Chapter 9
Chamber Systems and Buildings

Abstract Chamber systems over a type set I were defined by J. Tits as a family of partitions of a set of vertices called chambers. An equivalent representation as certain classes of graphs with edges labeled by non-empty subsets of I allows one to describe morphisms, truncations, and residues graph-theoretically. Residual connectedness is defined for chamber systems. For a residually connected chamber system, each edge is labeled by exactly one type, while chamber systems in which each chamber lies on infinitely many distinct panels are not residually connected at all. Generalized polygons are presented as both chamber systems and point-line geometries in order to introduce chamber systems of type M. Buildings are plucked out of the sea of all chamber systems of type M by any one of six equivalent conditions involving strong-gatedness of residues, or galleries of reduced type.

9.1 Introduction

In the first part of this book, we undertook to understand the projective spaces and polar spaces as geometries of points and lines. But there are other more complex spaces in which projective and polar spaces are small constituent entities whose interaction with the space as a whole utilizes only modest properties of these subspaces. Because of the natural way they are defined, many of these new spaces may be called "classical."

We now know that the basic classical geometries of this world can be pretty well understood from the unifying point of view of a *building*. Such an assertion is vague, of course. But there are many favorite objects which occur over and over again under the guise of classical mathematics.[1] In a few words, these can be described as groups

[1] For example, Lie groups arise as a certain class of automorphisms of locally Euclidean manifolds. These are associated with Lie Algebras over the real or complex numbers. Then there are algebraic groups which are full automomorphism groups of an affine or projective variety. The definition is very different, but it is true that a non-degenerate algebraic group possesses a group-structure called a (B, N)-pair. Until the concept of building, the similarities between simple Lie groups and indecomposable algebraic groups hung on the frail sign that both were determined by a Dynkin diagram.

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of Lie type acting on coset spaces of parabolic subgroups. The theory of buildings arose between 1964 and 1967 as an attempt to unite the objects of such a theory without assuming the presence of a group, a topology or a Lie Algebra. This great achievement is entirely due to Professor J. Tits, to whom this book is dedicated.

The projective spaces and polar spaces, together with an appropriate class of subspaces, are just two examples of buildings. But there are many more, associated with the exceptional groups of Lie types G_2, F_4, E_6, E_7, and E_8. All of these are generic schemes which can be defined over any field, finite or infinite. It is also important to note that these are geometries of rank equal to the subscript. Thus G_2 is a rank two geometry – that is, a point-line geometry (which happens to be a generalized hexagon). On the other hand, E_8 is a rank eight geometry – that is, an incidence geometry with 8 sorts of objects. I do not ask the reader to understand this all at once. Just let it be said that there is a very nice set of geometries that seems to come to us from the sky.

So, in order to continue, we must describe buildings. The traditional way of defining buildings involves axioms insuring that a chamber system C over I possesses a rather dense system of isometrically embedded Coxeter chamber systems (each isomorphic to $C(M)$ where M is a fixed Coxeter matrix) called *apartments*. Exacting properties of the building then rests heavily upon special properties of the Coxeter chamber system.

Of course many renditions merely need to *introduce* the topic in just enough detail to harvest the results. This is especially true of books on finite group theory [77, 78, 125, 126] or books and articles on groups of Lie type [27, 44]. Generally, they can live with the "apartment axioms" alone, or, at worst, with a few extra properties of Coxeter groups that they may need by presenting an abbreviated auxilliary section on Coxeter groups.

Then there is another class of books and articles which wish to present a theory of buildings which analytically displays and motivates the interlocking properties from a more or less pedagogical point of view. With only one exception (see last paragraph of this section), these books follow the format of Tits' original book: a development in the language of simplicial complexes leading to Coxeter complexes and finally the apartment axioms [14, 63, 106]. The idea is that the apartment axioms have a natural background in the language of simplicial complexes, foldings, etc., and one's understanding of concepts can be checked there.[2] These pedagogical treatments quoted here are excellent.

The basic paradigm is that algebraic properties concerning the factorizations of words that represent the same element of the Coxeter group can be used to deduce properties of the building.

In this book we take a different approach in which the basic properties of buildings are deduced directly from the strong-gatedness of their residues and nothing

[2] Of course Tits' book was probably intended for experts, but many early sections go to great lengths to give expositions about geometries, uniqueness of mappings, and pseudoquadratic forms that anyone can understand.

more than that. As a result, no special section about properties of Coxeter groups is required. This would then be a treatment independent of the type M hypothesis, and would not even once require mention of the unnecessary simplicial complexes which have nothing to do with any of the key properties.

There are reasons for producing an alternative view:

1. (A pedagogical one.) Some students (such as the author) who don't understand simplicial complexes, or think the whole topic a lot of unecessary baggage, shouldn't be prevented from being introduced to buildings. Everyone at least understands graphs (perhaps before they enter University) whether or not they understand simplicial complexes.
2. (More pedagogy.) The simpler the axioms one begins with, the easier it is to teach the subject and unfold the tree of implications. We don't want to start with something too elaborate.
3. (Pedagogy and boredom.) Why should a new customer always get the same menu?
4. (Theoretical economy.) Virtually every property of Coxeter groups that one assembles to prove properties of buildings can in fact be proved directly for all chamber systems of type M from a simpler hypothesis.
5. (More theoretical economy.) The approach given here entails a number of equivalent conditions never dispayed together in the cited expositions of this subject. Naturally that invites mathematical connections.

On p. 292 I mentioned that there was an expositional textbook which I thought to be an exception to the standard. That was the book by Mark Ronan which introduced buildings as a chamber system with a $W(M)$-valued measure. Looking back, I think it would be fair to say that Ronan's book and Rudolph Scharlau's historic insights more or less determined the path that I have taken here.

9.2 Chamber Systems

9.2.1 The Chamber System of a Geometry

Suppose $\Gamma = (V, \tau, *)$ is a geometry over the type set I. Recall that this means that $\tau : V \to I$ is a mapping assigning a *type* from the type-set I to each *object of* V. Then the symmetric binary relation $*$ on V occurs only between objects of different types. Thus $(V, *)$ is a multipartite graph (the *incidence graph of the geometry*) whose components are the (coclique) fibers of the type function τ.

A *flag* is simply any clique of $(V, *)$. One notes that the type function is injective when restricted to any flag. For any flag F, the subset $\tau(F) := \{\tau(x) | x \in F\}$ is called the *type of the flag* F. A *flag-chamber* of Γ is simply a flag whose type is I. Let $\mathcal{F}$ be the full collection of all flag-chambers of Γ. We are going to form a graph $C = (\mathcal{F}, E, \lambda)$ whose vertex set is $\mathcal{F}$ and whose edge set E is accompanied by an edge-labeling function $\lambda : E \to I$ as follows. Two flag-chambers form an edge e of C labeled $\lambda(e)$ if and only if they are distinct flags which differ only in their constituent object of type $\lambda(e)$. In this case we say the flag-chambers are $\lambda(e)$-adjacent: that is, flag-chambers F_1 and F_2 are said to be i-adjacent if and only if they differ only in their constituent objects of type i. Thus distance in this chambersystem graph C is nothing more or less than the induced Hamming distance obtained when one regards C to be a subgraph of the Cartesian product over the type-fibers. Note that if F_1, F_2, and F_3 are three chamber flags such that (F_1, F_2) and (F_2, F_3) are edges of C bearing the same label, then (F_1, F_3) is forced to be an edge of C with the same label. Thus for any $X \in \{\mathcal{P}, \mathcal{L}\}$, the relation of being equal or i-adjacent is an equivalence relation on the vertices of C (the set of flags $\mathcal{F}$).

9.2.2 Abstract Chamber Systems

Well, that is the starting point of the notion of an abstract chamber system. An *(abstract) chamber system over I* is a triple (V, E, λ) where (V, E) is a simple graph, and λ is a mapping from the 2-subsets of V into the collection $F(I)$ of finite subsets of I with these properties:

(Support axiom.) $\lambda(e)$ is non-empty if and only if $e \in E$.
(Triangle axiom.) For every two edges (a, b) and (b, c) of E,

$$\lambda(a, b) \cap \lambda(b, c) \subseteq \lambda(a, c).$$

Fix two vertices x and y of V. We say that vertex x is *i-adjacent to y* if and only if $(x, y) \in E$ and i is one of the labels to be found marking this edge – i.e., $i \in \lambda(x, y)$. Then the triangle axiom shows us that i-adjacency together with the identity relation form an equivalence relation on the vertices of V.

The cardinality of the type set I is called the *rank of the chamber system*. Note that $\lambda(E)$ may very well be a proper subset of I.[3]

We have seen in Chap. 2 that, although geometries are just multipartite graphs, it is customary to use a special language for these graphs: vertices are called "objects," edges are "incident pairs of objects," and cliques are "flags." Similarly in chamber systems it is customary to call the vertices *chambers* and to refer to a path $G = (v_0, v_1, \ldots, v_n)$ as a *gallery*. The sequence $(\lambda(v_0, v_1), \lambda(v_1, v_2), \ldots, \lambda(v_{n-1}, v_n))$ is then the *type of the gallery G*.

Let us begin with some simple examples. Our examples are somewhat atypical in that in each underlying graph, each edge bears only one label chosen from $I = \{1, 2, 3\}$, while for a general chamber system, an edge could bear any non-empty set

[3] Again, note the contrast with geometries. For geometries, the rank was the number of types of objects $|\tau(V)|$ in the geometry. But in chamber systems the rank is the number of elements of I whether or not each type appears as an edge label or not. There is a simple reason for this distinction. Geometries and chamber systems will soon be seen to be connected by two functors, and the definitions of rank are geared so that both of these functors are rank-preserving.

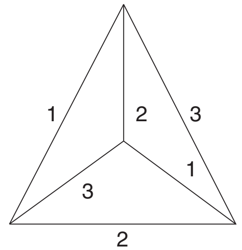

Fig. 9.1 A chamber system of rank three on the graph of the tetrahedron

of labels. Despite this special property, these examples will serve to illustrate many of the concepts which are to follow.

Example 1 (The edge-labeled tetrahedron.) Here the underlying graph is the graph of the vertices and edges of a tetrahedron – that is, the complete graph K_4. Notice that for each edge, there is a unique edge sharing no vertex with the former. In this way, the six edges are partitioned into three pairs, each pair forming a 1-factor of the graph.[4] The two edges of one 1-factor are all labeled "1," those of a second 1-factor are labeled "2," and those of the third are labeled "3." Thus each vertex of the tetrahedron lies on three edges bearing (one each) the three labels 1, 2, and 3. This example is illustrated in Fig. 9.1. We represent the labels attached to each edge by placing the label adjacent to the relevant edge in the figure.

Example 2 (The thin building of trigon type.) Here the underlying graph is the graph of the vertices and edges of the cube. In a normal embedding of the cube in the Euclidean plane, the edges come in three parallel classes, each class forming a 1-factor of the graph. We now label all the edges of one of these 1-factors by "1," the edges of the next 1-factor by "2," and the edges of the third 1-factor by "3." The result is the graph of Fig. 9.2a.

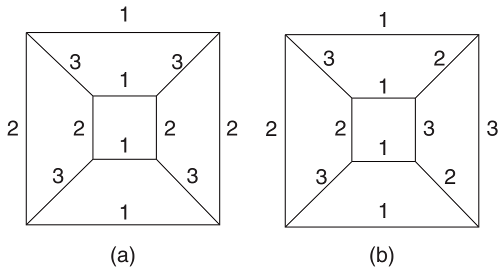

Fig. 9.2 Two chamber systems based on the graph of the cube

[4] This term "1-factor" was defined early in Chap. 8 in discussing an example of a near-polygon.

Example 3 (Another chamber system based on the cube.) Again we have the graph of the cube as our underlying graph, but this time the labels are attached differently as illustrated in Fig. 9.2b.

Example 4 (The shaved cube.) Imagine a perfect cube made of wood. Suppose one cut along each edge of the cube with a knife in a plane at a 135° angle with the planes of the two faces bordering the edge being cut. We do not cut deep enough to meet the plane of the cut in an edge opposite an edge in an original face of the cube. In other words, the shaving of the edges is rather slim. The result is a faceted figure with faces which are either squares or hexagons. In fact each edge of the cube is now a hexagonal face, while each face of the cube is still a square face, but one slightly smaller than the original. Our new graph has valence 3 and has 32 vertices.

Now there is (up to a permutation of I) just one way to label each edge by an element of I so that:

1. The three edges on a vertex bear distinct labels.
2. The edges of each external hexagonal or square face carry just two of the labels, alternating as one tours the face.

This chamber system is presented in Fig. 9.3.

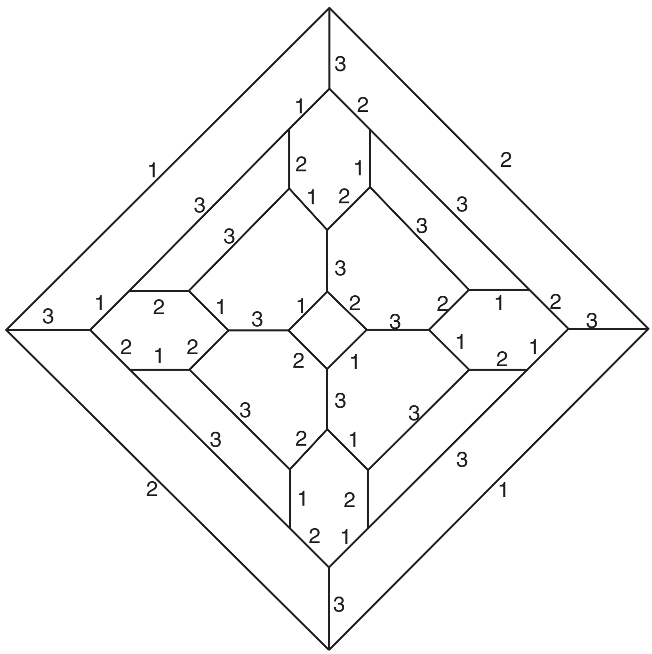

Fig. 9.3 The chamber system of the shaved cube

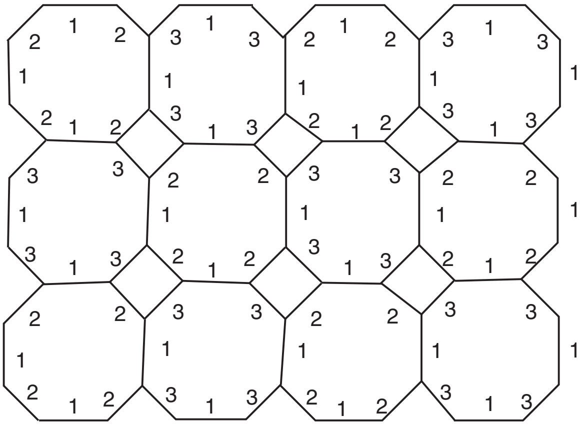

Fig. 9.4 A tiling of the Euclidean plane viewed as a chamber system

Example 5 (Tiling of the plane by squares and octagons: a thin building of type $\tilde{C}_2$.) This tiling of the Euclidean plane can be acheived as follows. At each integral point of the $\mathbf{Z} \times \mathbf{Z}$ lattice, one inserts a small diamond centered on the lattice point, with vertices pointed up and down and left and right. Each diamond should use $(1 - \sqrt{2}/2)$ of the line segment joining two lattice points. Now retain the middle $(\sqrt{2} - 1)$ of all the edges connecting adjacent lattice points. These segments are horizontal or vertical, connect vertices of adjacent diamonds, and are all labeled by "1." The diamonds alternate in their labeling. For adjacent diamonds, one has the label "2" on its northwest and southeast borders and "3" on its northeast and southwest borders, while it is the other way round for its adjacent partner diamond. This causes the octagonal faces to have their edges labeled by "1" and just one of the other labels. A portion of this infinite rank-three chamber system is given in Fig. 9.4.

Chamber Subsystems

Suppose $C := (C, E, \lambda)$ is a chamber system over I. For every subgraph $X := (X, E')$ of C (recall from Chap. 1 that this entails $X \subseteq C$, and $E' \subseteq E_X$, the subset of edges of E having both their vertices in X) we can define an edge-labeling $\lambda' : E' \to I$ which is just the restriction of λ to the edges of E'. Suppose, now that X is *triangle-closed* relative to (C, E), that is:

(ST) *If $\{a, b, c\}$ is a triangle of (C, E) (i.e., a 3-clique) with at least two of the three edges (a, b), (b, c), and (a, c) lying in E', then the third edge also lies in E'.*

Then the subgraph $X = (X, E')$ is converted into a chamber system $X = (X, E', \lambda')$ over J, where $J := \lambda(E')$. We call such a triangle-closed X a *label-induced chamber subsystem* of C. The adjective "label-induced" is there because all labels that are provided by the edge set E' are displayed.

More generally we call a chamber system (X, E', λ') over J a *chamber subsystem* of the chamber system $C = (C, E, \lambda)$ over I if and only if:

1. $X \subseteq C$ and $E' \subseteq E_X$ so (X, E') is a subgraph of (C, E).
2. For each edge $e \in E'$, $\lambda'(e) \subseteq \lambda(e)$; in particular $J = \lambda'(E') \subseteq \lambda(E) \subseteq I$.

