2

Inequalities for Functions Vanishing at the Boundary

The present chapter deals with the necessary and sufficient conditions for the validity of certain estimates for the norm $\|u\|_{L_q(\Omega,\mu)}$, where $u \in \mathscr{D}(\Omega)$ and μ is a measure in Ω. Here we consider inequalities with the integral

$$\int_\Omega \big[\Phi(x, \nabla u)\big]^p \, \mathrm{d}x,$$

on the right-hand side. The function $\Phi(x, \xi)$, defined for $x \in \Omega$ and $\xi \in \mathbb{R}^n$, is positive homogeneous of degree one in ξ. The conditions are stated in terms of isoperimetric (for $p = 1$ in Sect. 2.1) and isocapacitary (for $p \geq 1$, in Sects. 2.2–2.4) inequalities. For example, we give a complete answer to the question of validity of the inequality

$$\|u\|_{L_q(\Omega,\mu)} \leq C\Big(\int_\Omega \big[\Phi(x, \nabla u)\big]^p \, \mathrm{d}x\Big)^{1/p},$$

both for $q \geq p \geq 1$ and $0 < q < p$, $p \geq 1$. In particular, in the first case there hold sharp inequalities for the best constant C

$$\beta^{1/p} \leq C \leq p(p-1)^{(1-p)/p} \beta^{1/p},$$

where

$$\beta = \sup_{F \subset \Omega} \frac{\mu(F)^{p/q}}{(p, \Phi)\text{-cap}(F, \Omega)},$$

with the so-called (p, Φ)-capacity of a compact subset of Ω in the denominator. Actually, this is a special case of a more general assertion concerning Birnbaum–Orlicz spaces.

Among other definitive results we obtain criteria for the validity of multiplicative inequalities of the form

$$\|u\|_{L_p(\Omega,\mu)} \leq C\big\|\Phi(\cdot, \nabla u)\big\|^{\delta}_{L_p(\Omega)} \|u\|^{1-\delta}_{L_r(\Omega,\nu)}$$

V. Maz'ya, *Sobolev Spaces*,
Grundlehren der mathematischen Wissenschaften 342,
DOI 10.1007/978-3-642-15564-2_2,

as well as the necessary and sufficient conditions for compactness of related embedding operators.

In Sect. 2.5 we give applications of the results in Sect. 2.4 to the spectral theory of the multidimensional Schrödinger operator with a nonpositive potential. Here the necessary and sufficient conditions ensuring the positivity and semiboundedness of this operator, discreteness, and finiteness of its negative spectrum are obtained.

Certain properties of quadratic forms of the type

$$\int_{\mathbb{R}^n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \,\mathrm{d}x$$

are studied in Sects. 2.6.1 and 2.6.2. Finally, Sects. 2.7 and 2.8 are devoted to sharp constants in some multidimensional inequalities of the Hardy type.

2.1 Conditions for Validity of Integral Inequalities (the Case $p = 1$)

2.1.1 Criterion Formulated in Terms of Arbitrary Admissible Sets

A bounded open set $g \subset \mathbb{R}^n$ will be called admissible if $\bar{g} \subset \Omega$ and ∂g is a C^∞ manifold. In Chaps. 5–7 this definition will be replaced by a broader one.

Let $\mathscr{N}(x)$ denote the unit normal to the boundary of the admissible set g at a point x that is directed toward the interior of g. Let $\Phi(x, \xi)$ be a continuous function on $\Omega \times \mathbb{R}^n$ that is nonnegative and positive homogeneous of the first degree with respect to ξ. We introduce the weighted area of ∂g

$$\sigma(\partial g) = \int_{\partial g} \Phi\big(x, \mathscr{N}(x)\big) \,\mathrm{d}s(x). \tag{2.1.1}$$

Let μ and ν be measures in Ω and $\omega_n = s(\partial B_1)$.

The following theorem contains *a necessary and sufficient condition for the validity of the multiplicative inequality*:

$$\|u\|_{L_q(\Omega,\mu)} \le C \big\|\Phi(\cdot, \nabla u)\big\|^{\delta}_{L_1(\Omega)} \|u\|^{1-\delta}_{L_r(\Omega,\nu)} \tag{2.1.2}$$

for all $u \in \mathscr{D}(\Omega)$. This result will be proved using the same arguments as in Theorem 1.4.2/1.

Theorem. 1. *If for all admissible sets*

$$\mu(g)^{1/q} \le \alpha \sigma(\partial g)^{\delta} \nu(g)^{(1-\delta)/r}, \tag{2.1.3}$$

where $\alpha = \text{const} > 0$, $\delta \in [0, 1]$, r, $q > 0$, $\delta + (1-\delta) r^{-1} \ge q^{-1}$, *then* (2.1.2) *holds for all* $u \in \mathscr{D}(\Omega)$ *with* $C \le \alpha r^{\delta} (r\delta + 1 - \delta)^{-\delta - (1-\delta)/r}$.

2. *If* (2.1.2) *holds for all* $u \in \mathscr{D}(\Omega)$ *with* $q > 0$, $\delta \in [0,1]$, *then* (2.1.3) *holds for all admissible sets* g *and* $\alpha \le C$.

Proof. 1. First we note that by Theorem 1.2.4

$$\int_\Omega \Phi(x, \nabla u)\,\mathrm{d}x = \int_{\{x:|\nabla u|>0\}} \Phi\Big(x, \frac{\nabla u}{|\nabla u|}\Big)|\nabla u|\,\mathrm{d}x$$
$$= \int_0^\infty \mathrm{d}t \int_{\mathscr{E}_t} \Phi\Big(x, \frac{\nabla u}{|\nabla u|}\Big)\,\mathrm{d}s = \int_0^\infty \sigma(\partial\mathscr{L}_t)\,\mathrm{d}t. \tag{2.1.4}$$

Here we used the fact that $|\nabla u| \ne 0$ on $\mathscr{E}_t = \{x : |u(x)| = t\}$ for almost all t and that for such t the sets $\mathscr{L}_t = \{x : |u(x)| > t\}$ are bounded by C^∞ manifolds. By Lemma 1.2.3

$$\|u\|_{L_q(\Omega,\mu)} = \Big(\int_0^\infty \mu(\mathscr{L}_t)\,\mathrm{d}\big(t^q\big)\Big)^{1/q}.$$

Since $\mu(\mathscr{L}_t)$ is a nonincreasing function, then, applying (1.3.41), we obtain

$$\|u\|_{L_q(\Omega,\mu)} \le \Big(\int_0^\infty \mu(\mathscr{L}_t)^{\gamma/q}\,\mathrm{d}\big(t^\gamma\big)\Big)^{1/\gamma},$$

where $\gamma = r(r\delta+1-\delta)^{-1}$, $\gamma \le q$. Using the fact that the sets $\mathscr{L}_t$ are admissible for almost all t, from (2.1.3) we obtain

$$\|u\|_{L_q(\Omega,\mu)} \le \gamma^{1/\gamma}\alpha\Big(\int_0^\infty \sigma(\partial\mathscr{L}_t)^{\gamma\delta}\nu(\mathscr{L}_t)^{\gamma(1-\delta)/r}t^{\gamma-1}\,\mathrm{d}t\Big)^{1/\gamma}.$$

Since $\gamma\delta + \gamma(1-\delta)/r = 1$, then by Hölder's inequality

$$\|u\|_{L_q(\Omega,\mu)} \le \gamma^{1/\gamma}\alpha\Big(\int_0^\infty \sigma(\partial\mathscr{L}_t)\,\mathrm{d}t\Big)^\delta\Big(\int_0^\infty \nu(\mathscr{L}_t)t^{r-1}\,\mathrm{d}t\Big)^{(1-\delta)/r},$$

which by virtue of (2.1.4) and Lemma 1.2.3 is equivalent to (2.1.2).

2. Let g be any admissible subset of Ω and let $\mathrm{d}(x) = \mathrm{dist}(x, \mathbb{R}^n\backslash g)$, $g_t = \{x \in \Omega, \mathrm{d}(x) > t\}$. Let α denote a nondecreasing function, infinitely differentiable on $[0,\infty)$, equal to unity for $d \ge 2\varepsilon$ and to zero for $\mathrm{d} \le \varepsilon$, where ε is a sufficiently small positive number. Then we substitute $u_\varepsilon(x) = \alpha[\mathrm{d}(x)]$ into (2.1.2).

By Theorem 1.2.4,

$$\int_\Omega \Phi(x, \nabla u_\varepsilon)\,\mathrm{d}x = \int_0^{2\varepsilon} \alpha'(t)\int_{\partial g_t} \Phi\big(x, \mathscr{N}(x)\big)\,\mathrm{d}s(x),$$

where $\mathscr{N}(x)$ is the normal at $x \in \partial g_t$ directed toward the interior of g_t. Since

$$\int_{\partial g_t} \Phi\big(x, \mathscr{N}(x)\big)\,\mathrm{d}s(x) \xrightarrow{t\to 0} \sigma(\partial g),$$

we obtain

$$\int_{\Omega} \Phi(x, \nabla u_\varepsilon)\, \mathrm{d}x \xrightarrow{\varepsilon \to 0} \sigma(\partial g).$$

Let K be a compactum in g such that $\operatorname{dist}(K, \partial g) > 2\varepsilon$. Then $u_\varepsilon(x) = 1$ on K and

$$\|u_\varepsilon\|_{L_q(\Omega,\mu)} \geq \mu(K)^{1/q}.$$

Using $0 \leq u_\varepsilon(x) \leq 1$ and $\operatorname{supp} u_\varepsilon \subset g$, we see that

$$\|u_\varepsilon\|_{L_r(\Omega,\nu)} \leq \nu(g)^{1/r}.$$

Now from (2.1.2) we obtain

$$\mu(g)^{1/q} = \sup_{K \subset g} \mu(K)^{1/q} \leq C\sigma(\partial g)^{\delta} \nu(g)^{(1-\delta)/r}.$$

The result follows. □

2.1.2 Criterion Formulated in Terms of Balls for $\Omega = \mathbb{R}^n$

In the case $\Phi(x, \xi) = |\xi|$, $\Omega = \mathbb{R}^n$, $\nu = m_n$ it follows from (2.1.2) that for all balls $B_\varrho(x)$

$$\mu\big(B_\varrho(x)\big)^{1/q} \leq A\varrho^{\delta(n-1)+(1-\delta)n/r}. \tag{2.1.5}$$

With minor modification in the proof of Theorem 1.4.2/2 we arrive at the converse assertion.

Theorem. *If* (2.1.5) *holds with* $\delta \in [0, 1]$; q, $r > 0$, $\delta + (1-\delta)/r \geq 1/q$ *for all balls* $B_\varrho(x)$*, then*

$$\|u\|_{L_q(\mu)} \leq C\big\|\Phi(\cdot, \nabla u)\big\|_{L_1}^{\delta} \|u\|_{L_r}^{(1-\delta)} \tag{2.1.6}$$

holds for all $u \in \mathscr{D}(\mathbb{R}^n)$ *with* $C \leq cA$.

Proof. As already shown in the proof of Theorem 1.2.1/2, for any bounded open set g with a smooth boundary there exists a sequence $\{B_{\varrho_i}(x_i)\}_{i\geq 1}$ of disjoint balls with the properties

$$\begin{aligned} &(\alpha) \quad g \subset \bigcup_{i\geq 1} B_{3\varrho_i}(x_i), \\ &(\beta) \quad 2m_n\big(q \cap B_{\varrho_i}(x_i)\big) = v_n \varrho_i^n, \\ &(\gamma) \quad s(\partial g) \geq c \sum_{i\geq 1} \varrho_i^{n-1}. \end{aligned}$$

From (2.1.5) it follows that

$$\mu(g) \leq \sum_{i\geq 1} \mu\big(B_{3\varrho_i}(x_i)\big) \leq A^q \sum_{i\geq 1} (3\varrho_i)^{q[\delta(n-1)+(1-\delta)n/r]}. \tag{2.1.7}$$

Since $q\delta + (1-\delta)n/r \geq 1$, it follows from (2.1.7) that

$$\mu(g) \leq cA^q \Big(\sum_{i \geq 1} \varrho_i^{q \frac{\delta(n-1)+(1-\delta)n/r}{q\delta+(1-\delta)n/r}} \Big)^{q\delta+(1-\delta)n/r},$$

which by Hölder's inequality does not exceed

$$cA^q \Big(\sum_{i\geq 1} \varrho_i^{n-1} \Big)^{q\delta} \Big(\sum_{i \geq 1} \varrho_i^n \Big)^{(1-\delta)q/r}.$$

To conclude the proof it remains to apply Theorem 2.1.1. □

2.1.3 Inequality Involving the Norms in $L_q(\Omega, \mu)$ and $L_r(\Omega, \nu)$ (Case $p = 1$)

The next theorem is proved analogously to Theorem 2.1.1.

Theorem. 1. *If for all admissible sets $g \subset \Omega$*

$$\mu(g)^{1/q} \leq \alpha\sigma(\partial g) + \beta\nu(g)^{1/r}, \tag{2.1.8}$$

where $\alpha \geq 0$, $\beta \geq 0$, $q \geq 1 \geq r$, then

$$\|u\|_{L_q(\Omega,\mu)} \leq \alpha \big\| \varPhi(x, \nabla u) \big\|_{L(\Omega)} + \beta \|u\|_{L_r(\Omega,\nu)} \tag{2.1.9}$$

holds for all $u \in \mathscr{D}(\Omega)$.

2. *If* (2.1.9) *holds for all $u \in \mathscr{D}(\Omega)$, then* (2.1.8) *holds for all admissible sets g.*

2.1.4 Case $q \in (0, 1)$

Here we deal with the inequality

$$\|u\|_{L_q(\Omega,\mu)} \leq C \big\| \varPhi(\cdot, \nabla u) \big\|_{L_1(\Omega)} \tag{2.1.10}$$

for $u \in C_0^\infty(\Omega)$. As a particular case of (2.1.9), we obtain from Theorem 2.1.3 that (2.1.10) *holds with $q \geq 1$ if and only if for all admissible sets g*

$$\mu(g)^{1/q} \leq \alpha\sigma(\partial g) \tag{2.1.11}$$

and α is the best value of C.

We shall show that (2.1.10) can be completely characterized also for $q \in (0, 1)$. Let us start with the basic properties of the so-called nonincreasing rearrangement of a function.

Let u be a function in Ω measurable with respect to the measure μ. We associate with u its nonincreasing rearrangement u_μ^* on $(0, \infty)$, which is introduced by

$$u_\mu^*(t) = \inf\{s > 0 : \mu(\mathscr{L}_s) \le t\}, \tag{2.1.12}$$

where $\mathscr{L}_s = \{x \in \Omega : |u(x)| > s\}$.

Clearly u_μ^* is nonnegative and nonincreasing on $(0, \infty)$. We also have $u_\mu^*(t) = 0$ for $t \ge \mu(\Omega)$. Furthermore, it follows from the definition of u^* that

$$u_\mu^*\big(\mu(\mathscr{L}_s)\big) \le s \tag{2.1.13}$$

and

$$\mu(\mathscr{L}_{u^*(t)}) \le t, \tag{2.1.14}$$

the last because the function $s \to \mu(\mathscr{L}_s)$ is continuous from the right.

The nonincreasing rearrangement of a function has the following important property.

Lemma 1. *If* $q \in (0, \infty)$, *then*

$$\int_\Omega \big|u(x)\big|^q \, \mathrm{d}\mu = \int_0^\infty \big(u_\mu^*(t)\big)^q \, \mathrm{d}t.$$

Proof. The required equality is a consequence of the formula

$$\int_\Omega \big|u(x)\big|^q \, \mathrm{d}\mu = \int_0^\infty \mu(\mathscr{L}_t) \, \mathrm{d}\big(t^q\big)$$

and the identity

$$m_1\big(\mathscr{L}_s^*\big) = \mu(\mathscr{L}_s), \quad s \in (0, \infty) \tag{2.1.15}$$

in which $\mathscr{L}_s^* = \{t > 0 : u_\mu^*(t) > s\}$. To check (2.1.15) we first note that

$$m_1\big(\mathscr{L}_s^*\big) = \sup\{t > 0 : u_\mu^*(t) > s\} \tag{2.1.16}$$

by the monotonicity of u_μ^*. Hence, (2.1.13) yields

$$m_1\big(\mathscr{L}_s^*\big) \le \mu(\mathscr{L}_s).$$

For the inverse inequality, let $\varepsilon > 0$ and $t = m_1(\mathscr{L}_s^*) + \varepsilon$. Then (2.1.16) implies $u_\mu^*(t) \le s$ and therefore

$$m_1\big(\mathscr{L}_s^*\big) \le \mu(\mathscr{L}_{u_\mu^*(t)}) \le t$$

by (2.1.14). Thus $\mu(\mathscr{L}_s) \le m_1(\mathscr{L}_s^*)$ and (2.1.15) follows.

Definition. Let $\mathscr{C}(\varrho)$ denote the infimum $\sigma(\partial g)$ for all admissible sets such that $\mu(g) \ge \varrho$, where $\sigma(\partial g)$ is the weighted area defined by (2.1.1).

Theorem. *Let* Ω *be a domain in* $\mathbb{R}^n$ *and* $0 < q < 1$.

(i) (*Sufficiency*) *If*

$$D := \int_0^{\mu(\Omega)} \left(\frac{s^{1/q}}{\mathscr{C}(s)}\right)^{\frac{q}{1-q}} \frac{\mathrm{d}s}{s} < \infty, \tag{2.1.17}$$

then (2.1.10) *holds for all* $u \in C^\infty(\Omega)$. *The constant* C *satisfies the inequality* $C \le c_1(q) D^{(1-q)/q}$.

(ii) (*Necessity*) *If there is a constant* $C > 0$ *such that* (2.1.10) *holds for all* $u \in C^\infty(\Omega)$, *then* (2.1.17) *holds and* $C \ge c_2(q) D^{(1-q)/q}$.

Proof. (Sufficiency) Note that (2.1.17) implies $\mu(\Omega) < \infty$ and that $\mathscr{C}$ is a positive function. By monotonicity of $\mu(\mathscr{L}_t)$, one obtains

$$\begin{aligned}\int_\Omega |u|^q \,\mathrm{d}\mu &= \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \mu(\mathscr{L}_t)\,\mathrm{d}\big(t^q\big)\\ &\le \sum_{j=-\infty}^{\infty} \mu_j \big(2^{q(j+1)} - 2^{qj}\big),\end{aligned}$$

where $\mu_j = \mu(\mathscr{L}_{2^j})$. We claim that the estimate

$$\sum_{j=r}^{m} \mu_j \big(2^{q(j+1)} - 2^{qj}\big) \le c D^{1-q} \big\|\Phi(\cdot, \nabla u)\big\|^q_{L_q(\Omega)} \tag{2.1.18}$$

is true for any integers r, m, and $r < m$. Once (2.1.18) has been proved, (2.1.17) follows by letting $m \to \infty$ and $r \to -\infty$ in (2.1.18). Clearly, the sum on the left in (2.1.18) is not greater than

$$\mu_m 2^{q(m+1)} + \sum_{j=1+r}^{m} (\mu_{j-1} - \mu_j) 2^{jq}. \tag{2.1.19}$$

Let $S_{r,m}$ denote the sum over $1 + r \le j \le m$. Hölder's inequality implies

$$S_{r,m} \le \left[\sum_{j=1+r}^{m} 2^j \mathscr{C}(\mu_{j-1})\right]^q \left\{\sum_{j=1+r}^{m} \frac{(\mu_{j-1} - \mu_j)^{1/(1-q)}}{\mathscr{C}(\mu_{j-1})^{1/(1-q)}}\right\}^{1-q}. \tag{2.1.20}$$

We have

$$(\mu_{j-1} - \mu_j)^{1/(1-q)} \le \mu_{j-1}^{1/(1-q)} - \mu_j^{1/(1-q)}.$$

Hence, by the monotonicity of $\mathscr{C}$, the sum in curly braces is dominated by

$$\sum_{j=1+r}^{m} \int_{\mu_j}^{\mu_{j-1}} \mathscr{C}(t)^{q/(q-1)} \,\mathrm{d}\big(t^{1/(1-q)}\big),$$

which does not exceed $D/(1 - q)$. By (2.1.4) the sum in square brackets in (2.1.20) is not greater than

$$2\sum_{j=-\infty}^{\infty}\int_{\mathscr{L}_{2j-1}\setminus\mathscr{L}_{2j}}\Phi(x,\nabla u)\,\mathrm{d}x.$$

Thus

$$\sum_{j=1+r}^{m}(\mu_{j-1}-\mu_j)2^{qj}\le cD^{1-q}\big\|\Phi(\cdot,\nabla u)\big\|^q_{L_1(\Omega)}.$$

To conclude the proof of (2.1.18), we show that the first term in (2.1.19) is also dominated by the right part of (2.1.18). Indeed, if $\mu_m>0$, then

$$\begin{aligned}\mu_m 2^{mq}&\le\big(2^m\mathscr{C}(\mu_m)\big)^q\Big(\big(\mu_m/\mathscr{C}(\mu_m)\big)^{q/(1-q)}\mu_m\Big)^{1-q}\\&\le c\big\|\Phi(\cdot,\nabla u)\big\|^q_{L_1(\Omega)}\left(\int_0^{\mu_m}\left(\frac{t}{\mathscr{C}(t)}\right)^{q/(1-q)}\mathrm{d}t\right)^{1-q}.\end{aligned}$$

The sufficiency of (2.1.17) follows.

We turn to the necessity of (2.1.17). We shall use the following two auxiliary assertions.

Lemma 2. *Let $u\in C_0^{0,1}(\Omega)$. There exists a sequence $\{u_\nu\}_{\nu\ge1}$ of functions $u_\nu\in\mathscr{D}(\Omega)$ such that*

$$\int_\Omega\Phi\big(x,\nabla\big(u_\nu(x)-u(x)\big)\big)\,\mathrm{d}x\to0\quad\textit{as }\nu\to\infty.\tag{2.1.21}$$

Proof. Let $u_\nu=\mathcal{M}_{\nu^{-1}}u$, where $\mathcal{M}_\varepsilon$ stands for a mollification with radius ε. Let U be a neighborhood of supp u, $\bar U\subset\Omega$.

Clearly, supp u_ν is situated in U for all sufficiently large ν. Since $\Phi\in C(\Omega\times S^{n-1})$ and $u\in C_0^{(0,1)}(\Omega)$, it follows that

$$\Phi\big(x,\nabla\big(u_\nu(x)-u(x)\big)\big)=\Phi\left(x,\frac{\nabla(u_\nu(x)-u(x))}{|\nabla(u_\nu(x)-u(x))|}\right)\big|\nabla\big(u_\nu(x)-u(x)\big)\big|,$$

if $\nabla u_\nu(x)\ne\nabla u(x)$. Therefore, the left-hand side in (2.1.21) does not exceed

$$\max_{\bar U\times S^{n-1}}\Phi\int_\Omega\big|\nabla\big(u_\nu(x)-u(x)\big)\big|\,\mathrm{d}x\to0\quad\text{as }\nu\to\infty.$$

The proof is complete. □

Lemma 3. *Let $\{v_1,\dots,v_N\}$ be a finite collection of functions in the space $C(\Omega)\cap L_p^1(\Omega)$, $p\in[1,\infty)$. Then, for $x\in\Omega$, the function*

$$x\mapsto v(x)=\max\big\{v_1(x),\dots,v_N(x)\big\}$$

belongs to the same space and

$$\big\|\varPhi(\cdot,\nabla v)\big\|_{L_1(\varOmega)} \le \sum_{i=1}^{N}\big\|\varPhi(\cdot,\nabla v_i)\big\|_{L_1(\varOmega)}. \tag{2.1.22}$$

Proof. An induction argument reduces consideration to the case $N = 2$. Here

$$v(x) = \max\big\{v_1(x), v_2(x)\big\}.$$

The left-hand side in (2.1.22) is equal to

$$\int_{v_1 \ge v_2} \varPhi(x,\nabla v_1)\,\mathrm{d}x + \int_{v_1 < v_2} \varPhi(x,\nabla v_2)\,\mathrm{d}x,$$

which implies (2.1.22) for $N = 2$. $\square$

Continuation of the proof of Theorem. (Necessity) First we remark that the claim implies $\mu(\varOmega) < \infty$ and that $\mathscr{C}(t) > 0$ for all $t \in (0, \mu(\varOmega)]$. Let j be any integer satisfying $2^j \le \mu(\varOmega)$. Then there exists a subset g_j of $\varOmega$ such that

$$\mu(g_j) \ge 2^j, \quad \text{and} \quad \sigma(\varOmega \cap g_j) \le 2\mathscr{C}\big(2^j\big).$$

By the definition of $\mathscr{C}$ and by (2.1.4) there is a function $u_j \in C^\infty(\varOmega)$ subject to $u_j \ge 1$ on g_j, $u_j = 0$ on $\partial\varOmega$ and

$$\int_\varOmega \varPhi(x,\nabla u_j)\,\mathrm{d}x \le 4\mathscr{C}\big(2^j\big).$$

Let s be the integer for which $2^s < \mu(\varOmega) < 2^{s+1}$. For any integer $r < s$, we introduce the Lipschitz function

$$f_{r,s}(x) = \max_{r \le j \le s} \beta_j u_j(x), \quad x \in \varOmega,$$

where

$$\beta_j = \big(2^j/\mathscr{C}\big(2^j\big)\big)^{1/(1-q)}.$$

By Lemmas 2 and 3

$$\big\|\varPhi(\cdot,\nabla f_{r,s})\big\|_{L_1(\varOmega)} \le c\sum_{j=r}^{s} \beta_j \big\|\varPhi(\cdot,\nabla u_j)\big\|_{L_1(\varOmega)},$$

and one obtains the following upper bound:

$$\big\|\varPhi(\cdot,\nabla f_{r,s})\big\|_{L_1(\varOmega)} \le c\sum_{j=r}^{s} \beta_j \mathscr{C}\big(2^j\big). \tag{2.1.23}$$

We now derive a lower bound for the norm of $f_{r,s}$ in $L_q(\varOmega,\mu)$. Since $f_{r,s}(x) \ge \beta_j$ for $x \in g_j$, $r \le j \le s$, and $\mu(g_j) \ge 2^j$, the inequality

$$\mu\big(\{x \in \Omega : |f_{r,s}(x)| > r\}\big) < 2^j$$

implies $r \geq \beta_j$. Hence

$$f^*_{r,s}(t) \geq \beta_j \quad \text{for } t \in \big(0, 2^j\big),\ r \leq j \leq s,$$

where $f^*_{r,s}$ is the nonincreasing rearrangement of $f_{r,s}$. Then

$$\int_0^{\mu(\Omega)} \big(f^*_{r,s}(t)\big)^q \,\mathrm{d}t \geq \sum_{j=r}^{s} \int_{2^{j-1}}^{2^j} \big(f^*_{r,s}\big)^q \,\mathrm{d}t \geq \sum_{j=r}^{s} \beta_j^q 2^{j-1},$$

which implies

$$\|f_{r,s}\|^q_{L_q(\Omega,\mu)} \geq \sum_{j=r}^{s} \beta_j^q 2^{j-1}. \tag{2.1.24}$$

Next we note that by Lemma 2 if inequality (2.1.10) holds for all $u \in C_0^\infty(\Omega)$, then it holds for all Lipschitz u with compact supports in Ω. In particular,

$$\|f_{r,s}\|_{L_q(\Omega,\mu)} \leq C\big\|\Phi(\cdot, \nabla f_{r,s})\big\|_{L_1(\Omega)}.$$

Now (2.1.23) and (2.1.24) in combination with the last inequality give

$$C \geq c\frac{(\sum_{j=r}^{s} \beta_j^q 2^j)^{1/q}}{\sum_{j=r}^{s} \beta_j (2^j)} = c\Bigg(\sum_{j=r}^{s} \frac{2^{j/(1-q)}}{(\mathscr{C}(2^j))^{q/(1-q)}}\Bigg)^{(1-q)/q}.$$

By letting $r \to -\infty$ and by the monotonicity of $\mathscr{C}$, we obtain

$$C \geq c\Bigg(\sum_{j=-\infty}^{s} \bigg(\frac{2^j}{\mathscr{C}(t)}\bigg)^{\frac{q}{1-q}} 2^j\Bigg)^{\frac{1-q}{q}} \geq c\Bigg(\int_0^{\mu(\Omega)} \bigg(\frac{t}{\mathscr{C}(t)}\bigg)^{\frac{q}{1-q}} \mathrm{d}t\Bigg)^{\frac{1-q}{q}}.$$

This completes the proof of the Theorem. □

2.1.5 Inequality (2.1.10) Containing Particular Measures

We give two examples that illustrate applications of the inequality (2.1.10).

Example 1. Let $\Omega = \mathbb{R}^n$, $\mathbb{R}^{n-1} = \{x \in \mathbb{R}^n, x_n = 0\}$, $\mu(A) = m_{n-1}(A \cap \mathbb{R}^{n-1})$, where A is any Borel subset of $\mathbb{R}^n$. Obviously,

$$\mu(g) \leq \frac{1}{2} s(\partial g)$$

and hence

$$\|u\|_{L_1(\mathbb{R}^{n-1})} \leq \frac{1}{2}\|\nabla u\|_{L_1(\mathbb{R}^n)}$$

for all $u \in \mathscr{D}(\mathbb{R}^n)$.

Example 2. Let A be any Borel subset of $\mathbb{R}^n$ with $m_n(A) < \infty$ and let

$$\mu(A) = \int_A |x|^{-\alpha}\,\mathrm{d}x,$$

where $\alpha \in [0, 1]$. Further, let B_r be a ball centered at the origin, whose n-dimensional measure equals $m_n(A)$. In other words,

$$r = \Big(\frac{n}{\omega_n} m_n(A)\Big)^{1/n}.$$

Obviously,

$$\int_A |x|^{-\alpha}\,\mathrm{d}x \le \int_{A\cap B_r} |x|^{-\alpha}\,\mathrm{d}x + r^{-\alpha} m_n(B_r\backslash A) \le \int_{B_r} |x|^{-\alpha}\,\mathrm{d}x.$$

So

$$\mu(A)^{(n-1)/(n-\alpha)} \le (n-\alpha)^{(1-n)/(n-\alpha)} \omega_n^{\alpha(n-1)/n(n-\alpha)} \big[n m_n(A)\big]^{(n-1)/n}.$$

Let g be any admissible set in $\mathbb{R}^n$. By virtue of the isoperimetric inequality

$$\big[n m_n(g)\big]^{(n-1)/n} \le \omega_n^{-1/n} s(\partial g),$$

we have

$$\mu(g)^{(n-1)/(n-\alpha)} \le (n-\alpha)^{(1-n)/(n-\alpha)} \omega_n^{(\alpha-1)/(n-\alpha)} s(\partial g).$$

This inequality becomes an equality if g is a ball. Therefore

$$\sup_{\{g\}} \frac{\mu(g)^{(n-1)/(n-\alpha)}}{s(\partial g)} = (n-\alpha)^{(1-n)/(n-\alpha)} \omega_n^{(\alpha-1)/(n-\alpha)}$$

and *for all* $u \in \mathscr{D}(\mathbb{R}^n)$

$$\Big(\int_{\mathbb{R}^n} \big|u(x)\big|^{(n-\alpha)/(n-1)} |x|^{-\alpha}\,\mathrm{d}x\Big)^{(n-1)/(n-\alpha)} \le (n-\alpha)^{(1-n)/(n-\alpha)} \omega_n^{(\alpha-1)/(n-\alpha)} \|\nabla u\|_{L_1(\mathbb{R}^n)} \qquad (2.1.25)$$

with the best possible constant.

2.1.6 Power Weight Norm of the Gradient on the Right-Hand Side

In this subsection we denote by $z = (x, y)$ and $\zeta = (\xi, \eta)$ points in $\mathbb{R}^{n+m}$ with x, $\xi \in \mathbb{R}^n$, y, $\eta \in \mathbb{R}^m$, $m, n > 0$. Further, let $B_r^{(d)}(q)$ be the d-dimensional ball with center $q \in \mathbb{R}^d$.

Lemma 1. *Let g be an open subset of $\mathbb{R}^{n+m}$ with compact closure and smooth boundary ∂g, which satisfies*

$$\int_{B_r^{(n+m)}(z)\cap g} |\eta|^\alpha \, \mathrm{d}\zeta \Big/ \int_{B_r^{(n+m)}(z)} |\eta|^\alpha \, \mathrm{d}\zeta = \frac{1}{2}, \tag{2.1.26}$$

where $\alpha > -m$ for $m > 1$ and $0 \geq \alpha > -1$ for $m = 1$. Then

$$\int_{B_r^{(n+m)}(z)\cap \partial g} |\eta|^\alpha \, \mathrm{d}s(\zeta) \geq c r^{n+m-1} \big(r + |y|\big)^\alpha, \tag{2.1.27}$$

where s is the $(n+m-1)$-dimensional area.

The proof is based on the next lemma.

Lemma 2. *Let $\alpha > -m$ for $m > 1$ and $0 \geq \alpha > -1$ for $m = 1$. Then for any $v \in C^\infty(\overline{B_r^{(n+m)}})$ there exists a constant V such that*

$$\int_{B_r^{(n+m)}} \big|v(\zeta) - V\big| |\eta|^\alpha \, \mathrm{d}\zeta \leq c r \int_{B_r^{(n+m)}} \big|\nabla v(\zeta)\big| |\eta|^\alpha \, \mathrm{d}\zeta. \tag{2.1.28}$$

Proof. It suffices to derive (2.1.28) for $r = 1$. We put $B_1^{(n+m)} = B$ and $B_1^{(m)} \times B_1^{(n)} = Q$. Let $R(\zeta)$ denote the distance of a point $\zeta \in \partial Q$ from the origin, i.e., $R(\zeta) = (1 + |\zeta|^2)^{1/2}$ for $|\eta| = 1$, $|\xi| < 1$ and $R(\zeta) = (1 + |\eta|^2)^{1/2}$ for $|\zeta| = 1$, $|\eta| < 1$. Taking into account that B is the quasi-isometric image of Q under the mapping $\zeta \to \zeta/R(\zeta)$, we may deduce (2.1.28) from the inequality

$$\int_Q \big|v(\zeta) - V\big| |\eta|^\alpha \, \mathrm{d}\zeta \leq c \int_Q \big|\nabla v(\zeta)\big| |\eta|^\alpha \, \mathrm{d}\zeta, \tag{2.1.29}$$

which will be established now. Since $(m + \alpha)|\eta|^\alpha = \operatorname{div}(|\eta|^\alpha \eta)$, then, after integration by parts in the left-hand side of (2.1.29), we find that it does not exceed

$$(m+\alpha)^{-\alpha} \bigg(\int_Q |\nabla v| |\eta|^{\alpha+1} \, \mathrm{d}\xi + \int_{B_1^{(n)}} \mathrm{d}\xi \int_{\partial B_1^{(m)}} \big|v(\zeta) - V\big| \, \mathrm{d}s(\eta) \bigg). \tag{2.1.30}$$

For the sake of brevity we put $T = B_1^{(n)} \times (B_1^{(m)} \backslash B_{1/2}^{(m)})$. Let $m > 1$. The second summand in (2.1.30) is not greater than

$$c \int_T |\nabla v| \, \mathrm{d}\zeta + c \int_T |v - V| \, \mathrm{d}\zeta.$$

By Lemma 1.1.11, the last assertion and (2.1.30) imply (2.1.29), where V is the mean value of v in T. (Here it is essential that T is a domain for $m > 1$.)

If $m = 1$ then T has two components $T_+ = B_1^{(n)} \times (1/2, 1)$ and $T_- = B_1^{(n)} \times (-1, -1/2)$. Using the same argument as in the case $m > 1$, we obtain

$$\int_{B_1^{(n)}} \bigl|v(\xi, \pm 1) - V_\pm\bigr| \,\mathrm{d}\xi \le c \int_{T_\pm} \bigl|\nabla v(\zeta)\bigr| \,\mathrm{d}\zeta \le c \int_Q \bigl|\nabla v(\zeta)\bigr| |\eta|^\alpha \,\mathrm{d}\zeta,$$

where $V_\pm$ are the mean values of v in $T_\pm$. It remains to note that

$$|V_+ - V_-| \le c \int_{B_1^{(n)}} \mathrm{d}\xi \int_{-1}^{1} \left|\frac{\partial v}{\partial \eta}\right| \mathrm{d}\eta \le c \int_Q \bigl|\nabla v(\zeta)\bigr| |\eta|^\alpha \,\mathrm{d}\zeta,$$

provided $\alpha \le 0$. So for $m = 1$ we also have (2.1.29) with V replaced by V_+ or V_-. This concludes the proof of the lemma. □

Proof of Lemma 1. For the sake of brevity let $B = B_r^{(n+m)}(z)$. In (2.1.28) we replace v by a mollification of the characteristic function χ_ϱ of the set g with radius ϱ. Then the left-hand side in (2.1.28) is bounded from below by the sum

$$|1 - V| \int_{e_1} |\eta|^\alpha \,\mathrm{d}\zeta + |V| \int_{e_0} |\eta|^\alpha \,\mathrm{d}\zeta,$$

where $e_i = \{\zeta \in B : \chi_\varrho(\zeta) = i\}$, $i = 0, 1$.

Let ε be a sufficiently small positive number. By (2.1.26)

$$\left(\frac{1}{2} - \varepsilon\right)\bigl(|1 - V| + |V|\bigr) \int_B |\eta|^\alpha \,\mathrm{d}\zeta \le cr \int_B |\eta|^\alpha \bigl|\nabla \chi_\varrho(\zeta)\bigr| \,\mathrm{d}\zeta$$

for sufficiently small values of ϱ. Consequently,

$$\frac{1}{2} \int_B |\eta|^\alpha \,\mathrm{d}\zeta \le cr \limsup_{\varrho \to +0} \int_B |\eta|^\alpha \bigl|\nabla \chi_\varrho(\zeta)\bigr| \,\mathrm{d}\zeta = cr \int_{B \cap \partial g} |\eta|^\alpha \,\mathrm{d}s(\zeta).$$

It remains to note that

$$\int_B |\eta|^\alpha \,\mathrm{d}\zeta \ge c r^{m+n} \bigl(r + |y|\bigr)^\alpha.$$

The lemma is proved. □

Remark 1. Lemma 1 fails for $m = 1$, $\alpha > 0$. In fact, let $g = \{\zeta \in \mathbb{R}^{n+1} : \eta > \varepsilon \text{ or } 0 > \eta > -\varepsilon\}$, where $\varepsilon = \mathrm{const} > 0$. Obviously, (2.1.26) holds for this g. However,

$$\int_{B_r^{(n+1)} \cap \partial g} |\eta|^\alpha \,\mathrm{d}s(\zeta) \le c\varepsilon^\alpha,$$

which contradicts (2.1.27).

Theorem 1. *Let ν be a measure in $\mathbb{R}^{n+m}$, $q \ge 1$, $\alpha > -m$. The best constant in*

$$\|u\|_{L_q(\mathbb{R}^{n+m}, \nu)} \le C \int_{\mathbb{R}^{n+m}} |y|^\alpha |\nabla_z u| \,\mathrm{d}z, \quad u \in C_0^\infty\bigl(\mathbb{R}^{n+m}\bigr), \tag{2.1.31}$$

is equivalent to

$$K = \sup_{z;\varrho} \big(\varrho + |y|\big)^{-\alpha} \varrho^{1-n-m} \big[\nu\big(B_\varrho^{(n+m)}(z)\big)\big]^{1/q}. \tag{2.1.32}$$

Proof. 1. First, let $m > 1$ or $0 \geq \alpha > -1$, $m = 1$. According to Theorem 2.1.3

$$C = \sup_g \frac{[\nu(g)]^{1/q}}{\int_{\partial g} |y|^\alpha \, \mathrm{d}s(z)},$$

where g is an arbitrary subset of $\mathbb{R}^{n+m}$ with a compact closure and smooth boundary. We show that for each g there exists a covering by a sequence of balls $B_{\varrho_i}^{n+m}(z_i)$, $i = 1, 2, \dots$, such that

$$\sum_i \varrho_i^{n+m-1} \big(\varrho_i + |y_i|\big)^\alpha \leq c \int_{\partial g} |y|^\alpha \, \mathrm{d}s(z).$$

Each point $z \in g$ is the center of a ball $B_r^{(n+m)}(z)$ for which (2.1.26) is valid. In fact, the ratio in the left-hand side of (2.1.26) is a continuous function in r that equals unity for small values of r and converges to zero as $r \to \infty$. By Theorem 1.2.1 there exists a sequence of disjoint balls $B_{r_i}^{(n+m)}(z_i)$ such that

$$g \subset \bigcup_{i=1}^{\infty} B_{3r_i}^{(n+m)}(z_i).$$

According to Lemma 1,

$$\int_{B_{r_i}^{(n+m)}(z_i) \cap \partial g} |y|^\alpha \, \mathrm{d}s(z) \geq c r_i^{n+m-1} \big(r_i + |y_i|\big)^\alpha.$$

Thus $\{B_{3r_i}^{(n+m)}(z_i)\}_{i \geq 1}$ is the required covering.

Obviously,

$$\begin{aligned} \nu(g) &\leq \sum_i \nu\big(B_{3r_i}^{(n+m)}(z_i)\big) \leq \Big(\sum_i \big[\nu\big(B_{3r_i}^{(n+m)}(z_i)\big)\big]^{1/q}\Big)^q \\ &\leq cK^q \Big(\sum_i r_i^{n+m-1} \big(r_i + |y_i|\big)^\alpha\Big)^q \leq \Big(cK \int_{\partial g} |y|^\alpha \, \mathrm{d}s(z)\Big)^q. \end{aligned}$$

Therefore $C \leq cK$ for $m > 1$ and for $m = 1$, $0 \geq \alpha > -1$.

