Proof. We just give a sketch of the proof. We first remark that the process (e_t) solution of

$$
e_t = 1 + \sum_{k=1}^{J_t} \overline{c}(a_k) e_{v_k-} + \overline{g} \int_0^t e_s \mathrm{d}s,
$$

is given by $e_t = \prod_{k=1}^{J_t}(1 + \overline{c}(a_k))\mathrm{e}^{\overline{g}t}$. Now from H1, we deduce for $s \le t$

$$
\begin{aligned}
|X_s| & \le |x| + \sum_{k=1}^{J_s} \overline{c}(a_k)(1 + |X_{v_k-}|) + \int_0^s \overline{g}(1 + |X_u|)\mathrm{d}u, \\
& \le |x| + \sum_{k=1}^{J_t} \overline{c}(a_k) + \overline{g}t + \sum_{k=1}^{J_s} \overline{c}(a_k)|X_{v_k-}| + \int_0^s \overline{g}|X_u|\mathrm{d}u, \\
& \le \left(|x| + \sum_{k=1}^{J_t} \overline{c}(a_k) + \overline{g}t \right) e_s
\end{aligned}
$$

where the last inequality follows from Gronwall lemma. Then using the previous remark

$$\sup_{s\le t}|X_s| \le C_t(1+n_t(\overline{c}))\prod_{k=1}^{J_t}(1+\overline{c}(a_k)) \le C_t(1+n_t(\overline{c}))\mathrm{e}^{n_t(\overline{c})}. \tag{40}$$

We check easily that $|\alpha(t,a,x)| \le C(1+|x|)\overline{c}(a)$, and we get successively from (38) and (40)

$$\sup_{v_{k_l}\le s\le t}|U_{k_l}(s)| \le C_t(1+|X_{v_{k_l}-}|)\overline{c}(a_{k_l})(1+n_t(\overline{c}))\mathrm{e}^{n_t(\overline{c})} \le C_t(1+n_t(\overline{c}))^2\mathrm{e}^{2n_t(\overline{c})}.$$

Putting this in (39), we obtain a similar bound for $\sup_{v_{k_l}\le s\le t}|W_{k_l}(s)|$ and we end the proof of Lemma 3 by induction since we can derive equations for the higher order derivatives of $U_{k_l}(s)$ and $W_{k_l}(s)$ analogous to (39). □

We come back to the process (X_t^N) solution of (30). We recall that $F^N = X_t^N$ and we will check that F^N satisfies the hypotheses of Theorem 1.

Lemma 4. *(i) We assume that H1 holds. Then $\forall l\ge 1$, $\exists C_{t,l}, C_l$ independent of N such that*

$$|F^N|_l + |D_lF^N|_l \le C_{t,l}\left((1+N_t(\overline{c}))\mathrm{e}^{N_t(\overline{c})}\right)^{C_l},$$

with $N_t(\overline{c}) = \int_0^t\int_E \overline{c}(a)\mathrm{d}N(s,a)$.

(ii) Moreover, if we assume in addition that H2 and H3 hold and that $m_l = J_{t_l}^n - J_{t_{l-1}}^n \ge 1$, $\forall l\in\{1,\dots,L\}$, then we have $\forall 1\le l\le L$, $\forall i\in I_l$

$$|\partial_{V_i}F^N| \ge \left(\mathrm{e}^{2N_t(\overline{c})}N_t(1_{E_n}1/\underline{\alpha})\right)^{-1} := \gamma_n$$

and (16) holds.

We remark that on the non-degeneracy set Γ_L^n given by (36) we have at least one jump on (t_{l-1},t_l), that is $m_l\ge 1$, $\forall l\in\{1,\dots,L\}$. Moreover, we have $\Gamma_L^n\subset\{\gamma_n>0\}$.

