Chapter 2
Physical Effects in Gyroscopes

2.1 Sagnac Effect

Operating principle of all optical gyros is based on Sagnac effect [1]. It induces either a phase shift $\Delta\varphi$ between two optical signals propagating in opposite directions within a ring interferometer rotating around an axis perpendicular to the ring, or a frequency shift between two resonant modes propagating in clockwise (CW) and counter-clockwise (CCW) directions within an optical cavity rotating around an axis perpendicular to it.

To derive the analytic expression of the rotation induced phase shift between CW and CCW beams, a simple cinematic approach can be used [2]. We first consider a circular ring interferometer in which two waves counter-propagate in the vacuum (see Fig. 2.1). Light enters the interferometer at point P and is split into CW and CCW propagating signals by a beam splitter. When the interferometer is at rest with respect to a motionless inertial frame of reference, optical path lengths of the two optical signals propagating in opposite directions (CW and CCW signals) are equal. Also the speeds of the two signals are equal to c (c is light speed in the free space). After propagating in the loop both waves come back into the beam splitter after a time interval τ_r equal to:

$$\tau_r = \frac{2\pi R}{c} \tag{2.1}$$

where R is the ring interferometer radius.

If the ring interferometer is rotating at a rate Ω, the beam splitter located in P moves during the time interval τ_r by a length $\Delta l = \Omega R \tau_r$.

CW (co-directional with Ω) beam experiences a path length slightly greater than $2\pi R$ in order to complete one round trip, since the ring interferometer rotates through a small angle during the round-trip transit time. CCW beam experiences a path length slightly less than $2\pi R$ during one round trip. The difference between optical paths of CW (L_{CW}) and CCW (L_{CCW}) waves is given by:

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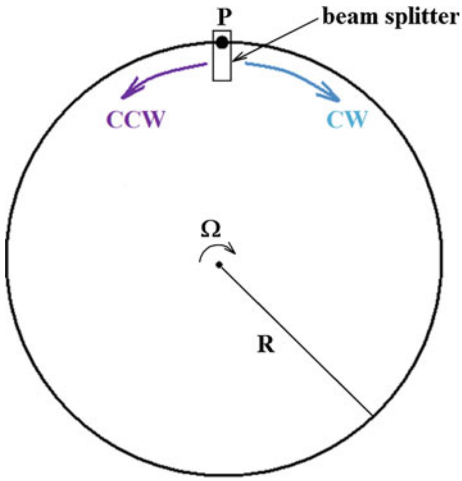

Fig. 2.1 Sagnac ring interferometer

$$\Delta L = L_{\mathrm{CW}} - L_{\mathrm{CCW}} = 2\Delta l = 2\Omega R \tau_r = \frac{4\pi\Omega R^2}{c} \tag{2.2}$$

Since CW and CCW waves propagate at the same speed (equal to the speed of light in vacuum c), the CCW wave arrives in P before the CW wave. The delay between the two optical signals is equal to:

$$\Delta t = \frac{\Delta L}{c} = \frac{4\pi\Omega R^2}{c^2} \tag{2.3}$$

The phase shift $\Delta\varphi$ between CW and CCW optical signals due to the interferometer rotation can be written as:

$$\Delta\varphi = \Delta t \frac{2\pi c}{\lambda} = \frac{8\pi^2 R^2}{c\lambda}\Omega \tag{2.4}$$

where λ is the optical signal wavelength.

Phase shift expression in Eq. 2.4 is valid for one circular loop. If optical path of CW and CCW beams consists of k turns, the phase shift $\Delta\varphi$ becomes:

$$\Delta\varphi = \frac{8\pi^2 R^2}{c\lambda} k\Omega \tag{2.5}$$

The time delay expression reported in Eq. 2.3 can be also derived in the framework of special relativity [3].

Now we consider a similar interferometer in which the vacuum is substituted by a homogeneous dielectric medium having a refractive index equal to n. While the interferometer is at rest, light travels at the speed c/n in both directions and propagation time around the loop of both waves is equal to $n \cdot \tau_r$. Both waves are still in phase after a propagation time equal to $n \cdot \tau_r$.

If the circular path is rotating, the beam splitter located in P has moved through a length $n\Delta l$ in the propagation time $n \cdot \tau_r$. So the optical path length of CW wave in one round trip is equal to:

$$L^*_{\mathrm{CW}} = 2\pi R + n\Delta l = 2\pi R + \frac{2\pi n\Omega R^2}{c} \tag{2.6}$$

whereas the optical path length of CCW wave in one round trip is given by:

$$L^*_{\mathrm{CCW}} = 2\pi R - n\Delta l = 2\pi R - \frac{2\pi n\Omega R^2}{c} \tag{2.7}$$

In this case the speed of light is no longer the same for both counter-propagating signals. In particular the speed of CW wave is equal to:

$$\upsilon_{\mathrm{CW}} = \frac{c}{n} + \alpha_d \Omega R \tag{2.8}$$

and the speed of CCW wave is:

$$\upsilon_{\mathrm{CCW}} = \frac{c}{n} - \alpha_d \Omega R \tag{2.9}$$

where α_d is the Fresnel–Fizeau drag coefficient, which is given by [4]:

$$\alpha_d = 1 - n^{-2} \tag{2.10}$$

The additive terms in light velocity expressions in Eqs. 2.8 and 2.9 are due to the drag of light propagating in a uniformly moving medium [5].