Morphisms of Chamber Systems

Suppose $C := (C, E, \lambda)$ and $C' := (C', E', \lambda')$ are two chamber systems over I. A *morphism of chamber systems over I* is a morphism $f : C \to C'$ of the underlying graphs $(C, E) \to (C', E')$ such that for every edge $e = (c_1, c_2)$ of E, for which $f(e)$ is an edge of E' (rather than a single vertex), we have

$$\lambda(e) \subseteq \lambda'(f(e)).$$

Consider a morphism f of chamber systems over I:

$$f : C = (C, E, \lambda) \to C' := (C', E', \lambda').$$

We say that f is *chamber surjective (injective, bijective)* if and only if the underlying graph morphism is vertex surjective (injective, bijective, resp.). The morphism is said to be a *full morphism* if and only if:

1. Every edge $e' \in E'$ is the image of an edge e of E.
2. For any $e \in E$ for which $f(e)$ is an edge e' in E', one has $\lambda(e) = \lambda'(e')$.

A bijective full morphism is called an *isomorphism of chamber systems*. As one might expect, an isomorphism of a chamber system with itself is called an *automorphism of the chamber system*, and such automorphisms of a chamber system C form a group which we denote by $\mathrm{Aut}(C)$.

Let us consider automorphisms of our examples.

In Example 1, the underlying graph is K_4, the graph of the tetrahedron, and the automorphism group of this graph is the symmetric group on four letters, Sym(4). But as a chamber system, its automorphism group is seen to be the Klein four-group acting regularly on the four chambers.

In Example 2, the trigon, the full automorphism group of the graph of the cube is $\mathrm{Sym}(4) \times Z_2$ of order 48. But the automorphism of the chamber system is easily seen to be the largest normal 2-subgroup of this group: $K \times Z_2$ where K is the Klein four subgroup of the Sym(4)-direct factor. Note that K acts in two regular orbits on chambers.

In Example 5, each diamond (involving only the labels 2 and 3) is centered on a lattice point of the integral lattice $L = \mathbf{Z} \times \mathbf{Z}$. Let σ denote the translation of the Euclidean plane which maps each vector (a, b) to $(a, b) + (1, 1) = (a + 1, b + 1)$ – that is translation by the vector $(1, 1)$. Similarly, let τ denote translation by the vector $(2, 0)$. Then the reader can verify that both σ and τ are automorphisms of L which induce automorphisms of the chamber system C of this example.

Now let $A = \langle \sigma, \tau \rangle \leq \mathrm{Aut}(C)$. Then it is easy to see that A acts freely (that is semiregularly) on chambers and that any "fundamental region" contains just eight chambers. For example, if we take two diamonds, one immediately to the right of another, their union is a "fundamental region" – that is, a subset F of C (now regarded as the set of chambers) such that (1) $F \cap F^a = \emptyset$ for each non-identity element a of A and (2) $C = \cup\{F^a | a \in A\}$.

Any time we have a subgroup A of the automorphism group of a chamber system (C, E, λ) over I, we can define a chamber-surjective morphism from C to a chamber system $C/A := (C/A, E/A, \lambda/A)$ as follows. First, regarding C as the set of chambers (the vertex set of (C, E)), let C/A denote the collection of A-orbits on C. An edge of E/A will be a pair of distinct A-orbits (K, L) for which at least one member of K and one member of L form an edge of E. Then one sees that the mapping $C \to C/A$, which maps each chamber to the A-orbit to which it belongs, is a morphism of graphs $(C, E) \to (C/A, E/A)$ given as an example of a graph-morphism in Chap. 1, p. 16. Finally we define the set of labels $(\lambda/A)(K, L)$ to be attached to the edge (K, L) as the set

$$\cup\{\lambda(k, l) | (k, l) \in K \times L, (k, l) \in E\}.$$

(Note that since K and L are A-orbits, the set above could be written $\cup\{\lambda(x, y) | y \in x^{\perp} \cap L\}$ for any fixed element x of K.) It is easy to check that the triangle axiom holds for λ/A and so $C/A := (C/A, E/A, \lambda/A)$ is indeed a chamber system and that the canonical graph morphism is a morphism of chamber systems.

In Example 2, let A be the subgroup of the automorphism group generated by the involution α which takes each vertex to its opposite vertex in the cube. This mapping is easily seen to preserve the labels and so is an automorphism of the chamber system. The canonical projection morphism $C \to C/A$ produces a morphism onto a chamber system C/A isomorphic to that of Example 1.

In the exercises, the student will be asked to show that if A is the group of automorphisms of the chamber system C of Example 5 generated by the automorphisms σ and τ described above, then its canonical image C/A is isomorphic to the chamber system C' of Example 3, displayed in Fig. 9.2b.

A morphism $f : (C, E, \lambda) \to (C', E', \lambda')$ of chamber systems over I is called a *fibering of chamber systems* if and only if f is a full morphism for which the underlying graph morphism is a fibering. Recall from Chap. 1, p. 16, that the latter entails these two properties: (1) f is chamber surjective and (2) if c is a chamber, then f bijectively maps the edges of C on c to the edges of C' on $f(c)$. In particular, all vertices of a fiber $f^{-1}(c')$ are pairwise at distance at least three in (C, E). Notice that in a fibering, given any gallery $g' = (c'_0, \ldots, c'_n)$ of the image, and a chamber c_0 of the fiber $f^{-1}(c'_0)$, there exists a unique lift $g = (c_0, \ldots, c_n)$ which is a gallery in C of the same type.

The canonical projection morphisms $C \to C/A$ from the chamber system of Example 2 onto the chamber system of Example 1 ($A = \langle \alpha \rangle$), and from Example 5 onto the chamber system of Example 3 ($A = \langle \sigma\tau \rangle$), are examples of fiberings.

The Functor C from Geometries to Chamber Systems

It is worth remarking that chamber systems over I (like geometries over I) form a category with respect to the chamber system morphisms (the label-preserving graph homomorphisms) that we have just defined. That means we have identity mappings and we can compose morphisms when the domains and ranges allow it.

Let $\Gamma = (V, E, \tau)$ be a geometry over I, where we suppose $I = \tau(V)$ is the full set of types realized by objects of the geometry. Our construction mentioned ealier produced a chamber system $\mathbf{C}(\Gamma)$ over I whose chambers are the flag-chambers of Γ, the set of flags of type I, two of them being i-adjacent if and only if they differ only by an object of type i. Note that if no chamber flags exist, there may be no chambers at all. Nonetheless, this empty set of chambers is considered to be a chamber system over I. Also, it is clear that if i and j are distinct types, then two flag-chambers cannot be both i- and j-adjacent at the same time. That means that in $\mathbf{C}(\Gamma)$, the labeling function λ assumes only singleton sets of I. Thus $\mathbf{C}(\Gamma)$ is not entirely a typical abstract chamber system.

Now consider a morphism $f : \Gamma \to \Gamma'$ of geometries over I, where I and Γ are as in the preceding paragraph. Then the geometry Γ' also has an object of each type in I. In addition, if F is a flag chamber of Γ, then $f(F)$ is a flag chamber of Γ'. Moreover, if F_1 and F_2 are i-adjacent flag chambers of Γ, then either $f(F_1) = f(F_2)$ (which happens if the objects of type i by which the F_i differ are mapped to a common chamber) or else $f(F_1)$ is i-adjacent to $f(F_2)$. Thus we see that the geometry morphism f induces a chamber system morphism $\mathbf{C}(f) : \mathbf{C}(\Gamma) \to \mathbf{C}(\Gamma')$. One verifies that if $f_1 \circ f_2$ is the composition of two morphisms of geometries over I, then $\mathbf{C}(f_1 \circ f_2) = \mathbf{C}(f_1) \circ \mathbf{C}(f_2)$. Also, if f is an identity mapping, then so is $\mathbf{C}(f)$.

Thus our construction of $\mathbf{C}(\Gamma)$ from Γ in fact provides a functor $\mathbf{C}$ from the category of geometries over I into the category of chamber systems over I.

Residues of a Chamber System

Let $C = (V, E, \lambda)$ be a chamber system over I. For each subset J of I, let E_J be the set of edges e of E for which $\lambda(e) \cap J$ is non-empty – that is, those edges whose non-empty set of labels contains at least one element of J. Each connected component $R := (W, E_W \cap E_J)$ of the graph (V, E_J) is called a *residue of type J of the chamber system C*.[5] The set $I - J$ is called the *cotype* of the residue. The cardinality of J and $I - J$ of a residue of type J is called the *rank* and *corank* of the residue $R := (W, E_W \cap E_J)$, respectively. Note that R may be regarded as a chamber system over J in its own right. Of couse, R is a chamber system over the possibly smaller set $\lambda(E_W \cap E_J)$, but is nevertheless called a residue of type J and taken to be a chamber system over J. A residue of rank one is just an i-adjacency

[5] The notation here is important. Since W is a subset of the vertex set V, the symbol E_W refers to the edges which have both their vertices in W (see Chap. 1). But when J is a subset of the index set I, E_J is the set of edges which bear a label in J.

equivalence class for some type i. Such residues of rank one are often called *panels*. Let us consider residues in our examples. In Example 1, for each type $i \in \{1, 2, 3\}$, there are two residues of type i, namely two edges labeled i forming a 1-factor of the tetrahedron. In fact, something like this is true for residues of type i in all of our examples: the residues of type i form a 1-factor of the graph.

In Example 1, for every 2-subset $\{i, j\}$ of I, the residue of type $\{i, j\}$ is unique and is a spanning subgraph which is a square. Note that these subgraphs are not induced subgraphs.

In Example 2, the subgraph on V obtained by using only edges which bear labels i or j $(i \neq j)$ has two connected components each of which is an induced subgraph which is a square; they are opposite faces of the cube.

In Example 3, there are again two residues of type $\{2, 3\}$ representing opposite faces of the cube. But here, the residues of type $\{1, 3\}$ and type $\{1, 2\}$ are spanning subgraphs which are octagons. Clearly neither is an induced subgraph. Moreover, these two residues intersect at the disconnected subgraph consisting of the 1-factor of all edges labeled '1.'

In Example 4, the residues of rank two correspond to the faces of the "shaved cube": 6 are squares (corresponding to the original faces of the cube) and 12 are hexagons (corresponding to the 12 edges of the original cube). The residues of type $\{i, j\}$, a 2-subset of $I = \{1, 2, 3\}$, consist of exactly two squares (corresponding to opposite faces of the original cube) and four are hexagons (corresponding to the parallel class of edges connecting these two faces in the original cube). Thus we see that the rank two residues of a given type do not belong to a fixed isomorphism type in this example. However all rank two residues are induced subgraphs and intersect pairwise at the empty set or a single rank one residue.

In Example 5, all residues of type $\{2, 3\}$ are squares; these are the "diamonds" which were centered over each integral lattice point. Otherwise the residues of type $\{i, j\}$ are octagons. All are induced graphs with any two meeting at the empty set or a rank one residue.

The Functor $\boldsymbol{\Gamma}$

Now one can define a geometry $\boldsymbol{\Gamma}(C)$ over I as follows. The objects of type i in $\boldsymbol{\Gamma}(C)$ are exactly the residues of C of cotype i – that is, the connected components of $(V, E_{I-\{i\}})$. Two objects (corank one residues) are incident in this geometry if and only if they contain a common chamber. Note that two distinct objects of the same type cannot be incident since these are chamber-disjoint connected components of the same subgraph.

Remark Then $\boldsymbol{\Gamma}$ is a functor from the category of chamber systems over I to the category of geometries. It isn't really important, but it does allow us to sling this word "functor" around, and that is a lot of fun.

Let us examine the geometries $\boldsymbol{\Gamma}(C)$ that are obtained as C ranges over our five examples. In Example 1, there is exactly one rank two residue of each type. Since these are the residues of corank one, we obtain a geometry with exactly three objects, one of each type. The objects are pairwise incident and so together they form the unique flag chamber of the geometry. Thus $\mathbf{\Gamma}(C)$ is the graph K_3 (viewed as a tripartite graph). Then $\mathbf{C}(\mathbf{\Gamma}(C))$ is still a chamber system over $I = \{1, 2, 3\}$ but it has only one chamber.

In Example 2 there are exactly two residues of corank one of each type, and such residues of different types always intersect in a residue of rank one. Thus the geometry $\mathbf{\Gamma}(C)$ is the complete tripartite graph $K_{2,2,2}$ with each component 2-coclique being the objects of a specific type.

In Example 3 the geometry $\mathbf{\Gamma}(C)$ is the complete tripartite graph $K_{2,1,1}$ where the unique 2-coclique represents the set of objects of type 1, and of the other two objects, one is type 2 and the other is type 3.

In Example 4 the geometry $\mathbf{\Gamma}(C)$ contains six objects of each type. The graph is clearly not a complete tripartite graph since there is a residue of type $\{1, 2\}$ meeting a residue of type $\{1, 3\}$ at the empty set. We leave it as an exercise for the student to work out the complete incidence graph.

In Example 5 the geometry $\mathbf{\Gamma}(C)$ has three sorts of objects: the diamonds, and the two types of suboctagons. If D is a diamond (type 1 object) it is incident in this geometry with exactly two objects of type 2 and two objects of type 3 and its residue in this geometry (the subgraph $D^{\perp} - \{D\}$) is the complete subgraph $K_{2,2}$, a subgeometry over $\{2, 3\}$.

Truncations of a Chamber System

Let $C = (V, E, \lambda)$ be a chamber system over I. Fix a subset J of I and set $K = I - J$. Let C/J denote the collection of all residues of C of type J (these are components of a partition of C). For each such pair, (A, B) of residues of type J, let $\lambda^K(A, B)$ denote the set of all types $k \in K$ for which $A \cup B$ lies in a residue of type $J \cup \{k\}$ (k-adjacency). Let E^K be the collection of distinct pairs (A, B) of residues of type J such that $\lambda^K(A, B) \neq \emptyset$. Then $C_K := (C/J, E^K, \lambda^K)$ is a chamber system over K, which we call the *truncation of C of type K*.

Remark Note that if C is connected, so is C_K. There is a simple reason for this. The mapping which takes each chamber c of C to the unique residue of type J which contains it, say $R_J(c)$, is a vertex surjective morphism of the underlying graphs of the two chamber systems.

Lemma 9.2.1 (Functorial properties of truncation.)

1. *If $f : C' \to C$ is a morphism of chamber systems over I and $K \subseteq I$, then there is an induced morphism $f_K : C'_K \to C_K$ as chamber systems over K in which, for each $R \in C'/(I - K)$, $f_K(R)$ is the unique residue of C of type K containing the connected set of chambers $f(R)$.*
2. *Suppose C is a chamber system over I and $K \subseteq I$. Then there is an isomorphism*

$$(\mathbf{\Gamma}(C))_K \to \mathbf{\Gamma}(C_K).$$

Chamber Systems Defined by Cosets

This is a simple construction which will help us describe the spherical buildings that are soon to arrive on the scene. Let G be a group and let B be a distinguished subgroup. Let $\mathcal{H} := \{H_1, H_2, \ldots, H_n\}$ be a collection of subgroups of G indexed by I, where each subgroup of $\mathcal{H}$ contains B. Just from this data, we may define a chamber system $C := C(G, B; \mathcal{H})$ whose chambers (i.e., vertices) are the right cosets G/B of the subgroup B. Coset Bg is i-adjacent to vertex Bh if and only both of these cosets lie in a common coset of H_i (this would be $H_i g = H_i h$). Thus i-adjacency (together with the identity relation) forms an equivalence relation, and so a chamber system over $I = \{1, \ldots, n\}$ is the result. Such a chamber system is called a *coset chamber system*.[6] (Note that, in this example, if the groups H_i do not meet pairwise at B, it is possible for an edge of the chamber system to bear multiple labels.) The notation is important: the first two key groups G and B appear first, followed by a distinguished semi-colon. Then the rest.

9.2.3 Residually Connected Chamber Systems

At the beginning of Sect. 9.2 we saw that if Γ is a geometry over I, then there is a well-defined chamber system of flag-chambers $\mathbf{C}(\Gamma)$ without multiply-labeled edges. Of course, if there are no flag chambers, $\mathbf{C}(\Gamma)$ could very well be a meagre landscape.

Conversely, given a chamber system C over I as given earlier Sect. 9.2, there is a geometry $\boldsymbol{\Gamma}(C)$ whose objects of type i are the residues of C of cotype i. Two such residues are incident if and only if they share a common chamber.

We have remarked that these constructions produce (1) a very natural functor $\boldsymbol{\Gamma}$ between the category of chamber systems and the category of geometries and (2) another functor $\mathbf{C}$ from the category of geometries over I to the category of chamber systems over I. Now there is a property called residual connectedness for geometries, and another notion by the same name for chamber systems which was first graph-theoretically formulated by Arjeh Cohen. When either one of these properties is present, the functors are inverses of each other. The theorem is this.

Theorem 9.2.2 (Arjeh Cohen, [10].)

1. *If geometry G is residually connected of finite rank, then so is $\mathbf{C}(G)$, and there is a geometry isomorphism $\boldsymbol{\Gamma}(\mathbf{C}(G)) \simeq G$.*
2. *If C is a residually connected chamber system, then $\boldsymbol{\Gamma}(C)$ is residually connected, and $\mathbf{C}(\boldsymbol{\Gamma}(C)) \simeq C$.*
3. *There exists an isomorphism between the subcategory of residually connected geometries over a finite typeset I, and the subcategory of residually connected chamber systems over the same finite I.*

[6] In analogy with the term "coset geometry."

In other words, the residually connected objects on both sides form subcategories where the restricted functor yields an isomorphism. However, it turns out that the conditions for residual connectedness can never be realized when the typeset is infinite and the edges leaving any chamber realize all types. So this is primarily a concept that will kick in only for chamber systems and geometries of finite rank. It will turn out that buildings of finite rank in fact possess this property – thus allowing us to speak of buildings as either geometries or chamber systems in the case of finite rank.