2. Now let $m = 1$, $\alpha > 0$. We construct a covering of the set $\{\zeta : \eta = 0\}$ by balls $\mathscr{B}_j$ with radii ϱ_j, equal to the distance of $\mathscr{B}_j$ from the hyperplane $\{\zeta : \xi = 0\}$. We assume that this covering has finite multiplicity. By $\{\varphi_j\}$ we denote a partition of unity subordinate to $\{\mathscr{B}_j\}$ and such that $|\nabla \varphi_j| \leq c/\varrho_j$ (see Stein [724], Chap. VI, §1.). Using the present theorem for the case $\alpha = 0$, which has already been considered (or equivalently, using Theorem 1.4.2/2), we arrive at

$$\|\varphi_j u\|_{L_q(\mathbb{R}^{n+1},\nu)} \le c \sup_{\varrho;z} \varrho^{-n} \big[\nu_j\big(B_\varrho^{(n+1)}(z)\big)\big]^{1/q} \big\|\nabla(\varphi_j u)\big\|_{L_1(\mathbb{R}^{n+1})},$$

where ν_j is the restriction of ν to $\mathscr{B}_j$. It is clear that

$$\sup_{\varrho;z} \varrho^{-n} \big[\nu_j\big(B_\varrho^{(n+1)}(z)\big)\big]^{1/q} \le c \sup_{\varrho\le\varrho_j, z\in\mathscr{B}_j} \varrho^{-n} \big[\nu\big(B_\varrho^{(n+1)}(z)\big)\big]^{1/q}.$$

Therefore,

$$\begin{aligned}&\|\varphi_j u\|_{L_q(\mathbb{R}^{n+1},\nu)}\\ &\quad\le c \sup_{\varrho\le\varrho_j, z\in\mathscr{B}_j} (\varrho+\varrho_j)^{-\alpha} \varrho^{-n} \big[\nu\big(B_\varrho^{(n+1)}(z)\big)\big]^{1/q} \int_{\mathbb{R}^{n+1}} \big|\nabla(\varphi_j u)\big||\eta|^\alpha \, \mathrm{d}\zeta.\end{aligned}$$

Summing over j and using (2.1.29), we obtain

$$\|u\|_{L_q(\mathbb{R}^{n+1},\nu)} \le cK\bigg(\int_{\mathbb{R}^{n+1}} |\nabla u||\eta|^\alpha \, \mathrm{d}\zeta + \int_{\mathbb{R}^{n+1}} |u||\eta|^{\alpha-1} \, \mathrm{d}\zeta\bigg).$$

Since

$$\int_{\mathbb{R}^{n+1}} |u||\eta|^{\alpha-1} \, \mathrm{d}\zeta \le \alpha^{-1} \int_{\mathbb{R}^{n+1}} |\nabla u||\eta|^\alpha \, \mathrm{d}\zeta$$

for $\alpha > 0$, then $C \le cK$ for $m = 1$, $\alpha > 0$.

3. To prove the reverse estimate, in (2.1.31) we put $U(\zeta) = \varphi(\varrho^{-1}(\zeta - z))$, where $\varphi \in C_0^\infty(B_2^{(n+m)})$, $\varphi = 1$ on $B_1^{(n+m)}$. Since

$$\int_{B_{2\varrho}^{(n+m)}(z)} |\eta|^\alpha |\nabla_\zeta u| \, \mathrm{d}\zeta \le c\varrho^{-1} \int_{B_{2\varrho}^{(n+m)}(z)} |\eta|^\alpha \, \mathrm{d}\zeta \le c\varrho^{n+m-1}\big(\varrho + |y|\big)^\alpha,$$

the result follows. □

Corollary. *Let ν be a measure in $\mathbb{R}^n$, $q \ge 1$, $\alpha > -m$. Then the best constant in* (2.1.31) *is equivalent to*

$$\sup_{x\in\mathbb{R}^n, \varrho>0} \varrho^{1-m-n-\alpha} \big[\nu\big(B_\varrho^{(n)}(x)\big)\big]^{1/q}.$$

For the proof it suffices to note that K, defined in (2.1.32), is equivalent to the preceding supremum if supp $\nu \subset \mathbb{R}^n$.

Remark 2. The part of the proof of Theorem 1 for the case $m = 1$, $\alpha > 0$ is also suitable for $m > 1$, $\alpha > 1 - m$ since for these values of α and for all $u \in C_0^\infty(\mathbb{R}^{n+m})$ we have

$$\int_{\mathbb{R}^{n+m}} |u||\eta|^{\alpha-1} \, \mathrm{d}\zeta \le (\alpha + m - 1)^{-1} \int_{\mathbb{R}^{n+m}} |\nabla u||\eta|^\alpha \, \mathrm{d}\zeta. \tag{2.1.33}$$

This implies that *the best constant C in* (2.1.31) *is equivalent to*

$$K_1 = \sup_{z \in \mathbb{R}^{n+m};\, \varrho < |y|/2} |y|^{-\alpha} \varrho^{1-n-m} \big[\nu\big(B_\varrho^{(n+m)}(z)\big)\big]^{1/q}$$

for $m \geq 1$, $\alpha > 1 - m$.

Since (2.1.33) is also valid for $\alpha < 1-m$ with the coefficient $(1-m-\alpha)^{-1}$ if u vanishes near the subspace $\eta = 0$, then following the arguments of the second and third parts of the proof of Theorem 1 with obvious changes, we arrive at the next theorem.

Theorem 2. *Let* ν *be a measure in* $\{\zeta \in \mathbb{R}^{n+m} : \eta \neq 0\}$, $q \geq 1$, $\alpha < 1-m$. *Then the best constant in* (2.1.31), *where* $u \in C_0^\infty(\{\zeta : \eta \neq 0\})$, *is equivalent to* K_1.

2.1.7 Inequalities of Hardy–Sobolev Type as Corollaries of Theorem 2.1.6/1

Here we derive certain inequalities for weighted norms which often occur in applications. Particular cases of them are the Hardy inequality

$$\big\| |x|^{-l} u \big\|_{L_p(\mathbb{R}^n)} \leq c \|\nabla_l u\|_{L_p(\mathbb{R}^n)}$$

and the Sobolev inequality

$$\|u\|_{L_{pn/(n-lp)}(\mathbb{R}^n)} \leq c \|\nabla_l u\|_{L_p(\mathbb{R}^n)},$$

where $lp < n$ and $u \in \mathscr{D}(\mathbb{R}^n)$. We retain the notation introduced in Sect. 2.1.5.

Corollary 1. *Let*

$$1 \leq q \leq (m+n)/(m+n-1), \qquad \beta = \alpha - 1 + \frac{q-1}{q}(m+n) > -\frac{m}{q}.$$

Then

$$\big\| |y|^\beta u \big\|_{L_q(\mathbb{R}^{n+m})} \leq c \big\| |y|^\alpha \nabla u \big\|_{L_1(\mathbb{R}^{n+m})} \tag{2.1.34}$$

for $u \in \mathscr{D}(\mathbb{R}^{n+m})$.

Proof. According to Theorem 2.1.6/1 it suffices to establish the uniform boundedness of the value

$$\big(\varrho + |y|\big)^{-\alpha} \varrho^{1-n-m} \bigg(\int_{|z-\zeta|<\varrho} |\eta|^{\beta q} \, \mathrm{d}\zeta \bigg)^{1/q}$$

with respect to ϱ and z. Obviously, it does not exceed

$$c\big(\varrho + |y|\big)^{-\alpha} \varrho^{1-n-m+n/q} \bigg(\int_{|\eta-y|<\varrho} |\eta|^{\beta q} \, \mathrm{d}\eta \bigg)^{1/q}.$$

This value is not greater than $c|y|^{\beta-\alpha}\varrho^{1-(m+n)(q-1)/q}$ for $\varrho \le c|y|$ and $c\varrho^{\beta-\alpha+1-(m+n)(q-1)/q}$ for $\varrho > c|y|$. The result follows. □

In (2.1.34) let us replace q^{-1}, α, and β by $1 - p^{-1} + q^{-1}$, $\alpha + (1-p)^{-1}q\beta$, and$((1-p^{-1})q+1)\beta$, respectively, and u by $|u|^s$ with $s = (p-1)qp^{-1}+1$. Then applying Hölder's inequality with exponents p and $p/(p-1)$ to its right-hand side we obtain the following assertion.

Corollary 2. *Let $m + n > p \ge 1$, $p \le q \le p(n+m)(n+m-p)^{-1}$, and $\beta = \alpha - 1 + (n+m)(1/p - 1/q) > -m/q$. Then*

$$\big\||y|^{\beta} u\big\|_{L_q(\mathbb{R}^{n+m})} \le c\big\||y|^{\alpha}\nabla u\big\|_{L_p(\mathbb{R}^{n+m})} \tag{2.1.35}$$

for all $u \in \mathscr{D}(\mathbb{R}^{n+m})$.

For $p = 2$, $\alpha = 1 - m/2$, $n > 0$, the substitution of $u(z) = |y|^{-\alpha}v(z)$ into (2.1.35) leads to the next corollary.

Corollary 3. *Let $m + n > 2$, $2 < q \le 2(n+m)/(n+m-2)$, and $\gamma = -1 + (n+m)(2^{-1} - q^{-1})$. Then*

$$\big\||y|^{\gamma} v\big\|^2_{L_q(\mathbb{R}^{n+m})} \le c\bigg(\int_{\mathbb{R}^{n+m}} (\nabla v)^2 \,\mathrm{d}z - \frac{(m-2)^2}{4}\int_{\mathbb{R}^{n+m}} \frac{v^2}{|y|^2}\,\mathrm{d}z\bigg) \tag{2.1.36}$$

for all $v \in \mathscr{D}(\mathbb{R}^{n+m})$, subject to the condition $v(x,0) = 0$ in the case $m = 1$.

In particular, the exponent γ vanishes for $q = 2(m+n)/(m+n-2)$ and we obtain

$$c\|v\|^2_{L_{\frac{2(m+n)}{m+n-2}}(\mathbb{R}^{n+m})} + \frac{(m-2)^2}{4}\int_{\mathbb{R}^{n+m}} \frac{v^2}{|y|^2}\,\mathrm{d}z \le \int_{\mathbb{R}^{n+m}} (\nabla v)^2\,\mathrm{d}z, \tag{2.1.37}$$

which is a refinement of both the Sobolev and the Hardy inequalities, the latter having the best constant.

To conclude this subsection we present a generalization of (2.1.35) for derivatives of arbitrary integer order l.

Corollary 4. *Let $m + n > lp$, $1 \le p \le q \le p(m+n-lp)^{-1}(m+n)$, and $\beta = \alpha - l + (m+n)(p^{-1} - q^{-1}) > -mq^{-1}$. Then*

$$\big\||y|^{\beta} u|\big\|_{L_q(\mathbb{R}^{n+m})} \le c\big\||y|^{\alpha}\nabla_l u\big\|_{L_p(\mathbb{R}^{n+m})} \tag{2.1.38}$$

for $u \in \mathscr{D}(\mathbb{R}^{n+m})$.

Proof. Let $p_j = p(n+m)(n+m-p(l-j))^{-1}$. Successively applying the inequalities

$$\big\||y|^{\beta} u\big\|_{L_q(\mathbb{R}^{n+m})} \le c\big\||y|^{\alpha}\nabla u\big\|_{L_{p_1}(\mathbb{R}^{n+m})},$$

$$\big\||y|^{\alpha}\nabla_j u\big\|_{L_{p_j}(\mathbb{R}^{n+m})} \le c\big\||y|^{\alpha}\nabla_{j+1} u\big\|_{L_{p_j+1}(\mathbb{R}^{n+m})}, \quad 1 \le j < l,$$

which follow from (2.1.35), we arrive at (2.1.38). □

Inequality (2.1.38) and its particular cases (2.1.34) and (2.1.35) obviously fail for $\alpha = l+nq^{-1}-(m+n)p^{-1}$. Nevertheless for this critical α we can obtain similar inequalities that are also invariant under similarity transformations in $\mathbb{R}^{n+m}$ by changing the weight function on the left-hand side.

2.1.8 Comments to Sect. 2.1

The results of Sects. 2.1.1–2.1.3 and 2.1.5 are borrowed from the author's paper [543] (see also [552]).

Properties of the weighted area minimizing function $\mathscr{C}$ introduced in Definition 2.1.4 were studied under the assumption that $\varPhi(x,\xi)$ does not depend on x and is convex. In particular, the sharp generalized isoperimetric inequality

$$\int_{\partial g} \varPhi\big(\mathscr{N}(x)\big)\,\mathrm{d}s(x) \geq n\varkappa_n^{1/n} m_n(g) \tag{2.1.39}$$

holds for all admissible sets $g \subset \mathbb{R}^n$. Here $\varkappa_n$ is the volume of the set $\{\xi \in \mathbb{R}^n : \varPsi(\xi) \leq 1\}$ with

$$\varPsi(\xi) = \sup_{x \neq 0} \frac{(x,\xi)_{\mathbb{R}^n}}{\varPhi(x)}$$

(see Busemann [158] and Burago, Zalgaller [151]). The surfaces minimizing the integrals of the form

$$\int_{\partial g} \varPhi\big(\mathscr{N}(x)\big)\,\mathrm{d}s(x)$$

over all sets g with a fixed volume, called Wulff shapes, appeared in 1901 (see Wulff [798]). The Wulff shape is called the crystal of the function $\varPhi$, which in its turn is called crystalline if its crystal is polyhedral (see, in particular, J.E. Taylor [745] for a theory of crystalline integrands as well as the bibliography).

The sharp constant

$$C_\alpha = \left(\frac{\alpha+1}{\alpha+2}\right)^{\frac{\alpha+1}{\alpha+2}} \left(2\int_0^\pi (\sin t)^\alpha\,\mathrm{d}t\right)^{-\frac{1}{\alpha+2}}, \quad \alpha \geq 0,$$

in the weighted isoperimetric inequality

$$m_2(g)^{\frac{\alpha+1}{\alpha+2}} \leq C_\alpha \int_{\partial g} \big(\mathscr{N}_1^2 + |x|^{2\alpha}\mathscr{N}_2^2\big)^{1/2}\,\mathrm{d}s,$$

where $(\mathscr{N}_1, \mathscr{N}_2) = \mathscr{N}$ was found by Monti and Morbidelli [611], which is equivalent to the sharp integral inequality

$$\|u\|_{L_{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^2)} \leq C_\alpha \int_{\mathbb{R}^2} \left(\left(\frac{\partial u}{\partial x}\right)^2 + |x|^{2\alpha}\left(\frac{\partial u}{\partial y}\right)^2\right)^{1/2} \mathrm{d}x$$

for all $u \in C_0^\infty(\mathbb{R}^2)$.

Theorem 2.1.4 is borrowed from Maz'ya [560], its proof being a modification of that in Maz'ya and Netrusov [572] relating to the case $p > 1$. For the contents of Sect. 2.1.6 see Sect. 2.1.5 in the author's book [556].

Obviously, in Theorem 2.1.6/1 the role of $|y|$ can be played by the distance to the m-dimensional Lipschitz manifold F supporting the measure ν. Horiuchi [383] proved the sufficiency in Theorem 2.1.6/1 for an absolutely continuous measure ν and for a more general class of sets F depending on the behavior as $\varepsilon \to 0$ of the n-dimensional Lebesgue measure of the tubular neighborhood of F, $\{z \in \mathbb{R}^{n+m} : \mathrm{dist}(z, F) < \varepsilon\}$.

The contents of Sect. 2.1.7 were published in [556], Sect. 2.1.6, for the first time. Estimates similar to (2.1.38) are generally well known (except, probably, for certain values of the parameters p, q, l, and α) but they were established by other methods (see Il'in [395]). The multiplicative inequality

$$\big\||x|^\gamma u\big\|_{L_r(\mathbb{R}^n)} \le C\big\||x|^\alpha|\nabla u|\big\|^a_{L_p(\mathbb{R}^n)}\big\||x|^\beta u\big\|^{1-a}_{L_q(\mathbb{R}^n)}$$

was studied in detail by Caffarelli, Kohn, and Nirenberg [162]. Lin [498] has generalized their results to include derivatives of any order.

The inequality (2.1.36) was proved by Maz'ya [556], Sect. 2.1.6. Tertikas and Tintarev [749] (see also Tintarev and Fieseler [753], Sect. 5.6, as well as Benguria, Frank, and Loss [83]) studied the existence and nonexistence of optimizers in (2.1.37) and found sharp constants in some cases. In one particular instance of (2.1.36), the sharp value of c will be given in Sect. 2.7.1. In [277], Filippas, Maz'ya, and Tertikas showed that for any convex domains $\Omega \subset \mathbb{R}^n$ the inequality

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{1}{4}\int \frac{u^2}{d^2}\,\mathrm{d}x \ge c(\Omega)\Big(\int_\Omega |u|^{\frac{2n}{n-2}}\,\mathrm{d}x\Big)^{\frac{n-2}{n}}$$

holds where $u \in C_0^\infty(\Omega)$ and $d = \mathrm{dist}(x, \partial\Omega)$. See Comments to Sect. 2.7 for other contributions to this area.

2.2 (p,Φ)-Capacity

2.2.1 Definition and Properties of the (p,Φ)-Capacity

Let e be a compactum in $\Omega \subset \mathbb{R}^n$ and let Φ be the function specified in Sect. 2.1.1. The number

$$\inf\Big\{\int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x : u \in \mathfrak{N}(e,\Omega)\Big\},$$

where $p \ge 1$, is called *the* (p,Φ)*-capacity of* e *relative to* Ω and is denoted by (p,Φ)-cap(e,Ω). Here

$$\mathfrak{N}(e,\Omega)=\big\{u\in\mathscr{D}(\Omega):u\geq 1 \text{ on } e\big\}.$$

If $\Omega=\mathbb{R}^n$, we omit Ω in the notations (p,Ω)-cap(e,Ω), $\mathfrak{N}(e,\Omega)$, and so on.

In the case $\Phi(x,\xi)=|\xi|$, we shall speak of *the p-capacity of a compactum e relative to Ω* and we shall use the notation $\mathrm{cap}_p(e,\Omega)$.

We present several properties of the (p,Φ)-capacity.

(i) *For compact sets $K\subset\Omega$, $F\subset\Omega$, the inclusion $K\subset F$ implies*

$$(p,\Phi)\text{-cap}(K,\Omega)\leq(p,\Phi)\text{-cap}(F,\Omega).$$

This is an obvious consequence of the definition of capacity. From the same definition it follows that the (p,Φ)-capacity of F relative to Ω does not increase under extension of Ω.

(ii) *The equality*

$$(p,\Phi)\text{-cap}(e,\Omega)=\inf\bigg\{\int_\Omega\big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x:u\in\mathfrak{P}(e,\Omega)\bigg\},\tag{2.2.1}$$

where $\mathfrak{P}(e,\Omega)=\{u:u\in\mathscr{D}(\Omega),u=1$ *in a neighborhood of* $e,0\leq u\leq 1$ *in* $\mathbb{R}^n\}$ *is valid.*

Proof. Since $\mathfrak{N}(e,\Omega)\subset\mathfrak{P}(e,\Omega)$ it is sufficient to estimate (p,Φ)-cap(e,Ω) from below. Let $\varepsilon\in(0,1)$ and let $f\in\mathfrak{N}(e,\Omega)$ be such that

$$\int_\Omega\big[\Phi(x,\nabla f)\big]^p\,\mathrm{d}x\leq(p,\Phi)\text{-cap}(e,\Omega)+\varepsilon.$$

Let $\{\lambda_m(t)\}_{m\geq 1}$ denote a sequence of functions in $C^\infty(\mathbb{R}^1)$ satisfying the conditions $0\leq\lambda_m'(t)\leq 1+m^{-1}$, $\lambda_m(t)=0$ in a neighborhood of $(-\infty,0]$ and $\lambda_m(t)=1$ in a neighborhood of $[1,\infty)$, $0\leq\lambda_m(t)\leq 1$ for all t. Since $\lambda_m(f(x))\in\mathfrak{P}(e,\Omega)$, then

$$\inf\bigg\{\int_\Omega\big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x:u\in\mathfrak{P}(e,\Omega)\bigg\}\leq\int_\Omega\big[\lambda_m'\big(f(x)\big)\big]^p\big[\Phi\big(x,\nabla f(x)\big)\big]^p\,\mathrm{d}x.$$

Passing to the limit as $m\to\infty$, we obtain

$$\begin{aligned}&\inf\bigg\{\int_\Omega\big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x:u\in\mathfrak{P}(e,\Omega)\bigg\}\\&\qquad\leq\int_\Omega\big[\Phi(x,\nabla f)\big]^p\,\mathrm{d}x\leq(p,\Phi)\text{-cap}(e,\Omega)+\varepsilon.\end{aligned}$$

□

(iii) *For any compactum $e\subset\Omega$ and $\varepsilon>0$ there exists a neighborhood G such that*

$$(p,\Phi)\text{-cap}(K,\Omega)\leq(p,\Phi)\text{-cap}(e,\Omega)+\varepsilon$$

for all compact sets K, $e\subset K\subset G$.

Proof. From (2.2.1) it follows that there exists a $u \in \mathfrak{P}(e, \Omega)$ such that

$$\int_{\Omega} \big[\Phi(x, \nabla u)\big]^p \, \mathrm{d}x \le (p, \Phi)\text{-cap}(e, \Omega) + \varepsilon.$$

Let G denote a neighborhood of e in which $u = 1$. It remains to note that

$$(p, \Phi)\text{-cap}(K, \Omega) \le \int_{\Omega} \big[\Phi(x, \nabla u)\big]^p \, \mathrm{d}x$$

for any compactum K, $e \subset K \subset G$. □

The next property is proved analogously.

(iv) *For any compactum $e \subset \Omega$ and any $\varepsilon > 0$ there exists an open set ω, $\bar{\omega} \subset \Omega$, such that*

$$(p, \Phi)\text{-cap}(e, \omega) \le (p, \Phi)\text{-cap}(e, \Omega) + \varepsilon.$$

(v) *The Choquet inequality*

$$\begin{aligned}&(p, \Phi)\text{-cap}(K \cup F, \Omega) + (p, \Phi)\text{-cap}(K \cap F, \Omega)\\ &\quad \le (p, \Phi)\text{-cap}(K, \Omega) + (p, \Phi)\text{-cap}(F, \Omega)\end{aligned}$$

holds for any compact sets K, $F \subset \Omega$.

Proof. Let u and v be arbitrary functions in $\mathfrak{P}(K, \Omega)$ and $\mathfrak{P}(F, \Omega)$, respectively. We put $\varphi = \max(u, v)$, $\psi = \min(u, v)$. Obviously, φ and ψ have compact supports and satisfy the Lipschitz condition in Ω, $\varphi = 1$ in the neighborhood of $K \cup F$ and $\psi = 1$ in a neighborhood of $K \cap F$. Since the set $\{x : u(x) \neq v(x)\}$ is the union of open sets on which either $u > v$ or $u < v$, and since $\nabla u(x) = \nabla v(x)$ almost everywhere on $\{x : u(x) = v(x)\}$, then

$$\begin{aligned}&\int_{\Omega} \big[\Phi(x, \nabla \varphi)\big]^p \, \mathrm{d}x + \int_{\Omega} \big[\Phi(x, \nabla \psi)\big]^p \, \mathrm{d}x\\ &\quad = \int_{\Omega} \big[\Phi(x, \nabla u)\big]^p \, \mathrm{d}x + \int_{\Omega} \big[\Phi(x, \nabla v)\big]^p \, \mathrm{d}x.\end{aligned}$$

Hence, having noted that mollifications of the functions φ and ψ belong to $\mathfrak{P}(K \cup F, \Omega)$ and $\mathfrak{P}(K \cap F, \Omega)$, respectively, we obtain the required inequality. □

A function of compact sets that satisfies conditions (i), (iii), and (v) is called a *Choquet capacity*.

Let E be an arbitrary subset of Ω. The number $(p, \Phi)\text{-cap}(E, \Omega) = \sup_{\{K\}} (p, \Phi)\text{-cap}(K, \Omega)$, where $\{K\}$ is a collection of compact sets contained in E, is called the (p, Φ) *capacity of E relative to Ω*. The number

$$\inf_{\{G\}} (p, \Phi)\text{-cap}(G, \Omega),$$

where $\{G\}$ is the collection of all open subsets of Ω containing E, is called the *outer capacity* (p,Φ)-$\overline{\text{cap}}(E,\Omega)$ of $E \subset \Omega$. A set E is called (p,Φ) *capacitable* if

$$(p,\Phi)\text{-cap}(E,\Omega) = (p,\Phi)\text{-}\overline{\text{cap}}(E,\Omega).$$

From these definitions it follows that any open subset of Ω is (p,Φ) capacitable. If e is a compactum in Ω, then by property (iii), given $\varepsilon > 0$, there exists an open set G such that

$$(p,\Phi)\text{-cap}(G,\Omega) \le (p,\Phi)\text{-cap}(e,\Omega) + \varepsilon.$$

Consequently, all compact subsets of Ω are (p,Φ) capacitable.

From the general theory of Choquet capacities it follows that analytic sets, and in particular, Borel sets are (p,Φ) capacitable (see Choquet [186]).

2.2.2 Expression for the (p,Φ)-Capacity Containing an Integral over Level Surfaces

Lemma 1. *For any compactum $F \subset \Omega$ the (p,Φ)-capacity (for $p > 1$) can be defined by*

$$(p,\Phi)\text{-cap}(F,\Omega) = \inf_{u\in\mathfrak{N}(F,\Omega)} \left\{ \int_0^1 \frac{\mathrm{d}\tau}{\left(\int_{\mathscr{E}_\tau} [\Phi(x,\nabla u)]^p \frac{\mathrm{d}s}{|\nabla u|}\right)^{1/(p-1)}} \right\}^{1-p}, \quad (2.2.2)$$

where $\mathscr{E}_t = \{x : |u(x)| = t\}$.

We introduce the following notation: Λ is the set of nondecreasing functions $\lambda \in C^\infty(\mathbb{R}^1)$, which satisfy the conditions $\lambda(t) = 0$ for $t \le 0$, $\lambda(t) = 1$ for $t \ge 1$, supp $\lambda' \subset (0,1)$; Λ_1 is the set of nondecreasing functions that are absolutely continuous on $\mathbb{R}^1$ and satisfy the conditions $\lambda(t) = 0$ for $t \le 0$, $\lambda(t) = 1$ for $t \ge 1$, $\lambda'(t)$ is bounded.

To prove Lemma 1 we shall use the following auxiliary assertion.

Lemma 2. *Let g be a nonnegative function that is integrable on $[0,1]$. Then*

$$\inf_{\lambda\in\Lambda} \int_0^1 (\lambda')^p g \,\mathrm{d}t = \left(\int_0^1 \frac{\mathrm{d}t}{g^{1/(p-1)}} \right)^{1-p}. \quad (2.2.3)$$

Proof. First we note that by Hölder's inequality

$$1 = \int_0^1 \lambda' \,\mathrm{d}t \le \left(\int_0^1 (\lambda')^p g \,\mathrm{d}t \right)^{1/p} \left(\int_0^1 \frac{\mathrm{d}t}{g^{1/(p-1)}} \right)^{1-1/p},$$

and hence the left-hand side of (2.2.3) is not smaller than the right.

Let $\lambda \in \Lambda_1$, $\zeta_\nu(t) = \lambda'(t)$ for $t \in [\nu^{-1}, 1-\nu^{-1}]$, supp $\zeta_\nu \subset [\nu^{-1}, 1-\nu^{-1}]$, $\nu = 1, 2, \dots$. We set

$$\eta_\nu(t)=\zeta_\nu(t)\biggl(\int_0^1\zeta_\nu\,\mathrm{d}\tau\biggr)^{-1}.$$

Since the sequence η_ν converges to λ' on (0,1) and is bounded, it follows by Lebesgue's theorem that

$$\int_0^1\eta_\nu^p g\,\mathrm{d}\tau\to\int_0^1(\lambda')^p g\,\mathrm{d}\tau.$$

Mollifying η_ν, we obtain the sequence $\{\gamma_\nu\}$, $\gamma_\nu\in C^\infty(\mathbb{R}^1)$, $\operatorname{supp}\gamma_\nu\subset(0,1)$,

$$\int_0^1\gamma_\nu\,\mathrm{d}\tau=1,\qquad\int_0^1\gamma_\nu^p g\,\mathrm{d}\tau\to\int_0^1(\lambda')^p g\,\mathrm{d}\tau.$$

Setting

$$\lambda_\nu(t)=\int_0^t\gamma_\nu\,\mathrm{d}\tau,$$

we obtain a sequence of functions in Λ such that

$$\int_0^1(\lambda_\nu')^p g\,\mathrm{d}\tau\to\int_0^1(\lambda')^p g\,\mathrm{d}\tau.$$

Hence,

$$\inf_{\Lambda}\int_0^1(\lambda')^p g\,\mathrm{d}\tau=\inf_{\Lambda_1}\int_0^1(\lambda')^p g\,\mathrm{d}\tau.\tag{2.2.4}$$

Let

$$M_\varepsilon=\bigl\{t:g(t)\geq\varepsilon\bigr\},\qquad\lambda_0(t)=\int_0^t\eta\,\mathrm{d}\tau,$$

where $\eta(t)=0$ on $\mathbb{R}^1\backslash M_\varepsilon$ and

$$\eta(t)=g(t)^{1/(1-p)}\biggl(\int_{M_\varepsilon}g^{1/(1-p)}\,\mathrm{d}\tau\biggr)^{-1}\quad\text{for }t\in M_\varepsilon.$$

Obviously, $\lambda_0\in\Lambda_1$, and

$$\int_0^1(\lambda_0')^p g\,\mathrm{d}\tau=\biggl(\int_{M_\varepsilon}g^{1/(1-p)}\,\mathrm{d}\tau\biggr)^{1-p}.$$

By (2.2.4) the left-hand side of (2.2.3) does not exceed

$$\biggl(\int_{M_\varepsilon}g^{1/(1-p)}\,\mathrm{d}\tau\biggr)^{1-p}.$$

We complete the proof by passing to the limit as $\varepsilon\to 0$. □

Proof of Lemma 1. Let $u\in\mathfrak{N}(F,\Omega)$, $\lambda\in\Lambda$. From the definition of capacity and Theorem 1.2.4 we obtain

$$(p,\Phi)\text{-cap}(F,\Omega) = \int_\Omega \big[\lambda'(u)\Phi(x,\nabla u)\big]^p \,\mathrm{d}x = \int_0^1 (\lambda')^p g \,\mathrm{d}t,$$

where

$$g(t) = \int_{\mathscr{E}_t} \big[\Phi(x,\nabla u)\big]^p \frac{\mathrm{d}s}{|\nabla u|}. \tag{2.2.5}$$

By Lemma 2

$$(p,\Phi)\text{-cap}(F,\Omega) \le \left(\int_0^1 g^{1/(1-p)}\,\mathrm{d}\tau\right)^{1-p}.$$

To prove the opposite inequality it is enough to note that

$$\int_\Omega \big[\Phi(x,\nabla u)\big]^p \,\mathrm{d}x \ge \int_0^1 g\,\mathrm{d}\tau \ge \left(\int_0^1 g^{1/(1-p)}\,\mathrm{d}\tau\right)^{1-p}.$$

The lemma is proved. □

Recalling the property (2.2.1) of the (p,Φ)-capacity, note that, in passing, we have proved here also the following lemma.

Lemma 3. *For any compactum $F \subset \Omega$ the (p,Φ)-capacity $(p > 1)$ can be defined as*

$$(p,\Phi)\text{-cap}(F,\Omega) = \inf_{u\in\mathfrak{P}(F,\Omega)} \left\{\int_0^1 \frac{\mathrm{d}t}{\big(\int_{\mathscr{E}_t}[\Phi(x,\nabla u)]^p \frac{\mathrm{d}s}{|\nabla u|}\big)^{1/(p-1)}}\right\}^{1-p}.$$

2.2.3 Lower Estimates for the (p,Φ)-Capacity

Lemma. *For any $u \in \mathscr{D}(\Omega)$ and almost all $t \ge 0$,*

$$\big[\sigma(\partial\mathscr{L}_t)\big]^{p/(p-1)} \le \left[-\frac{\mathrm{d}}{\mathrm{d}t} m_n(\mathscr{L}_t)\right]\left(\int_{\partial\mathscr{L}_t}\big[\Phi(x,\nabla u)\big]^p \frac{\mathrm{d}s}{|\nabla u|}\right)^{1/(p-1)}, \tag{2.2.6}$$

where, as usual, $\mathscr{L}_t = \{x \in \Omega : |u(x)| > t\}$.

Proof. By Hölder's inequality, for almost all t and T, $t < T$,

$$\left(\int_{\mathscr{L}_t\setminus\mathscr{L}_T} |u|^{p-1}\Phi(x,\nabla u)\,\mathrm{d}x\right)^{p/(p-1)} \le \int_{\mathscr{L}_t\setminus\mathscr{L}_T} |u|^p\,\mathrm{d}x \left(\int_{\mathscr{L}_t\setminus\mathscr{L}_T}\big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x\right)^{1/(p-1)}.$$

Using Theorem 1.2.4, we obtain

$$\left(\int_t^T \tau^{p-1}\sigma(\partial\mathscr{L}_\tau)\,\mathrm{d}\tau\right)^{p/(p-1)} \le \int_{\mathscr{L}_t\setminus\mathscr{L}_T} |u|^p\,\mathrm{d}x \left(\int_t^T \mathrm{d}\tau \int_{\mathscr{E}_\tau} \big[\Phi(x,\nabla u)\big]^p \frac{\mathrm{d}s}{|\nabla u|}\right)^{1/(p-1)}.$$

We divide both sides of the preceding inequality by $(T-t)^{p/(p-1)}$ and estimate the first factor on the right-hand side

$$\left(\frac{1}{T-t}\int_t^T \tau^{p-1}\sigma(\partial\mathscr{L}_\tau)\,\mathrm{d}\tau\right)^{p/(p-1)} \le T^p\frac{m_n(\mathscr{L}_t\setminus\mathscr{L}_T)}{T-t}\left(\frac{1}{T-t}\int_t^T \mathrm{d}\tau\int_{\partial\mathscr{L}_\tau}\big[\Phi(x,\nabla u)\big]^p\frac{\mathrm{d}s}{|\nabla u|}\right)^{1/(p-1)}.$$

Passing to the limit as $T\to t$, we obtain (2.2.6) for almost all $t>0$. The lemma is proved. □

From Lemma 2.2.2/3 and from the Lemma of the present subsection we immediately obtain the following corollary.

Corollary 1. *The inequality*

$$(p,\Phi)\text{-cap}(F,\Omega) \ge \inf_{u\in\mathfrak{P}(F,\Omega)}\left\{-\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}m_n(\mathscr{L}_\tau)\frac{\mathrm{d}\tau}{[\sigma(\partial\mathscr{L}_\tau)]^{p/(p-1)}}\right\}^{1-p} \tag{2.2.7}$$

holds.

Definition. In what follows we use the function $\mathscr{C}$ introduced in Definition 2.1.4 assuming $\mu=m_n$, that is, $\mathscr{C}$ stands for the infimum $\sigma(\partial g)$ for all admissible sets such that $m_n(g)\ge\varrho$. Then from (2.2.7) we obtain the next corollary, containing the so-called *isocapacitary inequalities*.

Corollary 2. *The inequality*

$$(p,\Phi)\text{-cap}(F,\Omega) \ge \left(\int_{m_n(F)}^{m_n(\Omega)} \frac{\mathrm{d}\varrho}{[\mathscr{C}(\varrho)]^{p/(p-1)}}\right)^{1-p} \tag{2.2.8}$$

is valid.

By virtue of the classical isoperimetric inequality

$$s(\partial g)\ge n^{(n-1)/n}\omega_n^{1/n}\big[m_n(g)\big]^{(n-1)/n}, \tag{2.2.9}$$

in the case $\Phi(x,\xi)=|\xi|$ we have

$$\mathscr{C}(\varrho)=n^{(n-1)/n}\omega_n^{1/n}\varrho^{(n-1)/n}.$$

Therefore,

$$\operatorname{cap}_p(F,\Omega) \geq \omega_n^{p/n} n^{(n-p)/n} \left|\frac{p-n}{p-1}\right|^{p-1} \Big| m_n(\Omega)^{(p-n)/n(p-1)} \\ - m_n(F)^{(p-n)/n(p-1)}\Big|^{1-p} \tag{2.2.10}$$

for $p \neq n$ and

$$\operatorname{cap}_p(F,\Omega) \geq n^{n-1}\omega_n \left(\log \frac{m_n(\Omega)}{m_n(F)}\right)^{1-n} \tag{2.2.11}$$

for $p = n$.

In particular, for $n > p$,

$$\operatorname{cap}_p(F) \geq \omega_n^{p/n} n^{(n-p)/n} \left(\frac{n-p}{p-1}\right)^{p-1} m_n(F)^{(n-p)/n}. \tag{2.2.12}$$

2.2.4 *p*-Capacity of a Ball

We show that the estimates (2.2.10) and (2.2.11) become equalities if Ω and F are concentric balls of radii R and r, $R > r$, i.e.,

$$\operatorname{cap}_p(F,\Omega) = \omega_n \left(\frac{|n-p|}{p-1}\right)^{p-1} \left|R^{(p-n)/(p-1)} - r^{(p-n)/(p-1)}\right|^{1-p} \tag{2.2.13}$$

for $n \neq p$ and

$$\operatorname{cap}_n(F,\Omega) = \omega_n \left(\log \frac{R}{r}\right)^{1-n} \tag{2.2.14}$$

for $n = p$.

Let the centers of the balls Ω and F coincide with the origin O of spherical coordinates (ϱ, ω), $|\omega| = 1$. Obviously,

$$\begin{aligned}\operatorname{cap}_p(F,\Omega) &\geq \inf_{u \in \mathfrak{N}(F,\Omega)} \int_{\partial B_1} \mathrm{d}\omega \int_r^R \left|\frac{\partial u}{\partial \varrho}\right|^p \varrho^{n-1}\, \mathrm{d}\varrho \\ &\geq \int_{\partial B_1} \mathrm{d}\omega \inf_{u \in \mathfrak{N}(F,\Omega)} \int_r^R \left|\frac{\partial u}{\partial \varrho}\right|^p \varrho^{n-1}\, \mathrm{d}\varrho.\end{aligned}$$

The inner integral attains its infimum at the function

$$[r,R] \in \varrho \to v(\varrho) = \begin{cases} \frac{R^{(p-n)/(p-1)} - \varrho^{(p-n)/(p-1)}}{R^{(p-n)/(p-1)} - r^{(p-n)/(p-1)}} & \text{for } p \neq n, \\ \frac{\log(\varrho R^{-1})}{\log(r R^{-1})} & \text{for } p = n. \end{cases}$$

This implies the required lower estimates for the p-capacity. The substitution of $v(\varrho)$ into the integral $\int_\Omega |\nabla u|^p\, \mathrm{d}x$ leads to (2.2.13) and (2.2.14).

In particular, the p-capacity of the n-dimensional ball B_r relative to $\mathbb{R}^n$ is equal to $\omega_n(\frac{n-p}{p-1})^{p-1} r^{n-p}$ for $n > p$ and to zero for $n \leq p$. Since the p-capacity is a monotone set function, then for any compactum $p\text{-cap}(F,\mathbb{R}^n) = 0$, if

$n \le p$. In the case $p \le n$ the capacity of a point relative to any open set Ω, containing this point, equals zero. If $p > n$, then the p-capacity of the center of the ball B_R relative to B_R equals $\omega_n(\frac{p-n}{p-1})^{p-1} R^{n-p}$. Therefore, in the last case, the p-capacity of any compactum relative to any bounded open set that contains this compactum is positive.

2.2.5 (p, Φ)-Capacity for $p = 1$

Lemma. *For any compactum $F \subset \Omega$*

$$(1, \Phi)\text{-cap}(F, \Omega) = \inf \sigma(\partial g),$$

where the infimum is taken over all admissible sets g in Ω containing F.

Proof. Let $u \in \mathfrak{N}(F, \Omega)$. Applying Theorem 1.2.4, we obtain

$$\int_\Omega \Phi(x, \nabla u)\, \mathrm{d}x = \int_0^1 \sigma(\partial \mathscr{L}_t)\, \mathrm{d}t \ge \inf_{g \supset F} \sigma(\partial g).$$

This implies the lower estimate for the capacity.

Let g be an admissible set containing F. The function $u_\varepsilon(x) = \alpha(\mathrm{d}(x))$ defined in the proof of the second part of Theorem 2.1.1 belongs to $\mathfrak{N}(F, \Omega)$ for sufficiently small $\varepsilon > 0$. So

$$(1, \Phi)\text{-cap}(F, \Omega) \le \int_\Omega \Phi(x, \nabla u_\varepsilon)\, \mathrm{d}x.$$

In the proof of the second part of Theorem 2.1.1, it was shown that the preceding integral converges to $\sigma(\partial g)$, which yields the required upper estimate for the capacity. The lemma is proved. □

2.2.6 The Measure m_{n-1} and 2-Capacity

Lemma. *If $B_\varrho^{(n-1)}$ is an $(n-1)$-dimensional ball in $\mathbb{R}^n$, $n > 2$, then*

$$\mathrm{cap}_2\big(B_\varrho^{(n-1)}, \mathbb{R}^n\big) = \frac{\omega_n}{c_n} \varrho^{n-2}, \tag{2.2.15}$$

where $c_3 = \frac{\pi}{3}$, $c_4 = 1$, and $c_n = (n-4)!!/(n-3)!!$ for odd $n \ge 5$ and $c_n = \frac{\pi}{2}(n-4)!!/(n-3)!!$ for even $n \ge 6$.