Proof. The proof of (i) is a straightforward consequence of Lemma 3, replacing $n_t(\overline{c})$ by $\sum_{p=1}^{J_t^N}\overline{c}(\Delta_p^N)$ and observing that

$$\sum_{p=1}^{J_t^N}\overline{c}(\Delta_p^N) = \int_0^t\int_{E_N}\overline{c}(a)\mathrm{d}N(s,a) \le \int_0^t\int_E \overline{c}(a)\mathrm{d}N(s,a) = N_t(\overline{c}).$$

Turning to (ii) we have from Lemma 2

$$\partial_{T_k^N} X_t^N = \alpha(T_k^N, \Delta_k^N, X_{T_k^N-}^N) + \sum_{p=k+1}^{J_t^N} \partial_x c(T_p^N, \Delta_p^N, X_{T_p^N-}^N) \partial_{T_k^N} X_{T_p^N-}^N$$
$$+ \int_{T_k^N}^t \partial_x g(s, X_s^N) \partial_{T_k^N} X_s \mathrm{d}s.$$

Assuming H2, we define $(Y_t^N)_t$ and $(Z_t^N)_t$ as the solutions of the equations

$$Y_t^N = 1 + \sum_{p=1}^{J_t^N} \partial_x c(T_p^N, \Delta_p^N, X_{T_p^N-}^N) Y_{T_k^N-} + \int_0^t \partial_x g(s, X_s^N) Y_s^N \mathrm{d}s,$$
$$Z_t^N = 1 - \sum_{p=1}^{J_t^N} \frac{\partial_x c(T_p^N, \Delta_p^N, X_{T_p^N-}^N)}{1 + \partial_x c(T_p^N, \Delta_p^N, X_{T_p^N-}^N)} Z_{T_k^N-} - \int_0^t \partial_x g(s, X_s^N) Z_s^N \mathrm{d}s.$$

We have $Y_t^N \times Z_t^N = 1$, $\forall t \geq 0$ and

$$|Y_t^N| \leq \mathrm{e}^{t\overline{g}} \mathrm{e}^{N_t(1_{E_N}\overline{c})} \leq \mathrm{e}^{N_t(\overline{c})}, \quad |Z_t^N| = \left| \frac{1}{Y_t^N} \right| \leq \mathrm{e}^{N_t(\overline{c})}.$$

Now one can easily check that

$$\partial_{T_k^N} X_t^N = \alpha(T_k^N, \Delta_k^N, X_{T_k^N-}^N) Y_t^N Z_{T_k^N}^N,$$

and using H3 and the preceding bound it yields

$$|\partial_{T_k^N} X_t^N| \geq \mathrm{e}^{-2N_t(\overline{c})} \underline{\alpha}(\Delta_k^N).$$

Recalling that we do not consider the derivatives with respect to all the variables (T_k^N) but only with respect to $(V_i) = (T_{l,2i+1}^n)_{l,i}$ with $n < N$ fixed, we have $\forall 1 \leq l \leq L$ and $\forall i \in I_l$

$$|\partial_{V_i} X_t^N| \geq \mathrm{e}^{-2N_t(\overline{c})} \left(\sum_{p=1}^{J_t^n} \frac{1}{\underline{\alpha}(\Delta_p^n)} \right)^{-1} = \left(\mathrm{e}^{2N_t(\overline{c})} N_t(1_{E_n} 1/\underline{\alpha}) \right)^{-1},$$

and Lemma 4 is proved. □

With this lemma we are at last able to prove Proposition 3.

Proof. From Theorem 1 we have since $\Gamma_L^n \subset \{\gamma_n > 0\}$

$$1_{\Gamma_L^n} |E_{\mathcal{G}} \Phi^{(L)}(F^N)| \leq C_L ||\Phi||_\infty 1_{\Gamma_L^n} E_{\mathcal{G}} (1 + |p_0|)^L \Pi_L(F^N).$$