Residual Connectedness of Geometries, Revisited

Let $\Gamma = (V, E, \tau)$ be a geomety over $I := \tau(V)$. In Chap. 2, we said that Γ was a *residually connected geometry* if and only if:

(RC1) Every flag of corank one has a non-empty residue (that is, it lies in a flag chamber).
(RC2) For every flag F of corank at least 2 (including the empty flag if appropriate), the residue $\mathrm{Res}_\Gamma(F)$ is a non-empty connected geometry.

In Exercise 3 of Chap. 2, we showed that this condition implies:

- Every residue of Γ is residually connected (Lemma 2.5.1 of Chap. 2).
- If the rank $|I|$ is finite, Γ is chamber connected (part 2 of Lemma 2.5.1. Exercise 3 of Chap. 2 is to show that this fails for infinite rank).
- If the rank $|I|$ is finite, then every truncation of rank at least two is residually connected (Corollary 2.5.3 of Chap. 2).

Residual Connectedness for Chamber Systems

Let $C = (C, E; \lambda)$ be a chamber system over I. The chamber system C is said to be *residually connected* if and only if:

(CRC1) For any family $\mathcal{F} = \{R_t\}$ of residues of C which intersect pairwise non-trivially, the global intersection $\cap\{R_t \in \mathcal{F}\}$ is non-empty and connected.
(CRC2) For any chamber c the intersection of all corank one residues of C which contain c is the set $\{c\}$ itself.

An immediate consequence of this property is recorded in the following.

Lemma 9.2.3 *If C is a residually connected chamber system over I, and J is a proper subset of I, then any residue of type J is the intersection of all the corank one residues which contain it.*

The proof of this lemma is left as Exercise 9.8 at the end of this chapter.

Theorem 9.2.4 *Assume $C = (C, E; \lambda)$ is a residually connected chamber system over I. Suppose $e = (x, y)$ is an edge bearing the label i – that is $e \in E$ and*

$i \in \lambda(e)$. *Suppose* $G = (x = x_0, x_1, \ldots, x_n = y)$ *is any gallery connecting to* x *to* y. *Then for some integer* j *in the interval* $[1, n]$, *we have*

$$\lambda(x_{j-1}, x_j) = \{i\}.$$

In particular:

1. *The type function* λ *never assumes multiple values – that is, for every edge* $e \in E$, $\lambda(e)$ *is a single-element subset* I.
2. *Each residue of cotype* i *is an induced subgraph of* (C, E).
3. *All residues are induced subgraphs.*

Proof Take C, e, $i \in \lambda(e)$, and G as in the hypothesis. Suppose each edge of G bears a label in $I - \{i\}$, and so x and y belong in the same residue of cotype i. On the other hand, since (x, y) is itself an edge bearing label i, x, and y lie together in a common residue of cotype t for each $t \in I$ distinct from i. It follows that every residue of corank one which contains x also contains y, and that contradicts (CRC2). So for some edge e of the gallery G, we must have $\lambda(e) = \{i\}$, as required.

The remaining three statements are immediate consequences of the first. If $\lambda(e)$ contained two distinct type-labels, say i and j, then the first statement would be contradicted with $G = (x, y)$. Thus λ cannot assume multiple values.

Suppose, for a residue R of cotype i, $(R, E_R \cap E_{I-\{i\}})$ (which by definition is connected) was not an induced graph. Then there must exist a pair of chambers, x and y in R, such that $e = (x, y)$ is an edge bearing no label of $I - \{i\}$ – forcing $\lambda(x, y) = \{i\}$. But since the graph for R is connected, there is a gallery G connecting x to y, each of whose edges bears a label not equal to i. That contradicts the first statement of the theorem. One can only conclude that the underlying graph of R is an induced graph.

Now statement 3 of the theorem follows from Lemma 9.2.3 upon observing than any arbitrary intersection of induced graphs is an induced graph. □

Residual Connectedness in Infinite Rank

The next theorem reveals that the residual connectedness property does not normally hold for a chamber system of infinite rank.

We say that a chamber system C over I is *firm* if and only if each panel contains at least two chambers. This could be stated another way: *for each chamber* c *and each type* i, *there exists a further chamber* c' *which is* i*-adjacent to* c.

Theorem 9.2.5 (Kasikova and Shult [84].) *No firm chamber system of infinite rank is residually connected.*

Proof Assume by way of contradiction that C is a residually connected firm chamber system of infinite rank $|I|$. Then each vertex c is i-adjacent to at least one other vertex, for each $i \in I$, and by Theorem 9.2.4, each edge bears a unique type label.

Thus any gallery $G = (c_0, c_1, \ldots)$ has a type $(t_1, t_2, \ldots)$ associated with it, where $t_i := \lambda(c_{i-1}, c_i)$. (The gallery can be finite or countably infinite.) We are particularly interested in those galleries in which *no type is repeated* – i.e., $t_i \neq t_j$ if $0 \leq i < j$. It is an easy consequence of firmness and infinite rank that any such gallery of finite length can be extended. Thus the following result holds:

Step 1. *There exists an infinite gallery* $G = (c_0, c_1, \ldots)$ *of type* $(t_1, t_2, \ldots)$ *where no type is repeated in the sequence* $\{t_i := \lambda(c_{i-1}, c_i) | i = 1, 2, \ldots\}$.

Now, for each chamber c_i in the gallery G, let $R_i := R_{I-\{i\}}(c_i)$ be the unique residue of cotype t_i containing c_i. Since the types t_j, $j > i$ are pairwise distinct, the residue R_i is forced to contain all of the chambers c_k, for all $k \geq i$. But is it possible for R_i to contain any of the earlier chambers in the sequence? The answer appears in the following step.

Step 2. *The residue* R_i *contains no chamber* c_j *of* G, *for* $j < i$.

Suppose by way of contradiction that $c_j \in R_i$ for $j < i$. Then the chambers $c_{j+1}, \ldots c_{i-1}$ are also in R_i. But now (c_{i-1}, c_i) is an edge connecting two vertices of R_i which is not itself an edge of R_i. Thus R_i is not an induced graph, contrary to Theorem 9.2.4. Thus the assertion of Step 2 is established.

[Note that Step 2 implies that the gallery G never crosses itself, that is, the chambers c_k are pairwise distinct.]

Step 3. *The intersection* $\cap_{i \in \mathbf{N}^+} R_i$ *is empty.*

Suppose c were a chamber in each R_i, $i = 1, 2, \ldots$. Clearly c is not one of the chambers c_k of the gallery G for R_{k+1} does not contain c_k by Step 2. Since C is connected (for it is assumed residually connected) there is a gallery of finite length $H = (c = h_0, h_1 \ldots h_d = c_0)$ connecting c to the initial vertex c_0 of the gallery G. Now select any chamber c_k, $k > 1$ of the gallery G. Since c and c_k both belong to the residue R_k, which by definition is connected, there exists a finite gallery H_k connecting c_k to c, having no edge of type k. Thus if H had no edge of type t_k, we should conclude that $c_0 \in R_k$ against Step 2. It follows that some edge of the finite gallery H bears the label t_k. But this statement must be true for *all* k greater than 1 – that is, for infinitely many values of k. On the other hand, since the residual connectedness of C forces each edge of C to bear exactly one type-label (Theorem 9.2.4), H can only accomodate d types. So we have a contradiction. It follows that no such c exists, and so the intersection of all the R_i is empty. That establishes Step 3.

Now we can complete the proof of the theorem. Clearly if $i < j$, $R_i \cap R_j$ contains c_j, so the R_i pairwise intersect non-trivially. On the other hand, their global intersection is empty. This contradicts (CRC1) and so C cannot be residually connected. □

Residual Connectedness and Functors C and Γ

We begin with the following.

Lemma 9.2.6 *Suppose C is a chamber system over I having the property (CRC1). Then $\mathbf{\Gamma}(C)$ is a residually connected geometry over I. If, in addition, C has (CRC2), then there is an isomorphism*

$$C \simeq \mathbf{C}(\mathbf{\Gamma}(C)).$$

Proof Let $F := \{R_\sigma | \sigma \in I - \{i\}\}$ be a flag of cotype i in $\mathbf{\Gamma}(C)$; here the R_σ are residues of C of cotype $\sigma \in I$, $\sigma \neq i$, which intersect pairwise non-trivially. By (CRC1), the intersection $\cap\{R_\sigma \in F\}$ is a (non-empty) panel P of type i. Then for any chamber $c \in P$, the unique residue R_i of cotype i containing c is an object in the residue of F in the geometry $\mathbf{\Gamma}(C)$. That proves (RC1).

Now suppose $H := \{R_\sigma | \sigma \in J\}$ is a flag of $\mathbf{\Gamma}(C)$ of type J, a proper subset of I. That means each R_σ can be assumed to be a residue of cotype σ, and any two members of H possess a non-empty intersection. It follows from (CRC1) that the intersection $S := \cap\{R_\sigma \in H\}$ is a residue of C of type $K := I - J$. Select any chamber $c \in S$, and type $k \in K$. Then the unique residue $R_k(c)$ of cotype k containing c is an object of type k in $\mathbf{\Gamma}(C)$ incident with every member of H. Thus the residue of H in this geometry contains an object of every type. Conversely, any object R of type k in the residue of H in $\mathbf{\Gamma}(C)$ has the form $R_k(c)$ for a chamber $c \in S \cap R$ since the latter intersection is, by (CRC1), a non-empty intersection of the family of corank one residues of $H \cup \{R\}$ which pairwise have a non-empty intersection. Since c lies in $R_k(c)$ as k ranges over K we see that any object $R = R_{k'}(c), c \in S$ of $\mathrm{Res}_{\mathbf{\Gamma}(C)}(H)$ lies in a flag-chamber $F_K(c) = \{R_k(c) | k \in K\}$ of this residue (of type K). Another application of (CRC1) shows that every flag chamber of $\mathrm{Res}_{\mathbf{\Gamma}(C)}(H)$ has the form $F_K(c)$ for some (not necessarily unique) chamber $c \in S$.

It is now easy to check that if c_1 and c_2 are ℓ-adjacent chambers of S ($\ell \in K$) then the corresponding flag-chambers $F_K(c_1)$ and $F_K(c_2)$ are ℓ-adjacent. It follows that the residue $\mathrm{Res}_{\mathbf{\Gamma}(C)}(H)$ is chamber connected, and so, by Exercise 2.5 of Chap. 2, p. 58, $\mathrm{Res}_{\mathbf{\Gamma}(C)}(H)$ is a non-empty connected geometry. Thus (RC2) holds for $\mathbf{\Gamma}(C)$, completing the proof of the first statement of the lemma.

Now suppose C is residually connected. For each type $k \in I$, and chamber c, we write $R_k(c)$ for the unique residue of cotype k containing c. Thus, for each chamber c we obtain a unique flag-chamber $F(c) := \{R_k(c) | k \in I\}$ of $\mathbf{\Gamma}(C)$ – that is, an element of $\mathbf{C}(\mathbf{\Gamma}(C))$. Note that if c and c' are distinct chambers, condition (CRC2) forces $F(c) \neq F(c')$. Conversely, if $H := \{R_k\}_{k \in I}$ is a pairwise non-trivially intersecting family of residues with R_k of cotype k, then by (CRC1) the intersection $\cap_{k \in I} R_k$ is non-empty and by (CRC2) consists of a single chamber c. Thus $H = F(c)$ so $F : C \to \mathbf{C}(\mathbf{\Gamma}(C))$ is surjective. Similarly, if c and c' are distinct chambers, they cannot both live in the same collection of corank one residues, so $F(c) \neq F(c')$. Thus F is a bijection.

Finally c and c' are i-adjacent if and only if $R_k(c) = R_k(c')$ for all $k \neq i$ and $R_i(c) \neq R_i(c')$ (for by (CRC2) the latter condition forces c and c' to belong to a common residue of type i). Thus F is an isomorphism of chamber systems. □

Lemma 9.2.7 *Let $\Gamma = (V, E; \tau)$ be a residually connected geometry over $I = \tau(V)$, where the rank $|I|$ is finite. Then $\mathbf{C}(\Gamma)$ is a residually connected chamber system. Moreover, there is a geometry isomorphism*

$$\Gamma \simeq \mathbf{\Gamma}(\mathbf{C}(\Gamma)).$$

Proof Assume $I = \{1, 2, \ldots, n\}$. By part 2 of Lemma 2.5.1, Chap. 2, p. 55, every flag of Γ lies in a chamber flag. If R is a residue of cotype k in $\mathbf{C}(\Gamma)$ then all flag-chambers of R contain a unique common object $f(R)$ of type k. More than that, since $|I|$ finite implies $\mathrm{Res}_\Gamma(f(R))$ is chamber-connected (Lemma 2.5.1, Chap. 2, once more), the flag chambers in R form the full set of flag-chambers of the form $f(R) + F$ where F ranges over all of the flag-chambers (of type $I - \{k\}$) of $\mathrm{Res}_\Gamma(f(R))$.

Suppose $\{R_k | k \in J\}$ is any family of residues of $\mathbf{C}(\Gamma)$ which pairwise possess a non-trivial intersection and with each R_k of cotype k, k ranging over J, a subset of I. Let X_k be the unique object of type k common to all flag-chambers of R_k. Then for any ℓ in J distinct from k, the existence of a flag-chamber in $R_k \cap R_\ell$ forces X_k to be incident with X_ℓ. Thus $F := \{X_k | k \in J\}$ is a flag of type J in Γ. There exists a flag chamber $\hat{F}$ containing F and clearly $\hat{F}$ is in the intersecion of all the R_k, $k \in J$ since any residue of cotype k containing F contains all flag chambers that contain F (by the argument at the end of the previous paragraph). Thus (CRC1) holds for $\mathbf{C}(\Gamma)$.

Now consider any flag chamber $c = (X_1, X_2, \ldots, X_n)$. Let $R_k(c)$ be the unique cotype k residue of $\mathbf{C}(\Gamma)$ containing c. As observed in the first paragraph above, $R_k(c)$ is the set of all flag chambers which contain X_k. Then the intersection $\cap_{k \in I} R_k$ is the flag chamber containing each X_k and there is only one – c itself. Thus (CRC2) holds for $\mathbf{C}(\Gamma)$.

It remains to exhibit the isomorphism. The function f given at the end of the first paragraph produces a mapping

$$\mathbf{\Gamma}(\mathbf{C}(\Gamma)) \to \Gamma,$$

taking each cotype k residue R of $\mathbf{C}(\Gamma)$ (itself a collection of flag-chambers agreeing in their type k object) to the unique object $f(R)$ of type k which they contain. Since $\mathrm{Res}_\Gamma(f(R))$ is chamber connected, any flag-chamber containing $f(R)$ belongs to R. Thus f is injective.

Also, for any object X_k of type k in Γ, and any flag chamber F' containing it, we have $X_k = f(R_k(F'))$. Thus f is surjective.

If R_k and R_ℓ contain a common flag chamber F', the objects $f(R_k)$ and $f(R_\ell)$ lie in F' and so are incident. Conversely, if X_k and X_ℓ are incident objects of Γ, then any flag-chamber F' containing F lies in both $R_k = f^{-1}(X_k)$ and $R_\ell = f^{-1}(X_\ell)$, the sets of all flag chambers containing X_k and X_ℓ, respectively. Thus f is an isomorphism of geometries. The proof is complete. □

Remark Now note that Theorem 9.2.2 is just a combined statement of Lemmas 9.2.6 and 9.2.7.

Corollary 9.2.8 (In the presence of residual connectedness, the functors $\mathbf{\Gamma}$ and $\mathbf{C}$ commute with the taking of residues.)

1. *Suppose* Γ *is a residually connected geometry of finite rank and let* $C = \mathbf{C}(\Gamma)$ *be its associated chamber system. Let* F *be a flag of cotype* $J \neq \emptyset$*. Let* $R(F)$ *be the set of all chamber flags of* Γ *which contain* F*. Then by residual connectedness and the finiteness of rank,* $R(F)$ *is a residue of* $\mathbf{C}(\Gamma)$ *of type* J*. Our assertion is that there is an isomorphism*

$$R(F) \simeq \mathbf{C}(Res_{\Gamma}(F))$$

   as chamber systems.
2. *Suppose* C *is a residually connected chamber system over* I *of arbitrary rank and let* $\Gamma = \mathbf{\Gamma}(C)$ *be its associated (residually connected) geometry. For every residue* R *of type* J*, there corresponds a flag* $F(R)$ *of cotype* J *of the associated geometry* $\mathbf{\Gamma}(C)$*, and there is a geometry isomophism*

$$Res_{\mathbf{\Gamma}(C)}(F(R)) \simeq \mathbf{\Gamma}(R).$$

Proof The isomorphisms in question are just the restrictions to residues of the two isomorphisms of Theorem 9.2.2. The beginning student is encouraged to make a formal proof of this corollary, noting the places at which the hypotheses of residual connectedness are used. □

Example 6 Let Γ be a geometry with exactly one object X_k of each type $k \in I = \{1, \ldots, n\}$. Then we see that Γ is indeed a residually connected geometry of finite rank n with exactly one flag chamber $c = (X_1, \ldots X_n)$. Then $\mathbf{C}(\Gamma)$ has one chamber and all residues of corank one coincide as sets of chambers. Nonetheless each such residue is attached to a unique type and so are distinct as residues. Thus $\mathbf{\Gamma}(\mathbf{C}(\Gamma))$ is indeed isomorphic to Γ. This example shows why one wants to take the rank of the chamber system as the cardinality of the relevent I rather than the set of types i for which i-adjacencies are actually exhibited. This way, the functor $\mathbf{\Gamma}$ preserves rank.