Proof. We introduce ellipsoidal coordinates in $\mathbb{R}^n$: $x_1 = \varrho \sinh \psi \cos \theta_1$, $x_j = \varrho \cosh \psi \sin \theta_1, \dots, \sin \theta_{j-1} \cos \theta_j, j = 2, \dots, n-1$, $x_n = \varrho \cosh \psi \sin \theta_1$, $\dots, \sin \theta_{n-1}$. A standard calculation leads to the formulas

$$\mathrm{d}x = \varrho^n \big(\cosh^2 \psi - \sin^2 \theta_1\big)(\cosh \psi)^{n-2}\, \mathrm{d}\psi\, \mathrm{d}\omega,$$

$$(\nabla u)^2 = \varrho^{-2} \left(\frac{\partial u}{\partial \psi}\right)^2 \big(\cosh^2 \psi - \sin^2 \theta_1\big)^{-1} + \cdots,$$

where $d\omega$ is the surface element of the unit ball in $\mathbb{R}^n$ and the dots denote a positive quadratic form of all first derivatives of u except $\partial u/\partial\psi$. The equation of the ball $B_\varrho^{(n-1)}$ in the new coordinates is $\psi = 0$. Therefore

$$\operatorname{cap}_2\big(B_\varrho^{(n-1)}, \mathbb{R}^n\big) \geq \varrho^{n-2} \int_{|\omega|=1} \left(\inf_{\{u\}} \int_0^\infty \left(\frac{\partial u}{\partial \psi} \right)^2 (\cosh\psi)^{n-2}\, d\psi \right) d\omega,$$

where $\{u\}$ is a set of smooth functions on $[0, \infty)$ with compact supports. The infimum on the right-hand side is equal to

$$\left(\int_0^\infty \frac{d\psi}{(\cosh\psi)^{n-2}} \right)^{-1} = c_n^{-1}.$$

This value is attained at the function

$$v = \int_\psi^\infty \frac{d\tau}{(\cosh\tau)^{n-2}} \left(\int_0^\infty \frac{d\tau}{(\cosh\tau)^{n-2}} \right)^{-1},$$

which equals unity on $B_\varrho^{(n-1)}$ and decreases sufficiently rapidly at infinity. Substituting v into the Dirichlet integral, we obtain

$$\operatorname{cap}_2\big(B_\varrho^{(n-1)}, \mathbb{R}^n\big) \leq \omega_n \varrho^{n-2} \int_0^\infty \left(\frac{\partial v}{\partial \psi} \right)^2 (\cosh\psi)^{n-2}\, d\psi = \frac{\omega_n}{c_n} \varrho^{n-2}.$$

This proves the lemma. □

We now recall the definition of the *symmetrization of a compact set K in $\mathbb{R}^n$ relative to the $(n-s)$-dimensional subspace* $\mathbb{R}^{n-s}$.

Let any point $x \in \mathbb{R}^n$ be denoted by (y, z), where $y \in \mathbb{R}^{n-s}$, $z \in \mathbb{R}^s$. The image K^* of the compact set K under symmetrization relative to the subspace $z = 0$ is defined by the following conditions:

1. The set K^* is symmetric relative to $z = 0$.

2. Any s-dimensional subspace, parallel to the subspace $y = 0$ and crossing either K or K^* also intersects the other one and the Lebesgue measures of both cross sections are equal.

3. The intersection of K^* with any s-dimensional subspace, which is parallel to the subspace $y = 0$, is a ball in $\mathbb{R}^s$ centered at the hyperplane $z = 0$.

Below we follow Pólya and Szegö [666] who established that the 2-capacity does not increase under the symmetrization relative to $\mathbb{R}^{n-1}$. Let π be an $(n-1)$-dimensional hyperplane and let $\mathrm{Pr}_\pi \mathscr{F}$ be the projection of $\mathscr{F}$ onto π. We choose π so that $m_{n-1}(\mathrm{Pr}_\pi\, \mathscr{F})$ attains its maximum value. We symmetrize $\mathscr{F}$ relative to π and obtain a compactum that is also symmetrized relative to a straight line perpendicular to π. So we obtain a body whose capacity does not exceed 2-cap $\mathscr{F}$ and whose intersection with π is an $(n-1)$-dimensional ball with volume $m_{n-1}(\mathrm{Pr}_\pi \mathscr{F})$. Thus the $(n-1)$-dimensional ball has the largest area of orthogonal projections onto an $(n-1)$-dimensional plane among all compacta with fixed 2-capacity.

This and the Lemma imply the isocapacitary inequality

$$\begin{aligned}&\left[m_{n-1}\left(\mathscr{F} \cap \mathbb{R}^{n-1}\right)\right]^{(n-2)/(n-1)}\\ &\quad\leq \left(\frac{\omega_{n-1}}{n-1}\right)^{(n-2)/(n-1)} \frac{c_n}{\omega_n} \operatorname{cap}_2\left(\mathscr{F}, \mathbb{R}^n\right),\end{aligned} \tag{2.2.16}$$

where c_n is the constant defined in the Lemma.

2.2.7 Comments to Sect. 2.2

The capacity generated by the integral

$$\int_\Omega f(x, u, \nabla u)\, \mathrm{d}x$$

was introduced by Choquet [186] where it served as an illustration of general capacity theory. Here the presentation follows the author's paper [543].

Lemma 2.2.2/1 for $p = 2$, $\Phi(x, \xi) = |\xi|$ is the so-called Dirichlet principle with prescribed level surfaces verified in the book by Pólya and Szegö [666] under rigid assumptions on level surfaces of the function u. As for the general case, their proof can be viewed as a convincing heuristic argument. The same book also contains isocapacitary inequalities, which are special cases of (2.2.10) and (2.2.11).

Lemma 2.2.3, leading to lower estimates for the capacity, was published for $\Phi(x, \xi) = |\xi|$ in 1969 by the author [538] and later by Talenti in [741], p. 709.

Properties of symmetrization are studied in the books by Pólya and Szegö [666] and by Hadwiger [334] et al. See, for instance, the book by Hayman [357] where the circular symmetrization and the symmetrization with respect to a straight line in $\mathbb{R}^2$ are considered. Nevertheless, Hayman's proofs can be easily generalized to the n-dimensional case. Lemma 2.2.5 is a straightforward generalization of a similar assertion due to Fleming [281] on 1-capacity.

In the early 1960s the p-capacity was used by the author to obtain the necessary and sufficient conditions for the validity of continuity and compactness properties of Sobolev-type embedding operators [527, 528, 530, 531].

Afterward various generalizations of the p-capacity proved to be useful in the theory of function spaces and nonlinear elliptic equations. A Muckenhoupt A_p-weighted capacity was studied in detail by Heinonen, Kilpeläinen, and Martio [375] and Nieminen [636] et al. A general capacity theory for monotone operators

$$W_p^1 \ni u \to -\operatorname{div}\big(a(x, Du)\big) \in \left(W_p^1\right)^*$$

was developed by Dal Maso and Skrypnik [220], whose results were extended to pseudomonotone operators by Casado-Díaz [174]. Biroli [104] studied the p-capacity related to the norm

$$\left(\sum_{i=1}^{m} \int_{\Omega} |X_i u|^p \, \mathrm{d}x + \int_{\Omega} |u|^p \, \mathrm{d}x \right)^{1/p},$$

where X_i are vector fields subject to Hörmander's condition: They and their commutators up to some order span at every point all $\mathbb{R}^n$. Properties of a capacity generated by the Sobolev space $W^1_{p(\cdot)}$ with the variable exponent $p : \mathbb{R}^n \to (1, \infty)$ were investigated by Harjulehto, Hästö, Koskenoja, and Varonen [353]. Another relevant area of research is the p-capacity on metric spaces with a measure (see, for instance, Kinnunen and Martio [425] and Gol'dshtein, Troyanov [317]), in particular, on the Carnot group and Heisenberg group (see Heinonen and Holopainen [374]).

A generalization of the inequality (2.2.8) was obtained by E. Milman [603] in a more general framework of measure metric spaces for the case $\Phi(x, \xi) = |\xi|$. Similarly to Sect. 2.3.8, if we introduce the p-capacity minimizing function

$$\nu_p(t) = \inf \operatorname{cap}_p(F, \Omega),$$

where the infimum is taken over compacta $F \subset \Omega$ with $m_n(F) \geq t$, then for any $p_1 \geq p_0 \geq 1$

$$\frac{1}{\nu_{p_1}(t)} \leq \frac{(\frac{q_0}{q_1} - 1)^{p_1/q_0}}{(1 - \frac{q_1}{q_0})^{p_1/q_1}} \left(\int_t^{m_n(\Omega)} \frac{\mathrm{d}s}{(s-t)^{q_1/q_0} \nu_{p_0}(s)^{q_1/p_0}} \right)^{p_1/q_1}, \tag{2.2.17}$$

where $q_i = p_i/(p_i - 1)$ denote the corresponding conjugate exponents. Clearly (2.2.17) coincides with (2.2.8) when $p_0 = 1$ by Lemma 2.2.5.

Under an appropriate curvature lower bound on the underlying space, it was also shown in [603, 605] that (2.2.17) may, in fact, be reversed, to within numeric constants. An application of this fact was given by E. Milman [604] to the stability of the first positive eigenvalue of the Neumann Laplacian on convex domains, with respect to perturbation of the domain.

2.3 Conditions for Validity of Integral Inequalities (the Case $p > 1$)

2.3.1 The (p, Φ)-Capacitary Inequality

Let $u \in \mathscr{D}(\Omega)$ and let g be the function defined by (2.2.5) with $p > 1$. Further, let

$$T \stackrel{\text{def}}{=} \sup\big\{t > 0 : (p, \Phi)\text{-cap}(\mathscr{N}_t, \Omega) > 0\big\} > 0, \tag{2.3.1}$$

where $\mathscr{N}_t = \{x \in \Omega : |u(x)| \geq t\}$. From (2.3.1) it follows that

$$\psi(t) \stackrel{\text{def}}{=} \int_0^t \frac{\mathrm{d}\tau}{[g(\tau)]^{1/(p-1)}} \leq \infty \tag{2.3.2}$$

for $0 < t < T$. In fact, let

$$v(x) = t^{-2}\big[u(x)\big]^2.$$

Since $v \in \mathfrak{N}(\mathscr{N}_t, \Omega)$, then from Lemma 2.2.2/1 and (2.3.1) we obtain

$$\int_0^1 \bigg(\int_{\{x:v(x)=\tau\}} \big[\Phi(x,\nabla v)\big]^p \frac{\mathrm{d}s}{|\nabla v|}\bigg)^{1/(1-p)} \mathrm{d}\tau \\ \le \big[(p,\Phi)\text{-cap}(\mathscr{N}_t,\Omega)\big]^{1/(1-p)} < \infty,$$

and it remains to note that

$$\int_0^1 \frac{\mathrm{d}\tau}{g(\tau)^{1/(p-1)}} = \int_0^1 \bigg(\int_{\{x:v(x)=\tau\}} \big[\Phi(x,\nabla v)\big]^p \frac{\mathrm{d}s}{|\nabla v|}\bigg)^{1/(p-1)} \mathrm{d}\tau.$$

Since by Theorem 1.2.4

$$\int_0^\infty g(\tau)\,\mathrm{d}\tau = \int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x < \infty,$$

it follows that $g(t) < \infty$ for almost all $t > 0$ and the function $\psi(t)$ is strictly monotonic. Consequently, on the interval $[0,\psi(T))$ the function $t(\psi)$, which is the inverse of $\psi(t)$, exists.

Lemma. *Let u be a function in $\mathscr{D}(\Omega)$ satisfying condition* (2.3.1)*. Then the function $t(\psi)$ is absolutely continuous on any segment $[0,\psi(T-\delta)]$, where $\delta \in (0,T)$, and*

$$\int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x \ge \int_0^{\psi(t)} \big[t'(\psi)\big]^p\,\mathrm{d}\psi. \tag{2.3.3}$$

If $T = \max|u|$, then we may write the equality sign in (2.3.3)*.*

Proof. Let $0 = \psi_0 < \psi_1 < \cdots < \psi_m = \psi(T-\delta)$ be an arbitrary partition of the segment $[0,\psi(T-\delta)]$. By Hölder's inequality,

$$\frac{[t(\psi_{k+1}) - t(\psi_k)]^p}{(\psi_{k+1}-\psi_k)^{p-1}} = \frac{[t(\psi_{k+1}) - t(\psi_k)]^p}{[\int_{t(\psi_k)}^{t(\psi_{k+1})} g(\tau)^{1/(1-p)}\,\mathrm{d}\tau]^{p-1}} \le \int_{t(\psi_k)}^{t(\psi_{k+1})} g(\tau)\,\mathrm{d}\tau,$$

and consequently,

$$\sum_{k=0}^{m-1} \frac{[t(\psi_{k+1}) - t(\psi_k)]^p}{(\psi_{k+1}-\psi_k)^{p-1}} \le \sum_{k=0}^{m-1} \int_{t(\psi_k)}^{t(\psi_{k+1})} g(\tau)\,\mathrm{d}\tau \\ = \int_0^{T-\delta} g(\tau)\,\mathrm{d}\tau \le \int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x. \tag{2.3.4}$$

The last inequality follows from Theorem 1.2.4. By (2.3.4) and F. Riesz's theorem (see Natanson [627]), the function $t(\psi)$ is absolutely continuous and its derivative belongs to $L_p(0,\psi(T-\delta))$. By Theorem 1.2.4,

$$\int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x \geq \lim_{\delta\to+0}\int_0^{T-\delta} g(\tau)\,\mathrm{d}\tau. \tag{2.3.5}$$

Since $t(\psi)$ is a monotonic absolutely continuous function, we can make the change of variable $\tau = t(\psi)$ in the last integral. Then

$$\int_0^{T-\delta} g(\tau)\,\mathrm{d}\tau = \int_0^{\psi(T-\delta)} t'(\psi)g(\psi)\,\mathrm{d}\psi = \int_0^{\psi(T-\delta)} \big[t'(\psi)\big]^p\,\mathrm{d}\psi,$$

which, along with (2.3.5), completes the proof. □

Theorem. (Capacitary Inequality) *Let $u \in \mathscr{D}(\Omega)$. Then for $p \geq 1$,*

$$\int_0^\infty (p,\Phi)\text{-cap}(\mathscr{N}_t,\Omega)\,\mathrm{d}\big(t^p\big) \leq \frac{p^p}{(p-1)^{p-1}}\int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x. \tag{2.3.6}$$

For $p = 1$ the coefficient in front of the integral on the right-hand side of (2.3.6) is equal to one. The constant $p^p(p-1)^{1-p}$ is optimal.

Proof. To prove (2.3.6) it is sufficient to assume that the number T, defined in (2.3.1), is positive. Since by Lemma 2.2.5

$$(1,\Phi)\text{-cap}(\mathscr{N}_t,\Omega) \leq \sigma(\partial\mathscr{L}_t)$$

for almost all $t > 0$, we see that (2.3.6) follows from (2.1.4) for $p = 1$.

Consider the case $p > 1$. Let $\psi(t)$ be a function defined by (2.3.2) and let $t(\psi)$ be the inverse of $\psi(t)$. We make the change of variable

$$\begin{aligned}\int_0^\infty (p,\Phi)\text{-cap}(\mathscr{N}_t,\Omega)\,\mathrm{d}\big(t^p\big) &= \int_0^T (p,\Phi)\text{-cap}(\mathscr{N}_t,\Omega)\,\mathrm{d}\big(t^p\big)\\ &= \int_0^{\psi(T)} (p,\Phi)\text{-cap}(\mathscr{N}_{t(\psi)},\Omega)\,\mathrm{d}\big(t(\psi)^p\big).\end{aligned}$$

Setting $v = t^{-2}u^2$, $\xi = t^{-2}\tau^2$ in (2.3.2), we obtain

$$\psi(t) = \int_0^1 \bigg(\int_{\{x:v(x)=\xi\}} \big[\Phi(x,\nabla v)\big]^p\frac{\mathrm{d}s}{|\nabla v|}\bigg)^{1/(1-p)}\mathrm{d}\xi. \tag{2.3.7}$$

Since $v \in \mathfrak{N}(\mathscr{N}_t,\Omega)$, then by Lemma 2.2.2/1 the right-hand side of (2.3.7) does not exceed

$$\big[(p,\Phi)\text{-cap}(\mathscr{N}_{t(\psi)},\Omega)\big]^{1/(1-p)}.$$

Consequently,

$$\int_0^\infty (p,\Phi)\text{-cap}(\mathscr{N}_t,\Omega)\,\mathrm{d}\big(t^p\big) \leq \int_0^{\psi(T)} \frac{\mathrm{d}[t(\psi)]^p}{\psi^{p-1}} = p\int_0^{\psi(T)}\bigg[\frac{t(\psi)}{\psi}\bigg]^{p-1} t'(\psi)\,\mathrm{d}\psi.$$

Applying the Hölder inequality and the Hardy inequality

$$\int_0^{\psi(T)} \frac{[t(\psi)]^p}{\psi^p}\,\mathrm{d}\psi \le \left(\frac{p}{p-1}\right)^p \int_0^{\psi(T)} \big[t'(\psi)\big]^p\,\mathrm{d}\psi, \tag{2.3.8}$$

we arrive at

$$\int_0^\infty (p,\Phi)\text{-cap}(\mathscr{N}_t,\Omega)\,\mathrm{d}\big(t^p\big) \le \frac{p^p}{(p-1)^{p-1}} \int_0^{\psi(T)} \big[t'(\psi)\big]^p\,\mathrm{d}\psi,$$

which together with Lemma 2.3.1 yields (2.3.6).

To show that the constant factor in the right-hand side of (2.3.6) is sharp, it suffices to put $\Phi(x,y) = |y|$ and $u(x) = f(|x|)$. Then (2.2.13) and (2.3.6) imply

$$\frac{|n-p|^p}{(p-1)^{p-1}} \int_0^\infty \frac{|f(r)|^p}{r^p} r^{n-1}\,\mathrm{d}r \le \frac{p^p}{(p-1)^{p-1}} \int_0^\infty \big|f'(r)\big|^p r^{n-1}\,\mathrm{d}r,$$

which is a particular case of the sharp Hardy inequality (1.3.1). □

Remark 1. The inequality

$$\int_0^\infty (p,\Phi)\text{-cap}(\mathscr{N}_t,\Omega)\,\mathrm{d}\big(t^p\big) \le C \int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x, \tag{2.3.9}$$

with a cruder constant than in (2.3.6) can be proved more simply in the following way. By the monotonicity of capacity, the integral in the left-hand side does not exceed

$$\Xi \stackrel{\text{def}}{=} \big(2^p - 1\big) \sum_{j=-\infty}^{+\infty} 2^{pj} (p,\Phi)\text{-cap}(\mathscr{N}_{2^j},\Omega).$$

Let $\lambda_\varepsilon \in C^\infty(\mathbb{R}^1)$, $\lambda_\varepsilon(t) = 1$ for $t \ge 1$, $\lambda_\varepsilon(t) = 0$ for $t \le 0$, $0 \le \lambda'_\varepsilon(t) \le 1+\varepsilon$, and let

$$u_j(x) = \lambda_\varepsilon\big(2^{1-j}\big|u(x)\big| - 1\big).$$

Since $u_j \in \mathfrak{N}(\mathscr{N}_{2^j},\Omega)$, we have

$$\begin{aligned}
\Xi &\le 2^{p-1} \sum_{j=-\infty}^{\infty} 2^{pj} \int_{\mathscr{N}_{2^{j-1}}\setminus\mathscr{N}_{2^j}} \big[\Phi(x,\nabla u_j)\big]^p\,\mathrm{d}x \\
&\le 2^{2p-1} \sum_{j=-\infty}^{\infty} \int_{\mathscr{N}_{2^{j-1}}\setminus\mathscr{N}_{2^j}} \big[\lambda'_\varepsilon\big(2^{1-j}|u| - 1\big)\big]^p \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x \\
&\le (1+\varepsilon)^p 2^{2p-1} \int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x.
\end{aligned}$$

Letting ε tend to zero, we obtain (2.3.9) with the constant $C = 2^{2p-1}$, which completes the proof.

Remark 2. In fact, the inequality just obtained is equivalent to the following one stronger than (2.3.9) (modulo the best constant)

$$\int_0^\infty (p,\phi)\text{-cap}(\mathscr{N}_{2t},\mathscr{L}_t)\, \mathrm{d}\big(t^p\big) \le c\int_\Omega \big[\Phi(x,\nabla u)\big]^p\, \mathrm{d}x. \tag{2.3.10}$$

Such *conductor inequalities* will be considered in Chap. 3.

2.3.2 Capacity Minimizing Function and Its Applications

Definition. Let $\nu_p(t)$ denote

$$\inf (p,\Phi)\text{-cap}(\bar{g},\Omega),$$

where the infimum is taken over all admissible sets g with

$$\mu(g) \ge t.$$

Note that by Lemma 2.2.5

$$\nu_1(t) = \mathscr{C}(t),$$

where $\mathscr{C}$ is the weighted area minimizing function introduced in Definition 2.1.4.

The following application of the capacity minimizing function ν_p is immediately deduced from the capacitary inequality (2.3.6)

$$\int_0^\infty \nu_p\big(\mu(\mathscr{N}_t)\big)\, \mathrm{d}\big(t^p\big) \le C\int \big[\Phi(x,\nabla u)\big]^p\, \mathrm{d}x, \tag{2.3.11}$$

where $C \ge p^p(p-1)^{1-p}$. Conversely, minimizing the integral in the right-hand side over $\mathfrak{P}(F,\Omega)$, we see that (2.3.11) gives the isocapacitary inequality

$$\nu_p\big(\mu(F)\big) \le C(p,\Phi)\text{-cap}(F,\Omega).$$

If for instance, $\mu = m_n$, then (2.2.8) leads to the estimate

$$\int_0^\infty \bigg(\int_{m_n(\mathscr{N}_t)}^{m_n(\Omega)} \frac{\mathrm{d}\rho}{[\mathscr{C}(\rho)]^{p/(p-1)}}\bigg)^{1-p} \mathrm{d}\big(t^p\big) \le \frac{p^p}{(p-1)^{p-1}}\int_\Omega \big[\Phi(x,\nabla u)\big]^p\, \mathrm{d}x. \tag{2.3.12}$$

In particular, being set into (2.3.6) with $p = n$ and $\Phi(x,\xi) = |\xi|$, the isocapacitary inequality (2.2.11) implies

$$\int_0^\infty \bigg(\log\frac{m_n(\Omega)}{m_n(\mathscr{N}_t)}\bigg)^{1-n} \mathrm{d}\big(t^n\big) \le \frac{n}{(n-1)^n\omega_n}\int_\Omega |\nabla u|^n\, \mathrm{d}x, \tag{2.3.13}$$

for all $u \in C_0^\infty(\Omega)$, where $C = n(n-1)^{1-n}\omega_n^{-1}$.

Clearly, the inequality (2.3.11) and its special cases (2.3.12) and (2.3.13) can be written in terms of the nonincreasing rearrangement u^*_μ of u introduced by (2.1.12):

$$\int_0^{m_n(\Omega)} \bigl[u^*_\mu(s)\bigr]^p \, \mathrm{d}\nu_p(s) \le C \int_\Omega \bigl[\Phi(x, \nabla u)\bigr]^p \, \mathrm{d}x,$$

where C is the same as in (2.3.11). In particular, (2.3.13) takes the form

$$\int_0^{m_n(\Omega)} \bigl[u^*_{m_n}(s)\bigr]^n \frac{\mathrm{d}s}{(\log \frac{m_n(\Omega)}{s})^n} \le C \int_\Omega |\nabla u|^n \, \mathrm{d}x \tag{2.3.14}$$

for all $u \in C_0^\infty(\Omega)$.

2.3.3 Estimate for a Norm in a Birnbaum–Orlicz Space

We recall the definition of a Birnbaum–Orlicz space (see Birnbaum and Orlicz [103], Krasnosel'skiĭ and Rutickiĭ [463], and Rao and Ren [671]).

On the axis $-\infty < u < \infty$, let the function $M(u)$ admit the representation

$$M(u) = \int_0^{|u|} \varphi(t) \, \mathrm{d}t,$$

where $\varphi(t)$ is a nondecreasing function, positive for $t > 0$, and continuous from the right for $t \ge 0$, satisfying the conditions $\varphi(0) = 0$, $\varphi \to \infty$ as $t \to \infty$. Such functions M are sometimes called Young functions. Further, let

$$\psi(s) = \sup\bigl\{t : \varphi(t) \le s\bigr\},$$

be the right inverse of $\varphi(t)$. The function

$$P(u) = \int_0^{|u|} \psi(s) \, \mathrm{d}s$$

is called the complementary function to $M(u)$.

Let $\mathscr{L}_M(\Omega, \mu)$ denote the space of μ-measurable functions for which

$$\|u\|_{\mathscr{L}_M(\Omega,\mu)} = \sup\biggl\{\biggl|\int_\Omega uv \, \mathrm{d}\mu\biggr| : \int_\Omega P(v) \, \mathrm{d}\mu \le 1\biggr\} < \infty.$$

In particular, if $M(u) = q^{-1}|u|^q$, $q > 1$, then $P(u) = (q')^{-1}|u|^{q'}$, $q' = q(q-1)^{-1}$ and

$$\|u\|_{\mathscr{L}_M(\Omega,\mu)} = (q')^{1/q'} \|u\|_{L_q(\Omega,\mu)}.$$

The norm in $\mathscr{L}_M(\Omega, \mu)$ of the characteristic function χ_E of the set E is

$$\|\chi_E\|_{\mathscr{L}_M(\Omega,\mu)} = \mu(E) P^{-1}\biggl(\frac{1}{\mu(E)}\biggr),$$

where P^{-1} is the inverse of the restriction of P to $[0, \infty)$.

In fact, if $v = P^{-1}(1/\mu(E))\chi_E$, then

$$\int_\Omega P(v)\,\mathrm{d}\mu = 1,$$

and the definition of the norm in $\mathscr{L}_M(\Omega, \mu)$ implies

$$\|\chi_E\|_{\mathscr{L}_M(\Omega,\mu)} \geq \int_\Omega \chi_E v\,\mathrm{d}\mu = \mu(E)P^{-1}\big(1/\mu(E)\big).$$

On the other hand, by Jensen's inequality,

$$\int_\Omega \chi_E v\,\mathrm{d}\mu \leq \mu(E)P^{-1}\bigg(\frac{1}{\mu(E)}\int_E P(v)\,\mathrm{d}\mu\bigg),$$

and if we assume that

$$\int_\Omega P(v)\,\mathrm{d}\mu \leq 1,$$

then the definition of the norm in $\mathscr{L}_M(\Omega, \mu)$ yields

$$\|\chi_E\|_{\mathscr{L}_M(\Omega,\mu)} \leq \mu(E)P^{-1}\big(1/\mu(E)\big).$$

Although formally $M(t) = |t|$ does not satisfy the definition of the Birnbaum–Orlicz space, all the subsequent results concerning $\mathscr{L}_M(\Omega, \mu)$ include this case provided we put $P^{-1}(t) = 1$. Then we have $\mathscr{L}_M(\Omega, \mu) = L_1(\Omega, \mu)$.

Theorem. 1. *If there exists a constant β such that for any compactum $F \subset \Omega$*

$$\mu(F)P^{-1}\big(1/\mu(F)\big) \leq \beta(p, \Phi)\text{-cap}(F, \Omega) \tag{2.3.15}$$

with $p \geq 1$, then for all $u \in \mathscr{D}(\Omega)$,

$$\big\||u|^p\big\|_{\mathscr{L}_M(\Omega,\mu)} \leq C\int_\Omega \big[\Phi(x, \nabla u)\big]^p\,\mathrm{d}x, \tag{2.3.16}$$

where $C \leq p^p(p-1)^{1-p}\beta$.

2. *If (2.3.16) is valid for any $u \in \mathscr{D}(\Omega)$, then (2.3.15) holds for all compacta $F \subset \Omega$ with $\beta \leq C$.*

Proof. 1. From Lemma 1.2.3 and the definition of the norm in $\mathscr{L}_M(\Omega, \mu)$ we obtain

$$\begin{aligned}\big\||u|^p\big\|_{\mathscr{L}_M(\Omega,\mu)} &= \sup\bigg\{\int_0^\infty \int_{\mathscr{N}_\tau} v\,\mathrm{d}\mu\,\mathrm{d}\big(\tau^p\big) : \int_\Omega P(v)\,\mathrm{d}\mu \leq 1\bigg\}\\ &\leq \int_0^\infty \sup\bigg\{\int_\Omega \chi_{\mathscr{N}_\tau} v\,\mathrm{d}\mu : \int_\Omega P(v)\,\mathrm{d}\mu \leq 1\bigg\}\,\mathrm{d}\big(\tau^p\big)\\ &= \int_0^\infty \|\chi_{\mathscr{N}_\tau}\|_{\mathscr{L}_m(\Omega,\mu)}\,\mathrm{d}\big(\tau^p\big).\end{aligned}$$

Consequently,

$$\big\| |u|^p \big\|_{\mathscr{L}_M(\Omega,\mu)} \le \int_0^\infty \mu(\mathscr{N}_\tau) P^{-1}\big(1/\mu(\mathscr{N}_\tau)\big)\, \mathrm{d}\big(\tau^p\big).$$

Using (2.3.15) and Theorem 2.3.1, we obtain

$$\begin{aligned} \big\| |u|^p \big\|_{\mathscr{L}_M(\Omega,\mu)} &\le \beta \int_0^\infty (p,\Phi)\text{-cap}(\mathscr{N}_\tau, \Omega)\, \mathrm{d}\big(\tau^p\big) \\ &\le \frac{p^p \beta}{(p-1)^{p-1}} \int_\Omega \big[\Phi(x, \nabla u)\big]^p \, \mathrm{d}x. \end{aligned}$$

2. Let u be any function in $\mathfrak{N}(F,\Omega)$. By (2.3.16),

$$\|\chi_F\|_{\mathscr{L}_M(\Omega,\mu)} \le C \int_\Omega \big[\Phi(x, \nabla u)\big]^p \, \mathrm{d}x.$$

Minimizing the right-hand side over the set $\mathfrak{N}(F,\Omega)$, we obtain (2.3.15). The theorem is proved. □

Remark 1. Obviously the isocapacitary inequality (2.3.15) can be written in terms of the capacity minimizing function ν_p as follows:

$$sP\big(s^{-1}\big) \le \beta\, \nu_p(s).$$

Remark 2. Let $\Phi(x,y)$ be a function satisfying the conditions stated in Sect. 2.1.1 and let the function $\Psi(x,u,y) : \Omega \times \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^1$, satisfy:

(i) the Caratheodory conditions: i.e., Ψ is measurable in x for all x, y, and continuous in x and y for almost all x.

(ii) The inequality

$$\Psi(x,u,y) \ge \big[\Phi(x,y)\big]^p$$

holds.

(iii) For all $u \in \mathscr{D}(\Omega)$

$$\liminf_{\lambda\to+\infty} \lambda^{-p} \int_\Omega \Psi(x, \lambda u, \lambda \nabla u)\, \mathrm{d}x \le K \int_\Omega \big[\Phi(x, \nabla u)\big]^p \, \mathrm{d}x.$$

Then (2.3.16) in the Theorem can be replaced by the following more general estimate:

$$\big\| |u|^p \big\|_{\mathscr{L}_M(\Omega,\mu)} \le C \int_\Omega \Psi(x, u, \nabla u)\, \mathrm{d}x. \tag{2.3.17}$$

As an illustration, note that Theorem 2.1.1 shows the equivalence of the inequality

$$\|u\|_{L_q(\mu)} \le \int_\Omega \sqrt{1 + (\nabla u)^2}\, \mathrm{d}x,$$

where $u \in \mathscr{D}(\Omega)$ and $q \ge 1$, and the isoperimetric inequality $\mu(g)^{1/q} \le \sigma(\partial g)$.

Here, to prove the necessity of (2.3.15) for (2.3.17), we must set $u = \lambda v$, where $v \in \mathfrak{N}(F,\Omega)$, in (2.3.17) and then pass to the limit as $\lambda \to \infty$. An analogous remark can be made regarding Theorems 2.1.1, 2.1.2, and others.

2.3.4 Sobolev Type Inequality as Corollary of Theorem 2.3.3

Theorem 2.3.3 contains the following assertion, which is of interest in itself.

Corollary. 1. *If there exists a constant β such that for any compactum $F \subset \Omega$*

$$\mu(F)^{\alpha p} \le \beta(p, \Phi)\text{-cap}(F, \Omega), \tag{2.3.18}$$

where $p \ge 1$, $\alpha > 0$, $\alpha p \le 1$, then for all $u \in \mathscr{D}(\Omega)$

$$\|u\|^p_{L_q(\Omega,\mu)} \le C \int_\Omega \big[\Phi(x, \nabla u)\big]^p \, \mathrm{d}x, \tag{2.3.19}$$

where $q = \alpha^{-1}$ and $C \le p^p(p-1)^{1-p}\beta$.

2. *If (2.3.19) holds for any $u \in \mathscr{D}(\Omega)$ and if the constant C does not depend on u, then (2.3.18) is valid for all compacta $F \subset \Omega$ with $\alpha = q^{-1}$ and $\beta \le C$.*

Remark. Obviously, the isocapacitary inequality (2.3.18) is equivalent to the weak-type integral inequality

$$\sup_{t>0}\big(t\mu(\mathscr{L}_t)^{1/q}\big) \le C^{1/p}\big\|\Phi(\cdot, \nabla u)\big\|_{L_p(\Omega)} \tag{2.3.20}$$

with $\mathscr{L}_t = \{x : |u(x)| > t\}$ and this, along with the Corollary, can be interpreted as the equivalence of the weak and the strong Sobolev-type estimates (2.3.20) and (2.3.19).

2.3.5 Best Constant in the Sobolev Inequality ($p > 1$)

From the previous corollary and the isoperimetric inequality (2.2.12) we obtain the Sobolev ($p > 1$)-Gagliardo ($p = 1$) inequality

$$\|u\|_{L_{pn/(n-p)}} \le C\|\nabla u\|_{L_p}, \tag{2.3.21}$$

where $n > p \ge 1$, $u \in \mathscr{D}(\mathbb{R}^n)$ and

$$C = p(n-p)^{(1-p)/p}\omega_n^{-1/n} n^{(p-n)/pn}.$$

The value of the constant C in (2.3.21) is sharp only for $p = 1$ (cf. Theorem 1.4.2/1). To obtain the best constant one can proceed in the following way.

By Lemma 2.3.1

$$\int_0^{\psi(\max|u|)} \big[t'(\psi)\big]^p \, \mathrm{d}\psi = \int_{\mathbb{R}^n} |\nabla u|^p \, \mathrm{d}x.$$

Putting

$$\psi = \frac{p-1}{\omega_n^{1/(p-1)}(n-p)} r^{(n-p)/(1-p)}, \qquad t(\psi) = \gamma(r),$$

and assuming $t(\psi) = \text{const}$ for $\psi \geq \psi(\max|u|)$, we obtain

$$\omega_n \int_0^\infty \left|\gamma'(r)\right|^p r^{n-1}\,\mathrm{d}r = \int_{\mathbb{R}^n} |\nabla u|^p\,\mathrm{d}x.$$

Furthermore, by Lemma 1.2.3,

$$\int_{\mathbb{R}^n} |u|^{pn/(n-p)}\,\mathrm{d}x = \int_0^{\max|u|} m_n(\mathscr{N}_t)\,\mathrm{d}\big(t^{pn/(n-p)}\big).$$

The definition of the function $\psi(t)$, Lemma 2.2.3, and the isoperimetric inequality (2.2.9) imply

$$\psi(t) \leq \omega_n^{1/(1-p)} \frac{p-1}{n-p} \left[\frac{n}{\omega_n} m_n(\mathscr{N}_t)\right]^{(n-p)/n(1-p)}.$$

Consequently,

$$m_n(\mathscr{N}_{t(\psi)}) \leq \omega_n n^{-1} r^n$$

and

$$\int_{\mathbb{R}^n} |u|^{pn/(n-p)}\,\mathrm{d}x \leq \frac{\omega_n}{n} \int_0^\infty r^n\,\mathrm{d}\big[\gamma(r)^{pn/(n-p)}\big].$$

Since

$$\int_0^\infty \left|\gamma'(r)\right|^p r^{n-1}\,\mathrm{d}r < \infty,$$

it follows that $\gamma(r)r^{(n-p)/p} \to 0$ as $r \to \infty$. After integration by parts, we obtain

$$\int_{\mathbb{R}^n} |u|^{pn/(n-p)}\,\mathrm{d}x \leq \omega_n \int_0^\infty \big[\gamma(r)\big]^{pn/(n-p)} r^{n-1}\,\mathrm{d}r.$$

Thus,

$$\sup_{u\in\mathscr{D}} \frac{\|u\|_{L_{pn/(n-p)}}}{\|\nabla u\|_{L_p}} = \omega_n^{-1/n} \sup_{\{\gamma\}} \frac{\big(\int_0^\infty [\gamma(r)]^{pn/(n-p)} r^{n-1}\,\mathrm{d}r\big)^{(n-p)/pn}}{\big(\int_0^\infty |\gamma'(r)|^p r^{n-1}\,\mathrm{d}r\big)^{1/p}}, \tag{2.3.22}$$

where $\{\gamma\}$ is the set of all nonincreasing nonnegative functions on $[0,\infty)$ such that $\gamma(r)r^{(n-p)/p} \to 0$ as $r \to \infty$. Thus, we reduced the question of the best constant in (2.3.21) to a one-dimensional variational problem that was solved explicitly by Bliss [109] as early as 1930 by classical methods of the calculus of variations. Paradoxically, the best constant in the Sobolev inequality had been obtained earlier than the inequality itself appeared. The sharp upper bound in (2.3.22) is attained at any function of the form

$$\gamma(r) = \big(a + br^{p/(p-1)}\big)^{1-n/p}, \quad a, b = \text{const} > 0,$$

and equals

$$n^{-1/p}\left(\frac{p-1}{n-p}\right)^{(p-1)/p}\left[\frac{p-1}{p}B\left(\frac{n}{p},\frac{n(p-1)}{p}\right)\right]^{-1/n}.$$

Finally, the sharp constant in (2.3.21) is given by

$$C=\pi^{-1/2}n^{-1/p}\left(\frac{p-1}{n-p}\right)^{(p-1)/p}\left\{\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)}\right\}^{1/n}, \tag{2.3.23}$$

and the equality sign can be written in (2.3.21) if

$$u(x)=\left[a+b|x|^{p/(p-1)}\right]^{1-n/p}, \tag{2.3.24}$$

where a and b are positive constants (although u does not belong to $\mathscr{D}$ it can be approximated by functions in $\mathscr{D}$ in the norm $\|\nabla u\|_{L_p(\mathbb{R}^n)}$).

2.3.6 Multiplicative Inequality (the Case $p \geq 1$)

The following theorem describes conditions for the equivalence of the generalized Sobolev-type inequality (2.3.19) and a multiplicative integral inequality.

We denote by β the best constant in the isocapacitary inequality (2.3.18).

Theorem. 1. *For any compactum $F\subset\Omega$ let the inequality* (2.3.18) *hold with $p\geq 1$, $\alpha>0$. Further, let q be a positive number satisfying one of the conditions* (i) $q\leq q^*=\alpha^{-1}$, *for $\alpha p\leq 1$, or* (ii) $q<q^*=\alpha^{-1}$, *for $\alpha p>1$.*

Then the inequality

$$\|u\|_{L_q(\Omega,\mu)}\leq C\left(\int_\Omega\left[\Phi(x,\nabla u)\right]^p\,\mathrm{d}x\right)^{(1-\varkappa)/p}\|u\|^{\varkappa}_{L_r(\Omega,\mu)} \tag{2.3.25}$$

holds for any $u\in\mathscr{D}(\Omega)$, where $r\in(0,q)$, $\varkappa=r(q^*-q)/q(q^*-r)$, $C\leq c\beta^{(1-\varkappa)/p}$.

2. *Let $p\geq 1$, $0<q^*<\infty$, $r\in(0,q^*]$ and for some $q\in(0,q^*]$ and any $u\in\mathscr{D}(\Omega)$ let the inequality* (2.3.25) *hold with $\varkappa=r(q^*-q)/q(q^*-r)$ and a constant C independent of u.*

Then (2.3.18) *holds for all compacta $F\subset\Omega$ with $\alpha=(q^*)^{-1}$ and $\beta\leq cC^{p/(1-\varkappa)}$.*

Proof. 1. Let $\alpha p\leq 1$. By Hölder's inequality,

$$\int_\Omega|u|^q\,\mathrm{d}\mu=\int_\Omega|u|^{q^*(q-r)/(q^*-r)}|u|^{r(q^*-q)/(q^*-r)}\,\mathrm{d}\mu$$
$$\leq\left(\int_\Omega|u|^{q^*}\,\mathrm{d}\mu\right)^{(q-r)/(q^*-r)}\left(\int_\Omega|u|^r\,\mathrm{d}\mu\right)^{(q^*-q)/(q^*-r)},$$

or equivalently,

$$\|u\|_{L_q(\Omega,\mu)} \leq \|u\|^{1-\varkappa}_{L_{q^*}(\Omega,\mu)} \|u\|^{\varkappa}_{L_r(\Omega,\mu)}.$$

Estimating the first factor by (2.3.19), we obtain (2.3.25) for $\alpha p \leq 1$. Let $\alpha p > 1$. By Lemma 1.2.3,

$$\int_\Omega |u|^q \, \mathrm{d}\mu = q \int_0^\infty \mu(\mathscr{N}_t) t^{q-1} \, \mathrm{d}t.$$

To the last integral we apply inequality (1.3.42), where $x = t^q$, $f(x) = \mu(\mathscr{N}_t)$, $b = p(q^*)^{-1} > 1$, $a > 1$ is an arbitrary number, $\lambda = a(q-r)q^{-1}$, $\mu = p(q^* - q)/q^* q$

$$\int_0^\infty \mu(\mathscr{N}_t) t^{q-1} \, \mathrm{d}t \leq c \Big(\int_0^\infty \big[\mu(\mathscr{N}_t)\big]^a t^{ar-1} \, \mathrm{d}t \Big)^{(q^*-q)/a(q^*-r)} \times \Big(\int_0^\infty \big[\mu(\mathscr{N}_t)\big]^{p/q^*} t^{p-1} \, \mathrm{d}t \Big)^{q^*(q-r)/p(q^*-r)}.$$

Since $a > 1$ and $\mu(\mathscr{N}_t)$ does not increase, we can apply (1.3.41) to the first factor in the following way:

$$\int_0^\infty \big[\mu(\mathscr{N}_t)\big]^a t^{ar-1} \, \mathrm{d}t \leq c \Big(\int_0^\infty \mu(\mathscr{N}_t) t^{r-1} \, \mathrm{d}t \Big)^a.$$

Thus,

$$\|u\|_{L_q(\Omega,\mu)} \leq c \Big(\int_0^\infty \big[\mu(\mathscr{N}_t)\big]^{p/q^*} t^{p-1} \, \mathrm{d}t \Big)^{(1-\varkappa)/p} \|u\|^{\varkappa}_{L_r(\Omega,\mu)}.$$

From condition (2.3.18) and Theorem 2.3.1 we obtain

$$\int_0^\infty \big[\mu(\mathscr{N}_t)\big]^{p/q^*} t^{p-1} \, \mathrm{d}t \leq c\beta \int_\Omega \big[\Phi(x, \nabla u)\big]^p \, \mathrm{d}x.$$

The proof of the first part of the theorem is complete.