Now from Lemma 1 (i) we have

$$|p_0| \leq 2LJ_t^n/t$$

and moreover we can check that $|\ln p|_1 = 0$. So we deduce from Lemma 4

$$\begin{aligned}
\Pi_L(F^N) &\leq \frac{C_{t,L}}{\gamma_n^{L(L+2)}} \left((1 + N_t(\overline{c}))\mathrm{e}^{N_t(\overline{c})}\right)^{C_L} \\
&\leq C_{t,L} N_t(1_{E_n} 1/\underline{\alpha})^{L(L+2)} \left((1 + N_t(\overline{c}))\mathrm{e}^{N_t(\overline{c})}\right)^{C_L}.
\end{aligned}$$

This finally gives

$$\begin{aligned}
&|E1_{\Gamma_L^n} \Phi^{(L)}(F^N)| \\
&\quad \leq ||\Phi||_\infty C_{t,L} E\left((J_t^N)^L N_t(1_{E_n} 1/\underline{\alpha})^{L(L+2)} \left((1 + N_t(\overline{c}))\mathrm{e}^{N_t(\overline{c})}\right)^{C_L}\right).
\end{aligned} \tag{41}$$

Now we know from a classical computation (see e.g., [2]) that the Laplace transform of $N_t(f)$ satisfies

$$E\mathrm{e}^{-sN_t(f)} = \mathrm{e}^{-t\alpha_f(s)}, \quad \alpha_f(s) = \int_E (1 - \mathrm{e}^{-sf(a)})\mathrm{d}\mu(a). \tag{42}$$

From H1, we have $\int_E \overline{c}(a)\mathrm{d}\mu(a) < \infty$, so we deduce using (42) with $f = \overline{c}$ that, $\forall q > 0$

$$E\left((1 + N_t(\overline{c}))\mathrm{e}^{N_t(\overline{c})}\right)^q \leq C_{t,q} < \infty.$$

Since J_t^n is a Poisson process with intensity $t\mu(E_n)$, we have $\forall q > 0$

$$E(J_t^n)^q \leq C_{t,q}\mu(E_n)^q.$$

Finally, using once again (42) with $f = 1_{E_n} 1/\underline{\alpha}$, we see easily that $\forall q > 0$

$$EN_t(1_{E_n} 1/\underline{\alpha})^q \leq C_{t,q} \left(\int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu(a)\right)^q.$$

Turning back to (41) and combining Cauchy–Schwarz inequality and the previous bounds, we deduce

$$\begin{aligned}
|E1_{\Gamma_L^n} \Phi^{(L)}(F^N)| &\leq ||\Phi||_\infty C_{t,L} \mu(E_n)^L \left(\int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu(a)\right)^{L(L+2)} \\
&= ||\Phi||_\infty C_{t,L} A_{n,L}.
\end{aligned} \tag{43}$$

We are now ready to give a bound for $\hat{p}_{X_t^N}(\xi)$. We have $\hat{p}_{X_t^N}(\xi) = E\Phi_\xi(F^N)$, with $\Phi_\xi(x) = \mathrm{e}^{i\xi x}$. Since $\Phi_\xi^{(L)}(x) = (i\xi)^L \Phi_\xi(x)$, we can write $|\hat{p}_{X_t^N}(\xi)| = |E\Phi_\xi^{(L)}(F^N)|/|\xi|^L$ and consequently we deduce from (43)

$$|\hat{p}_{X_t^N}(\xi)| \leq P((\Gamma_L^n)^c) + C_{t,L} A_{n,L}/|\xi|^L.$$

But from Lemma 1 (ii) we have

$$P((\Gamma_L^n)^c) \leq L\mathrm{e}^{-\mu(E_n)t/(2L)}$$

and finally

$$|\hat{p}_{X_t^N}(\xi)| \leq C_{L,t}\left(\mathrm{e}^{-\mu(E_n)t/(2L)} + A_{n,L}/|\xi|^L\right).$$

We achieve the proof of Proposition 3 by letting N go to infinity, keeping n fixed. □

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http://www.springer.com/978-3-642-15357-0

Stochastic Analysis 2010
Crisan, D. (Ed.)
2011, VIII, 299 p., Hardcover
ISBN: 978-3-642-15357-0