Example 7 Now suppose Γ is a geometry over $I = \{0, 1, \ldots\}$, the set of natural numbers with exactly two objects X_k and Y_k of each type k. We assume that any two objects of distinct types are incident. Then it is easy to see that Γ is connected, but is not chamber-connected since no gallery connects the flag chambers $X := \{X_k | k \in I\}$ and $Y = \{Y_k | k \in I\}$. (See Exercise 2.3 of Chap. 2, p. 58.) The arguments in the preceding Lemma 9.2.7 depended heavily on residues of flags being chamber connected. Thus the hypothesis that Γ has finite rank is crucial for this lemma.

9.3 Chamber Systems with Strongly Gated Residues

9.3.1 Introduction

In this section, C is a chamber system over I. It is not assumed that C is type M, or that it is even connected. (In Sect. 9.6 we shall resume the hypothesis that C is type M.) However, we do assume that C is a chamber system over I satisfying the condition (typ). We shall discuss three distinct hypotheses concerning a chamber system C:

(RG) Every residue is strongly gated in C.
(RG^1) Every residue of corank one is strongly gated.
(RG_2) (Scharlau's condition.) Every residue of C of rank at most 2 is strongly gated in C.

On the face of it, condition (RG) implies the other two. It is not difficult to show that (RG^1) implies (RG) (Corollary 9.3.6 below), and one would hope for an equally "metrical" proof that (RG_1) implies (RG). This is not quite as easy.[7] To keep the argument completely free of the assumption that C is type M, we have to introduce a new type of homotopy, called "Λ-homotopy," which tries to replace the notion of M-homotopy in arbitrary chamber systems which might not be of type M. The beauty of Λ-homotopy is that it always exists for any chamber system C satisfying (typ). It requires no assumptions whatsoever about rank two residues of C.

9.3.2 Basic Properties Concerning Strongly Gated Residues

To start with, the hypothesis that a subgraph $H = (H, E')$ of a graph $\Delta = (V, E)$ is *strongly gated* asserts that for every vertex $c \in V$ there exists a vertex $g = g(c)$ (to indicate that it depends on c) in H such that for *every* $x \in H$,

$$d_\Delta(c, x) = d_\Delta(c, g) + d_H(g, x). \tag{9.1}$$

When $d_\Delta(c, x)$ is finite, both terms on the right side are finite and $g(c)$ is in the same connected component as c and x. It is then easy to prove that the vertex $g(c)$ is uniquely determined by c.

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[7] The author's original proof in the unpublished Shult *Freiburg Notes* (1989) involved proving that the condition (RG_2) implies condition (P_c) which is an assertion about M-homotopy of galleries, a condition Tits proved was equivalent to the notion of "Building" for chamber systems of type M. The author's proof was hardly original. The arguments had already been outlined by Rudolf Scharlau except that "strongly gated" must replace "gated" in the arguments. But one sees that this proof invokes concepts completely dependent upon the hypothesis that the C is type M. One cannot even say that a gallery is of reduced type without this notion.

The reader might wonder what all this means when the graph Δ is not connected?[8] Well it makes perfect sense.

If two vertices x and y are in separate connected components of a graph G we say that they are "at infinite distance" and write $d_G(x, y) = \infty$.

If $d(c, x)$ is infinite in Eq. (9.1) then clearly at least one of the two terms on the right side of Eq. (9.1) is also infinite.

(1) If the first term is infinite then any vertex of H will serve as the gate and c lies in no common connected component with a single vertex of H.

(2) If the first term $d(c, g)$ is finite, then $d_H(g, x)$ is infinite. That means that x is not connected to g in the subgraph H. But it also means that x is not connected to c.

At first sight this does not seem very significant. But now suppose H is *strongly gated* – that is, it is strongly gated with respect to *any* vertex c. Then we see that two vertices of H are in the same connected component of Δ if and only if they are in the same connected component of H [Hint: Take c to be one of these vertices of H, deduce that it is its own gate $g(c)$, and apply the fundamental (9.1).]

Lemma 9.3.1 *Suppose $H = (H, E')$ is a strongly gated subgraph of a graph $\Delta = (V, E)$ and let $\Delta = \bigoplus_{\sigma \in \Sigma}(V_\sigma, E_\sigma)$ be a decomposition of Δ into its connected components. Then (as an induced subgraph of H) each non-empty intersection $V_\sigma \cap H$ is connected, and is strongly gated in the connected component $\Delta_\sigma := (V_\sigma, E_\sigma)$.*

Remark So, in a way, the discussion of strongly gated subgraphs of a graph reduces to the relationship of the connected components of both graphs.

Proof The proof is an easy exercise (Exercise 9.10). □

We now recall from Sect. 1.1.4 the following basic facts about subgraphs of a (possibly non-connected) graph.

Lemma 9.3.2 *1. Every strongly gated subgraph of a graph is a convex induced subgraph, and hence is an isometrically-embedded subgraph.*
2. Every convex induced subgraph of a connected graph is connected.
3. An arbitrary intersection of a family $\{R_\sigma | \sigma \in \mathcal{F}\}$ of connected convex induced subgraphs is either empty or is connected.

Proof Part 1 was detailed in Chap. 1. Part 2 is immediate from part 1, noting that the isometric embedding property forces finite distances. Part 3 follows from part 2 and the fact that the family of subgraphs which are convex and induced is closed under taking arbitrary intersections (Chap. 1). □

We wish to transfer these ideas to residues of a chamber system. For that purpose we need an analog of the concept of "induced subgraph" for residues. We say that a residue R of type $J \subseteq I$ in a chamber sytem C over I is an *induced residue of C* if

[8] The author even encountered one referee who insisted that it made no sense at all if Δ were not connected.

and only if, for any edge $e = (x, y)$ of C whose vertices x and y lie in R, we have $\lambda(e) \subseteq J$ – that is, the only adjacencies between vertices of R are the j-adjacencies for $j \in J$.

Note that if condition (*typ*) holds for C, any residue that is embedded as an induced subgraph C becomes an induced residue.

9.3.3 Intersections of Strongly Gated Residues

Corollary 9.3.3 *Let $\{R_\sigma|\sigma \in \mathcal{F}\}$ be a family of convex induced residues of a chamber system C. Let J_σ be the type of the residue R_σ. Then the gobal intersection $\cap\{R_\sigma|\sigma \in \mathcal{F}\}$ is either empty, or is an induced residue of type $\cap\{J_\sigma|\sigma \in \mathcal{F}\}$.*

Proof Set $T := \cap\{R_\sigma|\sigma \in \mathcal{F}\}$, and $J_T := \cap\{J_\sigma|\sigma \in \mathcal{F}\}$. If $T = \emptyset$ there is nothing to prove.

Suppose $e = (t, s)$ is an edge of C with $t \in T$. If $s \in C - T$, then $\lambda(e)$ cannot contain any label of J_T. On the other hand, if $s \in T$, then every type label of $\lambda(e)$ lies in J_σ since R_σ is an induced residue for all σ. Thus T is a union of induced residues of type J_T.

It remains only to show that T is connected. But since our hypothesis provides that each R_σ is connected, this property holds for T by part 3 of the preceding Lemma 9.3.2. □

Corollary 9.3.4 *Suppose $\{R_\sigma|\sigma \in \mathcal{F}\}$ is a family of strongly gated residues of a chamber system C satisfying condition (typ). Then the intersection $\cap\{R_\sigma|\sigma \in \mathcal{F}\}$ is either empty, or is a convex induced residue whose type is the intersection of all the types of the R_σ.*

Proof This is immediate from the preceding Corollary 9.3.3 and the observation that (typ) makes strongly gated residues induced. □

Theorem 9.3.5 (Shult [114].) *Suppose C is a connected chamber system with (typ) so each edge reflects just one type of adjacency. Suppose further that $\{R_\sigma|\sigma \in \mathcal{F}\}$ is a collection of strongly gated residues with a non-empty intersection R. Then R is strongly gated.*

Proof By Corollary 9.3.3, R is a convex induced residue of C of type $J_R := \cap\{J_\sigma|\sigma \in \mathcal{F}\}$ where J_σ is the type of the residue R_σ.

Suppose by way of contradiction that R is not strongly gated. Then there exists a vertex y such that R is not gated with respect to y. This failure cannot occur unless y is in the same connected component of the graph (C, E) as its connected subgraph R. So, among such y such that R is not gated with respect to R, we choose y so that the distance $d := d(y, R)$ – the length of a shortest geodesic connecting y to a vertex of R – is as small as possible. (Since y and R are in the same connected component, this distance d from y to a nearest vertex of R is finite.)

Let us first show that d is not zero – that is, y is not in R. Since R is a convex induced residue, its graph is isometrically embedded in the graph of C. Thus

$d_R(y, r) = d(y, r)$ where d_R is the internal metric of R, and so R is strongly gated with respect to y using y itself as the gate.

So we may assume that y is not in R and $d > 0$. Then there is at least one residue $M := R_\tau$ containing R but not containing y. Since M is strongly gated, there exists a vertex $g \in M$ such that for any vertex $r \in R$,

$$d(y, r) = d(y, g) + d_M(g, r). \tag{9.2}$$

It follows from this statement (universally quantified on r), and the fact that $d(y, g) > 0$, that

$$d(g, R) < d(y, R) = d.$$

By the minimal choice of d, R is strongly gated with respect to g. So, again, there is a gate $h \in R$ such that for any vertex s of R,

$$d(g, s) = d(g, h) + d_R(h, s), \text{ for all } s \in R. \tag{9.3}$$

But as a special case of (9.2):

$$d(y, h) = d(y, g) + d_M(g, h). \tag{9.4}$$

Now, as M is isometrically embedded, $d(g, s) = d_M(g, s)$, for all $s \in R$. By the substitution of bound variables we may replace s by r to get

$$\begin{aligned} d(y, r) &= d(y, g) + d_M(g, r) \\ &= d(y, g) + d(g, r) \\ &= d(y, g) + d(g, h) + d_R(h, r) \\ &= (d(y, g) + d_M(g, h)) + d_R(h, r) \\ &= d(y, h) + d_R(h, r) \end{aligned}$$

for all $r \in R$. Thus R is gated with respect to y after all. This contradiction to the choice of y completes the proof. □

Corollary 9.3.6 *In any chamber system satisfying condition (typ), all residues are strongly gated if and only if all residues of corank one are strongly gated.*

Proof Immediate from Theorem 9.3.5. □

Theorem 9.3.7 *Suppose C satisfies (typ). Suppose $\{R_\sigma | \sigma \in \Sigma\}$ is a finite family of strongly-gated residues of C. Suppose the R_σ pairwise intersect non-trivially – that is, for any $\sigma, \tau \in \Sigma$ we have $R_\sigma \cap R_\tau \neq \emptyset$.*

Then the global intersection is non-empty – that is

$$\cap\{R_\sigma | \sigma \in \Sigma\} \neq \emptyset.$$

Proof Since Σ is finite, we may represent the family as

$$\{R_1, R_2, \ldots, R_n\},$$

where each R_i is strongly gated in C and $R_i \cap R_j$ is non-empty for all $1 \leq i, j \leq n$. We may assume $n \geq 3$ and proceed by induction on n.

If $n \geq 4$ then by induction $R_3' := R_3 \cap \cdots \cap R_n$ is non-empty. By Corollary 9.3.4 R_3' is a strongly gated residue and by a second application of induction on n, $R_1 \cap R_3'$ and $R_2 \cap R_3'$ are non-empty. Then, by a third application of induction on n, we have

$$\emptyset \neq R_1 \cap R_2 \cap R_3' = R_1 \cap \cdots \cap R_n$$

and we are done. Thus we may assume $n = 3$ exactly.

By hypothesis, there is a chamber $y \in R_2 \cap R_3$. Also by hypothesis R_1 is strongly gated with respect to y. So there is a gate $g_1 \in R_1$ and by the convexity and isometric embedding of R_1 in C, for every $x \in R_1$ one can find a geodesic gallery from y to x that passes through g_1. Thus if we take $x \in R_1 \cap R_2$, there is a geodesic gallery from y to x passing through g_1. Then convexity of R_2 forces $g_1 \in R_2$. Similarly, if we take $x \in R_3 \cap R_1$, convexity of R_3 forces $g_1 \in R_3$. Now g_1 is a chamber in the global intersection $R_1 \cap R_2 \cap R_3$. The proof is complete. □

Corollary 9.3.8 *Suppose (RG) and (typ) hold. If C has finite rank, then C is a residually connected chamber system.*

Proof We must show three things. (1) The intersection over any family of pairwise non-trivially intersecting residues is non-empty. (2) Any non-empty intersection of residues is a residue – that is, it is connected. (3) The intersection of all residues containing a chamber c is the set $\{c\}$.

For (1) we can assume that the residues taking part in the intersection are pairwise distinct residues. Since the intersections are non-empty we also see that their types are pairwise distinct subsets of I. Since I is finite, there can only be a finite number of residues participating in the intersection. Then by (RG) and Theorem 9.3.7, the intersection of these residues is non-empty.

Assertion (2) follows from Corollary 9.3.4. Finally, conclusion (3) follows from (2) and the condition (typ) which says that for every pair of adjacent chambers there is a corank one residue containing one but not the other. □

9.3.4 2-Simply Connectedness is a Consequence of Strong Gatedness at Low Rank

In the next few sections, we shall be concerned with this property:

(RG$_2$) *Every residue of rank at most 2 is strongly gated.*

With a slight adjustment, this is essentially a notion introduced by R. Scharlau in [105].[9]

Theorem 9.3.9 *Suppose C is a chamber system satisfying (typ) and the hypothesis (RG_2) that all residues of rank at least 2 are strongly gated. Let $\mathcal{C}_2$ be the collection of all pointed circuits each of which is a circuit in some residue of rank two. Then every circuit of C is $\mathcal{C}_2$-contractible. Put another way, every connected component of C is 2-simply connected.*

Remark Of course (bowing to common usage) "2-simply connected" just means "$\mathcal{C}_2$-simply-connected" as defined in Chap. 1.

Proof Suppose, by way of contradiction, that G is a circular gallery which is not $\mathcal{C}_2$-contractible, chosen among such galleries to have minimal length $m = 2d$ or $2d+1$. Then $m > 4$, and any subsegment of G (that is, a subgallery of G) of length at most d is a geodesic path.

Select a chamber c_0 in G whose two edges in G bear labels i and j, and let R be the residue of type $\{i, j\}$ containing c_0. Let k and h be maximal so that the

$$(c_{-k}, c_{-k+1}, \ldots, c_0, c_1, \ldots, c_h)$$

is in a connected component of $R \cap G$ (actually a segment of G of type $\cdots ijijiji \cdots$ of maximal length containing c_0), and complete the notation so that

$$G = (c_0, \ldots, c_m = c_0)$$

with the convention that subscripts can be read modulo m. Then by the minimality of G, $d_C(c_0, c_d) = d$ and $d_C(c_{-1}, c_d) = d - 1$ or d according as m is even or odd.

If c_d were in the residue R, then the two geodesic paths

$$(c_0, \ldots, c_d) \text{ and } (c_{-1}, c_{-2}, \ldots, c_d)$$

would both connect vertices of R, and so would lie in R by convexity of the strongly gated R. That would force $G \subseteq R$ against G not being $\mathcal{C}_2$-contractible. Thus we may assume that c_d is not in R.

Now since R is strongly gated by hypothesis, there is a "gate" g in R so that

$$d(c_d, y) = d_C(c_d, g) + d_R(g, y) \tag{9.5}$$

for every chamber y of R.

Let A be a geodesic path from c_d to g and let H^+ and H^- be geodesic paths from g to c_h and c_{-k}, respectively. Also set

[9] In fact Scharlau's work was the starting point that inspired this entire section on strongly gated residues – most of it introduced in the Shult *Freiburg Notes* (1989) as early as 1988.

$$G^+ := (c_h, c_{h+1}, \ldots, c_d) \text{ (length } d-h\text{) and}$$
$$G^- := (c_{-k}, c_{-k-1}, \ldots, c_d) \text{ (length } m-d-k\text{)}.$$

We now have the arrangement of galleries given in Fig. 9.5.

Then by (9.5),

$$\ell(G^-) = m - d - k = \ell(A) + \ell(H^-)$$

and

$$\ell(G^+) = d - \ell = \ell(A) + \ell(H^+)$$

where, as usual, $\ell(X)$ denotes the length of the gallery X.

Then $A \circ H^+ \circ G^+$ is a circuit of length $2(d-h) = 2d - 2h$ and $A \circ H^- \circ G^-$ is a circuit of length $2(m-d-k)$.

If $m = 2d$, these lengths are $m - 2h$ and $m - 2k$, respectively. If $m = 2d + 1$, the first length is clearly less than m and the second length is $2(d + 1 - k)$. Since $\min(k, h) \geq 1$, in all cases the circuits have length less than m and so are $\mathcal{C}_2$-contractible by the minimality of G. It follows that G decomposes into three $\mathcal{C}_2$-contractible circuits (the two just mentioned and one circuit entirely in R), all visible in Fig. 9.5, so G is $\mathcal{C}_2$-contractible against the choice of G. The proof is complete. □

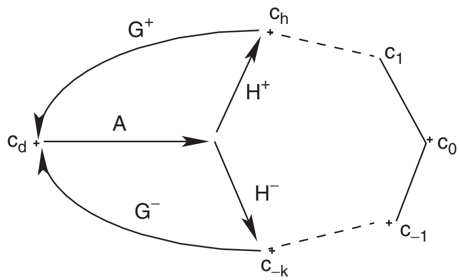

Fig. 9.5 The decomposition of a minimal circuit by condition (RG_2)

9.3.5 Λ-Homotopy

One might well imagine that there should be a corresponding "bottom up" theorem that should read something like this:

(Conjecture.) *Suppose C is a chamber system satisfying the condition (typ) as well as the condition.*

> *(RG_2) Every residue of rank at most two is strongly gated in C.*

Then every residue of C is strongly gated. Stated another way, in the presence of (typ), the condition (RG_2) implies (RG).