The CW and CCW waves arrive in P in different instants. The delay between these two time instants is equal to:

$$\Delta t^* = \frac{L^*_{\mathrm{CW}}}{\upsilon_{\mathrm{CW}}} - \frac{L^*_{\mathrm{CCW}}}{\upsilon_{\mathrm{CCW}}} = \frac{2\pi R + \frac{2\pi n\Omega R^2}{c}}{\frac{c}{n} + \alpha_d \Omega R} - \frac{2\pi R - \frac{2\pi n\Omega R^2}{c}}{\frac{c}{n} - \alpha_d \Omega R} \tag{2.11}$$

Rearranging Eq. 2.11 and assuming $c^2/n^2 \gg \alpha_d \Omega^2 R^2$ we obtain:

$$\Delta t^* \cong \frac{4\pi R^2 n^2 \Omega (1 - \alpha_d)}{c^2} = \frac{4\pi R^2 \Omega}{c^2} \tag{2.12}$$

By comparing Eqs. 2.12 and 2.3, one can conclude that $\Delta t = \Delta t^*$. Therefore, the phase shift induced by rotation is equal when optical propagation takes place in the vacuum or in a homogeneous medium having refractive index equal to n. The same result can be demonstrated using a more rigorous relativistic electrodynamic approach which consists of the derivation of the equation describing the optical propagation in a rotating frame and the application of a method of perturbation to calculate the rotation induced phase shift [6].

As previously pointed out, rotation induces a frequency difference between two counter-propagating resonant modes excited in an optical cavity.

An optical resonator at rest, in which optical propagation occurs in the vacuum, supports optical modes whose resonance frequencies $\nu_{q,0}$ satisfy the following relation:

$$qc = \nu_{q,0}\, p \tag{2.13}$$

where q is an integer number (resonance order) and p is the perimeter of the resonator. If two qth order counter-propagating resonant modes are excited in a ring cavity, their resonance frequencies are split by rotation being equal to:

$$\nu_q^{\mathrm{CW}} = \frac{qc}{p_+} \qquad \nu_q^{\mathrm{CCW}} = \frac{qc}{p_-} \tag{2.14}$$

where p_+ and p_- are perimeters of optical paths experienced by the two resonant modes. The difference $p_+ - p_-$ is denoted with Δp.

The frequency difference $\Delta\nu$ between the two qth order resonant modes is given by:

$$\Delta\nu = \nu_q^{\mathrm{CCW}} - \nu_q^{\mathrm{CW}} = qc\left(\frac{1}{p_-} - \frac{1}{p_+}\right) \cong qc\frac{\Delta p}{p^2} \tag{2.15}$$

Combining Eqs. 2.15 and 2.13, we obtain:

$$\Delta\nu = \nu_{q,0}\,\frac{\Delta p}{p} \tag{2.16}$$

The expression of the frequency splitting given by Eq. 2.16 does not change if optical propagation within the resonator occurs in a homogeneous medium having refractive index n or an optical waveguide having effective index n_{eff}.

For a circular ring resonator $\Delta p = 4\pi\Omega R^2/c$, p is obviously equal to $2\pi R$ and so $\Delta\nu$ can be written as:

$$\Delta\nu = \nu_{q,0}\frac{2R}{c}\Omega \tag{2.17}$$

where R is the resonator radius.

If the optical cavity has an arbitrary geometry, the frequency difference $\Delta\nu$ results [7]:

$$\Delta\nu = \frac{4a\nu_{q,0}}{pc}\Omega \tag{2.18}$$

where a is the area enclosed by the light path.

2.2 Coriolis Force Effect

The operating principle of all vibrating gyros is based on the effect of Coriolis force on a vibrating mass. A simple model of vibrating angular rate sensors is a two degree-of-freedom spring-mass-damper system shown in Fig. 2.2.