2. Let G be a bounded open set $\bar{G} \subset \Omega$. We fix a number $\delta > 0$ and we put

$$\beta_\delta = \sup \frac{\mu(F)^{p\alpha}}{(p,\Phi)\text{-cap}(F, G)}$$

on the set of all compacta F in G satisfying the condition (p,Φ)-cap$(F,G) \geq \delta$. (If (p,Φ)-cap$(F,G) = 0$ for any compactum $F \subset G$, then the substitution of an arbitrary $u \in \mathfrak{N}(F,G)$ into (2.3.25) immediately leads to $\mu = 0$.) Obviously,

$$\beta_\delta \leq \delta^{-1} \mu(G)^{p\alpha} < \infty.$$

Let v be an arbitrary function in $\mathfrak{N}(F,G)$ and let $\gamma = \max(pr^{-1}, q^* r^{-1})$. We substitute the function $u = v^\gamma$ into (2.3.25). Then

$$\mu(F)^{1/q} \le cC\left(\int_\Omega v^{p(\gamma-1)}\big[\Phi(x,\nabla v)\big]^p\,\mathrm{d}x\right)^{(1-\varkappa)/p}\big\|v^\gamma\big\|^{\varkappa}_{L_r(\Omega,\mu)}. \qquad (2.3.26)$$

Let $\psi(t)$ be the function defined in (2.3.2), where u is replaced by v. In our case $T = \max v = 1$. Clearly,

$$\int_G v^{\gamma r}\,\mathrm{d}\mu = \int_0^\infty \mu(\mathscr{N}_t)\,\mathrm{d}(t^{\gamma r}) = \int_0^1 \mu(\mathscr{N}_t)\big[\psi(t)\big]^{q^*/p'}\frac{\mathrm{d}(t^{\gamma r})}{[\psi(t)]^{q^*/p'}},$$

where $\mathscr{N}_t = \{x \in G : v(x) \ge t\}$. Since $\mathscr{N}_t \supset F$, we have by Lemma 2.2.2/3

$$\mu(\mathscr{N}_t)\psi(t)^{q^*/p'} \le \frac{\mu(\mathscr{N}_t)}{[(p,\Phi)\text{-cap}(\mathscr{N}_t,G)]^{q^*/p}} \le \beta_\delta^{q^*/p}.$$

Hence

$$\int_G v^{\gamma r} \le \beta_\delta^{q^*/p}\int_0^1\big[\psi(t)\big]^{-q^*/p'}\,\mathrm{d}\big(t^{\gamma r}\big).$$

Since $[\psi(t)]^{-q^*/p'}$ is a nonincreasing function, from (1.3.41) we obtain

$$\begin{aligned}\int_G v^{\gamma r}\,\mathrm{d}\mu &\le \beta_\delta^{q*/p}\left(\int_0^1\big[\psi(t)\big]^{q^*(1-p)/\gamma r}\,\mathrm{d}\big(t^p\big)\right)^{\gamma r/p}\\ &\le \beta_\delta^{q^*/p}\psi(1)^{(\gamma r-q^*)/p'}\left(\int_0^1\frac{\mathrm{d}(t^p)}{[\psi(t)]^{p-1}}\right)^{\gamma r/p}.\end{aligned}$$

Setting $t = t(\psi)$ in the last integral and applying the inequality (2.3.8) and Lemma 2.3.1, we obtain

$$\int_0^{\psi(1)}\frac{\mathrm{d}[t(\psi)]^p}{\psi^{p-1}} \le c\int_0^{\psi(1)}\big[t'(\psi)\big]^p\,\mathrm{d}\psi = c\int_G\big[\Phi(x,\nabla v)\big]^p\,\mathrm{d}x.$$

Thus,

$$\begin{aligned}\|v^\gamma\|_{L_r(\Omega,\mu)} &\le c\beta_\delta^{q^*/pr}\psi(1)^{(\gamma r-q^*)/rp'}\left(\int_G\big[\Phi(x,\nabla v)\big]^p\,\mathrm{d}x\right)^{\gamma/p}\\ &\le c\beta_\delta^{q^*/pr}\big[(p,\Phi)\text{-cap}(F,G)\big]^{(q^*-\gamma r)/rp}\left(\int_G\big[\Phi(x,\nabla v)\big]^p\,\mathrm{d}x\right)^{\gamma/p}.\end{aligned} \qquad (2.3.27)$$

The last inequality follows from the estimate

$$\big[\psi(1)\big]^{p-1} \le \big[(p,\Phi)\text{-cap}(F,G)\big]^{-1}$$

(see Lemma 2.2.2/3). Since $0 \le v \le 1$ and $\gamma \ge 1$, from (2.3.26) and (2.3.27) it follows that

$$\mu(F)^{1/q} \le cC\beta_\delta^{q^*\varkappa/pr}\big[(p,\Phi)\text{-cap}(F,G)\big]^{\varkappa(q^*-\gamma r)/rp} \times \left(\int_G \big[\Phi(x,\nabla v)\big]^p \,\mathrm{d}x\right)^{[1+\varkappa(\gamma-1)]/p}.$$

Minimizing

$$\int_G \big[\Phi(x,\nabla v)\big]^p \,\mathrm{d}x$$

on the set $\mathfrak{P}(F,G)$, we obtain

$$\begin{aligned}\mu(F)^{1/q} &\le cC\beta_\delta^{q^*\varkappa/pr}\big[(p,\Phi)\text{-cap}(F,G)\big]^{1/p+\varkappa(q^*-r)/pr}\\ &= cC\beta_\delta^{q^*\varkappa/pr}\big[(p,\Phi)\text{-cap}(F,G)\big]^{q^*/qp}.\end{aligned}$$

Hence

$$\mu(F)^{p/q^*} \le cC^{qp/q^*}\beta_\delta^{(q^*-q)/(q^*-r)}(p,\Phi)\text{-cap}(F,G).$$

Consequently,

$$\beta_\delta \le cC^{pq(q^*-r)/q^*(q-r)} = cC^{p/(1-\varkappa)}.$$

Since β_δ is majorized by a constant that depends neither on δ nor G, using the property (iv) of the (p,Φ)-capacity we obtain $\beta \le cC^{p/(1-\varkappa)}$. The theorem is proved. □

Remark. The theorem just proved shows, in particular, the equivalence of the multiplicative inequality (2.3.25) and the Sobolev-type inequality (2.3.19).

2.3.7 Estimate for the Norm in $L_q(\Omega,\mu)$ with $q < p$ (First Necessary and Sufficient Condition)

A characterization of (2.3.19) with $q \ge p$ was stated in Corollary 2.3.4. Now we obtain a condition for the validity of (2.3.19), which is sufficient if $p > q > 0$ and necessary if $p > q \ge 1$.

Definition. Let $S = \{g_j\}_{j=-\infty}^{\infty}$ be any sequence of admissible subsets of Ω with $\bar g_i \subset g_{i+1}$. We put $\mu_i = \mu(g_i)$, $\gamma_i = (p,\Phi)\text{-cap}(\bar g_i, g_{i+1})$, and

$$\varkappa = \sup_{\{S\}}\left[\sum_{i=-\infty}^{\infty}\left(\frac{\mu_i^{p/q}}{\gamma_i}\right)^{q/(p-q)}\right]^{(p-q)/q}. \qquad (2.3.28)$$

(The terms of the form $0/0$ are considered to be zeros.)

Theorem. (i) *If $\varkappa < \infty$, then*

$$\|u\|^p_{L_q(\Omega,\mu)} \le C\int_\Omega \big[\Phi(x,\nabla u)\big]^p \,\mathrm{d}x, \qquad (2.3.29)$$

where $u \in \mathscr{D}(\Omega)$ and $p > q > 0$, $C \le c\varkappa$.

(ii) *If there exists a constant C such that* (2.3.29) *holds for all* $u \in \mathscr{D}(\Omega)$ *with* $p > q \geq 1$, *then* $\varkappa \leq cC$.

Proof. (i) Let $t_j = 2^{-j} + \varepsilon_j$, $j = 0, \pm 1, \pm 2, \dots$, where ε_j is a decreasing sequence of positive numbers satisfying $\varepsilon_j 2^j \to 0$ as $j \to \pm\infty$. We assume further that the sets $\mathscr{L}_{t_j}$ are admissible. Obviously,

$$\|u\|^q_{L_q(\Omega,\mu)} = \sum_{j=-\infty}^{\infty} \int_{t_j}^{t_{j-1}} \mu(\mathscr{L}_t)\,\mathrm{d}\big(t^q\big) \leq c \sum_{j=-\infty}^{\infty} 2^{-qj} \mu(\mathscr{L}_{t_j}).$$

Let $g_j = \mathscr{L}_{t_j}$. We rewrite the last sum as

$$c \sum_{j=-\infty}^{\infty} \left(\frac{\mu_j^{p/q}}{\gamma_j}\right)^{q/p} \big(2^{-pj}\gamma_j\big)^{q/p}$$

and apply Hölder's inequality. Then

$$\|u\|^q_{L_q(\Omega,\mu)} \leq c\varkappa^{q/p} \left(\sum_{j=-\infty}^{\infty} 2^{-pj}\gamma_j\right)^{q/p}.$$

Let $\lambda_\varepsilon \in C^\infty(\mathbb{R}^1)$, $\lambda_\varepsilon(t) = 1$ for $t \geq 1$, $\lambda_\varepsilon(t) = 0$ for $t \leq 0$, $0 \leq \lambda'_\varepsilon(t) \leq 1 + \varepsilon$, $(\varepsilon > 0)$ and let

$$u_j(x) = \lambda_\varepsilon\left[\frac{|u(x)| - t_{j+1}}{t_j - t_{j+1}}\right].$$

Since $u_j \in \mathfrak{N}(\bar{g}_j, g_{j+1})$, it follows that

$$\begin{aligned}\sum_{j=-\infty}^{\infty} 2^{-pj}\gamma_j &\leq c \sum_{j=-\infty}^{\infty} (t_j - t_{j+1})^p \int_{g_{j+1}\setminus g_j} \big[\Phi(x, \nabla u_j)\big]^p \,\mathrm{d}x \\ &= c \sum_{j=-\infty}^{\infty} \int_{g_{j+1}\setminus g_j} \left[\lambda'_\varepsilon\left(\frac{u - t_{j+1}}{t_j - t_{j+1}}\right)\right]^p \big[\Phi(x, \nabla u)\big]^p \,\mathrm{d}x.\end{aligned}$$

Letting ε tend to zero, we obtain

$$\sum_{j=-\infty}^{\infty} 2^{-pj}\gamma_j \leq c \int_\Omega \big[\Phi(x, \nabla u)\big]^p \,\mathrm{d}x. \tag{2.3.30}$$

(ii) We introduce the sequence

$$S = \{g_j\}_{j=-\infty}^{\infty}$$

and put $\tau_{N+1} = 0$ and

$$\tau_k = \sum_{j=k}^{N} \left(\frac{\mu_j}{\gamma_j}\right)^{1/(p-q)}$$

for $k = -N, -N+1, \dots, 0, \dots, N-1, N$. By u_k we denote an arbitrary function in $\mathfrak{P}(\bar{g}_k, g_{k+1})$ and define the function

$$u_k = (\tau_k - \tau_{k+1})u_k + \tau_{k+1} \quad \text{on } g_{k+1} \backslash g_k,$$

$$u = \tau_{-N} \quad \text{on } g_{-N}, \qquad u = 0 \quad \text{on } \Omega \backslash g_{N+1}.$$

Since $u \in \mathscr{D}(\Omega)$, it satisfies (2.3.29). Obviously,

$$\int_\Omega u^q \, \mathrm{d}\mu = v \int_0^\infty \mu(\mathscr{L}_t) \, \mathrm{d}\big(t^q\big)$$

$$= \sum_{k=-N}^{N} \int_{\tau_{k+1}}^{\tau_k} \mu(\mathscr{L}_t) \, \mathrm{d}\big(t^q\big) \geq \sum_{k=-N}^{N} \mu_k \big(\tau_k^q - \tau_{k+1}^q\big).$$

Therefore, (2.3.29) and the inequality $(\tau_k - \tau_{k+1})^q \leq (\tau_k^q - \tau_{k+1}^q)$ implies

$$\left[\sum_{k=-N}^{N} \mu_k (\tau_k - \tau_{k+1})^q\right]^{p/q} \leq C \sum_{k=-N}^{N} \int_{g_{k+1} \backslash g_k} \big[\Phi(x, \nabla u_k)\big]^p \, \mathrm{d}x$$

$$= C \sum_{k=-N}^{N} (\tau_k - \tau_{k+1})^p \int_{g_{k+1} \backslash g_k} \big[\Phi(x, \nabla u_k)\big]^p \, \mathrm{d}x.$$

Since u_k is an auxiliary function in $\mathfrak{P}(\bar{g}_k, g_{k+1})$, it follows by minimizing the last sum that

$$\left[\sum_{k=-N}^{N} \mu(\tau_k - \tau_{k+1})^q\right]^{p/q} \leq C \sum_{k=-N}^{N} (\tau_k - \tau_{k+1})^p \gamma_k.$$

Putting here

$$\tau_k - \tau_{k+1} = \mu_k^{1/(p-q)} \gamma_k^{1/(q-p)},$$

we arrive at the result

$$\left|\sum_{k=-N}^{N} \big(\mu_k^{p/q} \gamma_k\big)^{q/(p-q)}\right|^{(p-q)/q} \leq C.$$

□

2.3.8 Estimate for the Norm in $L_q(\Omega, \mu)$ with $q < p$ (Second Necessary and Sufficient Condition)

Lemma. *Let g_1, g_2, and g_3 be admissible subsets of Ω such that $\bar{g}_1 \subset g_2$, $\bar{g}_2 \subset g_3$. We set*

$$\gamma_{ij} = (p, \Phi)\text{-cap}(\bar{g}_i, g_j),$$

where $i < j$. Then

$$\gamma_{12}^{-1/(p-1)} + \gamma_{23}^{-1/(p-1)} \le \gamma_{13}^{-1/(p-1)}.$$

Proof. Let ε be any positive number. We choose functions $u_k \in \mathfrak{P}(\bar{g}_k, g_{k+1})$, $k = 1, 2$, so that

$$\gamma_{k,k+1}^{-1/(p-1)} \le \int_0^1 \left[\int_{\mathscr{E}_\tau^k} \big[\Phi(x, \nabla u_k)\big]^p \frac{\mathrm{d}s}{|\nabla u_k|} \right]^{-1/(p-1)} \mathrm{d}\tau + \varepsilon,$$

where $\mathscr{E}_\tau^k = \{x : u_k(x) = \tau\}$. We put $u(x) = \frac{1}{2} u_2(x)$ for $x \in g_3 \backslash g_2$ and $u(x) = (u_1(x) + 1)/2$ for $x \in g_2$. Then

$$\begin{aligned}
&\int_0^1 \left[\int_{\mathscr{E}_\tau^1} \big[\Phi(x, \nabla u_1)\big]^p \frac{\mathrm{d}s}{|\nabla u_1|} \right]^{1/(1-p)} \mathrm{d}\tau \\
&\qquad = \int_{1/2}^1 \left[\int_{\mathscr{E}_\tau} \big[\Phi(x, \nabla u)\big]^p \frac{\mathrm{d}s}{|\nabla u|} \right]^{1/(1-p)} \mathrm{d}\tau, \\
&\int_0^1 \left[\int_{\mathscr{E}_\tau^2} \big[\Phi(x, \nabla u_2)\big]^p \frac{\mathrm{d}s}{|\nabla u_2|} \right]^{1/(1-p)} \mathrm{d}\tau \\
&\qquad = \int_0^{1/2} \left[\int_{\mathscr{E}_\tau} \big[\Phi(x, \nabla u)\big]^p \frac{\mathrm{d}s}{|\nabla u|} \right]^{1/(1-p)} \mathrm{d}\tau,
\end{aligned}$$

where $\mathscr{E}_\tau = \{x : u(x) = \tau\}$. Therefore,

$$\gamma_{12}^{1/(1-p)} + \gamma_{23}^{1/(1-p)} \le \int_0^1 \left(\int_{\mathscr{E}_\tau} \big[\Phi(x, \nabla u)\big]^p \frac{\mathrm{d}s}{|\nabla u|} \right)^{1/(1-p)} \mathrm{d}\tau + 2\varepsilon.$$

Since $u \in \mathfrak{P}(\bar{g}_1, g_3)$, by Lemma 2.2.2/3 the right-hand side of the last inequality does not exceed $\gamma_{13}^{1/(1-p)} + 2\varepsilon$. The lemma is proved. □

Let ν_p be the capacity minimizing function introduced in Definition 2.3.2. It can be easily checked that condition (2.3.15) is equivalent to

$$\beta \nu_p(t) \ge t P^{-1}(1/t)$$

and condition (2.3.18) to

$$\beta \nu_p(t) \ge t^{\alpha p}.$$

The theorem of the present subsection yields the following necessary and sufficient condition for the validity of (2.3.29) with $p > q > 0$:

$$K = \int_0^{\mu(\Omega)} \left[\frac{\tau}{\nu_p(\tau)} \right]^{q/(p-q)} \mathrm{d}\tau < \infty. \tag{2.3.31}$$

Theorem. *Let $p > q > 0$, $p > 1$.*
1. *If $K < \infty$, then (2.3.29) holds for all $u \in \mathscr{D}(\Omega)$ with $C \le cK^{(p-q)/q}$.*
2. *If (2.3.29) holds with $C < \infty$, then (2.3.31) is valid with $K^{(p-q)/q} \le cC$.*

Proof. 1. By Theorem 2.3.7 it suffices to prove the inequality

$$\sup_{\{S\}} \sum_{j=-\infty}^{\infty} \left(\frac{\mu_j^{p/q}}{\gamma_j} \right)^{q/(p-q)} \le \frac{p}{p-q} \int_0^{\mu(\Omega)} \left[\frac{\tau}{\nu_p(\tau)} \right]^{q/(p-q)} \mathrm{d}\tau, \tag{2.3.32}$$

where the notation of Sect. 2.3.7 is retained.

Let the integral in the right-hand side converge, let N be a positive integer, and let $\varGamma_j = (p, \varPhi)\text{-cap}(\bar{g}_j, g_{N+1})$ for $j \le N$, $\varGamma_{N+1} = \infty$. By the Lemma,

$$\gamma_j^{1/(1-p)} \le \varGamma_j^{1/(1-p)} - \varGamma_{j+1}^{1/(1-p)}, \quad j \le N.$$

Since $q(p-1)/(p-q) \ge 1$, then

$$|a-b|^{q(p-1)/(p-q)} \le \left| a^{q(p-1)/(p-q)} - b^{q(p-1)/(p-q)} \right|$$

and hence

$$\gamma_j^{-q/(p-q)} \le \varGamma_j^{-q/(p-q)} - \varGamma_{j+1}^{-q/(p-q)}.$$

This implies

$$\begin{aligned}
\sigma_N \overset{\text{def}}{=} \sum_{j=-N}^{N} \left(\frac{\mu_j^{p/q}}{\gamma_j} \right)^{q/(p-q)} &\le \sum_{j=-N}^{N} \mu_j^{p/(p-q)} \left(\varGamma_j^{-q/(p-q)} - \varGamma_{j+1}^{-q/(p-q)} \right) \\
&\le \sum_{j=-N+1}^{N} \left(\mu_j^{p/(p-q)} - \mu_{j-1}^{p/(p-q)} \right) \varGamma_j^{-q/(p-q)} + \mu_{-N}^{p/(p-q)} \varGamma_{-N}^{-q/(p-q)}.
\end{aligned}$$

It is clear that $\varGamma_j \ge (p, \varPhi)\text{-cap}(\bar{g}_j, \Omega) \ge \nu_p(\mu_j)$. Since the function ν_p does not decrease then

$$\mu_{-N}^{p/(p-q)} \left[\nu_p(\mu_{-N}) \right]^{q/(p-q)} \le \int_0^{\mu_{-N}} \frac{\mathrm{d}(\tau^{p/(p-q)})}{[\nu_p(\tau)]^{q/(p-q)}}.$$

Similarly,

$$\left(\mu_j^{p/(p-q)} - \mu_{j-1}^{p/(p-q)} \right) \left[\nu_p(\mu_j) \right]^{q/(p-q)} \le \int_{\mu_{j-1}}^{\mu_j} \frac{\mathrm{d}(\tau^{p/(p-q)})}{[\nu_p(\tau)]^{q/(p-q)}}.$$

Consequently,

$$\sigma_N \le \int_0^{\mu_N} \left[\nu_p(\tau) \right]^{q/(q-p)} \mathrm{d}\left(\tau^{p/(p-q)} \right).$$

The result follows.

2. With a μ-measurable function f we connect its nonincreasing rearrangement

$$f_\mu^*(t) = \inf\left\{ s : \mu\left\{ x \in \Omega : f(x) \ge s \right\} \le t \right\}. \tag{2.3.33}$$

By Lemma 2.1.4/1

$$\|f\|_{L_q(\Omega,\mu)} = \left(\int_0^{\mu(\Omega)} \bigl(f_\mu^*(t)\bigr)^q \,\mathrm{d}t\right)^{1/q}, \quad 0 < q < \infty. \tag{2.3.34}$$

We note that the inequality (2.3.29) implies that $\mu(\Omega) < \infty$ and that $\nu_p(t) > 0$ for all $t \in (0, \mu(\Omega)]$. Let l be any integer satisfying $2^l \le \mu(\Omega)$. We introduce an admissible subset g_l of Ω such that

$$\mu(\bar{g}_l) \ge 2^l, \qquad (p,\Phi)\text{-cap}(g_l,\Omega) \le 2\nu_p\bigl(2^l\bigr).$$

By u_l we denote a function in $\mathfrak{P}(\bar{g}_l, \Omega)$ satisfying

$$\int_\Omega \bigl[\Phi(x, \nabla u_l)\bigr]^p \,\mathrm{d}x \le 4\nu_p\bigl(2^l\bigr). \tag{2.3.35}$$

Let s be the integer for which $2^s \le \mu(\Omega) < 2^{s+1}$. We define the function in $C_0^{0,1}(\Omega)$

$$f_{r,s}(x) = \sup_{r \le l \le s} \beta_l u_l(x), \quad x \in \Omega,$$

where $r < s$ and the values β_l are defined by

$$\beta_l = \left(\frac{2^l}{\nu_p(2^l)}\right)^{1/(p-q)}.$$

By Lemma 2.1.4/2 and by Lemma 2.1.4/3 with Φ^p instead of Φ we have

$$\int_\Omega \bigl[\Phi(x, \nabla f_{r,s})\bigr]^p \,\mathrm{d}x \le \sum_{l=r}^s \beta_l^p \int_\Omega \bigl[\Phi(x, \nabla u_l)\bigr]^p \,\mathrm{d}x.$$

By (2.3.35) the right-hand side is majorized by

$$c \sum_{l=r}^s \beta_l^p \nu_p\bigl(2^l\bigr).$$

Now we obtain a lower estimate for the norm of $f_{r,s}$ in $L_q(\Omega, \mu)$. Since $f_{r,s}(x) \ge \beta_l$ on the set g_l, $r \le l \le s$, and $\mu(\bar{g}_l) \ge 2^l$, the inequality

$$\mu\bigl(\bigl\{x \in \Omega : \bigl|f_{r,s}(x)\bigr| > \tau\bigr\}\bigr) < 2^l$$

implies $\tau \ge \beta_l$. Hence

$$f_{r,s}^*(t) \ge \beta_l \quad \text{for } t \in \bigl(0, 2^l\bigr), r \le l \le s. \tag{2.3.36}$$

By (2.3.34) and (2.3.36)

$$\|f_{r,s}\|_{L_q(\Omega,\mu)}^q = \int_0^{\mu(\Omega)} \bigl((f_{r,s})_\mu^*(t)\bigr)^q \,\mathrm{d}t \ge c \sum_{l=r}^s \bigl((f_{r,s})_\mu^*(2^l)\bigr)^q 2^l \ge c \sum_{l=r}^s \beta_l^q 2^l.$$

Therefore,

$$B := \sup \frac{\|u\|_{L_q(\Omega,\mu)}}{(\int_\Omega [\Phi(x,\nabla u)]^p \,\mathrm{d}x)^{1/p}} \geq c \frac{(\sum_{l=r}^s \beta_l^q 2^l)^{1/q}}{(\sum_{l=r}^s \beta_l^p \nu_p(2^l))^{1/p}}$$

$$= c\left(\sum_{l=r}^s \frac{2^{lp/(p-q)}}{\nu_p(2^l)^{q/(p-q)}}\right)^{1/q-1/p}.$$

With $r \to -\infty$ we obtain

$$B \geq \left(\sum_{l=-\infty}^s \frac{2^{lp/(p-q)}}{\nu_p(2^l)^{q/(p-q)}}\right)^{1/q-1/p}$$

$$\geq c\left(\int_0^{2^{s-1}} \sum_{l=-\infty}^s \frac{t^{p/(p-q)}}{(\nu_p(t))^{q/(p-q)}} \frac{\mathrm{d}t}{t}\right)^{1/q-1/p}.$$

Hence by monotonicity of ν_p we obtain

$$B \geq c\left(\int_0^{\mu(\Omega)} \frac{t^{p/(p-q)}}{(\nu_p(t))^{q/(p-q)}} \frac{\mathrm{d}t}{t}\right)^{1/q-1/p}.$$

The proof is complete. □

We give a sufficient condition for inequality (2.3.29) with $\mu = m_n$ formulated in terms of the weighted isoperimetric function $\mathscr{C}$ introduced in Definition 2.2.3.

Corollary. *If $p > q > 0$, $p > 1$, and*

$$\int_0^{m_n(\Omega)} \left(\int_t^{m_n(\Omega)} \frac{\mathrm{d}\varrho}{(\mathscr{C}(\varrho))^{p/(p-1)}}\right)^{\frac{q(p-1)}{p-q}} t^{\frac{q}{p-q}} \,\mathrm{d}t < \infty,$$

then (2.3.29) with $\mu = m_n$ and any $u \in \mathscr{D}(\Omega)$ holds.

Proof. The result follows directly from the last Theorem and Corollary 2.7.2. □

2.3.9 Inequality with the Norms in $L_q(\Omega,\mu)$ and $L_r(\Omega,\nu)$ (the Case $p \geq 1$)

The next theorem gives conditions for the validity of the inequality

$$\|u\|_{L_q(\Omega,\mu)}^p \leq C\left(\int_\Omega \left[\Phi(x,\nabla u)\right]^p \mathrm{d}x + \|u\|_{L_r(\Omega,\nu)}^p\right) \tag{2.3.37}$$

for all $u \in \mathscr{D}(\Omega)$ with $q \geq p \geq r$, $p > 1$ (compare with Theorem 2.1.3).

Theorem. *Inequality* (2.3.37) *holds if and only if*

$$\mu(g)^{p/q} \le cC\big[(p,\Phi)\text{-cap}(\bar g, \mathscr{G}) + \big[\nu(\mathscr{G})\big]^{p/r}\big] \tag{2.3.38}$$

for all admissible sets g *and* $\mathscr{G}$ *with* $\bar g \subset \mathscr{G}$.

Proof. Sufficiency. By Lemma 1.2.3 and inequality (1.3.41),

$$\begin{aligned}\|u\|^p_{L_q(\Omega,\mu)} &= \Big[\int_0^\infty \mu(\mathscr{L}_t)\,\mathrm{d}\big(t^q\big)\Big]^{p/q}\\ &\le \int_0^\infty \big[\mu(\mathscr{L}_t)\big]^{p/q}\,\mathrm{d}\big(t^p\big) \le c\sum_{j=-\infty}^{\infty} 2^{-pj}\mu(g_j)^{p/q},\end{aligned}$$

where $g_j = \mathscr{L}_{t_j}$ and $\{t_j\}$ is the sequence of levels defined in the proof of part (i) of Theorem 2.3.7. We set $\gamma_j = (p,\Phi)\text{-cap}(\bar g_j, g_{j+1})$ and using the condition (2.3.38), we arrive at the inequality:

$$\|u\|^p_{L_q(\Omega,\mu)} \le cC\Big[\sum_{j=-\infty}^{\infty} 2^{-pj}\gamma_j + \sum_{j=-\infty}^{\infty} 2^{-pj}\nu(g_j)^{p/r}\Big]. \tag{2.3.39}$$

We can estimate the first sum on the right-hand side of this inequality by means of (2.3.30). The second sum does not exceed

$$c\int_0^\infty \big[\nu(\mathscr{L}_t)\big]^{p/r}\,\mathrm{d}\big(t^p\big) \le c\Big(\int_0^\infty \nu(\mathscr{L}_t)\,\mathrm{d}\big(t^r\big)\Big)^{p/r} = c\|u\|^p_{L_r(\Omega,\nu)}.$$

Necessity. Let g and $\mathscr{G}$ be admissible and let $\bar g \subset \mathscr{G}$. We substitute any function $u \in \mathfrak{P}(\bar g, \mathscr{G})$ into (2.3.37). Then

$$\mu(g)^{p/q} \le C\Big[\int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x + \nu(\mathscr{G})^{p/r}\Big].$$

Minimizing the first term on the right of the set $\mathfrak{P}(\bar g, G)$, we obtain (2.3.38). □

Remark. Obviously, a sufficient condition for the validity of (2.3.37) is the inequality

$$\mu(g)^{p/q} \le C_1\big[(p,\Phi)\text{-cap}(g,\Omega) + \nu(g)^{p/r}\big], \tag{2.3.40}$$

which is simpler than (2.3.38). In contrast to (2.3.37) it contains only one set g. However, as the following example shows, the last condition is not necessary.

Let $\Omega = \mathbb{R}^3$, $q = p = r = 2$, $\Phi(x,y) = |y|$, and let the measures μ and ν be defined as follows:

$$\begin{aligned}\mu(A) &= \sum_{k=0}^{\infty} s(A \cap \partial B_{2^k}),\\ \nu(A) &= \sum_{k=0}^{\infty} s(A \cap \partial B_{2^k+1}),\end{aligned}$$

where A is any Borel subset of $\mathbb{R}^3$ and s is a two-dimensional Hausdorff measure. The condition (2.3.40) is not fulfilled for these measures and the 2-capacity. Indeed, for the sets $g_k = B_{2^k+1} \backslash \bar{B}_{2^k-1}$, $k = 2, 3, \ldots$, we have $\mu(g_k) = \pi 4^{k+1}$, $\nu(g_k) = 0$, $2\text{-cap}(g_k, \mathbb{R}^3) = 4\pi(2^k + 1)$.

We shall show that (2.3.37) is true. Let $u \in \mathscr{D}(\mathbb{R}^3)$ and let (ϱ, ω) be spherical coordinates with center O. Obviously,

$$\big[u\big(2^k, \omega\big)\big]^2 \le 2 \int_{2^k}^{2^k+1} \left(\frac{\partial u}{\partial \varrho}(\varrho, \omega)\right)^2 \mathrm{d}\varrho + 2\big[u\big(2^k + 1, \omega\big)\big]^2.$$

Hence

$$4^k \int_{\partial B_1} \big[u\big(2^k, \omega\big)\big]^2 \mathrm{d}\omega \le 2 \int_{B_{2^k+1} \backslash B_{2^k}} \left(\frac{\partial u}{\partial \varrho}\right)^2 \mathrm{d}x + 2 \cdot 4^k \int_{\partial B_1} \big[u\big(2^k + 1, \omega\big)\big]^2 \mathrm{d}\omega.$$

Summing over k, we obtain

$$\int_{\mathbb{R}^3} u^2 \,\mathrm{d}\mu \le c\left(\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} u^2 \,\mathrm{d}\nu\right).$$

The proof is complete.

2.3.10 Estimate with a Charge σ on the Left-Hand Side

The following assertion yields a condition close in a certain sense to being necessary and sufficient for the validity of the inequality

$$\int_\Omega |u|^p \,\mathrm{d}\sigma \le c \int_\Omega \big[\Phi(x, \nabla u)\big]^p \,\mathrm{d}x, \quad u \in \mathscr{D}(\Omega), \tag{2.3.41}$$

where σ is an arbitrary charge in Ω, not a nonnegative measure as in Theorem 2.3.6. (Theorem 2.1.3 contains a stronger result for $p = 1$.)

Theorem. *Let σ^+ and σ^- be the positive and negative parts of the charge σ.*

1. *If for some $\varepsilon \in (0, 1)$ and for all admissible sets g and $\mathscr{G}$ with $\bar{g} \subset \mathscr{G}$ we have the inequality*

$$\sigma^-(g) \le C_\varepsilon(p, \Phi)\text{-cap}(\bar{g}, G) + (1 - \varepsilon)\sigma^-(G), \tag{2.3.42}$$

where $C_\varepsilon = \text{const}$, then (2.3.41) is valid with $C \le cC_\varepsilon$.

2. *If for all $u \in \mathscr{D}(\Omega)$ inequality (2.3.41) holds, then*

$$\sigma^+(g) \le C(p, \Phi)\text{-cap}(\bar{g}, \mathscr{G}) + \sigma^-(\mathscr{G}) \tag{2.3.43}$$

for all admissible sets g and $\mathscr{G}$, $\bar{g} \subset \mathscr{G}$.

Proof. Let $\delta = (1 - \varepsilon)^{-1/2p}$ and $g_j = \mathscr{L}_{\delta^j}$, $j = 0, \pm 1, \ldots$. By Lemma 1.2.3,

$$\|u\|^p_{L_p(\Omega,\sigma^+)} = \int_0^\infty \sigma^+(\mathscr{L}_t)\,\mathrm{d}\big(t^p\big) = \sum_{j=-\infty}^{\infty} \int_{\delta^j}^{\delta^{j+1}} \sigma^+(\mathscr{L}_t)\,\mathrm{d}\big(t^p\big)$$
$$\le \sum_{j=-\infty}^{\infty} \sigma^+(\mathscr{L}_{\delta^j})\big(\delta^{(j+1)p} - \delta^{jp}\big).$$

This and (2.3.42) imply

$$\|u\|^p_{L_p(\Omega,\sigma^+)} \le C_\varepsilon \sum_{j=-\infty}^{\infty} (p,\Phi)\text{-cap}(\bar{\mathscr{L}}_{\delta^j}, \mathscr{L}_{\delta^{j-1}})\big(\delta^{(j+1)p} - \delta^{jp}\big)$$
$$+ (1-\varepsilon) \sum_{j=-\infty}^{\infty} \sigma^-(\mathscr{L}_{\delta^{j-1}})\big(\delta^{(j+1)p} - \delta^{jp}\big). \qquad (2.3.44)$$

Using the same arguments as in the derivation of (2.3.30), we obtain that the first sum in (2.3.44) does not exceed

$$\frac{(\delta^p - 1)\delta^p}{(\delta - 1)^p} \int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x.$$

Since $\sigma^-(\mathscr{L}_t)$ is a nondecreasing function, then

$$\big(\delta^{(j-1)p} - \delta^{(j-2)p}\big)\sigma^-(\mathscr{L}_{\delta^{j-1}}) \le \int_{\delta^{j-2}}^{\delta^{j-1}} \sigma^-(\mathscr{L}_t)\,\mathrm{d}\big(t^p\big)$$

and hence

$$\sum_{j=-\infty}^{\infty} \sigma^-(\mathscr{L}_{\delta^{j-1}})\big(\delta^{(j+1)p} - \delta^{jp}\big) \le \delta^{2p} \int_0^\infty \sigma^-(\mathscr{L}_t)\,\mathrm{d}\big(t^p\big).$$

Thus

$$\|u\|^p_{L_p(\Omega,\sigma^+)} \le C_\varepsilon \frac{(\delta^p - 1)\delta^p}{(\delta - 1)^p} \int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x + \delta^{2p}(1-\varepsilon)\|u\|^p_{L_p(\Omega,\sigma^+)}.$$

It remains to note that $\delta^{2p}(1-\varepsilon) = 1$.

2. The proof of the second part of the theorem is the same as that of necessity in Theorem 2.3.9. The theorem is proved. □

2.3.11 Multiplicative Inequality with the Norms in $L_q(\Omega,\mu)$ and $L_r(\Omega,\nu)$ (Case $p \ge 1$)

The following assertion gives a necessary and sufficient condition for the validity of the multiplicative inequality

$$\|u\|^p_{L_q(\Omega,\mu)} \le C\Big\{\int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x\Big\}^\delta \|u\|^{p(1-\delta)}_{L_r(\Omega,\nu)} \qquad (2.3.45)$$

for $p \geq 1$ (cf. Theorem 2.1.1).

Theorem. 1. *Let g and $\mathscr{G}$ be any admissible sets such that $\bar{g} \subset \mathscr{G}$. If a constant α exists such that*

$$\mu(g)^{p/q} \leq \alpha\big[(p,\Phi)\text{-cap}(\bar{g},\mathscr{G})\big]^{\delta}\nu(\mathscr{G})^{(1-\delta)p/r}, \tag{2.3.46}$$

then (2.3.45) holds for all functions $u \in \mathscr{D}(\Omega)$ with $C \leq c\alpha$, $1/q \leq (1-\delta)/r + \delta/p$, $r, q > 0$.

2. *If (2.3.45) is true for all $u \in \mathscr{D}(\Omega)$, then (2.3.46) holds for all admissible sets g and $\mathscr{G}$ such that $\bar{g} \subset \mathscr{G}$. The constant α in (2.3.46) satisfies $\alpha \leq C$.*

Proof. 1. By Lemma 1.2.3 and inequality (1.3.41),

$$\|u\|_{L_q(\Omega,\mu)} = \left[\int_0^\infty \mu(\mathscr{L}_\tau)\,\mathrm{d}\big(\tau^q\big)\right]^{1/q} \leq \gamma^{1/\gamma}\left[\int_0^\infty \mu(\mathscr{L}_\tau)^{\gamma/q}\tau^{\gamma-1}\,\mathrm{d}\tau\right]^{1/\gamma},$$

where $\gamma = pr[p(1-\delta)+\delta r]^{-1}$, $\gamma \leq q$. Consequently,

$$\begin{aligned}\|u\|^p_{L_q(\Omega,\mu)} &\leq c\left[\sum_{j=-\infty}^{\infty} 2^{-\gamma j}\mu(g_j)^{\gamma/q}\right]^{p/\gamma}\\ &\leq c\alpha\left\{\sum_{j=-\infty}^{\infty} 2^{-\gamma j}\big[(p,\Phi)\text{-cap}(\bar{g}_j, g_{j+1})\big]^{\delta\gamma/p}\nu(g_{j+1})^{(1-\delta)\gamma/r}\right\}^{p/\gamma},\end{aligned}$$

where $g_j = \mathscr{L}_{t_j}$ and $\{t_j\}$ is the sequence of levels defined in the proof of the first part of Theorem 2.3.7. Hence,

$$\begin{aligned}\|u\|^p_{L_q(\Omega,\mu)} \leq c\alpha&\left[\sum_{j=-\infty}^{\infty} 2^{-pj}(p,\Phi)\text{-cap}(\bar{g}_j, g_{j+1})\right]^{\delta}\\ &\times\left[\sum_{j=-\infty}^{\infty} 2^{-rj}\nu(g_{j+1})\right]^{(1-\delta)p/r}.\end{aligned} \tag{2.3.47}$$

By (2.3.30),

$$\sum_{j=-\infty}^{\infty} 2^{-pj}(p,\Phi)\text{-cap}(\bar{g}_j, g_{j+1}) \leq c\int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x.$$

Obviously, the second sum in (2.3.47) does not exceed $c\|u\|^r_{L_r(\Omega,\nu)}$. Thus (2.3.45) follows.

2. Let g and $\mathscr{G}$ be admissible sets with $\bar{g} \subset \mathscr{G}$. We substitute any function $u \in \mathfrak{P}(\bar{g},\mathscr{G})$ into (2.3.45). Then

$$\mu(g)^{p/q} \leq C\left[\int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x\right]^{\delta}\nu(\mathscr{G})^{(1-\delta)p/r},$$

which yields (2.3.46). The theorem is proved. □

2.3.12 On Nash and Moser Multiplicative Inequalities

An important role in Nash's classical work on local regularity of solutions to second-order parabolic equations in divergence form with measurable bounded coefficients [625] is played by the multiplicative inequality

$$\left(\int_{\mathbb{R}^n} u^2\,\mathrm{d}x\right)^{1+2/n} \le C\int_{\mathbb{R}^n} |\nabla u|^2\,\mathrm{d}x\left(\int_{\mathbb{R}^n} |u|\,\mathrm{d}x\right)^{4/n}, \quad u \in C_0^\infty. \tag{2.3.48}$$

Another inequality of a similar nature

$$\|u\|_{L_{2+4/n}}^{1+2/n} \le c\|u\|_{L_2}^{2/n}\|\nabla u\|_{L_2}, \quad u \in C_0^\infty, \tag{2.3.49}$$

was used by Moser in his proof of the Harnack inequality for solutions of second-order elliptic equations with measurable bounded coefficients in divergence form [617].

These two inequalities are contained as very particular cases in the Gagliardo–Nirenberg inequality for all $u \in C_0^\infty(\mathbb{R}^n)$

$$\|\nabla_j u\|_{L_q} \le c\|\nabla_l u\|_{L_p}^{\alpha}\|u\|_{L_r}^{1-\alpha}, \tag{2.3.50}$$

where $1 \le p, r \le \infty$, $0 \le j < l$, $j/l \le \alpha, \le 1$, and

$$\frac{1}{q} = \frac{j}{n} + \alpha\left(\frac{1}{p} - \frac{l}{n}\right) + \frac{1-\alpha}{r}.$$

If $1 < p < \infty$ and $l - j - n/p$ is a nonnegative integer then (2.3.50) holds only for $\alpha \in [j/n, 1)$ (see Gagliardo [299] and Nirenberg [640]).