Unfortunately the author knows of no way to do this without invoking a more sensitive homotopy – one introduced by Tits in [139] in the context of chamber systems of type M – but one which works for arbitrary chamber systems satisfying the modest hypothesis (typ) (that distinct chambers are i-adjacent for at most one type-label i). We call it "Λ-homotopy," where the "Λ" stands for its key property: it is *length-preserving* on galleries.

Suppose R is a residue of rank two in a chamber system C with the property (typ). A $\Lambda(R)$-*homotopy* is a transformation of minimal galleries of R (viewed as a chamber system) $G \to H$ which connect the same two chambers of R. (Note that it is necessary to keep the residue "R" in view throughout this defintion. A minimal gallery is one of shortest length connecting two chambers of an assumed ambient chamber system. If we say that G is a *minimal gallery of a residue* R, we mean that all vertices and edges of G are vertices and edges of the residue R, and that one can discover no shorter gallery whose vertices and edges belong to the subgraph R connecting the initial and terminal vertices of G. But bear in mind that a minimal gallery of a residue R may not at all be a minimal gallery of C (what we have called a geodesic).

One should note the following simple consequences of the definition of a $\Lambda(R)$-homotopy.

Lemma 9.3.10 *We have:*

1. *Two $\Lambda(R)$-homotopic galleries of R have the same length. (Remember they are minimal galleries of R connecting the same two chambers.)*
2. *If R is type $\{i, j\}$, then two $\Lambda(R)$-homotopic galleries either:*
   (a) *they have length one – that is, $\ell(G) = \ell(H) = 1$, and $G = H$ is just recording an i-adjacency, or is recording a j-adjacency (condition (typ) prevents both from occurring), or*
   (b) *$\ell(G) = \ell(H) > 1$ and both i and j-adjacencies occur in each of the galleries.*

So much for R. Now let us say that a transformation $G \to H$ of two galleries of C form an *elementary* Λ-*homotopy* if and only if it has the form

$$A \circ U \circ B \to A \circ W \circ B,$$

where there exists a rank two residue R with respect to which the transformation $U \to W$ is a $\Lambda(R)$-homotopy.

Of course we then say that galleries G and H are Λ-*homotopic* if and only if H is obtained from G by a sequence of elementary Λ-homotopies.

Remark It is time to take note. Two galleries are $\Lambda(R)$-homotopic if and only if they are both minimal galleries in R connecting the same two chambers. But in the wider world of chamber system C, two Λ-homotopic galleries might not be geodesics of C.

Now it is only a matter of sorting through the definitions to deduce the following tautology.

Theorem 9.3.11 (Properties of Λ-homotopy.) *Suppose G and H are Λ-homotopic galleries. Then:*

1. *G and H begin and end at the same vertices.*
2. *G and H have exactly the same length.*
3. *The total collection of edge-labels exhibited in G are exactly the total collection of edge labels exhibited in H: we symbolically render this property as $Typ(G) = Typ(H)$.*

Corollary 9.3.12 *Suppose R is any sort of residue of a chamber system C satisfying (typ). Suppose G and H are two Λ-homotopic galleries connecting two chambers of R. If one of the galleries is in R so is the other.*

Proof If G is a gallery of R, then $Typ(G) = Typ(H)$ which implies the conclusion. □

Remark It is important to note that in defining Λ-homotopy we have not imposed any extra condition upon a chamber system C with (typ) – that is, we have not restricted the range of the definition in any way.

9.3.6 Further Consequences of the Hypothesis (RG_2)

We begin by proving that (RG_2) implies a property that would have been impossible to state, had we not had the diversion of the previous section.

Theorem 9.3.13 *Suppose C is a chamber system with the property (typ). Assume the following:*

> (RG_2) *All residues of C of rank at most two are strongly gated.*

Then C has this property:

> (Λ-min) *Any two minimal galleries of C which connect the same two chambers are Λ-homotopic.*

Proof Suppose by way of contradiction that C contains pairs (G, H), where G and H are two minimal galleries of C with the same initial and terminal vertices which are not Λ-homotopic. Then of course G and H have the same length, namely the distance in C from their common initial chamber to their common terminal chamber. We imagine the pair (G, H) chosen so that $d = \ell(G) = \ell(H)$ is minimal. We know that $d > 2$, otherwise G and H would belong to the same strongly-gated rank two residue and so would be Λ-homotopic by definition.

We may write $G = (s = g_0, g_1, \ldots, g_d = t)$ and $H = (s = h_0, \ldots, h_d = t)$. If $g_{d-1} = h_{d-1} = s'$, we would obtain factorizations

$$G = G_1 \circ (s', t) \text{ and } H_1 \circ (s', t).$$

Observing that G_1 and H_1 are minimal galleries of length $d-1$, one would conclude that G_1 and H_1 are Λ-homotopic, and then the definition would at once yield the Λ-homotopy of G and H, a contradiction.

Thus we may assume the "next-to-last" chambers of the galleries G and H are distinct. By condition (typ) we may let $i = \lambda(g_{d-1}, t)$ and $j = \lambda(h_{d-1}, t)$, the types attached to the last edge of each gallery.

Next we claim that $i \neq j$. If $i = j$ then g_{d-1}, h_{d-1}, and t would belong to a common panel P containing two distinct vertices at distance $d - 1$ from s and one at distance d. But that is impossible since (RG_2) forces panel P to be strongly gated with respect to s.

Let R be the residue of type $\{i, j\}$ containing g_{d-1}, h_{d-1}, and t, and let g be the gate of R with respect to the initial chamber s. As before we write

$$G = G_1 \circ (g_{d-1}, t) \text{ and } H_1 \circ (h_{d-1}, t).$$

Let G_2 and H_2 be minimal galleries in R from g to g_{d-1} and h_{d-1} respectively. Finally, fix a minimal gallery A of C from s to g. Then from the hypothesis of strong gatedness,

$$\begin{aligned} \ell(G_1) &= \ell(A) + \ell(G_2), \\ \ell(H_1) &= \ell(A) + \ell(H_2), \\ d &= \ell(A) + \ell(G_2 \circ (g_{d-1}, t) \\ &= \ell(A) + \ell(H_2 \circ (h_{d-1}, t). \end{aligned}$$

It follows that $A \circ G_2$, $A \circ H_1$, $G_2 \circ (g_{d-1}, t)$, and $H_2 \circ (h_{d-1}, t)$ are all minimal galleries of length less than d. We thus have a series of Λ-homotopies (indicated by "$\sim$"):

$$\begin{aligned} G &= G_1 \circ (g_{d-1}, t) \\ &\sim (A \circ G_2) \circ (g_{d-1}, t) \\ &\sim A \circ (H_2 \circ (h_{d-1}, t)) \\ &\sim H_1 \circ (h_{d-1}, t) \\ &= H. \end{aligned}$$

(We have freely applied the "associativity" of gallery concatenation without explicit intermediate equations.) The proof is complete at this point. □

Theorem 9.3.14 *Assume C is a chamber system with property (typ). If (RG_2) holds, then every residue of C is a convex induced residue. So it is isometrically embedded — that is, any geodesic in R is already a geodesic in C.*

Proof Suppose, among all residues R of C and all geodesics of these residues (their length is the distance between their extremities as measured by the internal metric of R) we chose a pair (R, G) so that G was not a geodesic of C and did this so G has minimal length d. Since (RG_2) makes all rank two residues isometrically embedded, we must have $d > 2$.

Let $G = (s = g_0, g_1, \ldots, g_{d-1}, g_d)$. By assumption

$$d_(s, g_i) = i, \text{ for } i \leq d - 1 \text{ and} \tag{9.6}$$

$$d(s, g_d) \leq d - 1. \tag{9.7}$$

Then the panel P (of type i, say) on the edge (g_{d-1}) contains a chamber t at distance $d - 2$ from the initial chamber s, and t may or may not be g_d. One thing is certain: (t, g_{d-1}) is labelled i. Let A be a geodesic of C from s to t. Then $A \circ (t, g_{d-1})$ and $(s = g_0, \ldots g_{d-1})$ are geodesics of C from s to $g_d - 1$. By Theorem 9.3.13, these two galleries are Λ-homotopic. By Corollary 9.3.12, A is a gallery in the residue R. Now if $t = g_d$, we have s and g_d connected by A; if $t \neq g_d$, then s and g_d are connected by $A \circ (t, g_d)$. In either case, s and g_d are connected by a gallery of the residue R having length at most $d - 1$. This contradicts our choice of G as a minimal gallery of R.

Thus no such pairs (R, G) exist, and the theorem is proved. □

Theorem 9.3.15 *Suppose C is a chamber system with property (typ). If (RG_2) holds, then every residue of C is strongly gated.*

Remark This theorem has been proved for chamber systems of type M using the fact that all residues of a Coxeter chamber system are strongly gated. Since we have not bothered to prove that property of a Coxeter chamber system and since we intend to prove it here without the type M hypothesis, the proof is essentially new. The induction used here is subtle and some care must be taken.

Proof Suppose the theorem is false. Then there exists a chamber system C satisfying (RG_2) and (typ), with a non-empty collection $\mathcal{Y}$ of triples (c, g, R) such that:

1. R is a residue and c is a chamber with $d(c, g) = d(c, R)$.
2. The subset

$$X(c, g, R) := \{x \in R | d(c, x) < d(c, g) + d_R(g, x)\}$$

is non-empty.

Suppose $(c, g, R) \in \mathcal{Y}$, so that the set $X(c, g, R)$ is non-empty. Clearly $g \notin X(c, g, R)$ by definition. We next consider geodesic paths from g to a member of $X(c, g, R)$ which encounters a member of $X(c, r, R)$ only at its terminus. (Clearly in our convex R, such a path is a path of R.) We then sort through such geodesic paths to find one such that the length of the path plus the distance of its terminus from c is as small as possible. Thus if

$$G = (g = g_0, g_1, \ldots, g_{d-1}, g_d = t)$$

is such a path, then for $i < d$, no vertex g_i lies in $X := X(c, g, R)$ and this is done so that $d + d(c, t)$ is as small as possible.

Then we have

$$d(c, g_i) = d(c, g) + i \text{ for } 0 \le i \le d - 1, \text{ and} \tag{9.8}$$

$$d(c, t) = d(c, g) + (d - 1) \text{ or } d - 2. \tag{9.9}$$

(Note that, by the preceding Theorem 9.3.14, every residue is isometrically embedded, so we can replace the $d_R(g, y)$s that would normally occur in the equations for the strongly-gated property, by $d(g, y)$s.) Let us write $d_1 := d(c, g)$, since this quantity makes a large number of appearances in this play.

Now suppose

$$d(c, t) = d(c, g_{d-1}) = d_1 + d - 1.$$

Then by (RG_2) the panel P on g_{d-1} and t contains an element t' with $d(c, t') = d_1 + d - 2$, and $d(g, t') \le d = d(g, t)$. Then as $t \ne t'$, the sum $d(c, t') + d(g(t')$ is smaller than the similar sum $d(c, t) + d(g, t)$ and $d(g, t) = d - 1$ shows that $(g = g_0, \ldots g_{d-2}, t')$ is a geodesic with all members except the last not in the set X. Thus by the minimality of the choice of the gallery G we get a contradiction.

Thus we must suppose that $t = t'$, so we have

$$d(c, t) = d_1 + d - 2 = d(c, g_{d-2}).$$

Moreover, $U := \{t, g_{d-1}, g_{d-2}\}$ cannot lie in a common panel (that would contradict $d(g, t) = d$), and so the edges (g_{d-1}, t) and (g_{d-1}, g_{d-2}) bear distinct labels i and j, respectively. Now let S be the unique residue of type $\{i, j\}$ containing the subgallery (g_{d-2}, g_{d-1}, t). By (RG_2) S is strongly gated with respect to c. Let p be its gate with respect to chamber c. Then (noting the isometric embeddings)

$$d(c, t) = d(c, p) + d(p, t) = d_1 + d - 2. \tag{9.10}$$

Now suppose p is not in X. Then

$$d(c, p) = d(c, g) + d(g, p). \tag{9.11}$$

Substitution of the right side of (9.11) for $d(c, p)$ in Eq. (9.10) yields

$$d_1 + d - 2 = d(c, t) = (d(c, g) + d(g, p)) + d(p, t) \tag{9.12}$$

$$= d_1 + d(g, p) + d(p, t). \tag{9.13}$$

So

$$d(g, p) + d(p, t) = d - 2. \tag{9.14}$$

But the latter is impossible since the "triangle inequality" forces

$$d = d(g, t) \leq d(g, p) + d(p, t). \tag{9.15}$$

So we must assume $p \in X$. Now from the gatedness of S, in addition to Eq. (9.10) we have

$$d(c, g_{d-2}) = d(c, p) + d(p, g_{d-2}) = d_1 + d - 2, \tag{9.16}$$

so

$$d(p, t) = d(p, g_{d-2}). \tag{9.17}$$

Now let

$$H := (d_{d-2} := t_0, t_1, \ldots, t_m := p)$$

be a geodesic in S. Since p is in X but g_{d-2} is not, there is a first index j such that $t_j \in X$. Thus for $0 \leq i < j$,

$$\begin{aligned} d(c, t_i) &= d(c, g) + d(g, t_i) && (9.18)\\ &= d(c, p) + d(p, t_i). && (9.19) \end{aligned}$$

So as i increases, by Eq. (9.19), $d(c, t_i)$ decreases, and so by (9.18) $d(g, t_i)$ also decreases. But as $t_j \in X$, we have

$$d(g, t_j) = d(g, t_{j-1}) + \epsilon \text{ where } \epsilon = 0 \text{ or } 1. \tag{9.20}$$

Thus

$$\begin{aligned} d(c, t_j) + d(g, t_j) &\leq d(c, t_{j-1}) + d(g, t_{j-1}) \text{ (by (9.20))} && (9.21)\\ &\leq d(c, g_{d-2}) + d(g, g_{d-2}) && (9.22)\\ &< d(c, t) + d(g, t). && (9.23) \end{aligned}$$

Now since t_{j-1} is not in X, one has

$$d(c, t_{j-1}) = d(c, g) + d(g, t_{j-1}), \tag{9.24}$$

so any geodesic from g to t_{j-1} concatenates with any geodesic from c to g to form a gallery which is a geodesic. This means that all chambers of a geodesic gallery E from g to t_{j-1} cannot belong to X. If $d(g, t_j) = d(c, t_{j-1}) + 1$ (so $\epsilon = 1$ in (9.20)), then $E \circ (t_{j-1}, t_j)$ is a minimal gallery from g to t_j with all chambers but its terminus not in X. On the other hand, if $d(g, t_j) = d(g, t_{j-1})$ (so $\epsilon = 0$), then the geodesic gallery E can be made to factor through a gate g_j of the unique panel

P_j on the edge (t_{j-1}, t_j) – that is $E = E_1 \circ (g_j, t_{j-1})$. But in that case, $E_1 \circ (g_j, t_j)$ is a geodesic from g to t_j with only its terminal chamber in X.

In short, $t_j \in X(c, g, R)$, like t, is approachable from g by a geodesic gallery all of whose other chambers are not in X, except that now (by (9.23)) the sum of the distances of t_j from c and g is now less than the sum of the distances of t from c and g. But t was supposed to be minimal in that respect. Thus the assumption $p \in X(g, c, R)$ also leads to a contradiction.

The proof is complete. □

9.3.7 Equivalence of Various Gatedness Conditions

The following theorem summarizes the results of this section.

Theorem 9.3.16 *We suppose C to be a chamber system over I with condition (typ).*

1. *The three strong-gatedness conditions (RG_2), (RG), and (RG^1) are equivalent to one another. They all imply the condition (Λ-min).*
2. *If C satisfies any one of these three conditions, it is 2-simply connected.*
3. *If C is firm and satisfies one of the equivalent gated conditions, then it is residually connected if and only if its rank $|I|$ is finite.*

Proof Clearly (RG) implies both (RG^1) and (RG_2) since the latter properties are particularizations of the former property. But (RG^1) implies (RG) by Corollary 9.3.6. Also, Theorem 9.3.15 asserts that (RG_2) implies (RG). So the three conditions are now equivalent. Finally, Theorem 9.3.13 shows that (RG_2) implies (Λ-min), so all parts of the first conclusion hold.

The second conclusion follows from the first upon noting that by Theorem 9.3.9, condition (RG_2) implies $\mathcal{C}_2$-simple-connectedness.

If (RG) holds, and C has finite rank, then C is residually connected. But by Theorem 9.2.5, if C has infinite rank and is firm, it cannot be residually connected.

9.4 Generalized Polygons

9.4.1 Panel Homotopy

Introduction

In Chap. 1, we saw that in any graph, a collection $\mathcal{C}$ of circuits defines a $\mathcal{C}$-homotopy. There are various homotopy theories that are useful for chamber systems, depending on the choice of $\mathcal{C}$. We have met one of these: $\mathcal{C}_2$-homotopy where $\mathcal{C}_2$ is the collection of all circuits of the chamber system each of which lies within some residue of rank two. In this section we shall meet another such theory: panel homotopy. The latter is very special, and concerns only the class $\mathcal{C}$ of circuits which are confined to panels (rank-one residues). We must introduce this notion in order to avoid some awkward terminology that has appeared in some of the literature. There one finds chamber systems that are generalized ∞-gons referred to as "trees." In fact they are not trees in the graph-theoretic sense if any of their panels are thick. The circuits that are important for generalized polygons are actually "panel-reduced" circuits. (One author calls them "proper circuits" without explanation.) The panel homotopy also has some relation to types of galleries that are useful in showing the equivalence of several definitions of "generalized polygon," so it is just something we have to go through. Actually, it is a very simple notion.