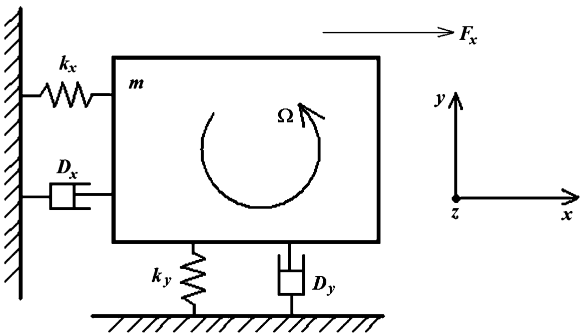

Fig. 2.2 Two degree-of-freedom spring-mass-damper system in a rotating reference frame

Coriolis force is a fictitious force experienced by a mass m moving in a rotating reference frame. It is equal to:

$$\overrightarrow{F_c} = 2m\left(\overrightarrow{v} \times \overrightarrow{\Omega}\right) \tag{2.19}$$

where $\overrightarrow{v}$ is the mass velocity in the rotating reference frame and $\overrightarrow{\Omega}$ is the angular velocity of the reference frame. The effect of Coriolis force on the two degree-of-freedom spring-mass-damper system shown in Fig. 2.2 can be derived starting from dynamic equations describing the motion of the system in a rotating reference frame.

In Fig. 2.2, the mass m can move along x and y axes and $\overrightarrow{\Omega}$ is directed along z. The oscillation along x, namely drive or primary oscillating mode, is excited by the force F_x directed along x whereas the oscillation along y, sense or secondary oscillating mode, is due to system rotation around z axis. Motion equations of the two degree-of-freedom system can be written in the form [8]:

$$\begin{cases} m\frac{d^2x}{dt^2} + D_x\frac{dx}{dt} + k_x x - 2\Omega m\frac{dy}{dt} = F_x \\ m\frac{d^2y}{dt^2} + D_y\frac{dy}{dt} + k_y y + 2\Omega m\frac{dx}{dt} = 0 \end{cases} \tag{2.20}$$

where Ω is the module of reference system angular rate, D_x and D_y are the damping coefficients along x and y axes, and k_x and k_y are the spring constants along x and y axes.

Usually the primary oscillating mode is excited by a sinusoidal force F_x and its amplitude is kept constant at a_x. To maximize a_x, the angular frequency of the exciting force ω_d is typically very close to the resonance frequency $\omega_x = \sqrt{k_x/m}$ of the primary resonator. So $x\,(t)$ can be written as:

$$x(t) = a_x \sin(\omega_d t) \cong a_x \sin(\omega_x t) \tag{2.21}$$

To calculate $y(t)$ we can use the second equation of the system reported in Eq. 2.20. This equation can be rewritten in the form:

$$\frac{d^2y}{dt^2} + \frac{\omega_y}{Q_y}\frac{d^2y}{dt^2} + \omega_y^2 y = -2a_x\Omega\omega_x\cos(\omega_x t) \tag{2.22}$$

where $\omega_y = \sqrt{k_y/m}$ is the resonance frequency of the secondary resonator and $Q_y = \sqrt{mk_y}/D_y$ is the quality factor of the sense mode. After the transient regime, $y(t)$ assumes the general form:

$$y(t) = a_y\cos\left[\omega_x t + \phi_y\right] \tag{2.23}$$

where a_y and ϕ_y are the amplitude and the phase response of the secondary resonator at ω_x, respectively.

Calculating dy/dt and d^2y/dt^2 and substituting them in Eq. 2.22, we obtain:

$$\begin{aligned}&\left[-a_y\omega_x^2\cos(\phi_y) + a_y\omega_y^2\cos(\phi_y) - \frac{a_y\omega_x\omega_y}{Q_y}\sin(\phi_y)\right]\cos(\omega_x t)\\&+\left[a_y\omega_x^2\sin(\phi_y) - a_y\omega_y^2\sin(\phi_y) - \frac{a_y\omega_x\omega_y}{Q_y}\cos(\phi_y)\right]\sin(\omega_x t)\\&= -2a_x\Omega\omega_x\cos(\omega_x t)\end{aligned} \tag{2.24}$$

Equation 2.24 provides the following algebraic system:

$$\begin{cases}\left[-a_y\omega_x^2\cos(\phi_y) + a_y\omega_y^2\cos(\phi_y) - \frac{a_y\omega_x\omega_y}{Q_y}\sin(\phi_y)\right] = -2a_x\Omega\omega_x\\ \left[a_y\omega_x^2\sin(\phi_y) - a_y\omega_y^2\sin(\phi_y) - \frac{a_y\omega_x\omega_y}{Q_y}\cos(\phi_y)\right] = 0\end{cases} \tag{2.25}$$

By solving the equation system (2.25), a_y and $y(t)$ are derived as follows:

$$a_y = -\frac{2a_x\Omega\omega_x}{\sqrt{\left(\omega_x^2 - \omega_y^2\right)^2 + \omega_x^2\omega_y^2/Q_y^2}} \tag{2.26}$$

$$y(t) = -\frac{2a_x\Omega\omega_x}{\sqrt{\left(\omega_x^2 - \omega_y^2\right)^2 + \omega_x^2\omega_y^2/Q_y^2}}\cos(\omega_x t + \phi_y) \tag{2.27}$$

Equation 2.27 shows that the amplitude of sense mode is directly proportional to the angular rate Ω. Then the angular rate of the two degree-of-freedom spring-mass-damper system can be easily estimated by measuring the amplitude of the oscillation along y.

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