If $n > 2$, the Nash and Moser inequalities follow directly by the Hölder inequality from the Sobolev inequality

$$\|u\|_{L_{2n/(n-2)}} \le c\|\nabla u\|_{L_2}, \quad u \in C_0^\infty. \tag{2.3.51}$$

We know by the second part of Theorem 2.3.6 that conversely, (2.3.48) and (2.3.49) imply (2.3.51). The just-mentioned theorem does not contain (2.3.48) and (2.3.49) for $n = 2$, which formally corresponds to the exceptional case $\alpha = 0$. However, we show here that both (2.3.48) and (2.3.49) with $n = 2$, and even the more general inequality

$$\int_{\mathbb{R}^n} |u|^q\,\mathrm{d}x \le c\left(\int_{\mathbb{R}^n} |\nabla u|^n\,\mathrm{d}x\right)^{\frac{q-r}{n}} \int_{\mathbb{R}^n} |u|^r\,\mathrm{d}x, \tag{2.3.52}$$

where $q \ge r > 0$ can be deduced from Theorem 2.3.11. In fact, by this theorem, (2.3.52) holds if and only if

$$m_n(g) \le \mathrm{const}\,\left(\mathrm{cap}_n(\bar{g}, G)\right)^{\frac{q-r}{n}} m_n(G), \tag{2.3.53}$$

where g and G are arbitrary bounded open sets with smooth boundaries, $\bar{g} \subset G$, and cap_n is the n capacity of $\bar{g}$ with respect to G. By the isocapacitary inequality (2.2.11),

$$\mathrm{cap}_n(\bar{g}, G) \geq n^{n-1}\omega_n \left(\log \frac{m_n(G)}{m_n(g)}\right)^{1-n}.$$

Hence (2.3.53) is a consequence of the boundedness of the function

$$(0,1) \ni x \to x\left(\log \frac{1}{x}\right)^{\frac{(q-r)(n-1)}{n}},$$

which, in its turn, implies the multiplicative inequality (2.3.52). □

The original proof of (2.3.52) (see Nirenberg [640], p. 129) is as follows. First, one notes that it suffices to obtain (2.3.52) for large q. Then (2.3.52) results by putting $|u|^{q(1-n)/n}$ instead of u into (1.4.49) and using an appropriate Hölder's inequality.

Extensions of Nash's inequality (2.3.48) to weighted inequalities with indefinite weights on the left-hand side were obtained by Maz'ya and Verbitsky [594] with simultaneously necessary and sufficient conditions on the weights.

2.3.13 Comments to Sect. 2.3

The basic results of Sects. 2.3.1–2.3.4 were obtained by the author in [531, 534] for $p = 2$, $\Phi(x, \xi) = |\xi|$, $M(u) = |u|$, and in [543] for the general case. Some of these results were repeated by Stredulinsky [729]. We shall return to capacitary inequalities similar to (2.3.6) in Chaps. 3 and 11. The inequality (2.3.14) can be found also in Brezis and Wainger [146] and Hansson [348].

Regarding the criterion in Sect. 2.3.3, see Comments to Sect. 2.4, where other optimal embeddings of Birnbaum–Orlicz–Sobolev spaces into C and Birnbaum–Orlicz spaces are discussed.

Inequality (2.3.21) is (up to a constant) the Sobolev ($p > 1$)-Gagliardo–Nirenberg ($p = 1$) inequality. The best constant for the case $p = 1$ (see (1.4.14)) was found independently by Federer and Fleming [273] and by the author [527] using the same method.

The best constant for $p > 1$, presented in Sect. 2.3.5 was obtained by Aubin [55] and Talenti [740] (the case $n = 3$, $p = 2$ was considered earlier by Rosen [682]), whose proofs are a combination of symmetrization and the one-dimensional Bliss inequality [109] (see Sect. 2.3.5). The uniqueness of the Bliss optimizer was proved by Gidas, Ni, and Nirenberg [307].

A different approach leading to the best constant in the Sobolev inequality, which is based on the geometric Brunn–Minkowski–Lyusternik inequality, was proposed in Bobkov and Ledoux [118].

The extremals exhibited in (2.3.24) of the Sobolev inequality (2.3.21) in the whole of $\mathbb{R}^n$, with sharp constant C, are the only ones—see Cordero-Erausquin, Nazaret, and Villani [212] who used the mass transportation techniques referred to in Comments to Sect. 1.4. Strengthened, quantitative versions of this inequality are also available. They involve a remainder term depending on the distance of the function u from the family of extremals. The first result in this connection was established by Bianchi and Egnell [96] for $p = 2$. The case when $p = 1$ was considered in Cianchi [199] and sharpened in Fusco, Maggi, and Pratelli [296] as far as the exponent in the remainder term is concerned. The general case when $1 < p < n$ is the object of Cianchi, Fusco, Maggi, and Pratelli [204]. Related results for $p > n$ are contained in Cianchi [202].

In [811], Zhang proved an improvement of the inequality (1.4.14), called the L_1 affine Sobolev inequality,

$$\int_{S^{n-1}} \|\nabla_u f\|_{L_1}^{-n} \,\mathrm{d}s_u \le n\left(\frac{\omega_n}{2\omega_{n-1}}\right)^n \|f\|_{L_{\frac{n}{n-1}}}^{-n}, \tag{2.3.54}$$

where $\nabla_u f$ is the partial derivative of f in direction u, $\mathrm{d}s_u$ is the surface measure on S^{n-1} and the constant factor on the right-hand side is sharp. Modifications of (2.3.54) for the L_p-gradient norm with $p > 1$ and for the Lorentz and Birnbaum–Orlicz settings are due to Zhang [811]; Lutwak, D. Yang, and Zhang [510]; Haberl and Schuster [333]; Werner and Ye [794]; and Cianchi, Lutwak, D. Yang, and Zhang [206].

A Sobolev-type trace inequality, which attracted much attention, is the following trace inequality:

$$\|f\|_{L_{\frac{p(n-1)}{n-p}}(\partial\mathbb{R}^n_+)} \le \mathscr{K}_{n,p}\|\nabla f\|_{L_p(\mathbb{R}^n_+)}, \tag{2.3.55}$$

where $n > p > 1$. In the case $p = 2$, Beckner [78] and Escobar [259], using different approaches, found the best value of $\mathscr{K}_{n,2}$. Xiao [799] generalized their result to the inequality

$$\|f\|_{L_{\frac{2(n-1)}{n-1-2\alpha}}(\partial\mathbb{R}^n_+)} \le C(n,\alpha)\int_{\mathbb{R}^n_+} \big|\nabla f(x)\big|^2 x_n^{1-2\alpha}\,\mathrm{d}x, \tag{2.3.56}$$

showing that

$$C(n,\alpha) = \left(\frac{2^{1-4\alpha}}{\pi^\alpha \Gamma(2(1-\alpha))}\right)\left(\frac{\Gamma((n-1-2\alpha)/2)}{\Gamma((n-1+2\alpha)/2)}\right)\left(\frac{\Gamma(n-1)}{\Gamma((n-1)/2)}\right)^{\frac{2\alpha}{n-1}}.$$

An idea in [799] is that it suffices to prove (2.3.56) for solutions of the Euler equation

$$\operatorname{div}\big(x_n^{1-2\alpha}\nabla u\big) = 0 \quad \text{on } \mathbb{R}^n_+.$$

Then by the Fourier transform with respect to $x' = (x_1, \dots, x_{n-1})$ the integral on the right-hand side of (2.3.56) takes the form

$$\text{const}\ \left\|(-\Delta_{x'})^{\alpha/2}u\right\|^2_{L_2(\mathbb{R}^{n-1})},$$

and the reference to the Lieb formula (1.4.48) gives the above value of $C(n,\alpha)$.

More recently, Nazaret proved, by using the mass transportation method mentioned in Comments to Sect. 1.4, that the only minimizer in (2.3.55) has the form

$$\text{const}\ \left((x_n+\lambda)^2+|x|^2\right)^{\frac{p-n}{2(p-1)}},$$

where $\lambda = \text{const} > 0$. Sharp Sobolev-type inequalities proved to be crucial in the study of partial differential equations and nowadays there is extensive literature dealing with them. To the works mentioned earlier we add the papers: Gidas, Ni, and Nirenberg [307]; Lieb [496]; Lions [501]; Han [335]; Beckner [78, 79]; Adimurthi and Yadava [29]; Hebey and Vaugon [362, 363]; Hebey [359]; Druet and Hebey [243]; Lieb and Loss [497]; Del Pino and Dolbeault [231]; Bonder, Rossi, and Ferreira [125]; Biezuner [99]; Ghoussoub and Kang [306]; Dem'yanov and A. Nazarov [233]; Bonder and Saintier [126]; et al.

The study of minimizers in the theory of Sobolev spaces based on the so-called concentration compactness is one of the topics in the book by Tintarev and Fieseler [753] where relevant historical information can be found as well.

The material of Sects. 2.3.6–2.3.11 is due to the author [543]. The sufficiency in Theorem 2.3.8 can be found in Maz'ya [543] and the necessity is due to Maz'ya and Netrusov [572].

The equivalence of the Nash and Moser inequalities (2.3.48) and (2.3.49) for $n > 2$ and Sobolev's inequality (2.3.51) is an obvious consequence of Theorem 2.3.6, which was proved by the author [543] (see also [552], Satz 4.3). This equivalence was rediscovered in the 1990s by Bakry, Coulhon, Ledoux, and Saloff-Coste [64] (see also Sect. 3.2 in the book by Saloff-Coste [687]) and by Delin [232]. The best constant in (2.3.48) was found by Carlen and Loss [167]:

$$C = 2n^{-1+2/n}(1+n/2)^{1+n/2}z_n^{-1}\omega_n^{-2/n},$$

where z_n is the smallest positive root of the equation

$$(1+n/2)J_{(n-2)/2}(z)+zJ'_{(n-2)/2}(z)=0.$$

The existence of the optimizer is proved in Tintarev and Fieseler [753], 10.3.

2.4 Continuity and Compactness of Embedding Operators of $\mathring{L}^1_p(\Omega)$ and $\mathring{W}^1_p(\Omega)$ into Birnbaum–Orlicz Spaces

Let $\mathring{L}^l_p(\Omega)$ and $\mathring{W}^l_p(\Omega)$ be completions of $\mathscr{D}(\Omega)$ with respect to the norms $\|\nabla_l u\|_{L_p(\Omega)}$ and $\|\nabla_l u\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega)}$.

Let μ be a measure in Ω. By $\mathscr{L}_M(\Omega,\mu)$ we denote the Birnbaum–Orlicz space generated by a convex function M, and by P we mean the complementary function of M (see Sect. 2.3.3).

The present section deals with some consequences of Theorem 2.3.3, containing the necessary and sufficient conditions for boundedness and compactness of embedding operators which map $\mathring{L}_p^1(\Omega)$ and $\mathring{W}_p^1(\Omega)$ into the space $\mathscr{L}_{p,M}(\Omega,\mu)$ with the norm $\||u|^p\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)}$, where μ is a measure in Ω. In the case $p=2$, $M(t)=|t|$ these results will be used in Sect. 2.5 in the study of the Dirichlet problem for the Schrödinger operator.

2.4.1 Conditions for Boundedness of Embedding Operators

With each compactum $F \subset \Omega$ we associate the number

$$\pi_{p,M}(F,\Omega)=\begin{cases}\frac{\mu(F)P^{-1}(1/\mu(F))}{\operatorname{cap}_p(F,\Omega)} & \text{for } \operatorname{cap}_p(F,\Omega)>0,\\ 0 & \text{for } \operatorname{cap}_p(F,\Omega)=0.\end{cases}$$

In the case $p=2$, $M(t)=|t|$, we shall use the notation $\pi(F,\Omega)$ instead of $\pi_{p,M}(F,\Omega)$.

The following assertion is a particular case of Theorem 2.3.3.

Theorem 1. 1. *Suppose that*

$$\pi_{p,M}(F,\Omega)\le\beta$$

for any compactum $F\subset\Omega$. Then, for all $u\in\mathscr{D}(\Omega)$,

$$\big\||u|^p\big\|_{\mathscr{L}_M(\Omega,\mu)}\le C\int_\Omega|\nabla u|^p\,\mathrm{d}x, \tag{2.4.1}$$

where $C\le p^p(p-1)^{1-p}\beta$.

2. *If (2.4.1) is valid for all $u\in\mathscr{D}(\Omega)$, then $\pi_{p,M}(F,\Omega)\le C$ for all compacta $F\subset\Omega$.*

Using this assertion we shall prove the following theorem.

Theorem 2. *The inequality*

$$\big\||u|\big\|^p_{\mathscr{L}_M(\Omega,\mu)}\le C\int_\Omega\big(|\nabla u|^p+|u|^p\big)\,\mathrm{d}x, \tag{2.4.2}$$

where $p<n$ is valid for all $u\in\mathscr{D}(\Omega)$ if and only if, for some $\delta>0$,

$$\sup\big\{\pi_{p,M}(F,\Omega): F\subset\Omega,\ \operatorname{diam}(F)\le\delta\big\}<\infty, \tag{2.4.3}$$

where, as usual, F is a compact subset of Ω.

Proof. Sufficiency. We construct a cubic grid in $\mathbb{R}^n$ with edge length $c\delta$, where c is a sufficiently small number depending only on n. With each cube $\mathscr{Q}_i$ of the grid we associate a concentric cube $2\mathscr{Q}_i$ with double the edge length and with faces parallel to those of $\mathscr{Q}_i$. We denote an arbitrary function in

$\mathscr{D}(\Omega)$ by u. Let η_i be an infinitely differentiable function in $\mathbb{R}^n$ that is equal to unity in $\mathscr{Q}_i$, to zero outside $2\mathscr{Q}_i$, and such that $|\nabla\eta_i| \le c_0/\delta$.

By Theorem 1,

$$\begin{aligned}\big\| |u\eta_i|^p \big\|_{\mathscr{L}_M(\Omega,\mu)} &\\ \le c\sup&\left\{\frac{\mu(F)P^{-1}(1/\mu(F))}{\operatorname{cap}_p(F, 2\mathscr{Q}_i\cap\Omega)} : F\subset 2\mathscr{Q}_i\cap\Omega\right\}\int_{2\mathscr{Q}_i\cap\Omega}\big|\nabla(u\eta_i)\big|^p\,\mathrm{d}x\\ \le c\sup&\big\{\pi_{p,M}(F,\Omega) : F\subset\Omega, \operatorname{diam}(F)\le\delta\big\}\int_{2\mathscr{Q}_i\cap\Omega}\big|\nabla(u\eta_i)\big|^p\,\mathrm{d}x.\end{aligned}$$

Summing over i and noting that

$$\big\||u|^p\big\|_{\mathscr{L}_M(\Omega,\mu)} \le \Big\|\sum_i |u\eta_i|^p\Big\|_{\mathscr{L}_M(\Omega,\mu)} \le \sum_i \big\||u\eta_i|^p\big\|_{\mathscr{L}_M(\Omega,\mu)},$$

we obtain the required inequality

$$\begin{aligned}\big\||u|^p\big\|_{L_M(\Omega,\mu)} \le c\sup&\big\{\pi_{p,M}(F,\Omega) : F\subset\Omega,\ \operatorname{diam}(F)\le\delta\big\}\\ &\times\int_\Omega\big(|\nabla u|^p + \delta^{-p}|u|^p\big)\,\mathrm{d}x.\end{aligned}\tag{2.4.4}$$

Necessity. Let F be any compactum in Ω and let $\operatorname{diam}(F)\le\delta<1$. We include F inside two open concentric balls B and $2B$ with diameters δ and 2δ, respectively. Then we substitute an arbitrary $u\in\mathfrak{P}(F, 2B\cap\Omega)$ into (2.4.2).

Since $u=1$ on F, then by (2.4.2)

$$\|\chi_F\|_{\mathscr{L}_M(\Omega,\mu)} \le C\bigg(\int_{2B}|\nabla u|^p\,\mathrm{d}x + \int_{2B}|u|^p\,\mathrm{d}x\bigg).$$

Consequently,

$$\mu(F)P^{-1}\big(1/\mu(F)\big) \le C\big(1+c\delta^p\big)\int_{2B}|\nabla u|^p\,\mathrm{d}x.$$

Minimizing the last integral over the set $\mathfrak{P}(F, 2B\cap\Omega)$ we obtain

$$\mu(F)P^{-1}\big(1/\mu(F)\big) \le C\big(1+c\delta^p\big)\operatorname{cap}_p(F, 2B\cap\Omega).$$

It remains to note that since $p<n$, it follows that

$$\operatorname{cap}_p(F, 2B\cap\Omega) \le c\operatorname{cap}_p(F,\Omega),\tag{2.4.5}$$

where c depends only on n and p.

In fact, if $u\in\mathfrak{N}(F,\Omega)$ and $\eta\in\mathscr{D}(2B)$, $\eta=1$ on B, $|\nabla\eta|\le c\delta$, then $u\eta\in\mathfrak{N}(F,\Omega\cap 2B)$ and hence

$$\begin{aligned}\mathrm{cap}_p(F, 2B\cap\Omega) &\le \int_{\Omega\cap 2B} \big|\nabla(u\eta)\big|^p\,\mathrm{d}x\\ &\le c\Big(\int_{2B}|\nabla u|^p\,\mathrm{d}x + \delta^{-p}\int_{2B}|u|^p\,\mathrm{d}x\Big)\\ &\le c\Big(\int_{\Omega}|\nabla u|^p\,\mathrm{d}x + \|u\|^p_{L_{pn/(n-p)}(\Omega)}\Big).\end{aligned}$$

This and the Sobolev theorem imply (2.4.5). The theorem is proved. □

2.4.2 Criteria for Compactness

The following two theorems give the necessary and sufficient conditions for the compactness of embedding operators that map $\mathring{L}^1_p(\Omega)$ and $\mathring{W}^1_p(\Omega)$ into $\mathscr{L}_{p,M}(\Omega,\mu)$.

Theorem 1. *The conditions*

$$\lim_{\delta\to 0}\sup\big\{\pi_{p,M}(F,\Omega) : F\subset\Omega,\ \mathrm{diam}(F)\le\delta\big\}=0 \tag{2.4.6}$$

and

$$\lim_{\varrho\to\infty}\sup\big\{\pi_{p,M}(F,\Omega) : F\subset\Omega\backslash B_\varrho\big\}=0 \tag{2.4.7}$$

are necessary and sufficient for any set of functions in $\mathscr{D}(\Omega)$, bounded in $\mathring{L}^1_p(\Omega)$ $(p<n)$, to be relatively compact in $\mathscr{L}_{p,M}(\Omega,\mu)$.

Theorem 2. *The condition* (2.4.6) *and*

$$\lim_{\varrho\to\infty}\sup\{\pi_{p,M}(F,\Omega) : F\subset\Omega\backslash B_\varrho,\ \mathrm{diam}(F)\le 1\}=0 \tag{2.4.8}$$

are necessary and sufficient for any set of functions in $\mathscr{D}(\Omega)$, bounded in $\mathring{W}^1_p(\Omega)$ $(p<n)$, to be relatively compact in $\mathscr{L}_{p,M}(\Omega,\mu)$.

To prove Theorems 1 and 2 we start with the following auxiliary assertion.

Lemma. *Let $\mu^{(\varrho)}$ be the restriction of μ to the ball B_ϱ. For an arbitrary set, bounded in $\mathring{L}^1_p(\Omega)$ or in $\mathring{W}^1_p(\Omega)$, $p<n$, to be relatively compact in $\mathscr{L}_{p,M}(\Omega,\mu^{(\varrho)})$ for all $\varrho>0$, it is necessary and sufficient that*

$$\lim_{\delta\to 0}\sup\big\{\pi_{p,M}(F,\Omega) : F\subset B_\varrho\cap\Omega,\ \mathrm{diam}(F)\le\delta\big\}=0, \tag{2.4.9}$$

for any $\varrho>0$.

Proof. Sufficiency. Since capacity does not increase under the extension of Ω, we see that for any compactum $F\subset B_\varrho\cap\Omega$,

$$\pi_{p,M}(F, B_\varrho\cap\Omega)\le\pi_{p,M}(F,\Omega).$$

This along with (2.4.9) implies

$$\lim_{\delta\to 0}\sup\bigl\{\pi_{p,M}(F,B_\varrho\cap\Omega): F\subset B_\varrho\cap\Omega,\ \operatorname{diam}(F)\le\delta\bigr\}=0$$

for all $\varrho>0$. This equality, together with (2.4.4), where the role of Ω is played by $B_{2\varrho}\cap\Omega$, yields

$$\bigl\||u|^p\bigr\|_{\mathscr{L}_M(\Omega,\mu^{(2\varrho)})}\le\varepsilon\int_{B_{2\varrho}\cap\Omega}|\nabla u|^p\,\mathrm{d}x+C_1(\varepsilon)\int_{B_{2\varrho}\cap\Omega}|u|^p\,\mathrm{d}x$$

for any $\varepsilon>0$ and for all $u\in\mathscr{D}(B_{2\varrho}\cap\Omega)$. Replacing u by $u\eta$, where η is a truncating function, equal to unity on B_ϱ and to zero outside $B_{2\varrho}$, we obtain

$$\bigl\||u|^p\bigr\|_{L_M(\Omega,\mu^{(\varrho)})}\le\varepsilon\int_{\Omega}|\nabla u|^p\,\mathrm{d}x+C_2(\varepsilon)\int_{B_{2\varrho}\cap\Omega}|u|^p\,\mathrm{d}x. \qquad (2.4.10)$$

It remains to note that in the case $p<n$ any set, bounded in $\mathring{L}^1_p(\Omega)$ (and a fortiori in $\mathring{W}^1_p(\Omega)$), is compact in $L_p(B_\varrho\cap\Omega)$ for any $\varrho>0$. The sufficiency of (2.4.8) is proved.

Necessity. Let $F\subset B_\varrho\cap\Omega$ be a compactum and let $\operatorname{diam}(F)\le\delta<1$. We include F inside concentric balls B and $2B$ with radii δ and 2δ, respectively. By u we denote an arbitrary function in $\mathfrak{P}(F,2B\cap\Omega)$. Since any set of functions in $\mathscr{D}(\Omega)$, bounded in $\mathring{W}^1_p(\Omega)$, is relatively compact in $\mathscr{L}_{p,M}(\Omega,\mu^{(\varrho)})$, then for all $v\in\mathscr{D}(\Omega)$

$$\bigl\|\chi_B|v|^p\bigr\|_{\mathscr{L}_M(\Omega,\mu^{(\varrho)})}\le\varepsilon(\delta)\int_\Omega\bigl(|\nabla v|^p+|v|^p\bigr)\,\mathrm{d}x,$$

where χ_B is the characteristic function of B and $\varepsilon(\delta)\to 0$ as $\delta\to 0$. To prove this inequality we must note that Theorem 2.4.1/2, applied to the measure $\mu^{(\varrho)}$, implies $\mu^{(\varrho)}(2B)\to 0$ as $\delta\to 0$. Since u equals zero outside $2B\cap\Omega$ we have

$$\int_\Omega|u|^p\,\mathrm{d}x\le c\delta^p\int_\Omega|\nabla u|^p\,\mathrm{d}x.$$

Therefore,

$$\mu(F)P^{-1}\bigl(1/\mu(F)\bigr)\le\bigl(1+c\delta^p\bigr)\varepsilon(\delta)\int_{2B}|\nabla u|^p\,\mathrm{d}x.$$

Minimizing the last integral over $\mathfrak{P}(F,2B\cap\Omega)$ and using (2.4.5), we arrive at

$$\pi_{p,M}(F,\Omega)\le\bigl(1+c\delta^p\bigr)\varepsilon(\delta).$$

The necessity of (2.4.9) follows. The lemma is proved. □

Proof of Theorem 1. Sufficiency. Let $\zeta\in C^\infty(\mathbb{R}^n)$, $0\le\zeta\le 1$, $|\nabla\zeta|\le c\varrho^{-1}$, $\zeta=0$ in a neighborhood of $B_{\varrho/2}$, $\zeta=1$ outside B_ϱ. It is clear that

$$\big\||u|^p\big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)} \le \big\|(1-\zeta)^p|u|^p\big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)} + \big\|\zeta^p|u|^p\big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)}$$
$$\le \big\||u|^p\big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu^{(\varrho)})} + \big\||\zeta u|^p\big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)}. \tag{2.4.11}$$

By the first part of Theorem 2.4.1/1, applied to the set $\Omega\backslash\bar{B}_{\varrho/2}$, by (2.4.7) and the inequality

$$\pi_{p,M}(F,\Omega\backslash\bar{B}_{\varrho/2}) \le \pi_{p,M}(F,\Omega),$$

given any ε, there exists a number $\varrho > 0$ such that

$$\big\||\zeta u|^p\big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)} \le \varepsilon\big\|\nabla(\zeta u)\big\|_{L_p(\Omega)}.$$

Since $|\nabla\zeta| \le c\varrho^{-1} \le c|x|^{-1}$ and

$$\big\||x|^{-1}u\big\|_{L_p(\Omega)} \le c\|\nabla u\|_{L_p(\Omega)},$$

we have

$$\big\||\zeta u|^p\big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)} \le c\varepsilon\|\nabla u\|_{L_p(\Omega)}.$$

The last inequality along with (2.4.11) implies

$$\big\||u|^p\big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)} \le \big\||u|^p\big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu^{(\varrho)})} + c\varepsilon\|\nabla u\|_{L_p(\Omega)}. \tag{2.4.12}$$

Obviously, (2.4.6) implies (2.4.9). Therefore, the lemma guarantees that any set of functions in $\mathscr{D}(\Omega)$, bounded in $\mathring{L}^1_p(\Omega)$, is compact in $\mathscr{L}_{p,M}(\Omega,\mu^{(\varrho)})$. This together with (2.4.12) completes the proof of the first part of the theorem.

Necessity. Let F be a compactum in Ω with $\operatorname{diam}(F) \le \delta < 1$. Duplicating the proof of necessity in the Lemma and replacing $\mu^{(\varrho)}$ there by μ, we arrive at the inequality $\pi_{p,M}(F,\Omega) \le (1+c\delta^p)\varepsilon(\delta)$ and hence at (2.4.6).

Now let $F \subset \Omega\backslash\bar{B}_\varrho$. Using the compactness in $\mathscr{L}_{p,M}(\Omega,\mu)$ of any set of functions in $\mathscr{D}(\Omega)$, which are bounded in $\mathring{L}^1_p(\Omega)$, we obtain

$$\big\|\chi_{\Omega\backslash B_\varrho}|u|^p\big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)} \le \varepsilon_\varrho\|\nabla u\|_{L_p(\Omega)},$$

where $\varepsilon_\varrho \to 0$ as $\varrho \to 0$ and u is an arbitrary function in $\mathscr{D}(\Omega)$. In particular, the last inequality holds for any $u \in \mathfrak{P}(F,\Omega)$ and therefore

$$\mu(F)P^{-1}\big(1/\mu(F)\big) \le \varepsilon^p_\varrho\|\nabla u\|^p_{L_p(\Omega)}.$$

Minimizing the right-hand side over the set $\mathfrak{P}(F,\Omega)$, we arrive at (2.4.7). The theorem is proved. □

Proof of Theorem 2. We shall use the same notation as in the proof of Theorem 1.

Sufficiency. From (2.4.4), where $\delta = 1$ and Ω is replaced by $\Omega\backslash\bar{B}_{\varrho/2}$, together with (2.4.8), it follows that given any $\varepsilon > 0$, there exists a $\varrho > 0$ such that

$$\big\| |\zeta u|^p \big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)} \le \varepsilon\big(\big\|\nabla(u\zeta)\big\|_{L_p(\Omega)} + \|\zeta u\|_{L_p(\Omega)}\big).$$

This together with (2.4.11) yields

$$\big\| |u|^p \big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)} \le \big\| |u|^p \big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu^{(\varrho)})} + c\varepsilon\|u\|_{W^1_p(\Omega)}.$$

The remainder of the proof is the same as the proof of sufficiency in the preceding theorem.

Necessity. The condition (2.4.6) can be derived in the same way as in the proof of necessity in Theorem 1.

Let $F \subset \Omega\backslash\bar{B}_\varrho$, $\varrho > 8$, $\operatorname{diam}(F) \le 1$. From the compactness in $\mathscr{L}_{p,M}(\Omega,\mu)$ of any set of functions in $\mathscr{D}(\Omega)$, bounded in $\mathring{W}^1_p(\Omega)$, it follows that

$$\big\|\chi_{\Omega\backslash B_{\varrho/2}}|u|^p\big\|_{\mathscr{L}_M(\Omega,\mu)} \le \varepsilon_\varrho\|u\|^p_{W^1_p(\Omega)},$$

where $\varepsilon_\varrho \to 0$ as $\varrho \to \infty$ and u is an arbitrary function in $\mathscr{D}(\Omega)$. We include F inside concentric balls B and $2B$ with radii 1 and 2 and let u denote any function in $\mathfrak{P}(F, 2B\cap\Omega)$. Using the same argument as in the proof of necessity in the Lemma we arrive at

$$\pi_{p,M}(F,\Omega) \le (1+c)\varepsilon_\varrho,$$

which is equivalent to (2.4.8). The theorem is proved. □

Remark. Let us compare (2.4.6) and (2.4.9). Clearly, (2.4.9) results from (2.4.6). The following example shows that the converse assertion is not valid. Consider a sequence of unit balls $\mathscr{B}^{(\nu)}$ ($\nu = 1, 2, \dots$), with dist $(\mathscr{B}^{(\nu)}, \mathscr{B}^{(\mu)}) \ge 1$ for $\mu \ne \nu$. Let $\Omega = \mathbb{R}^n$ and

$$\mu(F) = \int_F p(x)\,\mathrm{d}x,$$

where

$$p(x) = \begin{cases} \varrho^{-2+\nu^{-1}} & \text{for } x \in \mathscr{B}^{(\nu)}, \\ 0 & \text{for } x \notin \bigcup_{\nu=1}^\infty \mathscr{B}^{(\nu)}. \end{cases}$$

Here ϱ is the distance of x from the center of $\mathscr{B}^{(\nu)}$.

We shall show that the measure μ satisfies the condition (2.4.9) with $p = 2$, $M(t) = t$. First of all we note that for any compactum $F \subset \mathscr{B}^{(\nu)}$

$$\mu(F) = \int_F \varrho^{-2+1/\nu}\,\mathrm{d}x \le \int_{\partial B_1}\int_0^{r(F)} \varrho^{n-3+1/\nu}\,\mathrm{d}\varrho\,\mathrm{d}\omega,$$

where

$$r(F) = \Big[\frac{n}{\omega}m_n(F)\Big]^{1/n}.$$

To estimate $\operatorname{cap}(F)$, i.e., $\operatorname{cap}_2(F,\mathbb{R}^n)$, we apply the isoperimetric inequality (2.2.12)

$$\omega_n^{-1}(n-2)^{-1}\operatorname{cap}(F) \ge \left[\frac{n}{\omega_n} m_n(F)\right]^{(n-2)/n} = \left[r(F)\right]^{n-2}.$$

Now

$$\pi\big(F, \mathbb{R}^n\big) \le \frac{r(F)^{1/\nu}}{(n-2)(n-2+1/\nu)},$$

and (2.4.9) follows.

If F is the ball $\{x : \varrho \le \delta\}$, we have

$$\pi\big(F, \mathbb{R}^n\big) = \frac{\delta^{1/\nu}}{(n-2)(n-2+1/\nu)}.$$

Consequently,

$$\begin{aligned}&\sup\big\{\pi\big(F, \mathbb{R}^n\big) : F \subset \mathbb{R}^n, \operatorname{diam}(F) \le 2\delta\big\}\\ &\ge \lim_{\nu\to\infty} \frac{\delta^{1/\nu}}{(n-2)(n-2+1/\nu)} = (n-2)^{-2}\end{aligned}$$

and (2.4.6) is not valid.

2.4.3 Comments to Sect. 2.4

The material of this section is borrowed from Sect. 2.5 of the author's book [552]. Sharp embeddings of Birnbaum–Orlicz–Sobolev spaces of order one into the space $L_\infty(\Omega)$ will be considered in Chap. 7 of the present book (see also Maz'ya [528, 545]).

An optimal Sobolev embedding theorem in Birnbaum–Orlicz spaces was established by Cianchi in [194], and in alternative equivalent form, in [195]. A basic version of this result states that if Ω is an open set in $\mathbb{R}^n$ with finite measure, M is any Young function, and M_n is the Young function given by

$$M_n(t) = M\big(H^{-1}(t)\big) \quad \text{for } t > 0,$$

where

$$H(s) = \left(\int_0^s \left(\frac{t}{M(t)}\right)^{\frac{1}{n-1}} \mathrm{d}t\right)^{\frac{n-1}{n}} \quad \text{for } s \ge 0,$$

then there exists a constant C, depending on n, such that

$$\|u\|_{\mathscr{L}_{M_n}(\Omega)} \le C\|\nabla u\|_{\mathscr{L}_M(\Omega)}$$

for every weakly differentiable function u vanishing, in the appropriate sense, on $\partial\Omega$. Moreover, the Birnbaum–Orlicz space $\mathscr{L}_{M_n}(\Omega)$ is optimal. Analogous results for functions that need not vanish on $\partial\Omega$, and for domains Ω with infinite measure [194]. The case of higher-order derivatives was dealt with by Cianchi in [200].

Some necessary and sufficient conditions for embeddings of Sobolev-type spaces into Birnbaum–Orlicz spaces will be treated in Chap. 11 of this book.

Analog of certain results in the present section were obtained by Klimov [431–435] for the so-called *ideal function spaces*, for which the multiplication by any function α with $|\alpha(x)| \le 1$ a.e. is contractive.

A few words on the so-called *logarithmic Sobolev inequalities*. Let μ be a measure in Ω, $\mu(\Omega) = 1$, $p \ge 1$ and let ν_p be the capacity minimizing function generated by μ (see Definition 2.3.2). The inequality

$$\exp\Big(-\int_\Omega \log^+ \frac{1}{|u|}\,\mathrm{d}\mu\Big) \le 4\|\nabla u\|_{L_p(\Omega)} \exp\Big(-\frac{1}{p}\int_0^1 \log \nu_p(s)\,\mathrm{d}s\Big) \quad (2.4.13)$$

for all $u \in \mathring{L}_p^1(\Omega)$ was proved in 1968 by Maz'ya and Havin [568]. It shows, in particular, that

$$\int_0^1 \nu_p(s)\,\mathrm{d}s = +\infty$$

implies

$$\int_0^1 \log^+ \frac{1}{|u|}\,\mathrm{d}\mu = +\infty$$

for all $u \in L_p^1(\Omega)$. This fact allows for certain applications of (2.4.13) to complex function theory [568] (see also Sect. 14.3 of the present book for another logarithmic inequality of a similar nature).

Inequality (2.4.13) is completely different from the logarithmic Sobolev inequality obtained in 1978 by Weissler [793],

$$\exp\Big(\frac{4}{n}\int_{\mathbb{R}^n} |u|^2 \log|u|\,\mathrm{d}x\Big) \le \frac{2}{\pi e n}\int_{\mathbb{R}^n} |\nabla u|^2\,\mathrm{d}x,$$

where $\|u\|_{L_2(\mathbb{R}^n)} = 1$, which is equivalent (see Beckner and Pearson [81]) to the well-known Gross inequality of 1975 [327],

$$\int_{\mathbb{R}^n} u^2 \log\Big(u^2 \Big/ \int_{\mathbb{R}^n} u^2\,\mathrm{d}\mu\Big)\,\mathrm{d}\mu \le C\int_{\mathbb{R}^n} |\nabla u|^2\,\mathrm{d}\mu, \quad (2.4.14)$$

where

$$\mathrm{d}\mu = (2\pi)^{-n/2}\exp\big(-|x|^2/2\big)\,\mathrm{d}x.$$

Various extensions, proofs, and applications of (2.4.14) were the subject of many studies: R.A. Adams [24]; Stroock and Zegarlinski [730]; Holley and Stroock [381]; Davies [222]; Zegarlinski [810]; Beckner [77, 80]; Gross [328]; Aida, Masuda, and Shigekawa [32]; Aida and Stroock [33]; Bakry [63]; Bakry, Ledoux, and Qian [65]; Chen [185]; F.-Y. Wang [787–790]; Bodineau and Helffer [121]; Bobkov and Götze [114]; Ledoux [483–485]; Yosida [808]; Guionnet and Zegarlinski [330]; Xiao [799]; Lugiewicz and Zegarlinski [509]; Otto and Reznikoff [655]; Inglis and Papageorgiou [398]; Cianchi and Pick [207]; Martin and M. Milman [521]; et al.

2.5 Structure of the Negative Spectrum of the Multidimensional Schrödinger Operator

In this section we show how the method and results of Sect. 2.4 can be applied to the spectral theory of the Schrödinger operator.

2.5.1 Preliminaries and Notation

We start with some definitions from the theory of quadratic forms in a Hilbert space H. Let $\mathscr{L}$ be a dense linear subset of H and let $S[u,u]$ be a quadratic form defined on $\mathscr{L}$. If there exists a constant γ such that for all $u \in \mathscr{L}$

$$S[u,u] \geq \gamma \|u\|_H^2, \tag{2.5.1}$$

then the form S is called semibounded from below. The largest constant γ in (2.5.1) is called the greatest lower bound of the form S and is denoted by $\gamma(S)$. If $\gamma(S) > 0$, then S is called positive definite. For such a form the set $\mathscr{L}$ is a pre-Hilbert space with the inner product $S[u,u]$. If $\mathscr{L}$ is a Hilbert space the form S is called closed. If any Cauchy sequence in the metric $S[u,u]^{1/2}$ that converges to zero in H also converges to zero in the metric $S[u,u]^{1/2}$, then S is said to be closable. Completing $\mathscr{L}$ and extending S by continuity onto the completion $\bar{\mathscr{L}}$, we obtain the closure $\bar{S}$ of the form S.

Now, suppose that the form S is only semibounded from below. We do not assume $\gamma(S) > 0$. Then for any $c > -\gamma(S)$ the form

$$S[u,u] + c[u,u] \tag{2.5.2}$$

is positive definite. By definition, S is closable if the form (2.5.2) is closable for some, and therefore for any, $c > \gamma(S)$. The form $\overline{S + cE} - cE$ is called the closure $\bar{S}$ of S.

It is well known and can be easily checked that a semibounded closable form generates a unique self-adjoint operator $\tilde{S}$, for which

$$(\tilde{S}u, u) = S[u,u] \quad \text{for all } u \in \mathscr{L}.$$

Let Ω be an open subset of $\mathbb{R}^n$, $n > 2$, and let h be a positive number. We shall consider the quadratic form

$$S_h[u,u] = h \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int |u|^2 \,\mathrm{d}\mu(x)$$

defined on $\mathscr{D}(\Omega)$.

We shall study the operator $\tilde{S}_h$ generated by the form $S_h[u,u]$ under the condition that the latter is closable. If the measure μ is absolutely continuous with respect to the Lebesgue measure m_n and the derivative $p = \mathrm{d}\mu/\mathrm{d}m_n$ is locally square integrable, then the operator $\tilde{S}_h$ is the Friedrichs extension of the Schrödinger operator $-h\Delta - p(x)$.

In this section, when speaking of capacity, we mean the 2-capacity and use the notation cap.

Before we proceed to the study of the operator $\tilde{S}_h$ we formulate two lemmas on estimates for capacity that will be used later. For the proofs of these lemmas see the end of the section.

Lemma 1. *Let F be a compactum in $\Omega \cap B_r$. Then for $R > r$*

$$\operatorname{cap}(F, B_r \cap \Omega) \leq \begin{cases} (1 + \frac{2r}{R-r} \log \frac{Re^{1/2}}{r}) \operatorname{cap}(F, \Omega) & \text{for } n = 3, \\ (1 + \frac{2}{n-3} \frac{r}{R-r}) \operatorname{cap}(F, \Omega) & \text{for } n > 3. \end{cases}$$

Lemma 2. *Let F be a compactum in $\Omega \backslash \bar{B}_R$. Then for $r < R$*

$$\operatorname{cap}(F, \Omega \backslash \bar{B}_r) \leq \left(1 + \frac{1}{n-2} \frac{r}{R-r}\right) \operatorname{cap}(F, \Omega).$$

All the facts concerning the operator $\tilde{S}_h$ will be formulated in terms of the function

$$\pi(F, \Omega) = \begin{cases} \frac{\mu(F)}{\operatorname{cap}(F,\Omega)} & \text{for } \operatorname{cap}(F, \Omega) > 0, \\ 0 & \text{for } \operatorname{cap}(F, \Omega) = 0, \end{cases}$$

which is a particular case of the function $\pi_{p,M}(F, \Omega)$, introduced in Sect. 2.4, for $M(t) = |t|$, $p = 2$.

2.5.2 Positivity of the Form $S_1[u, u]$

The following assertion is a particular case of Theorem 2.4.1/1.

Theorem. 1. *If for any compactum $F \subset \Omega$*

$$\pi(F, \Omega) \leq \beta, \tag{2.5.3}$$

then for all $u \in \mathscr{D}(\Omega)$

$$\int_\Omega |u|^2 \mu(\mathrm{d}x) \leq C \int_\Omega |\nabla u|^2 \,\mathrm{d}x, \tag{2.5.4}$$

where $C \leq 4\beta$.

2. *If (2.5.4) holds for all $u \in \mathscr{D}(\Omega)$, then for any compactum $F \subset \Omega$*

$$\pi(F, \Omega) \leq C. \tag{2.5.5}$$

Corollary. *If*

$$\sup_{F \subset \Omega} \pi(F, \Omega) < \frac{1}{4},$$

then the form $S_1[u,u]$ is positive, closable in $L_2(\Omega)$, and hence it generates a self-adjoint positive operator $\tilde{S}_1$ in $L_2(\Omega)$.

Proof. The positiveness of $S_1[u,u]$ follows from the Theorem. Moreover, inequality (2.5.4) implies

$$S_1[u,u] \geq \Big[1 - 4 \sup_{F \subset \Omega} \pi(F,\Omega)\Big] \int_\Omega |\nabla u|^2 \, \mathrm{d}x. \tag{2.5.6}$$

Let $\{u_\nu\}_{\nu \geq 1}$, $u_\nu \in \mathscr{D}(\Omega)$, be a Cauchy sequence in the metric $S_1[u,u]^{1/2}$ and let u_ν converge to zero in $L_2(\Omega)$. Then by (2.5.6), u_ν converges to zero in $\mathring{L}_2^1(\Omega)$ and it is a Cauchy sequence in $L_2(\Omega,\mu)$. Since

$$\int_\Omega |u_\nu|^2 \, \mathrm{d}\mu \leq 4 \sup_{F \subset \Omega} \pi(F,\Omega) \int_\Omega |\nabla u_\nu|^2 \, \mathrm{d}x,$$

then $u_\nu \to 0$ in $L_2(\Omega,\mu)$. Thus, $S_1[u_\nu,u_\nu] \to 0$ and therefore the form $S_1[u,u]$ is closable in $L_2(\Omega)$. The corollary is proved. $\square$

We note that close necessary and sufficient conditions for the validity of the inequality

$$\int_\Omega |u|^2 \, \mathrm{d}\sigma \leq C \int_\Omega |\nabla u|^2 \, \mathrm{d}x, \quad u \in \mathscr{D}(\Omega),$$

where σ is an arbitrary charge in Ω, are contained in Theorem 2.3.10 for $\Phi(x,y) = |y|$, $p = 2$. The conditions in question coincide for $\sigma \geq 0$. They become the condition $\sup\{\pi(F,\Omega) : F \subset \Omega\} < \infty$, which follows from the Theorem.