Basic Terms

Any *gallery* is a walk $G = (c_0, \ldots c_n)$ where (c_{i-1}, c_i) is an edge. Its *length*, the natural number n, is denoted $\ell(G)$. We usually keep track of the *type of the gallery*, the sequence

$$\lambda(G) := \{\lambda(c_i, c_{i+1}), i = 0, 1, \ldots n - 1\}.$$

A *segment of length k of gallery G* is just a subsequence $(c_i, c_{i+1} \ldots, c_{i+k})$, $k \leq n$, itself regarded as a gallery from c_i to c_{i+k} with an inherited type.

A *circuit gallery* (or just plain "circuit") in a chamber system $C = (C, E; \lambda)$ over I is just a circuit in the underlying graph (C, E). The *type of a pointed circuit* $(c_0, c_1, \ldots, c_n)$ is the circular sequence of types

$$\lambda(G) := (\lambda(c_0, c_1), \cdots, \lambda(c_{n-1}, c_n = c_0), \lambda(c_0, c_1)).$$

We say that a gallery $G = (c_0, c_1, \ldots, c_n)$ is *p-reduced* (short for "panel-reduced") if and only if $\lambda(c_{i-1}, c_i) \cap \lambda(c_i, c_{i+1}) = \emptyset$ for $i = 1, \ldots n - 1$, and we include the extra requirement that if $c_n = c_0$, so that G is a circuit gallery, then also $\lambda(c_{n-1}, c_n) \cap \lambda(c_0, c_1) = \emptyset$. Thus, in p-reduced galleries and circuits, no two consecutive edges can lie in a common panel.

Panel Homotopy and Reduced Paths

Consider now the collection $\mathcal{C}_1$ of all circuit galleries of a chamber system $C = (C, E; \lambda)$, each of which lies in some residue of rank one (depending on the particular circuit). We call the homotopy with respect to this system of circuits $\mathcal{C}_1$, *panel homotopy* since it is defined by the assertion that all panels are contractible. An example of a panel homotopy of a gallery $G = (x_0, x_1, \ldots, x_{n-1}, x_n)$ is the replacement of G by a gallery

$$G' = (x_0, x_1, \ldots, x_{i-1}, y_i, x_i, \ldots x_{n-1}, x_n),$$

where the two edges (x_{i-1}, y_i) and (y_i, x_i) share a common type. Such a transformation $G \to G'$ or its reverse $G' \to G$ is called an *elementary panel homotopy*.

Like any $\mathcal{C}$-homotopy, it does not change the initial and terminal vertices of the path or gallery.

In the case that C is a chamber system with a *single-valued* type function (that is, exactly one type is assigned to each edge) the discussion is simpler. Every gallery $G = (x_0, x_1, \ldots, x_{n-1}, x_n)$ now has a type consisting of a sequence of types $(\lambda(x_0, x_1), \ldots, \lambda(x_{n-1}, x_n))$ which we render as a word $\lambda(g) := t_1 \cdots t_n$ where $t_i = \lambda(x_{i-1}, x_i)$, with the understanding that $\lambda(G) = \emptyset$ if G has length zero. [We let $\ell(G) = \ell(\lambda(G))$ so that "ℓ" may record the length of a word, as well as a gallery.] Then the type of a concatenation of galleries is just the concatenation of the sequences of types. We now see that if the types of two consecutive edges in a gallery are the same, we can shorten the length-two segment they comprise to one of length one by an elementary panel homotopy. The two consecutive edge types are then coalesced to give the types of the result of the homotopy. This process of coalescing consecutive types in a sequence eventually leads to a shorter sequence in which consecutive edges share no common type. When this is done to the type $\lambda(G)$ of a gallery G, we call the unique result $\hat{\lambda}(G)$ the *p-reduced type* of the gallery G. Thus a gallery of type (11454465776) has unique p-reduced type (14546576). Clearly a gallery G is panel-homotopic to a gallery $\hat{g}$ of p-reduced type (that is, $\hat{g}$ is p-reduced in the basic terminology of the previous section).

When λ is not single-valued, the coalescing process which converts a type to a p-reduced type exists but the p-reduced types obtained may no longer be unique. The lemma below states what we know.

Lemma 9.4.1 *In general, the following hold:*

1. *Every gallery (circuit) is panel-homotopic to a p-reduced gallery (circuit) (as defined in the previous section).*
2. *In the case that the type function λ is single-valued, every gallery is panel homotopic to a gallery of reduced type. Moreover, if gallery G is panel homotopic to gallery H then $\hat{\lambda}(G) = \hat{\lambda}(H)$.*

Proof 1. We prove this part in the case of circuits, but the proof is virtually the same for galleries that are not circuits – just one less edge to consider.

Suppose $G = (x_0, x_1, \ldots, x_{n-1}, x_n = x_0)$ is a circuit with edges $e_1 = (x_0, x_1), \ldots, e_n = (x_{n-1}, x_n)$ of minimal length with respect to not being panel-homotopic to a p-reduced circuit. Then $n > 2$ and G cannot be panel-homotopic to a circuit of shorter length. Then there is a smallest index k such that $\lambda(e_1) \cap \lambda(e_k) = \emptyset$. Then $e_1' := (x_0, x_{k-1})$ is an edge bearing a non-empty set of labels and G is panel-homotopic to the circuit G' defined by the edges $(e_1', e_k, e_{k+1}, \ldots, e_n)$. If $k > 2$, then G' is shorter than G which is impossible. Thus $k = 2$, and so $\lambda(e_1) \cap \lambda(e_2)$ is empty. By a similar argument, $\lambda(e_i) \cap \lambda(e_{i+1}) = \emptyset$, and so G itself is a p-reduced circuit.

2. The statements concerning chamber systems whose type function is single-valued are immediate from the first part and the discussion in the paragraph preceding the statement of the lemma. $\square$

Corollary 9.4.2 *Suppose* $(C, E; \lambda)$ *is a chamber system satisfying these hypotheses:*

(typ) *The type function* λ *assumes only single values on edges.*
(Cir(n)) *Except for the circuits of length zero,* C *has no circuits of p-reduced type of positive length less than* $2n$.

Then every gallery of p-reduced type of length at most n is a geodesic.

Proof We first observe that any geodesic path or gallery must have its successive edges labeled by distinct types. Otherwise, it would be panel-homotopic to a shorter gallery connecting the same initial and terminal vertices, and so could not be a geodesic.

Suppose $G = (x_0, \ldots, x_d)$ is a gallery of p-reduced type. Without loss of generality, one may suppose its type to be $\lambda(g) = t_1 t_2 \cdots t_d$, a word of length $\ell(g) = d$ with letters $t_i \in I$ with $t_i \neq t_{i+1}, i = 0, \ldots, d-1$. Now if G is not a geodesic, there is a geodesic gallery $H = (x_0, y_1, \ldots, y_e = x_d)$ of length $\ell(H) = e$ strictly less than d. Then as $d \leq n$, the circuit $F := G \circ H^{-1}$ has length less than $2n$. By Lemma 9.4.1, this circuit is panel-homotopic to a p-reduced circuit $\hat{F}$ of length $\lambda(F) = \ell(\hat{\lambda}(G \circ H^{-1}))$, where

$$\ell(\hat{\lambda}(G \circ H^{-1})) \leq \ell(G \circ H^{-1}) = d + e < n.$$

(Note that the first term is the length of a word, while the second is the length of a gallery.) Then, by (Cir(n)), $\hat{F}$ is length zero, so $G \circ H^{-1}$ must be panel-contractible. It follows from Theorem 1.3.3, p. 1.3.3, Chap. 1, that G is panel-homotopic to H, whence by Lemma 9.4.1, part 2,

$$\hat{\lambda}(G) = \hat{\lambda}(H). \tag{9.25}$$

But the left and right sides are respectively $\lambda(G)$ and $\lambda(H)$ since G and H are both of reduced type. Yet these words have lengths d and $e < d$ respectively, a contradiction. Thus x_d has distance d from x_0 so G is a geodesic gallery. □

Example 8 Consider the following simple chamber system over the typeset $\{1, 2\}$. The graph is a four circuit (a, b, c, d, a) with a diagonal edge (a, c) adjoined. The edges are assigned type sets as follows; edges (a, b) and (b, c) are type $\{1\}$, edges (c, d) and (d, a) are assigned type set $\{2\}$, and edge (a, c) is assigned type set $\{1, 2\}$. Then the panels of type 1 are $\{a, b, c\}$ and $\{d\}$, those of type 2 are $\{a, c, d\}$ and $\{b\}$. Note that there are no non-trivial circuits of p-reduced type of length greater than zero whatsoever.

9.4.2 The Chamber System of a Generalized Polygon

The First Definition of Generalized n-gon

Let $C = (C, E; \lambda)$ be a chamber system over $\{1, 2\}$. Then C is a *generalized n-gon* (n is an integer greater than one or the symbol ∞) if and only if:

(typ) The type function λ assumes only single values on edges.
(Dia(n)) The graph (C, E) has diameter at least n and is firm if $n = \infty$.
(Cir(n)) Except for the circuits of length zero, C has no circuits of p-reduced type of length less than $2n$.
(Ch(n)) For any geodesic gallery G of type $(1, 2, 1, 2, \ldots)$ connecting two chambers x and y at distance n, there is a second geodesic gallery of type $(2, 1, 2, 1, \ldots)$ connecting x to y. A similar statement with the types '1' and '2' transposed is also assumed.

We extend this definition to the case that $n = \infty$ by viewing (1) axiom (Dia(∞)) to mean that C is connected, (2) axiom (Ch(∞)) to be vacuous, and (3) axiom (Cir(∞)) to assert that there are no p-reduced circuits of positive length whatsoever.

If we wish to assert that chamber system C is a generalized n-gon but do not wish to specify n, we say that C is a *generalized polygon*.

Lemma 9.4.3 *Let $C = (C, E; \lambda)$ be a generalized n-gon over type set $\{1, 2\}$. Then the following statements hold:*

1. *C is residually connected.*
2. *Any gallery of length $d \leq n$ is of reduced type if and only if it is a geodesic path of length at most n. When $n = \infty$, all galleries of panel-reduced type are geodesics.*
3. *If $n \neq \infty$, circuits of reduced type and length $2n$ exist. All such circuits are isometrically embedded in the underlying graph (C, E). Every vertex lies in such a circuit. The chamber system is firm.*
4. *Every geodesic path of length d less than n can be extended to one of length $d + 1$. As a consequence, every edge is in a circuit of reduced type and length $2n$.*
5. *Suppose n is finite. Then, for any two chambers c and d, there exists a p-reduced circuit of length $2n$ containing them.*
6. *The graph $C = (C, E)$ has diameter n.*
7. *Every panel is strongly gated in (C, E).*

Proof 1. A panel of type 1 cannot intersect a panel of type 2 in two distinct chambers x and y, for otherwise (x, y) would become an edge of type $\{1, 2\}$ against λ being single-valued. Thus every chamber is the intersection of the panels that contain it. Since (Dia(n)) implies C is connected (even when $n = \infty$), we see that C satisfies all the requirements of being residually connected.

2. As commented at the beginning of the proof of Corollary 9.4.2, every geodesic gallery has p-reduced type. Conversely, in view of (typ) and (Cir(n)), Corollary 9.4.2 also tells that any gallery of length at most n is a geodesic if it has p-reduced type. When $n = \infty$, (Cir(n)) holds for every positive integer n, so in that case, the same corollary implies that every path of p-reduced type is a geodesic.

3. Now by (Dia(n)) there exists somewhere a geodesic of length n, and so by (Cir(n)) there is a circuit G of length $2n$ and p-reduced type $1212\cdots = (12)^n$. Any two antipodal vertices of this circuit – say x_i and x_{i+n} – are connected by a gallery of length n (a segment of G) of p-reduced type. By part 2 just proved, this gallery is a geodesic. It follows that G is isometrically embedded in (C, E).

Let y be any further vertex. We claim that y is distance n from some vertex and so by (Cir(n)) lies in a circuit of p-reduced type $(12)^n$. Now the circuit G of the previous paragraph was itself such a circuit and if y were one of its vertices our claim would be fulfilled. So assume $H = (y = y_0, \ldots, y_d = x_0)$ is a gallery of shortest length connecting y to a vertex x_0 of the circuit G. Then $d \geq 1$ and H has p-reduced type. By transposing the name of the types, if necessary, we assume the edge $(y_{d-1}, y_d = x_0)$ has type "2." Then we name the vertices of G so that $G = (x_0, x_1, \ldots, x_{2n-1}, x_n = x_0)$ in the orientation that produces the p-reduced type $1212\cdots$. Then we may extend H by concatenting it with an initial segment of G to produce a gallery

$$H' = (y_0, y_1, \ldots, y_d = x_0, x_1, \ldots, x_{n-d})$$

of p-reduced type and length n. By part 2, this gallery is a geodesic, and our claim is proved.

Since each vertex lies in a circuit of type $(12)^n$, the panels which contain it have at least two chambers. Thus C is firm.

4. Suppose $G = (x_0, \ldots, x_d)$ were a geodesic path of length d, $0 < d < n$. Then G has p-reduced type. Let $i \in \{1, 2\}$ be the type of its last edge (x_{d-1}, x_d). Since C is firm, x_d lies on an edge $e = (x_d, x_{d+1})$ of type j where $\{i, j\} = \{1, 2\}$. The concatenation $G \circ e$ now has p-reduced type and length at most n and so is a geodesic extending G by part 2.

5. By the diameter assumption, the two chambers c and d are connected by a geodesic path of length $d \leq n$. By iterating part 4, if necessary, this geodesic can be extended so that it is an initial segment P_{cd} of a geodesic path P_{cf} of length n. This extended path P_{cf} has type $12121\cdots$ (n factors), and so by (Ch(n)) there is another path Q_{cf} of type $21212\cdots$ (n factors) connecting c to f, and clearly $P_{cf} \circ (Q_{cf})^{-1}$ is a circuit of panel-reduced type containing the two chambers c and d.

6. Suppose $G = (x_0, \ldots x_n, x_{n+1})$ were a geodesic path of length $n + 1$. Without loss of generality, we may suppose it has type $1212\cdots ij$ where $\{i, j\} = \{1, 2\}$. Now, by (Ch(n)), there is also a gallery $H = (x_0 = y_0, y_1, \ldots y_n = x_n)$ of type $2121\cdots ij$ and length n connecting x_0 and x_n. Since x_n is i-adjacent to both y_{n-1} and x_{n+1}, the latter two are either equal or i-adjacent. But either possibility is prohibited by the fact that x_0 and x_{n+1} are at distance $n + 1$. Thus no such geodesic of length $n + 1$ can exist. So the diameter is at most n which, with (Dia(n)), proves the assertion 6.

7. Suppose P is a panel, say of type 1. Let y be any chamber. We must show that P is strongly gated with repect to y. Since P is a clique it is already isometrically embedded so it is enough to show that P is gated with respect to y when y is not in P. Suppose x_1 and x_2 are two distinct chambers of P achieving the minimal distance d from y to any element of P. Then there is geodesic gallery $G = (y, y_1, \ldots, y_{d-1}, y_d = x_1)$ from y to x_1. Since y_{d-1} is not in P, the last edge (y_{d-1}, x_1) must be type 2. Then the gallery $G \circ (x_1, x_2)$ is of p-reduced type and length $d + 1$. If $d < n$, then by part 2, $G \circ (x_1, x_2)$ is a geodesic, against x_2 being distance d from y. Thus n is finite and by part 5, $d = n$ exactly. In this case (Cir(n))

shows that there is a second gallery $G' = (y, y_1', \ldots, y_{n-1}', y_n = x_1)$ of reduced type with the last edge (y_{n-1}', x_1) being type 1. But in that case, $y_{n-1}' \in P$, against the minimality of d.

Thus P contains a unique chamber nearest y, for any y, and we are done.

All parts of the lemma have been proved. □

When one encounters an omnibus lemma following a definition, one can usually extract other equivalent definitions from various subsets of the listed properties. This case is no exception.

Equivalent Definitions of Generalized Polygon

We first record the following observation.

Lemma 9.4.4 *Let $C = (V, E; \lambda)$ be any chamber system of rank $|I| = 2$. The following statements are equivalent:*

1. *C is residually connected.*
2. *C is connected and each edge of C bears a single label – that is, two adjacent vertices are j-adjacent for a unique $j \in I$.*
3. *C is connected and any two distinct panels intersect in at most one chamber.*

The proof is an easy exercise (part of Exercise 9.10).

In [105] R. Scharlau considers the following axioms for a chamber system C over the type set $I = \{i, j\}$:

(CS0) The chamber system C is residually connected.

(CS1) For any vertex $c \in V$ and $k \in I$, there exists a chamber c' which is k-adjacent to c.

(CS$_n$2) C contains no p-reduced circuit of positive length less than $2n$.

(CS$_n$3) If a gallery from c to d has type type $ijij\cdots$ (n-factors,), then there is also a gallery of type $jiji\cdots$ (n factors) from c to d.

For the purpose of distinguishing axiom systems, we shall refer to the four axioms above as the "Scharlau presentation." Let us compare these axioms with the axioms for a chamber system of the previous section:

- The first axiom (CS0) asserts two things: (1) that C is connected and (2) that any two panels meet in at most one chamber. As a result, an edge connecting two chambers cannot bear both labels i and j. Thus

$$(\text{CS0}) \Rightarrow (\text{typ}).$$

- Axiom (CS1) is just the assertion that C is firm – that is, all panels contain at least two chambers.
- (CS$_n$2) is exactly the assertion (Cir(n)).

- Clearly (CS_n3) implies ($\mathrm{Ch}(n)$). But the latter is conceivably weaker than the former – at least if it is possible to have a gallery of type $ijij\cdots$ of length n which is not a geodesic.

For any chamber system C define the *p-reduced girth* to be the smallest length of a p-reduced circuit of positive length. Then axiom (CS_n2) just says that C either has no p-reduced circuits, or has p-reduced girth at least $2n$.

Now consider the following condition:

(CS_n4) For any two chambers c and d, there exists a p-reduced circuit of length $2n$ containing c and d.

When n is infinite, we interpret axiom ($\mathrm{CS}_\infty4$) to assert (CS0) and (CS1) together.

Note that if two chambers x and y of a chamber system over $\{i, j\}$ formed an edge e with $\lambda(e) = \{i, j\}$ — that is, x and y are both i- and j-adjacent – then the edge e could never participate in a p-reduced gallery of length two or more. It follows that (CS_n4) implies (typ) as well as the fact that (C, E) is connected. Thus the following arises.

Lemma 9.4.5 *For a rank two chamber system C, condition (CS_n4) implies condition (CS0) – that C is residually connected and (CS1) that C is firm. (Of course, if n is finite, it also implies that $2n$ is an upper bound for the p-reduced girth.)*

We now have the following.

Lemma 9.4.6 *For a chamber system C of type $\{i, j\}$, the following conditions are equivalent:*

1. *The Scharlau presentation: (CS0), (CS1), (CS_n2) and (CS_n3).*
2. *(CS_n2) and (CS_n4).*
3. *If $n < \infty$, C has p-reduced girth $2n$, and (CS_n4) holds for all n.*

Proof The equivalences are trivial if $n = \infty$. In that case they all assert that C is residually connected and firm and that there are no p-reduced circuits of positive finite length.

So we assume $n < \infty$.

Note that (CS_n2) asserts that the p-reduced girth is at least $2n$, while (CS_n4) implies it is at most $2n$. Thus 2 and 3 are equivalent statements.

First we show that 1 implies 3. Since (CS_n2) implies the p-reduced girth is at least $2n$, and (CS_n4) implies that it is no more than $2n$, it suffices to prove (CS_n4). Now choose two chambers c and d (possibly the same chamber). Then as C is residually connected (condition (CS0)) there is a geodesic G from c to d which is of p-reduced type. Since C is firm it can be extended to a gallery $G \circ H$ of length n of p-reduced type – say $ijij\cdots$ (n factors), transposing i and j, if necessary. Then by (CS_n3) there is a gallery F from c to the terminus of $G \circ H$ of type $jiji\cdots$ (n factors) and $G \circ H \circ F^{-1}$ is the desired circuit of length $2n$ of p-reduced type on c and d. So (CS_n4) holds.

Last, we show that 3 implies 1. Assume 3. By Lemma 9.4.5, (CS_n4) implies (CS0) and (CS1) Also the statement on p-reduced circuits in 3 implies (CS_n2). It remains only to show (CS_n3). Suppose c and d are chambers joined by a gallery of type $ijij\cdots$ (to exactly n factors). Applying (CS_n4) to c and d yields the conclusion of (CS_n3). The proof is complete. □

Now we have the following.

Corollary 9.4.7 *Any chamber system over* $I = \{i, j\}$ *satisfying any of the three equivalent conditions of Lemma 9.4.6 is a generalized* n*-gon.*

Proof Assume the Scharlau axioms. If n is infinite, C is connected and firm and (typ) holds. Then C is a generalized polygon. So we assume n is an integer larger than 2.

First, (CS0) implies (typ) and the fact that C is connected. (CS_n2) is the same as (Cir(n)) and (CS_n3) implies (Ch(n)). It only remains to be shown Dia(n). Since C is connected, it suffices to show that there exists a geodesic of length n and that no geodesic has length $n + 1$.

From firmness, any chamber x is the initial term of a gallery G of length exactly n and p-reduced type $ijij\cdots$. Since (typ) and (Circ(n)) hold, G is a geodesic of length n by Corollary 9.4.2.

Now suppose $H = (x_0, x_1, \ldots, x_n, x_{n+1})$ were a geodesic gallery of length $n+1$. Then we can write $H = G \circ (x_n, x_{n+1})$, where G is a geodesic of length n and p-reduced type — say, $ijij\cdots$ (n factors) and ending with type $k = \lambda(x_{n-1}, x_n) \in \{i, j\}$. By ($\mathrm{CS}_n3$) x_0 is also connected to x_n by a gallery $F := (x_0, y_1, \ldots, y_n = x_n)$ of type $jiji\cdots$ (n factors) ending in k', where $\{k, k'\} = \{i, j\}$. Now $\lambda(x_n, x_{n+1})$ is either k or k'. Thus x_{n+1} is distance one from either x_{n-1} or y_{n-1}. In either case, it cannot be distance $n + 1$ from x_0, contradicting the fact that H was a geodesic of length $n + 1$. Thus no such H exists and (C, E) has diameter n exactly. The proof is complete. □

A Metrical Definition of Generalized *n*-gon

Here is a very important variation due to R. Scharlau.

Lemma 9.4.8 (R. Scharlau [105].) *Suppose* $C = (C, E; \lambda)$ *is a rank two chamber system over* $I = \{i, j\}$*. Then* C *is a generalized* n*-gon,* $n > 1$ *if and only if the following hold:*

(typ) *The type function is single-valued.*
(F) C *is firm.*
(PG) C *is connected and each panel is gated.*

Remark When $n = \infty$ we interpret Diam(n) to assert that C is connected and that arbitrarily large distances between points occur.

Proof If C is a generalized n-gon, then (typ) holds *a fortiori* and the remaining properties hold by Lemma 9.4.1.

Suppose C is a rank-two chamber system with all of the properties, (typ), (F), (PG). If there are no p-reduced circuits of positive length, C is a generalized ∞-gon.

So we may assume there exists a p-reduced circuit of minimal possible length — say $E = (c_0, c_1, \dots c_{n-1}, c'_n, c'_{n-1}, \dots, c'_1, c_0)$ (the notation is chosen so that c_i and c'_{n-i} occupy antipodal positions in the circuit, $i = 0, \dots, n-1$. (Note that in rank two, any p-reduced circuit must have even length; so $2n$ is the p-reduced girth.) By the minimality of n, we have the condition Cir(n) = CS$_n$2. Since (typ) is assumed, Lemma 9.4.1 implies

Step 1. *Every p-reduced gallery of length at most n is a geodesic gallery.*

Next we shall show the following.

Step 2. *For any chamber d at distance at most n from c_0, the two chambers c_0 and d lie together in a circuit of p-reduced type and length $2n$.*

This result is trivial if d is on the circuit E. Suppose $G = (c_0, d_1, d_2, \dots, d_s)$ is a geodesic gallery from c_0 to d where we assume that $s = d_C(c_0, d) \leq n$. Now the type of (c_0, d_1) must agree either with that of (c_0, c_1) (say, type i), or with that of (c_0, c'_1) (say, type j). Without loss of generality we assume d_1 is i-adjacent to both c_0 and c_1, while (c'_1, c_0) and (c_1, c_2) are j-adjacencies. In particular, $d_1 \neq c'_1$. In Fig. 9.6 we have tried to represent the "general position" of this configuration. Note, however, that it is possible that $d_1 = c_2$ and after that $d_2 = c_3$, — and that in fact *the geodesic G from c_0 to d_s may wrap nearly around the bottom half of the circuit E in Fig. 9.6.*

Now, for each t with $1 \leq t \leq s$, the gallery

$$G_t := (d_t, d_{t-1}, \cdots, d_1, c_0, c'_1, c'_2, \cdots c'_{n-t})$$

is p-reduced of length n and hence by Step 1 is a geodesic. We make the following statement:

(Claim) *Given G_t, $1 \leq t \leq s$, there exists a gallery G^*_t from d_t to c'_{n-t} whose type is "opposite" G_t (that is i and j have been transposed).*

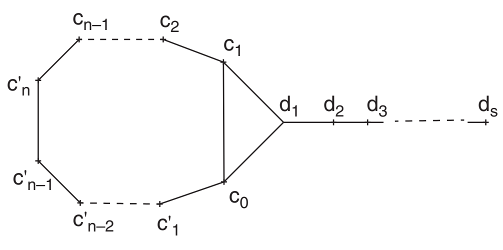

Fig. 9.6 The configuration for Step 2. Here the minimally chosen circuit for this step is $E = (c_0, c_1, \dots c_{n-1}, c'_n, c'_{n-1}, \dots, c'_1, c_0)$ (the notation is chosen so that c_i and c'_{n-i} occupy antipodal positions in the circuit, $i = 0, \dots, n-1$)

We prove the claim by induction on t. To prove it at the induction step t to $t+1$, we replace E by the circuit $E' := G_t \circ (G_t^*)^{-1}$, and apply induction. So it suffices to prove this assertion when $t = 1$.

Now if d_1 is in e there is nothing to prove. So we see that d_1 is distance one from both c_0 and c_1 by the same i-panel. Thus by Step 1, d_1 is distance n from both c'_n and c'_{n-1}, which are distinct elements of some k-panel P. Thus by (GP) there is a chamber p in P at distance $n-1$ from d_1. Let H be a geodesic of this length from p to d_1. Then $G_1 := (d_1, c_0, c'_1, \ldots c'_{n-1})$ and $H^{-1} \circ (p, c_{n-1})$ is the desired G_1^*.

Thus the claim is proved. But now the claim implies the assertion of Step 2.

Step 3. *Any two chambers at distance at most n from one another lie in a common p-reduced circuit of length* $2n$.

Suppose x and y are chambers at distance $s \leq n$. Since C is connected and $n \geq 1$, there is a finite sequence of chambers $(c_0 = x_0, x_1, \ldots, x_{m-1} = x, x_m = y)$ such that x_i has distance at most n from x_{i+1}. Now by Step 1 there is a p-reduced circuit of length $2n$ on both c_0 and x_1, and by iterating Step 1 with x_i in the role of c_0, there is such a circuit on $x_{m-1} = x$ and $x_m = y$.

Step 4. *Axiom (CS$_n$4) holds. That is, any two chambers lie in a p-reduced circuit of length* $2n$.

In view of Step 3, it suffices to show that there are no pairs of chambers at distance greater than n – equivalently (since C is connected), that no geodesic paths of length $n+1$ exist.

Suppose by way of contradiction that there are two chambers x and y at distance $d_C(x, y) = n + 1$. By interposing the penultimate chamber z of a geodesic from x to y, we have $d_C(x, z) = n$ and z is i-adjacent (say) to y. Now by Step 3, there is a p-reduced circuit on x and z, say

$$E := (x = x_0, x_1, \ldots, x_n = z, x_{n+1}, \ldots, x_{2n-1}, x_{2n} = x).$$

Now of the two edges of the circuit on z, one is labelled i and carries a chamber $z' = x_{n-1}$ or x_{n+1} at distance $n-1$ from x. But z' is i-adjacent to y by the fundamental property of a chamber system. That conclusion contradicts $d_C(x, y) = n + 1$. Thus no geodesics of that length exist, and the proof of Step 4 is complete.

We conclude at this stage that property 2 of Lemma 9.4.6 holds. It follows that C is a generalized polygon by Corollary 9.4.7. □

Remark The minimal p-reduced circuits of length $2n$ which appear throughout the proof of Scharlau's metrical characterization theorem are called *apartments* of the generalized polygon. Basically, that proof establishes that any two chambers lie in a common apartment. Later we shall meet this notion of "apartment" in the general context of buildings. Again we will be interested in the property that any two chambers lie in an apartment.

9.4.3 Generalized n-Gons as Geometries

Having noted that a generalized n-gon is a residually connected chamber system C, one may ask what special properties characterize the geometry $\mathbf{\Gamma}(C)$. Here the definition is especially simple.

In an arbitrary graph, the girth is the minimal length of a circular walk of positive length having no repeated edges.[10]

A geometry over $\{1, 2\}$ is called *a generalized n-gon geometry* if and only if:

(GP1) It is a bipartite graph of girth $2n$.
(GP2) It has diameter n.
(GP3) If n is infinite, then every vertex is on at least two edges.

When n is infinite we interpret the first two axioms in the following way: (1) the girth assumption (GP1) becomes the assertion that there are no circuits (circular tours) of positive length, and (2) the diameter assumption (GP2) is read as the assertion that the graph is connected. Thus if $n = \infty$, Γ is simply a tree with no degree-one vertices ("leaves").

Note that if $\Gamma = (\mathcal{P}, \mathcal{L})$ is a generalized n-gon geometry, then Γ is connected because it has diameter n (as understood allowing n to be infinite). The second observation is that the properties listed make no reference to any particular component of the bipartite graph. That is, if we choose to view this bipartite graph as a rank two incidence geometry of points and lines, then any consequence of the axioms stated in this language would also be true if lines and points were transposed, while preserving incidence. In a few words: *if* $(\mathcal{P}, \mathcal{L})$ *is a generalized polygon, then so is its dual geometry,* $(\mathcal{L}, \mathcal{P})$.

From now on, we view this graph as the incidence graph of a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$. If we wish to say that Γ is a generalized n-gon for some $n \in \mathbf{N} \cup \{\infty\}$, but otherwise do not wish to specify n, we say that Γ is a *generalized polygon geometry*.

We begin with an observation. Suppose Γ is any (simple) bipartite graph viewed as a rank two geometry. Then the chamber system $\mathbf{C}(\Gamma)$ is simply the edge graph of Γ with the edge-labelling recording in which component a vertex of two intersecting edges lies. Then, of course, axiom (typ) holds for $C(\Gamma)$.

Conversely, suppose C is a chamber system of rank two. Then, of course, $\mathbf{\Gamma}(C)$ is the bipartite graph recording when panels of distinct type may intersect non-trivially. We now notice the following very elementary facts.

Lemma 9.4.9 *Suppose Γ and C are respectively a rank two geometry and a rank two chamber system over the same index set $I = \{i, j\}$. Assume either (i) $C = \mathbf{C}(\Gamma)$, or (ii) $\Gamma = \mathbf{\Gamma}(C)$. Then the following statements hold:*

---

[10] Some books call such walks a circular tour. Note that the "backtracks" of Chap. 1 are excluded by this requirement. Otherwise all graphs would seem to have girth two!

1. *Γ is connected if and only if C is.*
2. *Assume that C has (typ). Then Γ has no vertices of degree one if and only if C is firm.*
3. *Assume C is residually connected and firm. Then there is a bijection between the p-reduced circuits of C and the circular walks of Γ whose succesive edges are distinct.*

Proof The first statement is trivial.

Consider the second statement. Suppose $C = \mathbf{C}(\Gamma)$. Then each panel of C is the collection of all edges on a vertex of Γ so the stated equivalence follows. Now suppose $\Gamma = \mathbf{\Gamma}(C)$. If P were a vertex of Γ on only one edge then it would be a panel of type i, say, intersecting only one panel R of type j. If C were firm each of its chambers would be j-adjacent to another distinct chamber forcing $P \subseteq R$. Since $|P| > 1$, the assumption (typ) (here artificially imposed) must fail. Thus (typ) and firmness of C imply that each vertex of Γ is on at least two edges. Conversely, suppose each vertex of Γ has degree at least two. Then each panel of one type meets at least two other panels of the other type, and so contains at least two chambers (even without assuming (typ)). Thus C is firm.

For the third statement, Γ and C are both derived from each other by the functors (see Lemmas 9.2.6 and 9.2.7). Now any circular walk with successive edges distinct becomes a sequence of edges with any two successive members of the sequence – but never three successive members – sharing a common vertex. This becomes a circuit in the edge graph which is a p-reduced circular gallery. Conversely, any circular p-reduced gallery of C has its edges corresponding to panels of types i and j successively alternating. These panels are thus vertices in Γ forming a circular walk with successive edges distinct. □

Theorem 9.4.10 *The following statements hold:*

1. *A generalized polygon geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is always residually connected, and its associated chamber system $\mathbf{C}(\Gamma)$ is a generalized polygon.*
2. *If the chamber system $C = (C, E; \lambda)$ is a generalized polygon, then its associated geometry $\mathbf{\Gamma}(C)$ is a generalized polygon geometry.*

Proof First we prove the result when the parameter n in each definition is infinite. In this case, a generalized polygon geometry Γ is a tree with no vertices of degree one. So it is connected and each vertex is in a flag-chamber (that is, an edge). Thus it is residually connected. Then by Lemma 9.4.9, $C = \mathbf{C}(\Gamma)$ is residually connected, firm, and has no p-reduced circuits of finite positive length.

Similarly, when $n = \infty$, a generalized polygon $C = (C, E; \lambda)$ is residually connected, firm, and possesses no p-reduced circuits of finite positive length. Then $\Gamma = \mathbf{\Gamma}(C)$ is residually connected and by Lemma 9.4.9 has no circular walks with successive edges distinct except for the trivial walk (and so is a tree) and has no vertices of degree one.

From now on we assume the parameter n is finite in both statements.

The proof of the first statement proceeds through a series of five steps.