2.5.3 Semiboundedness of the Schrödinger Operator

Theorem. 1. *If*

$$\lim_{\delta \to 0} \sup\big\{\pi(F,\Omega) : F \subset \Omega,\ \mathrm{diam}(F) \leq \delta\big\} < \frac{1}{4}, \tag{2.5.7}$$

then the form $S_1[u,u]$ is semibounded from below and closable in $L_2(\Omega)$.

2. *If the form $S_1[u,u]$ is semibounded from below in $L_2(\Omega)$, then*

$$\lim_{\delta \to 0} \sup\big\{\pi(F,\Omega) : F \subset \Omega,\ \mathrm{diam}(F) \leq \delta\big\} \leq 1. \tag{2.5.8}$$

Proof. 1. If Π is a sufficiently large integer, then there exists $\delta > 0$ such that

$$\sup\big\{\pi(F,\Omega) : F \subset \Omega,\ \mathrm{diam}(F) \leq \delta\big\} \leq \frac{1}{4}\left(\frac{\Pi - 1}{\Pi + 2}\right)^n. \tag{2.5.9}$$

We construct a cubic grid in $\mathbb{R}^n$ with edge length $H = \delta/(\varPi + 2)\sqrt{n}$. We include each cube $\mathscr{Q}_i$ of the grid inside concentric cubes $\mathscr{Q}_i^{(1)}$ and $\mathscr{Q}_i^{(2)}$ with faces parallel to those of $\mathscr{Q}_i$. Let the edge lengths of $\mathscr{Q}_i^{(1)}$ and $\mathscr{Q}_i^{(2)}$ be $(\varPi + 1)H$ and $(\varPi + 2)H$, respectively. Since $\operatorname{diam}(\mathscr{Q}_i^{(2)}) = \delta$ then for any compactum $F \subset \mathscr{Q}_i^{(2)} \cap \varOmega$

$$\pi\big(F, \varOmega \cap \mathscr{Q}_i^{(2)}\big) \le \pi(F, \varOmega) \le \frac{1}{4}\left(\frac{\varPi - 1}{\varPi + 2}\right)^n. \tag{2.5.10}$$

Let u denote an arbitrary function in $\mathscr{D}(\varOmega)$ and let η denote an infinitely differentiable function on $\mathbb{R}^n$ which is equal to unity in $\mathscr{Q}_i^{(1)}$ and to zero outside $\mathscr{Q}_i^{(2)}$. By (2.5.10) and Theorem 2.5.2 we have

$$\int_{\mathscr{Q}_i^{(2)}} |u\eta|^2 \,\mathrm{d}\mu \le \left(\frac{\varPi - 1}{\varPi + 2}\right)^n \int_{\mathscr{Q}_i^{(2)}} \big|\nabla(u\eta)\big|^2 \,\mathrm{d}x.$$

This implies

$$\int_{\mathscr{Q}_i^{(1)}} |u|^2 \,\mathrm{d}\mu \le \left(\frac{\varPi - 1}{\varPi + 2}\right)^n \int_{\mathscr{Q}_i^{(2)}} \left(|\nabla u|^2 + \frac{c_1}{H^2}|u|^2\right) \mathrm{d}x.$$

Summing over i and noting that the multiplicity of the covering $\{\mathscr{Q}_i^{(2)}\}$ does not exceed $(\varPi + 2)^n$ and that of $\{\mathscr{Q}_i^{(2)}\}$ is not less than $\varPi^n$, we obtain

$$\int_{\varOmega} |u|^2 \,\mathrm{d}\mu \le \big(1 - \varPi^{-n}\big) \int_{\varOmega} \left(|\nabla u|^2 + c\frac{\varPi^2}{\delta^2}|u|^2\right) \mathrm{d}x. \tag{2.5.11}$$

Thus, the form $S_1[u, u]$ is semibounded. Moreover, if K is a sufficiently large constant, then

$$S_1[u, u] + K\int_{\varOmega} |u|^2 \,\mathrm{d}x \ge \varepsilon \int_{\varOmega} |\nabla u|^2 \,\mathrm{d}x, \quad \varepsilon > 0.$$

Further, using the same argument as in the proof of Corollary 2.5.2, we can easily deduce that the form $S_1[u, u]$ is closable in $L_2(\varOmega)$.

2. Let F be an arbitrary compactum in $\varOmega$ with $\operatorname{diam}(F) \le \delta < 1$. We enclose F in a ball B with radius δ and construct the concentric ball B' with radius $\delta^{1/2}$.

We denote an arbitrary function in $\mathfrak{P}(F, B' \cap \varOmega)$ by u. In virtue of the semiboundedness of the form $S_1[u, u]$ there exists a constant K such that

$$\int_{B'} u^2 \,\mathrm{d}\mu \le \int_{B'} (\nabla u)^2 \,\mathrm{d}x + K\int_{B'} u^2 \,\mathrm{d}x.$$

Obviously, the right-hand side of this inequality does not exceed

$$\big(1+K\lambda^{-1}\delta\big)\int_{B'\cap\Omega}(\nabla u)^2\,\mathrm{d}x,$$

where λ is the first eigenvalue of the Dirichlet problem for the Laplace operator in the unit ball.

Minimizing the Dirichlet integral and taking into account that $u=1$ on F, we obtain

$$\mu(F)\le\big(1+K\lambda^{-1}\delta\big)\,\mathrm{cap}(F,B'\cap\Omega).$$

By Lemma 2.5.1,

$$\mathrm{cap}(F,B'\cap\Omega)\le\big(1+o(1)\big)\,\mathrm{cap}(F,\Omega),$$

where $o(1)\to 0$ as $\delta\to 0$. Hence

$$\sup\big\{\pi(F,\Omega): F\subset\Omega, \mathrm{diam}(F)\le\delta\big\}\le 1+o(1).$$

It remains to pass to the limit as $\delta\to 0$. The theorem is proved. □

The two assertions stated in the following are obvious corollaries of the Theorem. The second is a special case of Theorem 2.4.1/2.

Corollary 1. *The condition*

$$\lim_{\delta\to 0}\sup\big\{\pi(F,\Omega): F\subset\Omega, \mathrm{diam}(F)\le\delta\big\}=0 \tag{2.5.12}$$

is necessary and sufficient for the semiboundedness of the form $S_h[u,u]$ in $L_2(\Omega)$ for all $h>0$.

Corollary 2. *The inequality*

$$\int_\Omega |u|^2\,\mathrm{d}\mu\le C\int_\Omega\big(|\nabla u|^2+|u|^2\big)\,\mathrm{d}x,$$

where u is an arbitrary function in $\mathscr{D}(\Omega)$ and C is a constant independent of u, is valid if and only if

$$\sup\big\{\pi(F,\Omega): F\subset\Omega, \mathrm{diam}(F)\le\delta\big\}<\infty \tag{2.5.13}$$

for some $\delta>0$.

We shall give an example that illustrates an application of Theorem 2.5.2 and the theorem of the present subsection to the Schrödinger operator generated by a singular measure.

Example. Let M be a plane Borel subset of $\mathbb{R}^3$. We define the measure $\mu(F)=m_2(F\cap M)$ for any compactum $F\subset\mathbb{R}^3$. (In the sense of distribution theory the potential $p(x)$ is equal to the Dirac δ function concentrated on the plane set M.) Then

$$\pi\big(F,\mathbb{R}^3\big) = \frac{m_2(F\cap M)}{\operatorname{cap}(F)} \le \frac{m_2(F\cap M)}{\operatorname{cap}(F\cap M)}.$$

Since

$$\operatorname{cap}(F\cap M) \ge 8\pi^{-1/2}\big[m_2(F\cap M)\big]^{1/2}$$

(cf. (2.2.16)), we have

$$\pi\big(F,\mathbb{R}^3\big) \le \frac{\pi^{1/2}}{8}\big[m_2(F\cap M)\big]^{1/2}. \tag{2.5.14}$$

By Theorem 2.5.2 the form

$$S_1[u,u] = \int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x - \int_M |u|^2 m_2(\mathrm{d}x)$$

is positive if $m_2(M) \le 4\pi^{-1}$. Using Corollary 1, from (2.5.14) we obtain that the form $S_h[u,u]$ is semibounded and closable in $L_2(\mathbb{R}^3)$ for all $h>0$ for any plane set M.

2.5.4 Discreteness of the Negative Spectrum

Let ϱ be a fixed positive number and let $\mu^{(\varrho)}$ be the restriction of a measure μ to the ball $B_\varrho = \{x : |x| < \varrho\}$. Further, let $\mu_\varrho = \mu - \mu^{(\varrho)}$.

To exclude the influence of singularities of the measure μ, which are located at a finite distance, we shall assume that any subset of $\mathscr{D}(\Omega)$, bounded in $\mathring{W}_2^1(\Omega)$ (or in $\mathring{L}_2^1(\Omega)$), is compact in $L_2(\mu^{(\varrho)})$. In Lemma 2.4.2 it is shown that this condition is equivalent to

$$\lim_{\delta\to 0} \sup\big\{\pi(F,\Omega) : F\subset B_\varrho\cap\Omega,\ \operatorname{diam}(F)\le\delta\big\} = 0 \tag{2.5.15}$$

for any $\varrho > 0$.

Now we formulate two well-known general assertions that will be used in the following.

Lemma 1. (Friedrichs [292]). *Let $A[u,u]$ be a closed quadratic form in a Hilbert space H with the domain $D[A]$, $\gamma(A)$ being its positive greatest lower bound. Further, let $B[u,u]$ be a real form, compact in $D[A]$. Then the form $A-B$ is semibounded from below in H and closed in $D[A]$, and its spectrum is discrete to the left of $\gamma(A)$.*

Lemma 2. (Glazman [309]). *For the negative spectrum of a self-adjoint operator A to be infinite it is necessary and sufficient that there exists a linear manifold of infinite dimension on which $(Au,u)<0$.*

Now we proceed to the study of conditions for the spectrum of the Schrödinger operator to be discrete.

Theorem. *Let the condition (2.5.15) hold.*

1. *If*

$$\lim_{\delta\to\infty}\lim_{\varrho\to\infty}\sup\bigl\{\pi(F,\Omega): F\subset\Omega\backslash B_\varrho,\ \operatorname{diam}(F)\le\delta\bigr\}<\frac{1}{4}, \tag{2.5.16}$$

then the form $S_1[u,u]$ is semibounded from below closable in $L_2(\Omega)$, and the negative spectrum of the operator $\tilde{S}_1$ is discrete.

2. *If the form $S_1[u,u]$ is semibounded from below and closable in $L_2(\Omega)$, and the negative spectrum of the operator $\tilde{S}_1$ is discrete, then*

$$\lim_{\delta\to\infty}\lim_{\varrho\to\infty}\sup\bigl\{\pi(F,\Omega): F\subset\Omega\backslash B_\varrho,\ \operatorname{diam}(F)\le\delta\bigr\}\le 1. \tag{2.5.17}$$

Proof. 1. We show that the form $S_1[u,u]$ is semibounded from below and closable in $L_2(\Omega)$, and that for any positive γ the spectrum of the operator $\tilde{S}_1+2\gamma I$ is discrete to the left of γ. This will yield the first part of the theorem.

By (2.5.16), there exists a sufficiently large integer Π such that

$$\lim_{\varrho\to\infty}\sup\bigl\{\pi(F,\Omega): F\subset\Omega\backslash B_\varrho,\ \operatorname{diam}(F)\le\delta\bigr\}\le\frac{1}{4}\left(\frac{\Pi-2}{\Pi+2}\right)^n$$

for all $\delta>0$.

Given any δ, we can find a sufficiently large number $\varrho=\varrho(\delta)$ so that

$$\sup\bigl\{\pi(F,\Omega): F\subset\Omega\backslash B_\varrho,\operatorname{diam}(F)\le\delta\bigr\}\le\frac{1}{4}\left(\frac{\Pi-1}{\Pi+2}\right)^n.$$

Hence

$$\sup\left\{\frac{\mu_{(\varrho)}(F)}{\operatorname{cap}(F,\Omega)}: F\subset\Omega,\operatorname{diam}(F)\le\delta\right\}\le\frac{1}{4}\left(\frac{\Pi-1}{\Pi+2}\right)^n.$$

If we replace $\mu_{(\varrho)}$ here by μ, then we obtain the condition (2.5.9), which was used in the first part of Theorem 2.5.3 for the proof of inequality (2.5.11). We rewrite that inequality, replacing μ by $\mu_{(\varrho)}$:

$$\int_\Omega |u|^2\,\mathrm{d}\mu_{(\varrho)}\le\bigl(1-\Pi^{-n}\bigr)\int_\Omega\left(|\nabla u|^2+c\frac{\Pi^2}{\delta^2}|u|^2\right)\mathrm{d}x. \tag{2.5.18}$$

Let γ denote an arbitrary positive number. We specify $\delta>0$ by the equality $c\Pi^2(1-\Pi^{-n})\delta^{-2}=\gamma$ and find ϱ corresponding to δ. Then

$$\int_\Omega |u|^2\,\mathrm{d}\mu_{(\varrho)}\le\bigl(1-\Pi^{-n}\bigr)\int_\Omega|\nabla u|^2\,\mathrm{d}x+\gamma\int_\Omega|u|^2\,\mathrm{d}x.$$

Hence the form

$$A[u,u]=\int_\Omega|\nabla u|^2-\int_\Omega|u|^2\,\mathrm{d}\mu_{(\varrho)}+2\gamma\int_\Omega|u|^2\,\mathrm{d}x,$$

majorizes

$$\Pi^{-n}\int_\Omega |\nabla u|^2\,\mathrm{d}x+\gamma\int_\Omega |u|^2\,\mathrm{d}x.$$

This means that the form $A[u,u]$ has a positive lower bound γ and is closable in $L_2(\Omega)$. Let $\bar{A}[u,u]$ denote the closure of the form $A[u,u]$. Clearly, the domain of the form $\bar{A}[u,u]$ coincides with $\mathring{W}_2^1(\Omega)$.

By (2.5.15) and Corollary 2.5.3/2, the form

$$B[u,u]=\int_\Omega |u|^2\,\mathrm{d}\mu^{(\varrho)}$$

is continuous in $W_2^1(\Omega)$ and is closable in $\mathring{W}_2^1(\Omega)$. Lemma 2.4.2 ensures the compactness of the form $\bar{B}[u,u]$ in $\mathring{W}_2^1(\Omega)$. It remains to apply Lemma 1 to $\bar{A}[u,u]$ and $\bar{B}[u,u]$.

2. Suppose that

$$\lim_{\varrho\to\infty}\sup\bigl\{\pi(F,\Omega): F\subset\Omega\backslash B_\varrho,\ \operatorname{diam}\le\delta\bigr\}>1+\alpha,\quad \alpha>0,$$

for some δ. Then there exists a sequence of compacta F_ν with $\operatorname{diam}(F_\nu)\le\delta$, which tends to infinity and satisfies

$$\mu(F_\nu)>(1+\alpha)\operatorname{cap}(F_\nu,\Omega). \tag{2.5.19}$$

We include F_ν in a ball $B_\delta^{(\nu)}$ with radius δ. Let $B_\varrho^{(\nu)}$ denote a concentric ball with a sufficiently large radius ϱ that will be specified later. Without loss of generality, we may obviously assume that the balls $B_\varrho^{(\nu)}$ are disjoint.

By Lemma 2.5.1/1,

$$\operatorname{cap}\bigl(F_\nu,B_\varrho^{(\nu)}\cap\Omega\bigr)\le\bigl(1+\varepsilon(\varrho)\bigr)\operatorname{cap}(F_\nu,\Omega),$$

where $\varepsilon(\varrho)\to 0$ as $\varrho\to\infty$. This and (2.5.19) imply

$$\mu(F_\nu)>K\operatorname{cap}\bigl(F_\nu,B_\varrho^{(\nu)}\cap\Omega\bigr), \tag{2.5.20}$$

where

$$K=\frac{1+\alpha}{1+\varepsilon(\varrho)}.$$

Let ϱ be chosen so that the constant K exceeds 1. By (2.5.20) there exists a function u_ν in $\mathfrak{P}(F_\nu,B_\varrho^{(\nu)}\cap\Omega)$ such that

$$\int_{B_\varrho^{(\nu)}} u_\nu^2\,\mathrm{d}\mu>K\int_{B_\varrho^{(\nu)}}(\nabla u_\nu)^2\,\mathrm{d}x.$$

Hence

$$S_1[u_\nu,u_\nu]<-(K-1)\frac{\lambda}{\varrho^2}\int_\Omega u_\nu^2\,\mathrm{d}x,$$

where λ is the first eigenvalue of the Dirichlet problem for the Laplace operator in the unit ball.

Now, Lemma 2 implies that the spectrum of the operator $\tilde{S}_1$ has a limit point to the left of $-(K-1)\lambda\varrho^{-2}$. So we arrive at a contradiction. $\square$

2.5.5 Discreteness of the Negative Spectrum of the Operator $\tilde{S}_h$ for all h

The following assertion contains a necessary and sufficient condition for the discreteness of the negative spectrum of the operator $\tilde{S}_h$ for all $h > 0$. We note that although the measure μ in Theorem 2.5.4 is supposed to have no strong singularities at a finite distance (condition (2.5.17)), the corresponding criterion for the family of all operators $\{\tilde{S}_h\}_{h>0}$ is obtained for an arbitrary nonnegative measure.

Corollary. *The conditions*

$$\lim_{\delta\to 0} \sup\bigl\{\pi(F,\Omega) : F \subset \Omega,\ \operatorname{diam}(F) \le \delta\bigr\} = 0 \tag{2.5.21}$$

and

$$\lim_{\varrho\to\infty} \sup\bigl\{\pi(F,\Omega) : F \subset \Omega\backslash B_\varrho, \operatorname{diam}(F) \le 1\bigr\} = 0 \tag{2.5.22}$$

are necessary and sufficient for the semiboundedness of the form $S_h[u,u]$ in $L_2(\Omega)$ and for the discreteness of the negative spectrum of the operator $\tilde{S}_h$ for all $h > 0$.

We also note that the semiboundedness of the form $S_h[u,u]$ for all $h > 0$ implies that $S_h[u,u]$ is closable in $L_2(\Omega)$ for all $h > 0$.

Proof. Sufficiency. We introduce the notation

$$l(\delta) = \lim_{\varrho\to\infty} \sup\bigl\{\pi(F,\Omega) : F \subset \Omega\backslash B_\varrho,\ \operatorname{diam}(F) \le \delta\bigr\}.$$

First we note that (2.5.21) implies (2.5.15). Therefore, according to Theorem 2.5.4, the condition $l(\delta) \equiv 0$, combined with (2.5.21), is sufficient for the semiboundedness of the form $S_h[u,u]$ and for the discreteness of the negative spectrum of the operator $\tilde{S}_h$ for all $h > 0$.

To prove the sufficiency of the conditions $l(1) = 0$ and (2.5.21) we represent an arbitrary compactum F with $\operatorname{diam}(F) \le \delta'$, $\delta' > \delta$, as the union $\bigcup_{\nu=1}^{N} F_\nu$, where $\operatorname{diam}(F_\nu) \le \delta$ and N depends only on δ'/δ and n. Since $\operatorname{cap}(F,\Omega)$ is a nondecreasing function of F, then

$$\frac{\mu(F)}{\operatorname{cap}(F,\Omega)} \le \sum_{\nu=1}^{N} \frac{\mu(F_\nu)}{\operatorname{cap}(F_\nu,\Omega)}.$$

This and the monotonicity of $l(\delta)$ immediately imply $l(\delta) \le l(\delta') \le N l(\delta)$, which proves the equivalence of the conditions $l(\delta) \equiv 0$ and $l(1) = 0$.

Necessity. If the form $S_h[u,u]$ is semibounded for all $h > 0$, then by Corollary 2.5.3/1 the condition (2.5.21) holds together with (2.5.15). Under (2.5.15) Theorem 2.5.4 implies the necessity of $l(\delta) \equiv 0$ which is equivalent to $l(1) = 0$. The corollary is proved. □

2.5.6 Finiteness of the Negative Spectrum

Theorem. *Suppose that the condition* (2.5.15) *holds.*

1. *If*

$$\lim_{\varrho\to\infty} \sup\bigl\{\pi(F,\Omega) : F\subset \Omega\backslash B_\varrho\bigr\}< \frac{1}{4}, \tag{2.5.23}$$

then the form $S_1[u,u]$ is semibounded from below and closable in $L_2(\Omega)$, and the negative spectrum of the operator $\tilde{S}_1$ is finite.

2. *If the form $S_1[u,u]$ is semibounded from below and closable in $L_2(\Omega)$, and the negative spectrum of the operator $\tilde{S}_1$ is finite, then*

$$\lim_{\varrho\to\infty} \sup\bigl\{\pi(F,\Omega) : F\subset \Omega\backslash B_\varrho\bigr\}\le 1. \tag{2.5.24}$$

Proof. 1. Since for any compactum $F\subset\Omega$

$$\frac{\mu(F)}{\mathrm{cap}(F,\Omega)} \le \frac{\mu(F\backslash B_\varrho)}{\mathrm{cap}(F\backslash B_\varrho,\Omega)} + \frac{\mu(F\cap \bar{B}_\varrho)}{\mathrm{cap}(F\cap\bar{B}_\varrho,\Omega)}, \tag{2.5.25}$$

conditions (2.5.15) and (2.5.23) imply

$$\lim_{\delta\to 0}\bigl\{\pi(F,\Omega) : F\subset\Omega, \mathrm{diam}(F)\le\delta\bigr\}< \frac{1}{4}.$$

According to the last inequality and Theorem 2.5.3, the form $S_1[u,u]$ is semibounded and closable in $L_2(\Omega)$. From (2.5.11) it follows that the metric

$$C\int_\Omega |u|^2\,\mathrm{d}x + S_1[u,u]$$

is equivalent to the metric of the space $\mathring{W}{}_2^1(\Omega)$ for C large enough.

Turning to condition (2.5.23), we note that there exists a positive constant α such that

$$\sup\bigl\{\pi(F,\Omega) : F\subset\Omega, F\subset\Omega\backslash B_{\varrho_0}\bigr\}< \frac{1}{4}-\alpha$$

for sufficiently large ϱ_0. Hence

$$\sup\left\{\frac{\mu_{(\varrho_0)}(F)}{\mathrm{cap}(F,\Omega)} : F\subset\Omega\right\} < \frac{1}{4}-\alpha,$$

and by Theorem 2.5.2 the form

$$(1-4\alpha)\int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_\Omega |u|^2\mu_{(\varrho_0)}(\mathrm{d}x)$$

is positive. Therefore for any $u\in\mathscr{D}(\Omega)$

$$S_1[u,u] \geq 4\alpha \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega |u|^2 \mu^{(\varrho_0)}(\mathrm{d}x).$$

We estimate the right-hand side by inequality (2.4.10) with $\varepsilon = 2\alpha$, $p = 2$, $M(t) = |t|$:

$$S_1[u,u] \geq 2\alpha \int_\Omega |\nabla u|^2 \,\mathrm{d}x - K \int_{B_{2\varrho_0} \cap \Omega} |u|^2 \,\mathrm{d}x. \tag{2.5.26}$$

Passing to the closure of the form $S_1[u,u]$, we obtain (2.5.26) for all $u \in \mathring{W}_2^1(\Omega)$.

Since any set, bounded in $\mathring{L}_2^1(\Omega)$, is compact in the metric

$$\left(\int_{B_\varrho \cap \Omega} |u|^2 \,\mathrm{d}x \right)^{1/2}$$

for any $\varrho > 0$, the form

$$2\alpha \int_\Omega |u|^2 \,\mathrm{d}x - K \int_{B_{2\varrho_0} \cap \Omega} |u|^2 \,\mathrm{d}x,$$

is nonnegative up to a finite-dimensional manifold. Taking (2.5.26) into account, we may say the same for the form $S_1[u,u]$. Now the result follows from Lemma 2.5.4/2.

2. Suppose

$$\lim_{\varrho \to \infty} \sup\{\pi(F,\Omega) : F \subset \Omega \backslash B_\varrho\} > 1 + \alpha,$$

where $\alpha > 0$.

Let $\{\varrho_k\}_{k \geq 1}$ denote an increasing sequence of positive numbers such that

$$\varrho_k \varrho_{k+1}^{-1} \xrightarrow{k \to \infty} 0. \tag{2.5.27}$$

We construct the subsequence $\{\varrho_{k_\nu}\}_{\nu \geq 1}$, defined as follows: Let $k_1 = 1$. We find a compactum F_1, contained in $\Omega \backslash \bar{B}_{\varrho_{k_1}}$, such that $\pi(F_1, \Omega) > 1 + \alpha$. Further we select k_2 to be so large that F_1 is contained in $B_{\varrho_{k_2}}$. Let F_2 denote a compactum in $\Omega \backslash B_{\varrho_{k_2+1}}$ such that $\pi(F_2, \Omega) > 1 + \alpha$. If numbers $k_1, \dots, k_\nu$ and compacta $F_1, \dots, F_\nu$ have already been chosen, then $k_{\nu+1}$ is defined by the condition $F_\nu \subset B_{\varrho_{k_{\nu+1}}}$. The set $F_{\nu+1} \subset \Omega \backslash B_{\varrho_{k_{\nu+1}+1}}$ must be chosen to satisfy the inequality

$$\pi(F_{\nu+1}, \Omega) > 1 + \alpha.$$

Thus we obtained a sequence of compacta $F_\nu \subset \Omega$ with F_ν in the spherical layer $B_{\varrho_{k_{\nu+1}}} \backslash \bar{B}_{\varrho_{k_\nu+1}}$ and subject to the condition

$$\mu(F_\nu) > (1+\alpha)\,\mathrm{cap}(F_\nu, \Omega). \tag{2.5.28}$$

We introduce the notation $R_\nu = B_{\varrho_{k_{\nu+1}+1}} \backslash \bar{B}_{\varrho_{k_\nu}}$. By Lemma 2.5.1/2, where $r = \varrho_{k_\nu}$ and $R = \varrho_{k_\nu+1}$,

$$\operatorname{cap}(F_\nu, \Omega \cap R_\nu) \le \left(1 + (n-2)^{-1} \frac{\varrho_{k_\nu}}{\varrho_{k_\nu+1} - \varrho_{k_\nu}}\right) \operatorname{cap}(F_\nu, \Omega \cap B_{\varrho_{k_{\nu+1}+1}}),$$

which along with condition (2.5.27) implies

$$\operatorname{cap}(F_\nu, \Omega \cap R_\nu) \le \left[1 + o(1)\right] \operatorname{cap}(F_\nu, \Omega \cap B_{\varrho_{k_{\nu+1}+1}}). \tag{2.5.29}$$

From Lemma 2.5.1/1 with $r = \varrho_{k_\nu+1}$ and $R = \varrho_{k_{\nu+1}+1}$ it follows that

$$\operatorname{cap}(F_\nu, \Omega \cap B_{\varrho_{k_{\nu+1}+1}}) \le \left[1 + o(1)\right] \operatorname{cap}(F_\nu, \Omega).$$

According to (2.5.29),

$$\operatorname{cap}(F_\nu, \Omega \cap R_\nu) \le \left[1 + o(1)\right] \operatorname{cap}(F_\nu, \Omega).$$

Hence by (2.5.28), for sufficiently large ν,

$$\mu(F_\nu) > (1 + \alpha') \operatorname{cap}(F_\nu, \Omega \cap R_\nu),$$

where α' is a positive constant.

Now we can find a sequence of functions $u_\nu \in \mathfrak{P}(F_\nu, \Omega \cap R_\nu)$ such that

$$\int_{R_\nu \cap \Omega} u_\nu^2 \mu(\mathrm{d}x) > (1 + \alpha') \int_{R_\nu \cap \Omega} (\nabla u_\nu)^2 \,\mathrm{d}x,$$

which yields the inequality $S_1[u_\nu, u_\nu] < 0$. It remains to note that the supports of the functions u_ν are disjoint and therefore the last inequality holds for all linear combinations of u_ν. This and Lemma 2.5.4/2 imply that the negative spectrum of the operator S_1 is infinite. The theorem is proved. □

2.5.7 Infiniteness and Finiteness of the Negative Spectrum of the Operator $\tilde{S}_h$ for all h

We shall find criteria for the infiniteness and for the finiteness of the negative spectrum of the operator $\tilde{S}_h$ for all h. We underline that here, as in the proof of the discreteness criterion in Corollary 2.5.5, we obtain the necessary and sufficient conditions without additional assumptions on the measure μ.

Corollary 1. *Conditions* (2.5.21) *and*

$$\sup\left\{\pi(F, \Omega) : F \subset \Omega\right\} = \infty \tag{2.5.30}$$

are necessary and sufficient for the semiboundedness of the form $S_h[u, u]$ in $L_2(\Omega)$ and for the infiniteness of the spectrum of the operator $\tilde{S}_h$ for all $h > 0$.

Proof. By Corollary 2.5.3/1, (2.5.21) is equivalent to the semiboundedness of the form $S_h[u, u]$ for all $h > 0$.

We must prove that the criterion

$$\lim_{\varrho\to\infty} \sup\{\pi(F,\Omega) : F \subset \Omega\backslash B_\varrho\} = \infty, \tag{2.5.31}$$

which follows from Theorem 2.5.6, is equivalent to (2.5.30). Obviously, (2.5.30) is a consequence of (2.5.31). Assume that the condition (2.5.30) is valid. Taking into account (2.5.21), we obtain

$$\sup\{\pi(F,\Omega) : F \subset B_\varrho \cap \Omega\} < \infty$$

for any ϱ. On the other hand, (2.5.30) implies

$$\lim_{\varrho\to\infty} \sup\{\pi(F,\Omega) : F \subset B_\varrho \cap \Omega\} = \infty.$$

We choose a sequence $\varrho_\nu \to \infty$ such that

$$\sup\{\pi(F,\Omega) : F \subset B_{\varrho_{\nu+1}} \cap \Omega\} > 2\sup\{\pi(F,\Omega) : F \subset B_{\varrho_\nu} \cap \Omega\}.$$

From this and inequality (2.5.25) we obtain

$$\sup\{\pi(F,\Omega) : F \subset R_{\varrho_\nu,\varrho_{\nu+1}} \cap \Omega\} \geq \sup\{\pi(F,\Omega) : F \subset B_{\varrho_\nu} \cap \Omega\},$$

where $R_{\varrho,\varrho'} = B_{\varrho'}\backslash \bar{B}_\varrho$. Hence

$$\sup\{\pi(F,\Omega) : F \subset R_{\varrho_\nu,\varrho_{\nu+1}} \cap \Omega\} \xrightarrow{\nu\to\infty} \infty,$$

and the result follows. □

Corollary 2. *Conditions* (2.5.21) *and*

$$\lim_{\varrho\to\infty} \sup\{\pi(F,\Omega) : F \subset \Omega\backslash B_\varrho\} = 0 \tag{2.5.32}$$

are necessary and sufficient for the semiboundedness of $\tilde{S}_h$ and for the finiteness of the negative spectrum of $\tilde{S}_h$ for all $h > 0$.

The necessity and sufficiency of conditions (2.5.21) and (2.5.32) immediately follow from Theorem 2.5.6.

2.5.8 Proofs of Lemmas 2.5.1/1 and 2.5.1/2

The following facts are well known (cf. Landkof [477]). For $n \geq 3$ and for any open set $\Omega \subset \mathbb{R}^n$ there exists a unique Green function $G(x,y)$ of the Dirichlet problem for the Laplace operator.

Let μ be a nonnegative measure in Ω. Let V^μ denote the Green potential of the measure μ, i.e.,

$$V^\mu(x) = \int_\Omega G(x,y)\mu(\mathrm{d}y).$$

Obviously, V^{μ} is a harmonic function outside the support of the measure μ. There exists a unique capacitary distribution of a compactum F with respect to Ω, i.e., a measure μ_F, supported on F, such that $V^{\mu_F}(x) \le 1$ in Ω and

$$\mu_F(F) = (n-2)^{-1}\omega_n^{-1} \operatorname{cap}(F, \Omega).$$

The potential V^{μ_F} is called the capacitary potential of F relative to Ω. If F is the closure of an open set with C^∞-smooth boundary, then V^{μ_F} is a smooth function in $\overline{\Omega\backslash F}$, equal to unity on F, and continuous in Ω.

Proof of Lemma 2.5.1/1. Using the continuity of the capacity from the right, we can easily reduce the proof for an arbitrary compactum to the consideration of a compactum $F \subset B_r \cap \Omega$, which is the closure of an open set with a C^∞-smooth boundary.

Let V^{μ_F} denote the capacitary potential of F relative to Ω and let η denote a continuous piecewise linear function, equal to unity on $[0, r]$, and to zero outside $[0, R]$.

The function $u(x) = \eta(|x|)V^{\mu_F}(x)$ can be approximated in $\mathring{L}_2^1(\Omega \cap B_R)$ by functions in $\mathfrak{N}(F, B_R \cap \Omega)$. Hence

$$\operatorname{cap}(F, B_R \cap \Omega) \le \int_{B_R\cap\Omega} |\nabla u|^2 \,\mathrm{d}x. \tag{2.5.33}$$

We extend V^{μ_F} to be zero outside Ω. It is readily checked that

$$\int_{B_R\cap\Omega} |\nabla u|^2 \,\mathrm{d}x = \int_{B_R} \left|\nabla V^{\mu_F}\right|^2 \eta^2 \,\mathrm{d}x + A + B, \tag{2.5.34}$$

where

$$A = \frac{1}{R-r}\int_{\partial B_r} \left(V^{\mu_F}\right)^2 s(\mathrm{d}x),$$
$$B = \frac{n-1}{(R-r)^2}\int_{B_R\backslash B_r} \left(V^{\mu_F}\right)^2 \frac{R-|x|}{|x|} \,\mathrm{d}x.$$

Obviously,

$$\int_{\Omega} \left|\nabla V^{\mu_F}\right|^2 \eta^2 \,\mathrm{d}x \le \operatorname{cap}(F, \Omega). \tag{2.5.35}$$

Now we note that

$$\int_{\partial B_\varrho} V^{\mu_F} s(\mathrm{d}x) = \int_F \mu_F(\mathrm{d}y) \int_{\partial B_\varrho} G(x,y) s(\mathrm{d}x) \le \int_F \mu_F(\mathrm{d}y) \int_{\partial B_\varrho} \frac{s(\mathrm{d}x)}{|x-y|^{n-2}}.$$

The integral over ∂B_ϱ is a single-layer potential and it is equal to a constant on ∂B_ϱ. Hence, for $y \in B_\varrho$,

$$\int_{\partial B_\varrho} \frac{s(\mathrm{d}x)}{|x-y|^{n-2}} = \omega_n \varrho.$$

Thus

$$\int_{\partial B_\varrho} V^{\mu_F} s(\mathrm{d}x) \le (n-2)^{-1} \varrho \operatorname{cap}(F, \Omega). \tag{2.5.36}$$

The following inequality is a direct consequence of the maximum principle

$$V^{\mu_F}(x) \le \frac{r^{n-2}}{|x|^{n-2}} \quad \text{for } |x| \ge r.$$

Now, the bound for A is

$$A \le (R-r)^{-1} \int_{\partial B_r} V^{\mu_F} s(\mathrm{d}x) \le \frac{r}{(n-2)(R-r)} \operatorname{cap}(F, \Omega). \tag{2.5.37}$$

We introduce spherical coordinates (ϱ, ω) in the integral B. Then

$$B = \frac{n-1}{(R-r)^2} \int_r^R \varrho^{n-2}(R-\varrho)\,\mathrm{d}\varrho \int_{\partial B_\varrho} \left(V^{\mu_F}\right)^2 \omega(\mathrm{d}x).$$

Hence

$$B \le \frac{(n-1)r^{n-2}}{R-r} \int_r^R \mathrm{d}\varrho \int_{\partial B_\varrho} V^{\mu_F} \omega(\mathrm{d}x).$$

Using (2.5.36), we obtain

$$B \le \frac{n-1}{n-2} \frac{r^{n-2}}{R-r} \int_r^R \varrho^{2-n}\,\mathrm{d}\varrho \operatorname{cap}(F, \Omega),$$

which along with (2.5.33)–(2.5.35) and (2.5.37) gives the final result. □

Proof of Lemma 2.5.1/2. The general case can be easily reduced to the consideration of a compactum $F \subset \Omega \backslash \bar{B}_R$, which is the closure of an open set with a smooth boundary. Let V^{μ_F} denote the capacitary potential of F relative to Ω, extended by zero outside Ω.

The function

$$u(x) = \begin{cases} V^{\mu_F}(x) & \text{for } x \in \Omega \backslash B_R, \\ \frac{R(|x|-r)}{|x|(R-r)} V^{\mu_F}(x) & \text{for } x \in \Omega \cap (B_R \backslash B_r), \\ 0 & \text{for } x \in \Omega \cap B_r, \end{cases}$$

can be approximated in $\mathring{L}_2^1(\Omega \backslash \bar{B}_r)$ by the functions in $\mathfrak{N}(F, \Omega \backslash \bar{B}_r)$. Therefore,

$$\operatorname{cap}(F, \Omega \backslash \bar{B}_r) \le \int_{\Omega \backslash \bar{B}_r} (\nabla u)^2\,\mathrm{d}x.$$

This implies

$$\operatorname{cap}(F, \Omega \backslash \bar{B}_r) \le \int_\Omega \left(\nabla V^{\mu_F}\right)^2 \mathrm{d}x + \frac{r}{R(R-r)} \int_{\partial B_R} (V^{\mu_F})^2 s(\mathrm{d}x). \tag{2.5.38}$$

Since $V^{\mu_F} \le 1$ and

$$\int_{\partial B_R} V^{\mu_F} s(\mathrm{d}x) \le (n-2)^{-1} R \,\mathrm{cap}(F, \Omega)$$

(cf. (2.5.36)), it follows that

$$\frac{r}{R(R-r)} \int_{\partial B_r} \left(V^{\mu_F}\right)^2 s(\mathrm{d}x) \le \frac{r}{R-r}(n-2)^{-1} \,\mathrm{cap}(F, \Omega),$$

which together with (2.5.38) completes the proof. □

2.5.9 Comments to Sect. 2.5

The presentation follows the author's paper [534] and the main results were announced in Maz'ya [531]. A number of results on the spectrum of the Schrödinger operator are presented in the monograph by Glazman [309] who used the so-called splitting method. Birman [100, 101] established some important results in the perturbation theory of quadratic forms in Hilbert spaces. In particular, he proved that the discreteness (the finiteness) of the negative spectrum of the operator $S_h = -h\Delta - p(x)$ in $\mathbb{R}^n$ for $p(x) \ge 0$ and for all $h > 0$ is equivalent to the compactness of the embedding of $W_2^1(\mathbb{R}^n)(\mathring{L}_2^1(\mathbb{R}^n))$ into the space with the norm

$$\left(\int_{\mathbb{R}^n} |u|^2 p(x) \,\mathrm{d}x\right)^{1/2}.$$

Using such criteria, Birman derived the necessary or sufficient conditions for the discreteness, finiteness, or infiniteness of the negative spectrum of S_h for all $h > 0$. The statement of these conditions makes no use of the capacity. The results of Birman's paper [101] were developed in the author's paper [534] the content of which is followed here.

The theorems of Sect. 2.5 turned out to be useful in the study of the asymptotic behavior of eigenvalues of the Dirichlet problem for the Schrödinger operator. Rozenblum [683] considered the operator $H = -\Delta + q(x)$ in $\mathbb{R}^n$ with $q = q_+ - q_-$, where $q_- \in L_{n/2,\mathrm{loc}}, n \ge 3$. We state one of his results. Let a cubic grid be constructed in $\mathbb{R}^n$ with d as the edge length of each cube and let $F(d)$ be the union of those cubes $\mathscr{Q}$ of the grid that satisfies the condition

$$\sup\left\{\frac{\int_E q_-(x)\,\mathrm{d}x}{\mathrm{cap}(E)} : E \subset 2\mathscr{Q}\right\} > \gamma,$$

where $2\mathscr{Q}$ is the concentric homothetic cube having edge length $2d$, $\gamma = \gamma(n)$ is a large enough number.

Then, for $\lambda > 0$, the number $\mathscr{N}(-\lambda, H)$ of eigenvalues of H that are less than $-\lambda$ satisfies the inequality

$$\mathscr{N}(-\lambda, H) \le c_1 \int_{F(c_2\lambda^{1/2})} \big(c_3\lambda - q(x)\big)_+^{n/2}\,\mathrm{d}x,$$

where c_1, c_2, and c_3 are certain constants depending only on n.

In the case $\Omega = \mathbb{R}^n$ by Theorem 2.5.2, the inequality (2.5.4) with $\Omega = \mathbb{R}^n$ holds if and only if

$$\sup_F \frac{\mu(F)}{\operatorname{cap}(F)} < \infty,$$

where F is an arbitrary compact set in $\mathbb{R}^n$. For the same case, other criteria for the validity of (2.5.4) are known. The following one is due to Kerman and Sawyer [420] (see Theorem 11.5/1 of the present book):

For every open ball B in $\mathbb{R}^n$,

$$\int_B \int_B \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{|x-y|^{n-2}} \le c\mu(B).$$

Another two criteria for (2.5.4) were obtained by Maz'ya and Verbitsky [591]:

(i) *The pointwise inequality*

$$I_1(I_1\mu)^2(x) \le cI_1(\mu)(x) < \infty \quad \text{a.e.}$$

holds, where I_1 stands for the Riesz potential of order 1, *i.e.*, $I_1\mu = |x|^{1-n} \star \mu$.

(ii) *For every compact set $F \subset \mathbb{R}^n$,*

$$\int_F (I_1\mu)^2\,\mathrm{d}x \le c\operatorname{cap}(F).$$

One more condition necessary and sufficient for (2.5.4) was found by Verbitsky [775]:

For every dyadic cube P in $\mathbb{R}^n$,

$$\sum_{Q\subset P} \left[\frac{\mu(Q)}{|Q|^{1-1/n}}\right]^2 |Q| \le c\mu(P),$$

where the sum is taken over all dyadic cubes Q contained in P and c does not depend on P.

We now state the main result of the paper [592] by the author and Verbitsky, characterizing arbitrary complex-valued distributions V subject to the inequality

$$\left|\int_{\mathbb{R}^n} |u|^2 V\,\mathrm{d}x\right| \le c\int_{\mathbb{R}^n} |\nabla u|^2\,\mathrm{d}x \quad \text{for all } u \in \mathscr{D}. \tag{2.5.39}$$

This characterization reduces the case of distributional potentials V to that of nonnegative absolutely continuous weights. (Cf. Sect. 1.3.4, where similar statements are established for functions of one variable.)

Theorem. *Let $V \in \mathscr{D}'$, $n > 2$. Then the inequality (2.5.39) holds, if and only if there is a vector field $\mathbf{\Gamma} \in L_2(\mathbb{R}^n, \mathrm{loc})$ such that $V = \operatorname{div} \mathbf{\Gamma}$ and*

$$\int_{\mathbb{R}^n} \bigl|u(x)\bigr|^2 \bigl|\mathbf{\Gamma}(x)\bigr|^2 \,\mathrm{d}x \le C \int_{\mathbb{R}^n} \bigl|\nabla u(x)\bigr|^2 \,\mathrm{d}x$$

for all $u \in \mathscr{D}$. The vector field $\mathbf{\Gamma}$ can be chosen in the form $\mathbf{\Gamma} = \nabla \Delta^{-1} V$.

2.6 Properties of Sobolev Spaces Generated by Quadratic Forms with Variable Coefficients

2.6.1 Degenerate Quadratic Form

In the preceding sections of the present chapter we showed that rather general inequalities, containing the integral $\int_\Omega [\Phi(x, \nabla u)]^p \,\mathrm{d}x$, are equivalent to isocapacitary inequalities that relate (p, Φ)-capacity and measures. Although such criteria are of primary interest, we should note that their verification in particular cases is often difficult. Even for rather simple quadratic forms

$$\bigl[\Phi(x, \xi)\bigr]^2 = \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j,$$

the estimates for the corresponding capacities by measures are unknown.

Thus, the general necessary and sufficient conditions obtained in the present chapter cannot diminish the value of straightforward methods of investigation of integral inequalities without using capacity. In the present section this will be illustrated, using as an example the quadratic form

$$\bigl[\Phi(x, \xi)\bigr]^2 = \bigl(|x_n| + |x'|^2\bigr)\xi_n^2 + |\xi'|^2,$$

where $x' = (x_1, \dots, x_{n-1})$, $\xi' = (\xi_1, \dots, \xi_{n-1})$.

By Corollary 2.3.4, the inequality

$$\int_{\mathbb{R}^{n-1}} \bigl[u(x', 0)\bigr]^2 \,\mathrm{d}x' \le c \int_{\mathbb{R}^n} \bigl[\Phi(x, \nabla u)\bigr]^2 \,\mathrm{d}x \tag{2.6.1}$$

holds for all $u \in \mathscr{D}(\mathbb{R}^n)$ if and only if

$$m_{n-1}\bigl(\{x \in g,\ x_n = 0\}\bigr) \le c(2, \Phi)\text{-cap}(g)$$

for any admissible set g. A straightforward proof of the preceding isoperimetric inequality is unknown to the author. Nevertheless, the estimate (2.6.1) is true and will be proved in the sequel.

Theorem 1. *Let*

$$\big[\varPhi(x,\nabla u)\big]^2 = \big(|x_n| + |x'|^2\big)(\partial u/\partial x_n)^2 + \sum_{i=1}^{n-1}(\partial u/\partial x_i)^2.$$

Then (2.6.1) *is valid for all* $u \in \mathscr{D}(\mathbb{R}^n)$.

Proof. Let the integral in the right-hand side of (2.6.1) be denoted by $Q(u)$. For any $\delta \in (0,1/2)$ we have

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \big|u(x',0)\big|^2\,\mathrm{d}x' &\le 2\int_{\mathbb{R}^n} \frac{(|x_n|+|x'|^2)^{1/2}}{|x_n|^{(1-\delta)/2}|x'|^\delta}\left|u\frac{\partial u}{\partial x_n}\right|\mathrm{d}x\\ &\le 2\big[Q(u)\big]^{1/2}\left(\int_{\mathbb{R}^n}|x_n|^{\delta-1}|x'|^{-2\delta}|u|^2\,\mathrm{d}x\right)^{1/2}. \quad (2.6.2)\end{aligned}$$

To give a bound for the last integral we use the following well-known generalization of the Hardy–Littlewood inequality:

$$\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}^{n-1}}\frac{f(y)\,\mathrm{d}y}{|x'-y|^{n-1-\delta}}\right)^2\frac{\mathrm{d}x'}{|x'|^{2\delta}} \le c\int_{\mathbb{R}^{n-1}}\big[f(y)\big]^2\,\mathrm{d}y. \quad (2.6.3)$$

(For the proof of this estimate see Lizorkin [505]. It can also be derived as a corollary to Theorem 1.4.1/2.) Since the convolution with the kernel $|x'|^{\delta+1-n}$ corresponds to the multiplication by $|\xi'|^{-\delta}$ of the Fourier transform, (2.6.3) can be written as

$$\int_{\mathbb{R}^{n-1}}|u|^2|x'|^{-2\delta}\,\mathrm{d}x' \le c\int_{\mathbb{R}^{n-1}}\big[(-\varDelta_{x'})^{\delta/2}u\big]^2\,\mathrm{d}x',$$

where $(-\varDelta_{x'})^{\delta/2}$ is the fractional power of the Laplace operator. Now we find that the right-hand side in (2.6.2) does not exceed

$$c\left(Q(u) + \int_{\mathbb{R}^n}|x_n|^{\delta-1}\big[(-\varDelta_{x'})^{\delta/2}u\big]^2\,\mathrm{d}x\right). \quad (2.6.4)$$

From the almost obvious estimate

$$\int_0^\infty g^2t^{\delta-1}\,\mathrm{d}t \le c\left(\int_0^\infty (g')^2t\,\mathrm{d}t + \int_0^\infty g^2\,\mathrm{d}t\right),$$

it follows that

$$\begin{aligned}&|\xi'|^{2\delta}\int_{\mathbb{R}^n}\big|(F_{x'\to\xi'}u)(\xi',x_n)\big|^2|x_n|^{\delta-1}\,\mathrm{d}x_n\\ &\quad\le c\left(\int_{\mathbb{R}^1}\left|\left(F_{x'\to\xi'}\frac{\partial u}{\partial x_n}\right)(\xi',x_n)\right|^2|x_n|\,\mathrm{d}x_n\right.\\ &\qquad\left. + |\xi'|^2\int_{\mathbb{R}^1}\big|(F_{x'\to\xi'}u)(\xi',x_n)\big|^2\,\mathrm{d}x_n\right),\end{aligned}$$

where $F_{x'\to\xi'}$ is the Fourier transform in $\mathbb{R}^{n-1}$. So the second integral in (2.6.4) does not exceed

$$c\int_{\mathbb{R}^n}\big(|x_n|(\partial u/\partial x_n)^2+(\nabla_{x'}u)^2\big)\,\mathrm{d}x.$$

The result follows. □

The next assertion shows that Theorem 1 is exact in a certain sense.

Theorem 2. *The space of restrictions to* $\mathbb{R}^{n-1}=\{x\in\mathbb{R}^n : x_n=0\}$ *of functions in the set* $\{u\in\mathscr{D}(\mathbb{R}^n) : Q(u)+\|u\|^2_{L_2(\mathbb{R}^n)}\le 1\}$ *is not relatively compact in* $L_2(B_1^{(n-1)})$, *where* $B_\varrho^{(n-1)}=\{x'\in\mathbb{R}^{n-1} : |x'|<\varrho\}$.

Proof. Let φ denote a function in $C_0^\infty(B_1^{(n-1)})$ such that $\varphi(y)=\varphi(-y)$, $\|\varphi\|_{L_2(\mathbb{R}^{n-1})}=1$ and introduce the sequence $\{\varphi_m\}_{m=1}^\infty$ defined by $\varphi_m(y)=m^{(n-1)/2}\varphi(my)$. Since this sequence is normalized and weakly convergent to zero in $L_2(B_1^{(n-1)})$, it contains no subsequences converging in $L_2(B_1^{(n-1)})$. Further, let $\{v_m\}_{m=1}^\infty$ be the sequence of functions in $\mathbb{R}^n$ defined by

$$v_m(x)=F^{-1}_{\eta'\to x'}\exp\big\{-\langle\eta\rangle^2|x_n|\big\}F_{x'\to\eta'}\varphi_m,$$

where $\eta\in\mathbb{R}^{n-1}$, $\langle\eta\rangle=(|\eta|^2+1)^{1/2}$.

Consider the quadratic form

$$T(u)=\int_{\mathbb{R}^n}\left[\big(|x_n|+|x'|^2\big)\left|\frac{\partial u}{\partial x_n}\right|^2+|\nabla_{x'}u|^2+|u|^2\right]\mathrm{d}x.$$

It is clear that

$$T(u)=(2\pi)^{1-n}\int_{\mathbb{R}^n}\left(|x_n|\left|\frac{\partial Fu}{\partial t}\right|^2+\left|\frac{\partial}{\partial t}\nabla_\eta Fu\right|^2+\langle\eta\rangle^2|Fu|^2\right)\mathrm{d}\eta\,\mathrm{d}x_n.$$

Differentiating the function $T(v_m)$, we obtain from the last equality that $T(v_m)$ does not exceed

$$c\int_{\mathbb{R}^n}\big[\big(1+\langle\eta\rangle^2|x_n|+\langle\eta\rangle^4|x_n|^3\big)\langle\eta\rangle^2|F\varphi_m|^2+\langle\eta\rangle^4|\nabla F\varphi_m|^2\big]$$
$$\times\exp\big(-2\langle\eta\rangle^2|x_n|\big)\,\mathrm{d}\eta\,\mathrm{d}x_n.$$

Thus we obtain

$$\begin{aligned}T(v_m)&\le c\int_{\mathbb{R}^{n-1}}\big(\langle\eta\rangle^2|\nabla F\varphi_m|^2+|F\varphi_m|^2\big)\,\mathrm{d}\eta\\&=c_1\left(\sum_{i=1}^{n-1}\|x_i\varphi_m\|^2_{W_2^1(\mathbb{R}^{n-1})}+\|\varphi_m\|^2_{L_2(\mathbb{R}^{n-1})}\right)\le\text{const}.\end{aligned}$$

Let $\psi\in C_0^\infty(B_2^{(n-1)})$, $\psi=1$ on $B_1^{(n-1)}$. It is clear that $(v_m\psi)|_{\mathbb{R}^{n-1}}=\varphi_m$ and $T(v_m\psi)\le$ const. The sequence $\{v_m\psi/(T(v_m\psi))^{1/2}\}_{m=1}^\infty$ is the required counterexample. The theorem is proved. □

2.6.2 Completion in the Metric of a Generalized Dirichlet Integral

Consider the quadratic form

$$S[u,u] = \int_{\mathbb{R}^n} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + u^2 \right) \mathrm{d}x,$$

where $\|a_{ij}(x)\|_{i,j=1}^n$ is a uniformly positive definite matrix, whose elements $a_{ij}(x)$ are smooth real functions.

Let the completion of $C_0^{0,1}$ with respect to the norm $(S[u,u])^{1/2}$ be denoted by $\overset{\circ}{H}(S)$. Further, we introduce the space $H(S)$ obtained as the completion with respect to this norm of the set of functions in $C^{0,1}$ with the finite integral $S[u,u]$.

If the elements of the matrix $\|a_{ij}\|_{i,j=1}^n$ are bounded functions, then $\overset{\circ}{H}(S) = \overset{\circ}{W}{}_2^1$, $H(S) = W_2^1$ and both spaces, obviously, coincide. It is also known that $\overset{\circ}{H}(S) = H(S)$ if the functions a_{ij} do not grow too rapidly at infinity. Here we consider the problem of the coincidence of $\overset{\circ}{H}(S)$ and $H(S)$ in the general case.

Definition. Let $E \subset \mathbb{R}^n$. In the present subsection the set E is said to have finite $H(S)$ capacity if there exists a function $u \in C^{0,1} \cap H(S)$ that is equal to 1 on E.

Theorem 1. *The spaces $\overset{\circ}{H}(S)$ and $H(S)$ coincide if and only if, for an arbitrary domain G with finite $H(S)$ capacity, there exists a sequence of functions $\{\varphi_m\}_{m\geq 1}$ in $C_0^{0,1}$ that converges in measure to unity on G and is such that*

$$\lim_{m\to\infty} \int_G a_{ij}(x) \frac{\partial \varphi_m}{\partial x_i} \frac{\partial \varphi_m}{\partial x_j} \, \mathrm{d}x = 0. \tag{2.6.5}$$

Before we proceed to the proof, we note that if G is a bounded domain then the sequence $\{\varphi_m\}_{m\geq 1}$ always exists. We can put $\varphi_m = \varphi$ where $\varphi \in C_0^{0,1}$, $\varphi = 1$ on G.

Proof. Sufficiency. We show that any function $u \in C^{0,1} \cap H(S)$ can be approximated in $H(S)$ by functions in $\overset{\circ}{H}(S)$. Without loss of generality we may assume that $u \geq 0$.

First, we note that if $t > 0$ then $\mathscr{L}_t = \{x : u(x) > t\}$ is a set of finite $H(S)$ capacity. In fact, the function $v(x) = t^{-1} \min\{u(x), t\}$ equals unity on $\mathscr{L}_t$, satisfies a Lipschitz condition, and $S[v,v] \leq t^{-2} S[u,u] < \infty$.

From the Lebesgue theorem it follows that the sequences $\min\{u, m\}$, $(u - m^{-1})_+$, $m = 1, 2, \ldots$, converge to u in $H(S)$ (see Sect. 5.1.2). So we may assume from the very beginning that u is bounded and vanishes on the exterior of a bounded set G of finite $H(S)$ capacity.

We denote the complements of the set G by G_j and then define the sequence

$$u^{(m)}(x) = \begin{cases} u(x) & \text{for } x \in \bigcup_{j\le m} G_j, \\ 0 & \text{for } x \in \mathbb{R}^n \setminus \bigcup_{j\le m} G_j, \end{cases}$$

$j = 1, 2, \ldots$. It is clear that $u^{(m)} \to u$ in $H(S)$ as $m \to \infty$. Since each $u^{(m)}$ vanishes on the exterior of a finite number of domains, we may assume without loss of generality that G is a domain.

Let $\{\varphi_m\}$ be the sequence of functions specified for the domain G in the statement of the theorem. Replacing $\{\varphi_m\}$ by the sequence $\{\psi_m\}$, defined by $|\psi_m| = \min\{2, |\varphi_m|\}$, sgn ψ_m = sgn φ_m, we obtain a bounded sequence with the same properties. Obviously, $\psi_m u \in \mathring{H}(S)$ and $\psi_m u \to u$ in L_2. Moreover,

$$\begin{aligned}\int_{\mathbb{R}^n} a_{ij} \frac{\partial}{\partial x_i}(u - u\psi_m)\frac{\partial}{\partial x_j}(u - u\psi_m)\,\mathrm{d}x \\ \le 2\int_G (1-\psi_m)^2 a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,\mathrm{d}x + 2\int_G u^2 a_{ij}\frac{\partial \psi_m}{\partial x_i}\frac{\partial \psi_m}{\partial x_j}\,\mathrm{d}x. \end{aligned} \tag{2.6.6}$$

Since the sequence

$$(1-\psi_m)^2 \sum_{i,j} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j},$$

converges to zero in G with respect to the measure m_n and is majorized by the integrable function

$$9\sum_{i,j} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j},$$

the first integral on the right in (2.6.6) converges to zero. The convergence to zero of the second integral follows from the boundedness of u and equality (2.6.5). Thus $u\psi_m \to u$ in $H(S)$. The required approximation is constructed.

Necessity. Let G be an arbitrary domain in $\mathbb{R}^n$ with finite $H(S)$ capacity. Let u denote a function in $C^{0,1} \cap H(S)$, which is equal to unity on G. Since $H(S)$ and $\mathring{H}(S)$ coincide u can be approximated in $H(S)$ by the sequence $\{\varphi_m\}_{m\ge1}$ contained in $C_0^{0,1}$. Noting that $u = 1$ on G and $\varphi_m \to u$ in $L_2(G)$, we obtain that $\varphi_m \to 1$ in G in measure. Furthermore,

$$\int_G a_{ij}\frac{\partial \varphi_m}{\partial x_i}\frac{\partial \varphi_m}{\partial x_j}\,\mathrm{d}x = \int_G a_{ij}\frac{\partial}{\partial x_i}(u-\varphi_m)\frac{\partial}{\partial x_j}(u-\varphi_m)\,\mathrm{d}x \xrightarrow{m\to\infty} 0.$$

So the theorem is proved. □

Although the above result is not very descriptive, it facilitates verification of concrete conditions for coincidence or noncoincidence of $H(S)$ and $\mathring{H}(S)$. We now present some of them.

Theorem 2. (cf. Maz'ya [536]). *The spaces $H(S)$ and $\mathring{H}(S)$ coincide provided $n = 1$ or $n = 2$.*

Proof. Taking into account Theorem 1 and the discussion that follows its statement, we arrive at the equality $H(S) = \mathring{H}(S)$ if we show that any domain G with finite $H(S)$ capacity is bounded. The case $n = 1$ is obvious. Let $n = 2$ and $u \in C^{0,1} \cap H(S)$, $u = 1$ on G.

Let O and P denote arbitrary points in G and let the axis Ox_2 be directed from O to P. Then

$$S[u,u] \geq c \int_0^{|P|} \mathrm{d}x_2 \int_{\mathbb{R}^1} \left(\left(\frac{\partial u}{\partial x_1} \right)^2 + u^2 \right) \mathrm{d}x_1 \geq c_1 \int_0^{|P|} \max_{x_1} \big[u(x_1, x_2) \big]^2 \, \mathrm{d}x_2.$$

Taking into account that G is a domain and $u = 1$ on G we arrive at

$$\max_{x_1} \big[u(x_1, x_2) \big]^2 \geq 1.$$

Therefore $\operatorname{diam}(G) \leq cS[u,u]$, which completes the proof. □

The following assertion shows that for $n \geq 3$ the form $S[u,u]$ must be subjected to certain conditions by necessity. The result is due to Uraltseva [769]. Our proof, though different, is based on the same idea.

Theorem 3. *Let $n > 2$. Then there exists a form $S[u,u]$ for which $H(S) \neq \mathring{H}(S)$.*

Proof. 1. Consider the domain $G = \{x : 0 < x_n < \infty, |x'| < f(x_n)\}$ where $x' = (x_1, \ldots, x_{n-1})$ and f is a positive decreasing function in $C^\infty[0,\infty)$, $f(0) < 1$. For $x \notin G$ we put $a_{ij}(x) = \delta_i^j$.

For arbitrary functions a_{ij} on G, for any $u \in C^{0,1}$, $u = 1$ on G, we have $S[u,u] = \|u\|^2_{W_2^1}$. This implies that G is a domain with finite $H(S)$ capacity if and only if $\operatorname{cap}(G) < \infty$ (here, as before, cap is the Wiener capacity, i.e., 2-cap). Clearly,

$$\begin{aligned}
\operatorname{cap}(G) &\leq \sum_{j=0}^{\infty} \operatorname{cap}\big(\{x \in G : j \leq x_n \leq j+1\}\big) \\
&\leq \sum_{j=0}^{\infty} \operatorname{cap}\big(\{x : |x'| \leq f(j), j \leq x_n \leq j+1\}\big).
\end{aligned}$$

This and the well-known estimates for the capacity of the cylinder (cf. Landkof [477] or Proposition 13.1.3/1 of the present book) yield

$$\operatorname{cap}(G) \leq c \sum_{j=0}^{\infty} \big[f(j) \big]^{n-3} \quad \text{for } n > 3,$$

$$\operatorname{cap}(G) \leq c \sum_{j=0}^{\infty} \big| \log f(j) \big|^{-1} \quad \text{for } n = 3.$$

Therefore G is a domain with finite $H(S)$ capacity provided

$$\int_0^\infty \big[f(t)\big]^{n-3}\,\mathrm{d}t < \infty \quad \text{for } n > 3,$$
$$\int_0^\infty \big|\log f(t)\big|^{-1}\,\mathrm{d}t < \infty \quad \text{for } n = 3.$$

2. In the interior of G we define the quadratic form $a_{ij}(x)\xi_i\xi_j$ by

$$a_{ij}(x)\xi_i\xi_j = \xi^2 + \left(\frac{g(x_n)}{f(x_n)}\right)^{n-1}\eta(x)\left(f'(x_n)\sum_{i=1}^{n-1} x_i\xi_i + \xi_n\right)^2,$$

where $\eta \in C_0^\infty(G)$, $0 \le \eta \le 1$, $\eta(x) = 1$ on the set $\{x : 1 < x_n < \infty, |x'| < \frac{1}{2}f(x_n)\}$, and g is an arbitrary positive function on $[0,\infty)$ satisfying the condition

$$\int_0^\infty \big[g(t)\big]^{1-n}\,\mathrm{d}t < \infty.$$

Using the change of variable $x_n = y_n$, $x_i = f(y_n)y_i$, $1 \le i \le n-1$, we map G onto the cylinder $\{y : 0 < y_n < \infty, |y'| < 1\}$. Obviously,

$$\int_G \left(\frac{g(x_n)}{f(x_n)}\right)^{n-1}\eta(x)\left(f'(x_n)\sum_{i=1}^{n-1} x_i\frac{\partial\varphi}{\partial x_i} + \frac{\partial\varphi}{\partial x_n}\right)^2\mathrm{d}x$$
$$\ge \int_C \big[g(y_n)\big]^{n-1}\left(\frac{\partial\varphi}{\partial y_n}\right)^2\mathrm{d}y,$$

where $C = \{y : 1 < y_n < \infty, |y'| < \frac{1}{2}\}$. Applying the Cauchy inequality to the last integral we obtain

$$\int_1^\infty \big[g(t)\big]^{1-n}\,\mathrm{d}t\int_G a_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\,\mathrm{d}x \ge \int_{|y'|\le 1/2}\left(\int_1^\infty\left|\frac{\partial\varphi}{\partial y_n}\right|\mathrm{d}y_n\right)^2\mathrm{d}y'.$$

If $\varphi \in C_0^{0,1}$ then the right-hand side exceeds

$$\int_{|y'|<1/2}\max_{1<y_n<\infty}\big[\varphi(y',y_n)\big]^2\,\mathrm{d}y' \ge \int_{C_1}\varphi^2\,\mathrm{d}y,$$

where $C_1 = \{y \in C : y_n < 2\}$. Passing to the variables $x_1,\dots,x_n$ on the right, we arrive at

$$\int_1^\infty \big[g(t)\big]^{1-n}\,\mathrm{d}t\int_G a_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\,\mathrm{d}x \ge \int_{G_1}\varphi^2\frac{\mathrm{d}x}{[f(x_n)]^{n-1}},$$

where $G_1 = \{x : |x'| < \frac{1}{2}f(x_n), 1 < x_n < 2\}$. Thus, for any sequence $\{\varphi_m\}_{m\ge1}$ of functions in $C_0^{0,1}$ converging in measure to unity in G we have

$$\liminf_{m\to\infty}\int_G a_{ij}\partial\varphi_m/\partial x_i\cdot\partial\varphi_m/\partial x_j\,\mathrm{d}x > 0.$$

To conclude the proof, it remains to make use of Theorem 1. □

Theorem 3 has an interesting application to the problem of the self-adjointness of an elliptic operator in $L_2(\mathbb{R}^n)$, $n \geq 3$ (cf. Uraltseva [769]). Let the operator

$$u \to S_0 u = -\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + u$$

be defined on C_0^∞. If $\|a_{ij}\|_{i,j=1}^n$ is the matrix constructed in Theorem 3, then $H(S) = \overset{\circ}{H}(S)$ and hence there exists a function $w \in H(S)$, which does not vanish identically and is orthogonal to any $v \in C_0^\infty$ in $H(S)$, i.e.,

$$0 = \int_{\mathbb{R}^n}\left(a_{ij}\frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j} + wv\right)\mathrm{d}x = \int_{\mathbb{R}^n} wS_0 v\,\mathrm{d}x.$$

Therefore the range of the closure $\bar{S}_0$ does not coincide with L_2. If $\bar{S}_0$ is self-adjoint then $w \in \mathrm{Dom}(\bar{S}_0)$ and $\bar{S}_0 w = 0$. This obviously implies $w = 0$. We arrived at a contradiction, which means that $\bar{S}_0$ is not self-adjoint. Thus, the condition of the uniform positive definiteness of the matrix $\|a_{ij}\|_{i,j=1}^n$ alone is insufficient for the self-adjointness of $\bar{S}_0$.

2.6.3 Comments to Sect. 2.6

The results of Sect. 2.6.1 are due to the author [556], Sect. 2.6. We note that the proof of Theorem 2.6.1/2 implies nondiscreteness of the spectrum of the Steklov problem

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a(x)u = 0 \quad \text{in } \Omega,$$
$$\sum_{i,j=1}^n a_{ij}\cos(\nu, x_j)\frac{\partial u}{\partial x_i} = \lambda u \quad \text{on } \partial\Omega,$$

under the condition that $\partial\Omega$ is characteristic at least at one point. Here ν is a normal to $\partial\Omega$ and the matrix $\|a_{ij}\|_{i,j=1}^n$ is nonnegative $a(x) > 0$. The coefficients a_{ij}, a, and the surface $\partial\Omega$ are assumed to be smooth.

In conclusion, we note that the topic of Sect. 2.6.2 was also considered in the paper by S. Laptev [482] who studied the form

$$S[u, u] = \int_{\mathbb{R}^n}\left(\alpha(x)(\nabla u)^2 + u^2\right)\mathrm{d}x,$$

where $\alpha(x) \geq \text{const} > 0$. He presented an example of a function α for which $H(S) \neq \overset{\circ}{H}(S)$ and showed that $H(S)$ and $\overset{\circ}{H}(S)$ coincide in each of the following three cases: (i) α is a nondecreasing function in $|x|$, (ii) $\alpha(x) = O(|x|^2+1)$, and (iii) $n = 3$ and α depends only on $|x|$.

2.7 Dilation Invariant Sharp Hardy's Inequalities

2.7.1 Hardy's Inequality with Sharp Sobolev Remainder Term

Here we find the best value of C for a particular case of the inequality

$$\int_{\mathbb{R}^n_+} |\nabla v|^2 \,\mathrm{d}x \geq \frac{1}{4}\int_{\mathbb{R}^n_+} \frac{|v|^2}{x_n^2}\,\mathrm{d}x + C\|x_n^\gamma v\|^2_{L_q(\mathbb{R}^n_+)}, \tag{2.7.1}$$

which is equivalent to (2.1.36) with $m = 1$.

Theorem. *For all $u \in C^\infty(\overline{\mathbb{R}^n_+})$, $u = 0$ on $\mathbb{R}^{n-1}$, the sharp inequality*

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 \,\mathrm{d}x \geq \frac{1}{4}\int_{\mathbb{R}^n_+} \frac{|u|^2}{x_n^2}\,\mathrm{d}x + \frac{\pi^{n/(n+1)}(n^2-1)}{4(\Gamma(\frac{n}{2}+1))^{2/(n+1)}}\left\|x_n^{-1/(n+1)}u\right\|^2_{L_{\frac{2(n+1)}{n-1}}(\mathbb{R}^n_+)} \tag{2.7.2}$$

holds.

Proof. We start with the Sobolev inequality

$$\int_{\mathbb{R}^{n+1}} |\nabla w|^2 \,\mathrm{d}z \geq \mathcal{S}_{n+1}\|w\|^2_{L_{\frac{2(n+1)}{n-1}}(\mathbb{R}^{n+1})} \tag{2.7.3}$$

with the best constant

$$\mathcal{S}_{n+1} = \frac{\pi^{(n+2)/(n+1)}(n^2-1)}{4^{n/(n+1)}(\Gamma(\frac{n}{2}+1))^{2/(n+1)}} \tag{2.7.4}$$

(see (2.3.23)).

Let us introduce the cylindrical coordinates (r, φ, x'), where $r \geq 0$, $\varphi \in [0, 2\pi)$, and $x' \in \mathbb{R}^{n-1}$. Assuming that w does not depend on φ, we write (2.7.3) in the form

$$2\pi\int_{\mathbb{R}^{n-1}}\int_0^\infty \left(\left|\frac{\partial w}{\partial r}\right|^2 + |\nabla_{x'}w|^2\right) r\,\mathrm{d}r\,\mathrm{d}x' \geq (2\pi)^{(n-1)/(n+1)}\mathcal{S}_{n+1}\left(\int_{\mathbb{R}^{n-1}}\int_0^\infty |w|^{2(n+1)/(n-1)} r\,\mathrm{d}r\,\mathrm{d}x'\right)^{(n-1)/(n+1)}.$$

Replacing r by x_n, we obtain

$$\int_{\mathbb{R}^n_+} |\nabla w|^2 x_n \,\mathrm{d}x \geq (2\pi)^{-2/(n+1)}\mathcal{S}_{n+1}\left(\int_{\mathbb{R}^n_+} |w|^{2(n+1)/(n-1)} x_n\,\mathrm{d}x\right)^{(n-1)/(n+1)}.$$

It remains to set here $w = x_n^{1/2}v$ and to use (2.7.4). $\square$

2.7.2 Two-Weight Hardy's Inequalities

As usual, here and elsewhere $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n, x_n > 0\}$ and $C_0^\infty(\mathbb{R}^n_+)$ and $C_0^\infty(\overline{\mathbb{R}^n_+})$ stand for the spaces of infinitely differentiable functions with compact support in $\mathbb{R}^n_+$ and $\overline{\mathbb{R}^n_+}$, respectively.

Theorem 1. *The inequality*

$$\int_{\mathbb{R}^n_+} \frac{|u(x)|^p}{(x_{n-1}^2 + x_n^2)^{1/2}} \leq (2p)^p \int_{\mathbb{R}^n_+} x_n^{p-1} \big|\nabla u(x)\big|^p \, \mathrm{d}x \tag{2.7.5}$$

holds for all $u \in C_0^\infty(\overline{\mathbb{R}^n_+})$.

Proof. We put $\varrho^2 = x_{n-1}^2 + x_n^2$ and denote the integrals on the left- and right-hand sides by $\mathscr{I}$ and $\mathscr{J}$, respectively. Integrating by parts, we obtain

$$\mathscr{I} = -p \int_{\mathbb{R}^n} x_n \varrho^{-1} |u|^{p-1} \operatorname{sgn} u \frac{\partial u}{\partial x_n} \, \mathrm{d}x + \int_{\mathbb{R}^n} x_n^2 \varrho^{-3} |u|^p \, \mathrm{d}x.$$

We denote two summands in the right-hand side by $\mathscr{I}_1$ and $\mathscr{I}_2$. Clearly, by Hölder's inequality we have $|\mathscr{I}_1| \leq {}^{(p-1)/p} \mathscr{J}^{1/p}$. To obtain a bound for $\mathscr{I}_2$ we introduce cylindrical coordinates (z, ϱ, θ) with $z \in \mathbb{R}^{n-2}$, $x_{n-1} + ix_n = \varrho \exp(i\theta)$. Then

$$\mathscr{I}_2 = -p \int_{\mathbb{R}^{n-2}} \mathrm{d}z \int_0^\pi \sin^2\theta \, \mathrm{d}\theta \int_0^\infty |u|^{p-1} \operatorname{sgn} u \frac{\partial u}{\partial \varrho} \varrho \, \mathrm{d}\varrho \leq \varrho \mathscr{I}^{(p-1)/p} \mathscr{J}^{1/p}.$$

Thus $\mathscr{I} \leq 2p \mathscr{I}^{(p-1)/p} \mathscr{J}^{1/p}$ and (2.7.5) follows. □

In this section we are concerned with generalizations of the inequality

$$\int_{\mathbb{R}^n_+} x_n |\nabla u|^2 \, \mathrm{d}x \geq \Lambda \int_{\mathbb{R}^n_+} \frac{|u|^2}{(x_{n-1}^2 + x_n^2)^{1/2}} \, \mathrm{d}x, \quad u \in C_0^\infty(\overline{\mathbb{R}^n_+}). \tag{2.7.6}$$

By substituting $u(x) = x_n^{-1/2} v(x)$ into (2.7.6), one arrives at the improved Hardy inequality

$$\int_{\mathbb{R}^n_+} |\nabla v|^2 \, \mathrm{d}x - \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{|v|^2 \, \mathrm{d}x}{x_n^2} \geq \Lambda \int_{\mathbb{R}^n_+} \frac{|v|^2 \, \mathrm{d}x}{x_n (x_{n-1}^2 + x_n^2)^{1/2}} \tag{2.7.7}$$

for all $v \in C_0^\infty(\mathbb{R}^n_+)$.

More generally, replacing u by $x_n^{-1/2} v(x)$ in the next theorem, we find a condition on the function q that is necessary and sufficient for the inequality

$$\int_{\mathbb{R}^n_+} |\nabla v|^2 \, \mathrm{d}x - \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{|v|^2 \, \mathrm{d}x}{x_n^2} \geq C \int_{\mathbb{R}^n_+} q\left(\frac{x_n}{(x_{n-1}^2 + x_n^2)^{1/2}} \right) \frac{|v|^2 \, \mathrm{d}x}{x_n (x_{n-1}^2 + x_n^2)^{1/2}}, \tag{2.7.8}$$

where v is an arbitrary function in $C_0^\infty(\mathbb{R}^n_+)$. This condition implies, in particular, that the right-hand side of (2.7.7) can be replaced by

$$C\int_{\mathbb{R}^n_+} \frac{|v|^2\,\mathrm{d}x}{x_n^2\left(1-\log\frac{x_n}{(x_{n-1}^2+x_n^2)^{1/2}}\right)^2}.$$

Theorem 2. (i) *Let q denote a locally integrable nonnegative function on $(0,1)$. The best constant in the inequality*

$$\int_{\mathbb{R}^n_+} x_n|\nabla u|^2\,\mathrm{d}x \geq C\int_{\mathbb{R}^n_+} q\biggl(\frac{x_n}{(x_{n-1}^2+x_n^2)^{1/2}}\biggr)\frac{|u|^2}{(x_{n-1}^2+x_n^2)^{1/2}}\,\mathrm{d}x, \tag{2.7.9}$$

for all $u\in C_0^\infty(\overline{\mathbb{R}^n_+})$, is given by

$$\lambda := \inf\frac{\int_0^{\pi/2}(|y'(\varphi)|^2+\frac{1}{4}|y(\varphi)|^2)\sin\varphi\,\mathrm{d}\varphi}{\int_0^{\pi/2}|y(\varphi)|^2 q(\sin\varphi)\,\mathrm{d}\varphi}, \tag{2.7.10}$$

where the infimum is taken over all smooth functions on $[0,\pi/2]$.

(ii) *Inequalities (2.7.9) and (2.7.8) with a positive C hold if and only if*

$$\sup_{t\in(0,1)}(1-\log t)\int_0^t q(\tau)\,\mathrm{d}\tau<\infty. \tag{2.7.11}$$

Moreover,

$$\lambda\sim\biggl(\sup_{t\in(0,1)}(1-\log t)\int_0^t q(\tau)\,\mathrm{d}\tau\biggr)^{-1}, \tag{2.7.12}$$

where $a\sim b$ means that $c_1a\leq b\leq c_2a$ with absolute positive constants c_1 and c_2.

Proof. (i) Let $U\in C_0^\infty(\overline{\mathbb{R}^2_+})$, $\zeta\in C_0^\infty(\mathbb{R}^{n-2})$, $x'=(x_1,\dots,x_{n-2})$, and let $N=\text{const}>0$. Putting

$$u(x)=N^{(2-n)/2}\zeta\bigl(N^{-1}x'\bigr)U(x_{n-1},x_n)$$

into (2.7.9) and passing to the limit as $N\to\infty$, we see that (2.7.9) is equivalent to the inequality

$$\begin{aligned}\int_{\mathbb{R}^2_+} x_2\bigl(|U_{x_1}|^2+|U_{x_2}|^2\bigr)\,\mathrm{d}x_1\,\mathrm{d}x_2&\\ \geq C\int_{\mathbb{R}^2_+} q\biggl(\frac{x_2}{(x_1^2+x_2^2)^{1/2}}\biggr)&\frac{|U|^2\,\mathrm{d}x_1\,\mathrm{d}x_2}{(x_1^2+x_2^2)^{1/2}},\end{aligned} \tag{2.7.13}$$

where $U\in C_0^\infty(\overline{\mathbb{R}^2_+})$. Let (ρ,φ) be the polar coordinates of the point $(x_1,x_2)\in\mathbb{R}^2_+$. Then (2.7.13) can be written as

$$\int_0^\infty \int_0^\pi \big(|U_\rho|^2 + \rho^{-2}|U_\varphi|^2\big) \sin\varphi \, \mathrm{d}\varphi \, \rho^2 \, \mathrm{d}\rho \geq C \int_0^\infty \int_0^\pi |U|^2 q(\sin\varphi) \, \mathrm{d}\varphi \, \mathrm{d}\rho.$$

By the substitution

$$U(\rho,\varphi) = \rho^{-1/2} v(\rho,\varphi)$$

the left-hand side becomes

$$\int_0^\infty \int_0^\pi \left(|\rho v_\rho|^2 + |v_\varphi|^2 + \frac{1}{4}|v|^2\right) \sin\varphi \, \mathrm{d}\varphi \, \frac{\mathrm{d}\rho}{\rho}$$
$$- \operatorname{Re} \int_0^\pi \int_0^\infty \overline{v} v_\rho \, \mathrm{d}\rho \sin\varphi \, \mathrm{d}\varphi. \qquad (2.7.14)$$

Since $v(0) = 0$, the second term in (2.7.14) vanishes. Therefore, (2.7.13) can be written in the form

$$\int_0^\infty \int_0^\pi \left(|\rho v_\rho|^2 + |v_\varphi|^2 + \frac{1}{4}|v|^2\right) \sin\varphi \, \mathrm{d}\varphi \, \frac{\mathrm{d}\rho}{\rho}$$
$$\geq C \int_0^\infty \int_0^\pi |v|^2 q(\sin\varphi) \, \mathrm{d}\varphi \, \frac{\mathrm{d}\rho}{\rho}. \qquad (2.7.15)$$

Now, the definition (2.7.10) of λ shows that (2.7.9) holds with $C = \lambda$.

To show the optimality of this value of C, put $t = \log\rho$ and $v(\rho,\varphi) = w(t,\varphi)$. Then (2.7.9) is equivalent to

$$\int_{\mathbb{R}^1} \int_0^\pi \left(|w_t|^2 + |w_\varphi|^2 + \frac{1}{4}|w|^2\right) \sin\varphi \, \mathrm{d}\varphi \, \mathrm{d}t$$
$$\geq C \int_{\mathbb{R}^1} \int_0^\pi |w|^2 q(\sin\varphi) \, \mathrm{d}\varphi \, \mathrm{d}t. \qquad (2.7.16)$$

Applying the Fourier transform $w(t,\varphi) \to \hat{w}(s,\varphi)$, we obtain

$$\int_{\mathbb{R}^1} \int_0^\pi \left(|\hat{w}_\varphi|^2 + \left(|s|^2 + \frac{1}{4}\right)|\hat{w}|^2\right) \sin\varphi \, \mathrm{d}\varphi \, \mathrm{d}s$$
$$\geq C \int_{\mathbb{R}^1} \int_0^\pi |\hat{w}|^2 q(\sin\varphi) \, \mathrm{d}\varphi \, \mathrm{d}s. \qquad (2.7.17)$$

Putting here

$$\hat{w}(s,\varphi) = \varepsilon^{-1/2} \eta(s/\varepsilon) y(\varphi),$$

where $\eta \in C_0^\infty(\mathbb{R}^1)$, $\|\eta\|_{L^2(\mathbb{R}^1)} = 1$, and y is a function on $C^\infty([0,\pi])$, and passing to the limit as $\varepsilon \to 0$, we arrive at the estimate

$$\int_0^\pi \left(\left|y'(\varphi)\right|^2 + \frac{1}{4}\left|y(\varphi)\right|^2\right) \sin\varphi \, \mathrm{d}\varphi \geq C \int_0^\pi \left|y(\varphi)\right|^2 q(\sin\varphi) \, \mathrm{d}\varphi, \qquad (2.7.18)$$

where π can be changed for $\pi/2$ by symmetry. This together with (2.7.10) implies $\Lambda \leq \lambda$. The proof of (i) is complete.

(ii) Introducing the new variable $\xi = \log\cot\frac{\varphi}{2}$, we write (2.7.10) as

$$\lambda = \inf_z \frac{\int_0^\infty (|z'(\xi)|^2 + \frac{|z(\xi)|^2}{4(\cosh\xi)^2})\,\mathrm{d}\xi}{\int_0^\infty |z(\xi)|^2 q(\frac{1}{\cosh\xi})\frac{\mathrm{d}\xi}{\cosh\xi}}. \tag{2.7.19}$$

Since

$$\big|z(0)\big|^2 \le 2\int_0^1 \big(\big|z'(\xi)\big|^2 + \big|z(\xi)\big|^2\big)\,\mathrm{d}\xi$$

and

$$\begin{aligned}
&\int_0^\infty \big|z(\xi)\big|^2 \frac{e^{2\xi}}{(1+e^{2\xi})^2}\,\mathrm{d}\xi \\
&\quad\le 2\int_0^\infty \big|z(\xi)-z(0)\big|^2\frac{\mathrm{d}\xi}{\xi^2} + 2\big|z(0)\big|^2\int_0^\infty \frac{e^{2\xi}}{(1+e^{2\xi})^2}\,\mathrm{d}\xi \\
&\quad\le 8\int_0^\infty \big|z'(\xi)\big|^2\,\mathrm{d}\xi + \big|z(0)\big|^2
\end{aligned}$$

it follows from (2.7.19) that

$$\lambda \sim \inf_z \frac{\int_0^\infty |z'(\xi)|^2\,\mathrm{d}\xi + |z(0)|^2}{\int_0^\infty |z(\xi)|^2 q(\frac{1}{\cosh\xi})\frac{\mathrm{d}\xi}{\cosh\xi}}. \tag{2.7.20}$$

Setting $z(\xi) = 1$ and $z(\xi) = \min\{\eta^{-1}\xi, 1\}$ for all positive ξ and fixed $\eta > 0$ into the ratio of quadratic forms in (2.7.20), we deduce that

$$\lambda \le \min\biggl\{\biggl(\int_0^\infty q\Bigl(\frac{1}{\cosh\xi}\Bigr)\frac{\mathrm{d}\xi}{\cosh\xi}\biggr)^{-1}, \biggl(\sup_{\eta>0}\eta\int_\eta^\infty q\Bigl(\frac{1}{\cosh\xi}\Bigr)\frac{\mathrm{d}\xi}{\cosh\xi}\biggr)^{-1}\biggr\}.$$

Hence,

$$\lambda \le c\biggl(\sup_{t\in(0,1)}(1-\log t)\int_0^t q(\tau)\,\mathrm{d}\tau\biggr)^{-1}.$$

To obtain the converse estimate, note that

$$\begin{aligned}
&\int_0^\infty \big|z(\xi)\big|^2 q\Bigl(\frac{1}{\cosh\xi}\Bigr)\frac{\mathrm{d}\xi}{\cosh\xi} \\
&\quad\le 2\big|z(0)\big|^2\int_0^\infty q\Bigl(\frac{1}{\cosh\xi}\Bigr)\frac{\mathrm{d}\xi}{\cosh\xi} + 2\int_0^\infty \big|z(\xi)-z(0)\big|^2 q\Bigl(\frac{1}{\cosh\xi}\Bigr)\frac{\mathrm{d}\xi}{\cosh\xi}.
\end{aligned}$$

The second term in the right-hand side is dominated by

$$8\sup_{\eta>0}\biggl(\eta\int_\eta^\infty q\Bigl(\frac{1}{\cosh\xi}\Bigr)\frac{\mathrm{d}\xi}{\cosh\xi}\biggr)\int_0^\infty \big|z'(\xi)\big|^2\,\mathrm{d}\xi$$

(see Sect. 1.3.2). Therefore,

$$\int_0^\infty \big|z(\xi)\big|^2 q\Big(\frac{1}{\cosh\xi}\Big)\frac{\mathrm{d}\xi}{\cosh\xi}$$
$$\le 8\max\Big\{\int_0^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{\mathrm{d}\xi}{\cosh\xi},\ \sup_{\eta>0}\eta\int_\eta^\infty q\Big(\frac{1}{\cosh\sigma}\Big)\frac{\mathrm{d}\sigma}{\cosh\sigma}\Big\}$$
$$\times\Big(\int_0^\infty \big|z'(\xi)\big|^2\,\mathrm{d}\xi + \big|z(0)\big|^2\Big),$$

which together with (2.7.20) leads to the lower estimate

$$\lambda \ge \min\Big\{\Big(\int_0^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{\mathrm{d}\xi}{\cosh\xi}\Big)^{-1}, \Big(\sup_{\eta>0}\eta\int_\eta^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{\mathrm{d}\xi}{\cosh\xi}\Big)^{-1}\Big\}.$$

Hence,

$$\lambda \ge c\Big(\sup_{t\in(0,1)}(1-\log t)\int_0^t q(\tau)\,\mathrm{d}\tau\Big)^{-1}.$$

The proof of (ii) is complete. □

Since (2.7.11) holds for $q(t) = t^{-1}(1-\log t)^{-2}$, Theorem 2(ii) leads to the following assertion.

Corollary 1. *There exists an absolute constant $C > 0$ such that the inequality*

$$\int_{\mathbb{R}^n_+} |\nabla v|^2\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^n_+}\frac{|v|^2\,\mathrm{d}x}{x_n^2} \ge C\int_{\mathbb{R}^n_+}\frac{|v|^2\,\mathrm{d}x}{x_n^2(1-\log\frac{x_n}{(x_{n-1}^2+x_n^2)^{1/2}})^2} \qquad (2.7.21)$$

holds for all $v\in C_0^\infty(\mathbb{R}^n_+)$. The best value of C is equal to

$$\lambda := \inf\frac{\int_0^\pi [|y'(\varphi)|^2 + \frac{1}{4}|y(\varphi)|^2]\sin\varphi\,\mathrm{d}\varphi}{\int_0^\pi |y(\varphi)|^2(\sin\varphi)^{-1}(1-\log\sin\varphi)^{-2}\,\mathrm{d}\varphi}, \qquad (2.7.22)$$

where the infimum is taken over all smooth functions on $[0,\pi/2]$. By numerical approximation, $\lambda = 0.16,\dots$.

A particular case of Theorem 2 corresponding to $q = 1$ is the following assertion.

Corollary 2. *The sharp value of Λ in (2.7.6) and (2.7.7) is equal to*

$$\lambda := \inf\frac{\int_0^\pi [|y'(\varphi)|^2 + \frac{1}{4}|y(\varphi)|^2]\sin\varphi\,\mathrm{d}\varphi}{\int_0^\pi |y(\varphi)|^2\,\mathrm{d}\varphi}, \qquad (2.7.23)$$

where the infimum is taken over all smooth functions on $[0,\pi]$. By numerical approximation, $\lambda = 0.1564,\dots$.

Remark 1. Let us consider the Friedrichs extension $\tilde{\mathcal{L}}$ of the operator

$$\mathcal{L} : z \to -\big((\sin\varphi)z'\big)' + \frac{\sin\varphi}{4} z, \tag{2.7.24}$$

defined on smooth functions on $[0,\pi]$. It is a simple exercise to show that the energy space of $\tilde{\mathcal{L}}$ is compactly embedded into $L^2(0,\pi)$. Hence, the spectrum of $\tilde{\mathcal{L}}$ is discrete and λ defined by (2.7.23) is the smallest eigenvalue of $\tilde{\mathcal{L}}$.

Remark 2. The argument used in the proof of Theorem 2(i) with obvious changes enables one to obtain the following more general fact. Let P and Q be measurable nonnegative functions in $\mathbb{R}^n$, positive homogeneous of degrees 2μ and $2\mu - 2$, respectively. The sharp value of C in

$$\int_{\mathbb{R}^n} P(x)|\nabla u|^2 \,\mathrm{d}x \geq C \int_{\mathbb{R}^n} Q(x)|u|^2 \,\mathrm{d}x, \quad u \in C_0^\infty\big(\mathbb{R}^n\big), \tag{2.7.25}$$

is equal to

$$\lambda := \inf \frac{\int_{S^{n-1}} P(\omega)(|\nabla_\omega Y|^2 + (\mu - 1 + \frac{n}{2})^2 |Y|^2) \,\mathrm{d}s_\omega}{\int_{S^{n-1}} Q(\omega)|Y|^2 \,\mathrm{d}s_\omega},$$

where the infimum is taken over all smooth functions on the unit sphere S^{n-1}.

2.7.3 Comments to Sect. 2.7

The material of this subsection is borrowed from Maz'ya and Shaposhnikova [587]. In Sect. 2.7.1 we are concerned with the inequality (2.7.1) which is a special case of (2.1.36). Another inequality of a similar nature, whose generalizations are dealt with in Sect. 2.7.2, is (2.7.7). It is equivalent to (2.7.6) and was obtained in 1972 by the author, proving to be useful in the study of the generic case of degeneration in the oblique derivative problem for second-order elliptic differential operators [541].

Without the second term on the right-hand sides of (2.7.1) and (2.7.7), these inequalities reduce to the classical Hardy inequality with the sharp constant 1/4. An interesting feature of (2.7.1) and (2.7.7) is their dilation invariance. The value $\Lambda = 1/16$ in (2.7.7) obtained in Maz'ya [541] is not the best possible. Tidblom replaced it by 1/8 in [752]. As a corollary of Theorem 2.7.2/2, we find an expression for the optimal value of Λ (see Corollary 2.7.2/2).

Sharp constants in Hardy-type inequalities as well as variants, extensions, and refinements of (2.7.1) and (2.7.7), usually called Hardy's inequalities with remainder term, became a theme of many subsequent studies (Davies [225, 226]; Brezis and Marcus [142]; Brezis and Vázquez [145]; Matskewich and Sobolevskii [526]; Sobolevskii [715]; Davies and Hinz [227]; Marcus, Mizel, and Pinchover [516]; Laptev and Weidl [481]; Weidl [792]; Yafaev [801]; Brezis, Marcus, and Shafrir [143]; Vázquez and Zuazua [773]; Eilertsen [256];

Adimurthi [26]; Adimurthi, Chaudhuri, and Ramaswamy [27]; Filippas and Tertikas [278]; M. Hoffman-Ostenhof, T. Hoffman-Ostenhof, and Laptev [380]; Barbatis, Filippas, and Tertikas [73, 74]; Balinsky [67]; Barbatis, Filippas, and Tertikas [72]; Chaudhuri [178]; Z.-Q. Wang and Meijun [791]; Balinsky, Laptev, and A. Sobolev [68]; Dávila and Dupaigne [228]; Dolbeault, Esteban, Loss, and Vega [237]; Filippas, Maz'ya, and Tertikas [275–277]; Gazzola, Grunau, and Mitidieri [303]; Tidblom [751, 752]; Colin [211]; Edmunds and Hurri-Syrjänen [252]; Galaktionov [301, 302]; Samko [689]; Yaotian and Zhihui [804]; Adimurthi, Grossi, and Santra [28]; Alvino, Ferone, and Trombetti [42]; Barbatis [70, 71]; Brandolini, Chiacchio, and Trombetti [140]; Dou, Niu, and Yuan [241]; Evans and Lewis [264]; Tertikas and Tintarev [749]; Tertikas and Zographopoulos [750]; Benguria, Frank, and Loss [83]; Bosi, Dolbeault, and Esteban [127]; Frank and Seiringer [289, 290]; Frank, Lieb, and Seiringer [288]; A. Laptev and A. Sobolev [480]; Cianchi and Ferone [203]; Kombe and Özaydin [446]; Filippas, Tertikas, and Tidblom [279]; Pinchover and Tintarev [661]; Avkhadiev and Laptev [58] et al.).

2.8 Sharp Hardy–Leray Inequality for Axisymmetric Divergence-Free Fields

2.8.1 Statement of Results

Let $\mathbf{u}$ denote a $C_0^\infty(\mathbb{R}^n)$ vector field in $\mathbb{R}^n$. The following n-dimensional generalization of the one-dimensional Hardy inequality,

$$\int_{\mathbb{R}^n} \frac{|\mathbf{u}|^2}{|x|^2}\,\mathrm{d}x \le \frac{4}{(n-2)^2}\int_{\mathbb{R}^n} |\nabla \mathbf{u}|^2\,\mathrm{d}x \tag{2.8.1}$$

appears for $n = 3$ in the pioneering paper by Leray on the Navier–Stokes equations [487]. The constant factor on the right-hand side is sharp. Since one frequently deals with divergence-free fields in hydrodynamics, it is natural to ask whether this restriction can improve the constant in (2.8.1).

We show in the present section that this is the case indeed if $n > 2$ and the vector field $\mathbf{u}$ is axisymmetric by proving that the aforementioned constant can be replaced by the (smaller) optimal value

$$\frac{4}{(n-2)^2}\left(1 - \frac{8}{(n+2)^2}\right), \tag{2.8.2}$$

which, in particular, evaluates to $68/25$ in three dimensions. This result is a special case of a more general one concerning a divergence-free improvement of the multidimensional sharp Hardy inequality

$$\int_{\mathbb{R}^n} |x|^{2\gamma-2}|\mathbf{u}|^2\,\mathrm{d}x \le \frac{4}{(2\gamma+n-2)^2}\int_{\mathbb{R}^n} |x|^{2\gamma}|\nabla \mathbf{u}|^2\,\mathrm{d}x. \tag{2.8.3}$$

Let $\boldsymbol{\phi}$ be a point on the $(n-2)$-dimensional unit sphere S^{n-2} with spherical coordinates $\{\theta_j\}_{j=1,\dots,n-3}$ and ϕ, where $\theta_j \in (0,\pi)$ and $\varphi \in [0,2\pi)$. A point $x \in \mathbb{R}^n$ is represented as a triple $(\rho, \theta, \boldsymbol{\phi})$, where $\rho > 0$ and $\theta \in [0,\pi]$. Correspondingly, we write $\mathbf{u} = (u_\rho, u_\theta, \mathbf{u}_{\boldsymbol{\phi}})$ with $\mathbf{u}_{\boldsymbol{\phi}} = (u_{\theta_{n-3}}, \dots, u_{\theta_1}, u_{\boldsymbol{\phi}})$.

The *condition of axial symmetry* means that $\mathbf{u}$ depends only on ρ and θ.

For higher dimensions, our result is as follows.

Theorem 1. *Let $\gamma \neq 1 - n/2$, $n > 2$, and let $\mathbf{u}$ be an axisymmetric divergence-free vector field in $C_0^\infty(\mathbb{R}^n)$. We assume that $\mathbf{u}(\mathbf{0}) = \mathbf{0}$ for $\gamma < 1 - n/2$. Then*

$$\int_{\mathbb{R}^n} |x|^{2\gamma-2}|\mathbf{u}|^2 \,\mathrm{d}x \le C_{n,\gamma} \int_{\mathbb{R}^n} |x|^{2\gamma}|\nabla \mathbf{u}|^2 \,\mathrm{d}x \tag{2.8.4}$$

with the best value of $C_{n,\gamma}$ given by

$$C_{n,\gamma} = \frac{4}{(2\gamma+n-2)^2}\left(1 - \frac{2}{n+1+(\gamma-n/2)^2}\right), \tag{2.8.5}$$

for $\gamma \le 1$, and by

$$C_{n,\gamma}^{-1} = \left(\frac{n}{2} + \gamma - 1\right)^2 + \min\left\{n-1,\ 2 + \min_{x\ge 0}\left(x + \frac{4(n-1)(\gamma-1)}{x+n-1+(\gamma-n/2)^2}\right)\right\} \tag{2.8.6}$$

for $\gamma > 1$.

The two minima in (2.8.6) can be calculated in closed form, but their expressions for arbitrary dimensions turn out to be unwieldy and we omit them.

However, the formula for $C_{3,\gamma}$ is simple.

Corollary. *For $n = 3$ inequality (2.8.4) holds with the best constant*

$$C_{3,\gamma} = \begin{cases} \frac{4}{(2\gamma+1)^2} \cdot \frac{2+(\gamma-3/2)^2}{4+(\gamma-3/2)^2} & \text{for } \gamma \le 1, \\ \frac{4}{8+(1+2\gamma)^2} & \text{for } \gamma > 1. \end{cases} \tag{2.8.7}$$

For $n = 2$, we obtain the sharp constant in (2.8.4) without axial symmetry of the vector field.

Theorem 2. *Let $\gamma \neq 0$, $n = 2$, and let $\mathbf{u}$ be a divergence-free vector field in $C_0^\infty(\mathbb{R}^2)$. We assume that $\mathbf{u}(\mathbf{0}) = \mathbf{0}$ for $\gamma < 0$. Then inequality (2.8.4) holds with the best constant*

$$C_{2,\gamma} = \begin{cases} \gamma^{-2}\frac{1+(1-\gamma)^2}{3+(1-\gamma)^2} & \text{for } \gamma \in [-\sqrt{3}-1, \sqrt{3}-1], \\ (\gamma^2+1)^{-1} & \text{otherwise.} \end{cases} \tag{2.8.8}$$

2.8.2 Proof of Theorem 1

In the spherical coordinates introduced previously, we have

$$\begin{aligned}\operatorname{div}\mathbf{u} &= \rho^{1-n}\frac{\partial}{\partial\rho}\big(\rho^{n-1}u_\rho\big) + \rho^{-1}(\sin\theta)^{2-n}\frac{\partial}{\partial\theta}\big((\sin\theta)^{n-2}u_\theta\big)\\ &\quad+\sum_{k=1}^{n-3}(\rho\sin\theta\sin\theta_{n-3}\cdots\sin\theta_{k+1})^{-1}(\sin\theta_k)^{-k}\frac{\partial}{\partial\theta_k}\big((\sin\theta_k)^k u_{\theta_k}\big)\\ &\quad+(\rho\sin\theta\sin\theta_{n-3}\cdots\sin\theta_1)^{-1}\frac{\partial u_\varphi}{\partial\varphi}. \end{aligned}\tag{2.8.9}$$

Since the components u_φ and u_{θ_k}, $k=1,\dots,n-3$, depend only on ρ and θ, (2.8.9) becomes

$$\begin{aligned}\operatorname{div}\mathbf{u} &= \rho^{1-n}\frac{\partial}{\partial\rho}\big(\rho^{n-1}u_\rho(\rho,\theta)\big) + \rho^{-1}(\sin\theta)^{2-n}\frac{\partial}{\partial\theta}\big((\sin\theta)^{n-2}u_\theta(\rho,\theta)\big)\\ &\quad+\sum_{k=1}^{n-3}k(\sin\theta_{n-3}\cdots\sin\theta_{k+1})^{-1}\cot\theta_k\,\frac{u_{\theta_k}(\rho,\theta)}{\rho\sin\theta}. \end{aligned}\tag{2.8.10}$$

By the linear independence of the functions

$$1,\quad (\sin\theta_{n-3}\cdots\sin\theta_{k+1})^{-1}\cot\theta_k,\quad k=1,\dots,n-3,$$

the divergence-free condition is equivalent to the collection of $n-2$ identities

$$\rho\frac{\partial u_\rho}{\partial\rho}+(n-1)u_\rho+\left(\frac{\partial}{\partial\theta}+(n-2)\cot\theta\right)u_\theta=0,\tag{2.8.11}$$

$$u_{\theta_k}=0,\quad k=1,\dots,n-3.\tag{2.8.12}$$

If the right-hand side of (2.8.4) diverges, there is nothing to prove. Otherwise, the matrix $\nabla\mathbf{u}$ is $O(|x|^m)$, with $m>-\gamma-n/2$, as $x\to 0$. Since $\mathbf{u}(\mathbf{0})=\mathbf{0}$, we have $\mathbf{u}(x)=O(|x|^{m+1})$ ensuring the convergence of the integral on the left-hand side of (2.8.4). We introduce the vector field

$$\mathbf{v}(x)=\mathbf{u}(x)|x|^{\gamma-1+n/2}.\tag{2.8.13}$$

The inequality (2.8.4) becomes

$$\left(\frac{1}{C_{n,\gamma}}-\left(\frac{n}{2}+\gamma-1\right)^2\right)\int_{\mathbb{R}^n}\frac{|\mathbf{v}|^2}{|x|^n}\,\mathrm{d}x\le\int_{\mathbb{R}^n}\frac{|\nabla\mathbf{v}|^2}{|x|^{n-2}}\,\mathrm{d}x.\tag{2.8.14}$$

The condition $\operatorname{div}\mathbf{u}=0$ is equivalent to

$$\rho\operatorname{div}\mathbf{v}=\left(\frac{n-2}{2}+\gamma\right)v_\rho.\tag{2.8.15}$$

To simplify the exposition, we assume first that $\mathbf{v}_{\boldsymbol{\varphi}} = \mathbf{0}$. Now, (2.8.15) can be written as

$$\rho \frac{\partial v_\rho}{\partial \rho} + \left(\frac{n}{2} - \gamma\right) v_\rho + \mathcal{D} v_\theta = 0, \tag{2.8.16}$$

where

$$\mathcal{D} := \frac{\partial}{\partial \theta} + (n-2)\cot\theta. \tag{2.8.17}$$

Note that $\mathcal{D}$ is the adjoint of $-\partial/\partial\theta$ with respect to the scalar product

$$\int_0^\pi f(\theta)\overline{g(\theta)}(\sin\theta)^{n-2}\, \mathrm{d}\theta.$$

A straightforward, though lengthy calculation yields

$$\begin{aligned}\rho^2|\nabla \mathbf{v}|^2 = \rho^2\left(\frac{\partial v_\rho}{\partial \rho}\right)^2 + \rho^2\left(\frac{\partial v_\theta}{\partial \rho}\right)^2 + \left(\frac{\partial v_\rho}{\partial \theta}\right)^2 + \left(\frac{\partial v_\theta}{\partial \theta}\right)^2 \\ + v_\theta^2 + (n-1)v_\rho^2 + (n-2)(\cot\theta)^2 v_\theta^2 + 2\left(v_\rho \mathcal{D} v_\theta - v_\theta \frac{\partial v_\rho}{\partial \theta}\right).\end{aligned} \tag{2.8.18}$$

Hence

$$\begin{aligned}\rho^2 \int_{S^{n-1}} |\nabla \mathbf{v}|^2\, \mathrm{d}s = \int_{S^{n-1}} \Bigg\{\rho^2\left(\frac{\partial v_\rho}{\partial \rho}\right)^2 + \left(\frac{\partial v_\theta}{\partial \theta}\right)^2 + \rho^2\left(\frac{\partial v_\theta}{\partial \rho}\right)^2 + \left(\frac{\partial v_\rho}{\partial \theta}\right)^2 \\ + v_\theta^2 + (n-1)v_\rho^2 + (n-2)(\cot\theta)^2 v_\theta^2 + 4 v_\rho \mathcal{D} v_\theta\Bigg\}\, \mathrm{d}s.\end{aligned} \tag{2.8.19}$$

Changing the variable ρ to $t = \log\rho$ and applying the Fourier transform with respect to t,

$$\mathbf{v}(t, \theta) \mapsto \mathbf{w}(\lambda, \theta),$$

we derive

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}}\, \mathrm{d}x = \int_{\mathbb{R}} \int_{S^{n-1}} \Bigg\{ \big(l^2 + n - 1\big)|w_\rho|^2 + \big(l^2 - n + 3\big)|w_\theta|^2 \\ + \left|\frac{\partial w_\rho}{\partial \theta}\right|^2 + \left|\frac{\partial w_\theta}{\partial \theta}\right|^2 + (n-2)(\sin\theta)^{-2}|w_\theta|^2 \\ + 4\Re(\overline{w}_\rho \mathcal{D} w_\theta)\Bigg\}\, \mathrm{d}s\, \mathrm{d}\lambda\end{aligned} \tag{2.8.20}$$

and

$$\int_{\mathbb{R}^n} \frac{|\mathbf{v}|^2}{|x|^n}\, \mathrm{d}x = \int_{\mathbb{R}} \int_{S^{n-1}} |\mathbf{w}|^2\, \mathrm{d}s\, \mathrm{d}\lambda. \tag{2.8.21}$$

From (2.8.15), we obtain

$$w_\rho = -\frac{\mathcal{D}w_\theta}{i\lambda + n/2 - \gamma}, \tag{2.8.22}$$

which implies

$$|w_\rho|^2 = \frac{|\mathcal{D}w_\theta|^2}{\lambda^2 + (n/2-\gamma)^2} \tag{2.8.23}$$

and

$$\Re(\overline{w}_\rho \mathcal{D}w_\theta) = -\frac{(n/2-\gamma)|\mathcal{D}w_\theta|^2}{\lambda^2 + (n/2-\gamma)^2}. \tag{2.8.24}$$

Introducing this into (2.8.20), we arrive at the identity

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}}\,\mathrm{d}x = \int_0^\infty \int_{S^{n-1}} \Big\{ & \big(\lambda^2+n-1\big)\frac{|\mathcal{D}w_\theta|^2}{\lambda^2+(n/2-\gamma)^2} \\ & + \big(\lambda^2 - n + 3\big)|w_\theta|^2 + \left|\frac{\partial w_\theta}{\partial \theta}\right|^2 + (n-2)(\sin\theta)^{-2}|w_\theta|^2 \\ & + \frac{1}{\lambda^2+(n/2-\gamma)^2}\left|\frac{\partial}{\partial\theta}\mathcal{D}w_\theta\right|^2 \\ & - 4\Big(\frac{n}{2}-\gamma\Big)\frac{|\mathcal{D}w_\theta|^2}{\lambda^2+(n/2-\gamma)^2}\Big\}\,\mathrm{d}s\,\mathrm{d}\lambda.\end{aligned}$$

We simplify the right-hand side to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}}\,\mathrm{d}x = \int_0^\infty \int_{S^{n-1}} \Big\{ & \Big(\frac{-n-1+\lambda^2+4\gamma}{\lambda^2+(n/2-\gamma)^2}+1\Big)|\mathcal{D}w_\theta|^2 \\ & + \big(\lambda^2-n+3\big)|w_\theta|^2 + \frac{1}{\lambda^2+(n/2-\gamma)^2}\left|\frac{\partial}{\partial\theta}\mathcal{D}w_\theta\right|^2\Big\}\,\mathrm{d}s\,\mathrm{d}\lambda.\end{aligned} \tag{2.8.25}$$

On the other hand, by (2.8.21) and (2.8.22)

$$\int_{\mathbb{R}^n} \frac{|\mathbf{v}|^2}{|x|^{n-2}}\,\mathrm{d}x = \int_0^\infty \int_{S^{n-1}} \Big(\frac{|\mathcal{D}w_\theta|^2}{\lambda^2+(n/2-\gamma)^2} + |w_\theta|^2\Big)\,\mathrm{d}s\,\mathrm{d}\lambda. \tag{2.8.26}$$

Defining the self-adjoint operator

$$T := -\frac{\partial}{\partial\theta}\mathcal{D}, \tag{2.8.27}$$

or equivalently,

$$T = -\delta_\theta + \frac{n-2}{(\sin\theta)^2}, \tag{2.8.28}$$

where δ_θ is the θ part of the Laplace–Beltrami operator on S^{n-1}, we write (2.8.25) and (2.8.26) as

$$\int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}}\,\mathrm{d}x = \int_{\mathbb{R}} \int_{S^{n-1}} Q(\lambda, w_\theta)\,\mathrm{d}s\,\mathrm{d}\lambda \tag{2.8.29}$$

and

$$\int_{\mathbb{R}^n} \frac{|\mathbf{v}|^2}{|x|^n}\,\mathrm{d}x = \int_{\mathbb{R}} \int_{S^{n-1}} q(\lambda, w_\theta)\,\mathrm{d}s\,\mathrm{d}\lambda, \tag{2.8.30}$$

respectively, where Q and q are sesquilinear forms in w_θ, defined by

$$\begin{aligned} Q(\lambda, w_\theta) = &\left(\frac{-n-1+\lambda^2+4\gamma}{\lambda^2+(n/2-\gamma)^2}+1\right) T w_\theta \cdot \overline{w_\theta} \\ &+ \big(\lambda^2-n+3\big)|w_\theta|^2 + \frac{1}{\lambda^2+(n/2-\gamma)^2}|T w_\theta|^2 \end{aligned}$$

and

$$q(\lambda, w_\theta) = \frac{T w_\theta \cdot \overline{w_\theta}}{\lambda^2+(n/2-\gamma)^2} + |w_\theta|^2. \tag{2.8.31}$$

The eigenvalues of T are $\alpha_\nu = \nu(\nu+n-2)$, $\nu \in \mathbb{Z}^+$. Representing w_θ as an expansion in eigenfunctions of T, we find

$$\begin{aligned} &\inf_{w_\theta} \frac{\int_{\mathbb{R}} \int_{S^{n-1}} Q(\lambda, w_\theta)\,\mathrm{d}s\,\mathrm{d}\lambda}{\int_{\mathbb{R}} \int_{S^{n-1}} q(\lambda, w_\theta)\,\mathrm{d}s\,\mathrm{d}\lambda} \\ &\quad = \inf_{\lambda\in\mathbb{R}} \inf_{\nu\in\mathbb{N}} \frac{\left(\frac{-n-1+\lambda^2+4\gamma}{\lambda^2+(n/2-\gamma)^2}+1\right)\alpha_\nu + \lambda^2 - n + 3 + \frac{\alpha_\nu^2}{\lambda^2+(n/2-\gamma)^2}}{\frac{\alpha_\nu}{\lambda^2+(n/2-\gamma)^2+1}}. \end{aligned} \tag{2.8.32}$$

Thus our minimization problem reduces to finding

$$\inf_{x\geq 0} \inf_{\nu\in\mathbb{N}} f(x, \alpha_\nu, \gamma), \tag{2.8.33}$$

where

$$f(x, \alpha_\nu, \gamma) = x - n + 3 + \alpha_\nu \left(1 - \frac{16(1-\gamma)}{4x+4\alpha_\nu+(n-2\gamma)^2}\right). \tag{2.8.34}$$

Since $\gamma \leq 1$, it is clear that f is increasing in x, so the value (2.8.33) is equal to

$$\inf_{\nu\in\mathbb{N}} f(0, \alpha_\nu, \gamma) = \inf_{\nu\in\mathbb{N}} \left(3 - n + \alpha_\nu \left(1 - \frac{16(1-\gamma)}{4\alpha_\nu+(n-2\gamma)^2}\right)\right). \tag{2.8.35}$$

We have

$$\frac{\partial}{\partial \alpha_\nu} f(0, \alpha_\nu, \gamma) = 1 - \frac{16(1-\gamma)(n-2\gamma)}{(4\alpha_\nu+(n-2\gamma)^2)^2}. \tag{2.8.36}$$

Noting that

$$4\alpha_\nu + (n-2\gamma)^2 \geq 4(n-1) + (n-2\gamma)^2 \geq 4\sqrt{n-1}(n-2\gamma), \tag{2.8.37}$$

we see that

$$\frac{\partial}{\partial \alpha_\nu} f(0, \alpha_\nu, \gamma) \geq 1 - \frac{1-\gamma}{(n-1)(n-2\gamma)} > 0. \tag{2.8.38}$$

Thus the minimum of $f(0, \alpha_\nu, \gamma)$ is attained at $\alpha_1 = n-1$ and equals

$$3 - n + (n-1)\left(1 - \frac{16(1-\gamma)}{4(n-1) + (n-2\gamma)^2}\right) = \frac{2(\gamma - 1 + n/2)^2}{n - 1 + (\gamma - n/2)^2}. \quad (2.8.39)$$

This completes the proof for the case $\mathbf{v}_{\boldsymbol{\varphi}} = \mathbf{0}$.

If we drop the assumption $\mathbf{v}_{\boldsymbol{\varphi}} = \mathbf{0}$, then, to the integrand on the right-hand side of (2.8.19), we should add the terms

$$\rho^2 \left(\frac{\partial v_\varphi}{\partial \rho}\right)^2 + \left(\frac{\partial v_\varphi}{\partial \theta}\right)^2 + (\sin\theta \sin\theta_{n-3} \cdots \sin\theta_1)^{-2} v_\varphi^2. \quad (2.8.40)$$

The expression in (2.8.40) equals

$$\rho^2 \left|\nabla\left(v_\varphi \mathrm{e}^{\mathrm{i}\varphi}\right)\right|^2. \quad (2.8.41)$$

As a result, the right-hand side of (2.8.29) is augmented by

$$\int_{\mathbb{R}} \int_{S^{n-1}} R(\lambda, w_\varphi)\, \mathrm{d}s\, \mathrm{d}\lambda, \quad (2.8.42)$$

where

$$R(\lambda, w_\varphi) = \lambda^2 |w_\varphi|^2 + \left|\nabla_\omega\left(w_\varphi \mathrm{e}^{\mathrm{i}\varphi}\right)\right|^2 \quad (2.8.43)$$

with $\omega = (\theta, \theta_{n-3}, \dots, \varphi)$. Hence,

$$\inf_{\mathbf{v}} \frac{\int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}}\, \mathrm{d}x}{\int_{\mathbb{R}^n} \frac{|\mathbf{v}|^2}{|x|^n}\, \mathrm{d}x} = \inf_{w_\theta, w_\varphi} \frac{\int_{\mathbb{R}} \int_{S^{n-1}} (Q(\lambda, w_\theta) + R(\lambda, w_\varphi))\, \mathrm{d}s\, \mathrm{d}\lambda}{\int_{\mathbb{R}} \int_{S^{n-1}} (q(\lambda, w_\theta) + |w_\varphi|^2)\, \mathrm{d}s\, \mathrm{d}\lambda}. \quad (2.8.44)$$

Using the fact that w_θ and w_φ are independent, the right-hand side is the minimum of (2.8.32) and

$$\inf_{w_\varphi} \frac{\int_{\mathbb{R}} \int_{S^{n-1}} R(\lambda, w_\varphi)\, \mathrm{d}s\, \mathrm{d}\lambda}{\int_{\mathbb{R}} \int_{S^{n-1}} |w_\varphi|^2\, \mathrm{d}s\, \mathrm{d}\lambda}. \quad (2.8.45)$$

Since $w_\varphi \mathrm{e}^{\mathrm{i}\varphi}$ is orthogonal to one on S^{n-1}, we have

$$\int_{S^{n-1}} \left|\nabla_\omega\left(w_\varphi \mathrm{e}^{\mathrm{i}\varphi}\right)\right|^2 \mathrm{d}s \geq (n-1) \int_{S^{n-1}} |w_\varphi|^2\, \mathrm{d}s. \quad (2.8.46)$$

Hence the infimum in (2.8.45) is at most $n-1$, which exceeds the value in (2.8.39). The result follows for $\gamma \leq 1$.

For $\gamma > 1$ the proof is similar. Differentiation of f in α_ν gives

$$1 + \frac{16(\gamma - 1)((n-2\gamma)^2 + 4x)}{(4x + 4\alpha_\nu + (n-2\gamma)^2)^2}, \quad (2.8.47)$$

which is positive. Hence the role of the value (2.8.39) is played by the smallest value of $f(\cdot, n-1, \gamma)$ on $\mathbb{R}^+$. Therefore,

$$\inf_{\mathbf{v}} \frac{\int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}}\,\mathrm{d}x}{\int_{\mathbb{R}^n} \frac{|\mathbf{v}|^2}{|x|^n}\,\mathrm{d}x} = 2 + \min_{x\geq 0}\left(x + \frac{4(n-1)(\gamma-1)}{x+n-1+(\gamma-n/2)^2}\right). \tag{2.8.48}$$

The proof is complete. □

Proof of Corollary 1. We need to consider only $\gamma > 1$. It follows directly from (2.8.6) that

$$C_{3,\gamma}^{-1} = \left(\frac{3}{2} + \gamma - 1\right)^2 + 2,$$

which gives the result.

Remark. Using (2.8.22), we see that a minimizing sequence $\{\mathbf{v}_k\}_{k\geq 1}$, which shows the sharpness of inequality (2.8.4) with the constant (2.8.5), can be obtained by taking $\mathbf{v}_k = (v_{\rho,k}, v_{\theta,k}, \mathbf{0})$ with the Fourier transform $\mathbf{w}_k = (w_{\rho,k}, w_{\theta,k}, \mathbf{0})$ chosen as follows:

$$w_{\theta,k}(\lambda,\theta) = h_k(\lambda)\sin\theta, \qquad w_{\rho,k}(\lambda,\theta) = \frac{1-n}{i\lambda + n/2 - \gamma} h_k(\lambda)\cos\theta. \tag{2.8.49}$$

The sequence $\{|h_k|^2\}_{k\geq 1}$ converges in distributions to the delta function at $\lambda = 0$. The minimizing sequence that gives the value (2.8.7) of $C_{3,\gamma}$ is

$$w_{\theta,k}(\lambda, 0) = 0, \quad w_{\rho,k}(\lambda,\theta) = 0, \quad \text{and} \quad w_{\phi,k}(\lambda,\theta) = h_k(\lambda)\sin\theta,$$

where $\{|h_k|^2\}_{k\geq 1}$ is as previously.

2.8.3 Proof of Theorem 2

The calculations are similar but simpler than those in the previous section. We start with the substitution $\mathbf{v}(x) = \mathbf{u}(x)|x|^{2\gamma}$ and write (2.8.14) in the form

$$\frac{1}{C_{2,\gamma}} = \gamma^2 + \inf_{\mathbf{v}} \frac{\int_{\mathbb{R}^2} |\nabla \mathbf{v}|^2\,\mathrm{d}x}{\int_{\mathbb{R}^2} |\mathbf{v}|^2 |x|^{-2}\,\mathrm{d}x}. \tag{2.8.50}$$

In polar coordinates ρ and φ, with $\varphi \in [0, 2\pi)$, we have

$$\int_{\mathbb{R}^2} |\nabla \mathbf{v}|^2\,\mathrm{d}x = \int_{\mathbb{R}^2} \left\{ |\nabla v_\rho|^2 + |\nabla v_\varphi|^2 + \rho^{-2}\left(v_\rho^2 + v_\varphi^2 - 4v_\rho(\partial_\varphi v_\varphi)\right)\right\}\mathrm{d}x. \tag{2.8.51}$$

Changing the variable ρ to $t = \log\rho$ and applying the Fourier transform $\mathbf{v}(\rho,\varphi) \to \mathbf{w}(\lambda,\varphi)$, we obtain that the right-hand side is

$$\begin{aligned} \int_{\mathbb{R}} \int_0^{2\pi} &\left\{ \left(\lambda^2+1\right)\left(|w_\rho|^2 + |w_\varphi|^2\right) + |\partial_\varphi w_\varphi|^2 \right. \\ &\left. + |\partial_\varphi w_\rho|^2 - 4(\partial_\varphi w_\varphi)\overline{w_\rho} \right\} \mathrm{d}\varphi\,\mathrm{d}\lambda. \end{aligned} \tag{2.8.52}$$

The divergence-free condition for u becomes

$$w_\rho = -\frac{\partial_\varphi w_\varphi}{i\lambda + 1 - \gamma}, \tag{2.8.53}$$

which yields

$$\int_{\mathbb{R}^2} |\nabla \mathbf{v}|^2 \,\mathrm{d}x = \int_{\mathbb{R}} \int_0^{2\pi} \Big\{ \Big(\frac{\lambda^2 + 4\gamma - 3}{\lambda^2 + (1-\gamma)^2} + 1 \Big) |\partial_\varphi w_\varphi|^2 + \frac{|\partial_\varphi^2 w_\varphi|^2}{\lambda^2 + (1-\gamma)^2} + \big(\lambda^2 + 1\big)|w_\varphi|^2 \Big\} \,\mathrm{d}\varphi \,\mathrm{d}\lambda. \tag{2.8.54}$$

Analogously,

$$\begin{aligned} \int_{\mathbb{R}^2} |\mathbf{v}|^2 |x|^{-2} \,\mathrm{d}x &= \int_{\mathbb{R}} \int_0^{2\pi} \big(|w_\rho|^2 + |w_\varphi|^2\big) \,\mathrm{d}\varphi \,\mathrm{d}\lambda \\ &= \int_{\mathbb{R}} \int_0^{2\pi} \Big(\frac{|\partial_\varphi w_\varphi|^2}{\lambda^2 + (1-\gamma)^2} + |w_\varphi|^2 \Big) \,\mathrm{d}\varphi \,\mathrm{d}\lambda. \end{aligned} \tag{2.8.55}$$

Therefore, by (2.8.50)

$$\frac{1}{C_{2,\gamma}} = \gamma^2 + \inf_{x \ge 0} \inf_{\nu \in \mathbb{N} \cup 0} f(x, \nu, \gamma), \tag{2.8.56}$$

where

$$f(x, \nu, \gamma) = x + 1 + \nu \Big(1 - \frac{4(1-\gamma)}{x + \nu + (1-\gamma)^2} \Big). \tag{2.8.57}$$

Let first $\gamma \le 1$. Then f is increasing in x, which implies $f(x, \nu, \gamma) \ge f(0, \nu, \gamma)$. Since the derivative

$$\frac{\partial}{\partial \nu} f(0, \nu, \gamma) = 1 - \frac{4(1-\gamma)^3}{(\nu + (1-\gamma)^2)^2}, \tag{2.8.58}$$

is positive for $\nu \ge 2$, we need to compare only the values $f(0, 0, \gamma)$, $f(0, 1, \gamma)$, and $f(0, 2, \gamma)$. An elementary calculation shows that both $f(0, 0, \gamma)$ and $f(0, 2, \gamma)$ exceed $f(0, 1, \gamma)$ for $\gamma \notin (-1 - \sqrt{3}, -1 + \sqrt{3})$.

Let now $\gamma > 1$. We have

$$\frac{\partial}{\partial \nu} f(x, \nu, \gamma) = 1 + \frac{4(\gamma - 1)(x + (1-\gamma)^2)}{(x + \nu + (1 - \gamma^2))^2} > 0 \tag{2.8.59}$$

and therefore $f(x, \nu, \gamma) \ge f(x, 0, \gamma) = x + 1 \ge 1$. The proof of Theorem 2 is complete. □

Remark. Minimizing sequences that give $C_{2,\gamma}$ in (2.8.8) can be chosen as follows:

$$w_{\rho,k}(\lambda, \varphi) = 0, \qquad w_{\varphi,k}(\lambda, \varphi) = h_k(\lambda),$$

for $\gamma \notin (-1 - \sqrt{3}, -1 + \sqrt{3})$, and

$$w_{\rho,k} = \frac{h_k(\lambda)\sin(\varphi - \varphi_0)}{i\lambda + 1 - \gamma}, \qquad w_{\varphi,k} = h_k(\lambda)\cos(\varphi - \varphi_0),$$

when $\gamma \in (-1-\sqrt{3}, -1+\sqrt{3})$, for any constant φ_0. Here $\{|h_k|^2\}_{k\geq 1}$ converges in distributions to the delta function at 0.

Corollary. *Let $\gamma \neq 0$. Denote by ψ a real-valued scalar function in $C_0^\infty(\mathbb{R}^2)$ and assume, in addition, that $\nabla\psi(\mathbf{0}) = \mathbf{0}$ if $\gamma < 0$. Then the sharp inequality*

$$\int_{\mathbb{R}^2} |\nabla\psi|^2 |x|^{2(\gamma-1)}\, \mathrm{d}x \leq C_{2,\gamma} \int_{\mathbb{R}^2} \big(\psi_{x_1x_1}^2 + 2\psi_{x_1x_2}^2 + \psi_{x_2x_2}^2\big)|x|^{2\gamma}\, \mathrm{d}x \quad (2.8.60)$$

holds with $C_{2,\gamma}$ given in (2.8.8).

Indeed, for $n = 2$, inequality (2.8.4) becomes (2.8.60) if ψ is interpreted as a stream function of the vector field $\mathbf{u}$, i.e., $\mathbf{u} = \nabla \times \psi$.

2.8.4 Comments to Sect. 2.8

The results of this section are borrowed from the paper by Costin and Maz'ya [214]. In [715], Sobolevskii stated that the sharp constant in Hardy's inequality for arbitrary solenoidal vector functions in a convex domain coincides with the same constant in the classical one-dimensional case.

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Sobolev Spaces
with Applications to Elliptic Partial Differential Equations
Maz'ya, V.
2011, XXVIII, 866 p., Hardcover
ISBN: 978-3-642-15563-5