Assume Γ is a generalized polygon geometry with parts $\mathcal{P}$ and $\mathcal{L}$. Let $e = (a_1, a_2)$ and $f = (b_1, b_2)$ be two distinct edges of the bipartite graph Γ. Of the four possible distances $d_\Gamma(a_i, b_j)$, choose notation for these vertices so that $d_\Gamma(a_1, b_1)$ is the smallest distance d. Since Γ is bipartite, $d_\Gamma(a_1, b_2) = d_\Gamma(a_2, b_1) = d + 1$. Then there are two cases: (1) $d_\Gamma(a_2, b_2) = d + 2$ and (2) $d_\Gamma(a_2, b_2) = d$. In case (1), for any geodesic path G from a_1 to b_1, the path $(a_2, a_1) \circ G \circ (b_1, b_2)$ is a geodesic path of length $d + 2$, and so $d \leq n - 2$. In case (2), there are geodesic paths G and H of length d connecting a_1 to b_1 and a_2 to b_2, respectively and $W = (a_2, a_1) \circ G \circ (b_1, b_2) \circ H^{-1}$ is a circular walk with at least two distinct edges. It follows that some subset of the edges forms a walk with consecutive edges distinct, which necessarily has length at least $2n$ by the girth assumption. Thus the length of W, which is $2d+2$, is bounded below by $2n$. On the other hand $d \leq n-1$, since the graph diameter n is at least $d_\Gamma(a_1, b_2)$. Thus $d = n - 1$ exactly. Then the girth assumption reveals that the circular walk W is itself a minimal circuit of length $2n$. Thus we conclude the following.

Step 1. *Given any two distinct edges e_1 and e_2 of a generalized polygon geometry Γ of diameter $n < \infty$, either (1) they are the first and last edges of a geodesic path, or (2) they are antipodal edges of a minimal circuit of length $2n$.*

Next, observe the following.

Step 2. *In a generalized n-gon geometry every vertex is on at least two edges.*

We had assumed this when n is infinite, but must now prove it in the finite diameter case. Because we then have finite girth, the collection T of vertices on at least two edges is non-empty. Suppose by way of contradiction that x is a vertex on a unique edge $e = (x, y)$ (there is at least one edge on x since Γ is connected and $n \geq 1$). Choose $t \in T$ so that the distance $d = d_\Gamma(x, t)$ is maximal. Now t is on two distinct edges, $e_i := (t, s_i)$, $i = 1, 2$, and each s_i is at distance $d - 1$ from x and distance $d - 2$ from y from the minimality of d and the uniqueness of y. That means there is a circular walk of length $2(d - 2) + 2 = 2d - 2$ incorporating the distinct edges e_i each exactly once and so there is some circular walk with consecutive edges distinct of length smaller than this, but length at least four. The girth assumption forces $2d - 2 \geq 2n$ so $d > n$, contrary to the assumption on diameter.

Step 3. *In a generalized n-gon geometry any geodesic path of length $d < n$ can be extended to one of length $d + 1$.*

This is an easy step. Consider a geodesic path G from u to v of length $d < n$. By Step 2, there are at least two edges on v. If G cannot be extended, v is adjacent to two vertices, each of distance $d - 1$ from u. Their geodesics to u, together with the path of length 2 from one to another through v, can be assembled into a circular walk of length $2(d - 1) + 2 = 2d$ having two edges used just once. Thus the girth assumption yields $2d \geq 2n$, against $d < n$.

Now let us return to Step 1. Suppose e_1 and e_2 are two edges of Γ which are the first and last edges of a geodesic path G. If this path G has length less than d, then by Step 3, it can be extended to a geodesic path $P := G \circ H$ of length n. Otherwise it already has length n and we write $G = P$. In either case P is a geodesic path with terminal vertex t and initial vertex s incident with edge e_1. By Step 2, t is on an edge $f = (t, r)$ not in path P, and it is easy to see that edges e_1 and f are in Case (2) of Step 1. This means there is a minimal circuit of length $2n$ incorporating the path P. We have just proved the following.

Step 4. *If Γ is a generalized n-gon geometry with n finite, then the rank two chamber system $\mathbf{C}(\Gamma)$ satisfies axiom (CS$_n$4) – any two chambers are on a p-reduced circuit of length $2n$.*

We can now conclude with the following.

Step 5. *If Γ is a generalized n-gon geometry with n finite, then the rank two chamber system $\mathbf{C}(\Gamma)$ is a generalized n-gon.*

Since Γ has girth $2n$ the bijection in Lemma 9.4.9, part 3, shows that a minimal p-reduced circuit of $\mathbf{C}(\Gamma)$ has length $2n$. Thus $\mathbf{C}(\Gamma)$ now satisfies (CS$_n$2) and (CS$_n$4). By Lemma 9.4.6, $\mathbf{C}(\Gamma)$ is a generalized n-gon.

The second statement of the theorem is easier to prove. Assume the chamber system C over $I = \{1, 2\}$ is a generalized n-gon, for finite n. Then C is residually connected and the geometry $\Gamma := \mathbf{\Gamma}(C)$ is also residually connected. Since C contains panel-reduced girth $2n$, the bipartite graph Γ also has girth $2n$ by Lemma 9.4.6. It follows that Γ has diameter at least n, and it remains to be shown that it is no larger. By way of contradiction, suppose $g = (x_0, x_1, \dots, x_n, x_{n+1})$ were a geodesic path of length $n + 1$ in the graph Γ. Letting e_i denote the edge (x_i, x_{i+1}), we obtain a gallery $G = (e_0, e_1, \dots, e_n)$ of length n and alternating type – say $1212 \cdots i$. Then by (CS$_n$3) there is a second gallery $G' := (e_0, f_1, f_2, \dots, f_{n-1}, e_n)$ of type $2121 \cdots j$ where $\{i, j\} = \{1, 2\}$. Now the f_i are edges of Γ and, because e_0 and f_1 are 2-adjacent, f_1 contains point x_0 rather than x_1. Similarly, the edge f_{n-1} cannot contain x_n, and so must contain x_{n+1}. Thus putting $f_1 = (x_0, y_1)$, $f_i = (y_{i-1}, y_i)$, $i = 2, \dots n - 2$ and $f_{n-1} := (y_{n-2}, x_{n+1})$, we have a walk $(x_0, y_1, \dots y_{n-2}, x_{n+1})$ of length $n - 1$, a contradiction to the assumption that x_0 and x_{n+1} were at distance $n + 1$. Thus Γ has diameter exactly n and Γ is a generalized n-gon geometry.

The proof is complete. $\square$

Generalized Polygons as Point-Line Geometries

As previously noted, a generalized polygon geometry can be regarded as the bipartite incidence graph Γ of a geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ of points and lines. The use of the symbol Γ in both cases is not an abuse of notation, for the expressions attached to the symbol in both cases really represent the same thing. The difference in the two views is just a matter of metaphysical constructions. Normally, we think of G

as a bipartite graph. But if there are no repeated lines, we can also view it as a set of points, together with a distinguished collection of subsets of the point set, namely the lines. This change in viewpoint introduces a change in the way we describe things. For example, one now has notions such as partial linear space, subspace, and more importantly, the metrical features of a new graph – the point-collinearity graph $\Delta = (\mathcal{P}, \sim)$. In this section we wish to reconsider generalized n-gon geometries for $n > 2$ from the point-line point of view.

If the parameter n is infinite, a generalized n-gon geometry Γ is a tree with no leaves. In that case each point is on at least two lines, and each line has at least two points, but otherwise these local incidence numbers may vary wildly from point to point and from line to line.

If $n = 2$, the graph Γ is a complete bipartite graph, so every point of Γ is incident with every line and vice versa. Then we say that Γ is a *generalized digon*.

If $n = 3$, any two distinct lines of Γ are vertices of Γ at distance 2 – the same is true of two points of Γ. Thus any two lines are incident with at least one common point and any two points are incident with at least one line. But since there are no 4-circuits in Γ (that is, no circular walks of length 4 which are not backtracks) the adjective phrases "at least one" of the previous sentence can be replaced by "a unique." Thus Γ is a partial linear space in which any two distinct points are on a unique line (i.e., it is a linear space) and any two distinct lines are together incident with a unique point. By now you will recognize that Γ is a "generalized projective plane" as defined in Chap. 5.

Similarly, if $n = 4$, Γ is a generalized quadrangle as defined in Chap. 7.

For general finite n, a point p and a line L are vertices of the bipartite graph Γ at odd distance from one another. If n is even this distance $d_\Gamma(p, L)$ is less than n so there is a unique geodesic path of Γ connecting them. Thus we have the following.

Theorem 9.4.11 *If n is an even number $2k$, and $\Gamma = (\mathcal{P} \cup \mathcal{L}, *)$ is a generalized polygon geometry, then, as a point-line geometry, $\Gamma = (\mathcal{P}, \mathcal{L})$ has these properties:*

(GPL0) Every point is on at least two lines and each line has at least two points (the latter is required of any point-line geometry).
(GPL1) The point-collinearity graph $\Delta = (\mathcal{P}, \sim)$ of Γ has diameter k.
(GPL2) If p is a point, and L is a line, then there is a unique point q of L nearest p in the metric of Δ and the geodesic path in Δ from p to q is unique.

These properties should remind the reader of the form of the axioms for a (nondegenerate) generalized quadrangle. Similarly we also have the following.

Theorem 9.4.12 *Any point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ satisfying the axioms (GPL0), (GPL1), and (GPL2) is a generalized $2n$-gon geometry.*

The proofs of both of these theorems are left as exercises.

A Characterization of Generalized Polygons

P. Abramenko and H. Van Maldegham [1] proved the following theorem characterizing generalized polygons among the classes of rank-two geometries.

Theorem 9.4.13 *Let* $\Gamma = (\mathcal{P} \cup \mathcal{L}, *)$ *be (the incidence graph of) a connected firm rank two geometry of finite diameter* n, $n \geq 3$. *Suppose:*

(AV) *For each pair of vertices* x *and* y *at distance* $n-1$ *in* Γ, *there is a unique vertex adjacent to* y *which is at distance* $n-2$ *from* x.

Then Γ *is a generalized* n*-gon.*

Corollary 9.4.14 *Suppose a point-line geometry* $(\mathcal{P}, \mathcal{L})$ *and its dual* $(\mathcal{L}, \mathcal{P})$ *are both (connected) near* n*-gons. Then both are in fact generalized* n*-gons.*

Remarks (a) At first sight, the condition (AV) seems to do little to prevent the bipartite graph Γ from possessing circuits of length $2d$ smaller than $2n$. But, as we shall see, it is actually a powerful hypothesis.

(b) The proof in the Abramenko-Van Maldeghem paper [1] involves 1-twinning[11] (a special "opposite" relationship among flags) in the "flag graph for Γ" – that is, the chamber system graph $\mathbf{C}(\Gamma)$ with the edge-labels ignored. In order to avoid the introduction of concepts which would be used in this book in only one instance, we venture to present below an alternative proof of the Abramenko–Van Maldeghem theorem, using only the incidence graph $\Gamma = (\mathcal{P} \cup \mathcal{L}, *)$.

Proof of Theorem 9.4.13 Throughout, $\Gamma = (\mathcal{P} \cup \mathcal{L}, *)$ is a connected bipartite graph (with parts $\mathcal{P}$ and $\mathcal{L}$) of diameter n, satisfying the axiom (AV), and having each vertex on at least two edges (firmness). The term "geodesic path" will refer to a path whose length is the graph-theoretic distance between its initial and terminal vertices.

In order to indicate exactly how the axiom (AV) is being used in each instance, for a pair of vertices x and y with $d_\Gamma(x, y) = n - 1$, we consider the following assertion:

$[AV](x, y)$: *There is exactly one vertex at distance* $n-2$ *from* x *that is adjacent to* y.

The proof proceeds by a series of steps.

Step 1. *Suppose* $p = (x_0, x_1, \ldots, x_n)$ *is a geodesic path of length* n. *Suppose* $y_1 \in x_0{}^{\perp} - \{x_1\}$. *Then* $d(y_1, x_{n-1}) = n$.

[11] The important notion of 1-twinning, due to Mühlherr [90], is studied in a wider context in the beautiful paper of Abremenko and Van Maldeghem. This rich paper contains much more than Theorem 9.4.13 and its Corollary 9.4.14.

Proof Since Γ is bipartite, we know $d(y_1, x_n) = n-1$ so $d(y_1, x_{n-1}) = n$ or $n-2$. If $d(y, x_{n-1}) = n-2$, then, noting that $d(x_0, x_{n-1}) = n-1$ and that $x_1 \neq y_1$, the axiom $[AV](x_{n-1}, x_0)$ is violated. □

Step 2. *Every vertex is distance n from some other vertex.*

Proof Immediate from Step 1 and the connectedness of Γ. □

Step 3. *Let* $p = (x_0, x_1, \ldots, x_n)$ *and* y_1 *be as in Step* 1. *Then there exists an isometrically embedded circuit C of length* $2n$ *containing* $(y_1, x_0, \ldots, x_n)$ *as a segment.*

Proof In general, a (pointed) circuit $Z = (z_0, \ldots, z_{2n} = z_0)$ of length $2n$ is isometrically embedded in Γ if and only if, for each vertex x_i, $i = 0, \ldots, n-1$, one has $d_\Gamma(x_i, x_{i+n}) = n$.

Now we repeatedly apply Step 1. We presently have, $d(y_1, x_{n-1}) = n$. Next choose $y_2 \in {y_1}^\perp$ so that $d(y_2, x_n) = n-2$ (this is possible because there is a geodesic path of length $n-1$ proceeding from y_1 to x_n). Then $y_2 \neq x_0$ and by Step 1 (with y_2, y_1, and $(y_1, x_0, \ldots, x_{n-1})$ respectively replacing y_1, x_0 and path p) we have $d(y_2, x_{n-2}) = n$.

That was only a model for the following inductive step. Suppose, for some integer d, with $2 < d < n-1$, we have obtained a path $(y_1, y_2, \ldots, y_d)$ with $d(y_i, x_n) = n-i$, and $d(y_i, x_{n-i}) = n$, for $i = 1, 2, \ldots d$. Since $d(y_d, x_n) = n-d$, there exists a vertex y_{d+1} adjacent to y_d and at distance $n-d-1$ from x_n. Now apply Step 1 with y_{d+1}, y_d and $(y_d, \ldots, y_1, x_0, \ldots, x_{n-d})$ in the respective roles of y_1, x_0, and p, to conclude that $d(y_{d+1}, x_{n-d-1}) = n$.

In this way one forms a circuit,

$$C := (x_0, x_1, \ldots, x_n, y_{n-1}, y_{n-2}, \ldots y_2, y_1, y_0 = x_0),$$

in which antipodal pairs of points (y_i, x_{n-i}), $i = 0, 1, \ldots x_{n-i}$ are all at distance n. Thus C is isometrically embedded and contains $(y_1, x_0, \ldots, x_n)$ as a continuous segment. □

Step 4. *Every geodesic path of length k,* $1 \leq k < n$, *is a segment of an isometrically embedded* $2n$*-circuit.*

Proof First consider the case $k = 1$. Suppose (x_0, x_1) is an edge. By Step 2, there is a geodesic path of length n, having x_0 as an initial vertex, say $r = (x_0, z_1, z_2, \ldots, z_n)$. If $x_1 = z_2$, then p is an initial segment of r. Otherwise by Step 1, $d(x_1, z_{n-1}) = n$, and $p^{-1} = (x_1, x_0)$ is an initial segment of $r' := (x_1, x_0, z_1, \ldots, z_{n-1})$. Thus either p or p^{-1} is a segment of a geodesic path of length n, which by Step 3 is a segment of an isometrically embedded circuit of length $2n$.

Now suppose, by way of contradiction, that the set $\mathcal{Y}$ of geodesic paths that are not segments of any isometrically embedded $2n$-circuit is not empty. Among such geodesic paths, choose one, say $p = (y, x_0, \ldots, x_{d-1})$, of minimal length d. From the previous paragraph, we know that $d > 1$. By the minimality of d, the segment $s = (x_0, x_2, \ldots x_{d-1})$ lies in an isometrically embedded $2n$-circuit $C = (x_0, \ldots, x_{2n} = x_0)$. If y is a vertex of C, then $y = x_{2n-1}$, and p is a segment of C contrary to our choice. Thus $y \in x_0{}^{\perp} - C$ and we may apply Step 1 (with y, x_0, and $(x_0, \ldots, x_n)$ in the the respective roles of y_1, x_0, and p of Step 1) to conclude that $(y, x_0, \ldots, x_{n-1})$ is a geodesic of length n, which by Step 3, is a segment of an isometrically embedded $2n$-circuit, C'. Thus p is a segment of C', contrary to the choice p as a member of $\mathcal{Y}$. One must conclude that $\mathcal{Y}$ is empty, and Step 4 is proved. □

Step 5. *The graph* Γ *has girth* $2n$.

Proof From Step 3, Γ contains circular tours (that is, pointed circuits with all vertices distinct) of length $2n$. Suppose, by way of contradiction that Γ had girth less than $2n$. Then there exist circular tours with lengths between 2 and $2n$. Among these, select a circular tour $D := (z_0, \ldots, z_2 d)$ of minimal possible length $2d$ (the length is even since Γ is bipartite), where $2 \leq d < n$.

We claim that $d(z_i, z_i + d) = d$ (where the indices are to be read modulo $2d$) so that D is isometrically embedded. Without loss of generality, we may take $i = 0$. Suppose, by way of contradiction, that $d(x_0, x_d) = e < d$. Then there is a geodesic path, $q := (x_0, t_1, \ldots, t_e = x_d)$, as well as the "half-cycle" path $D_1 := (x_0, x_2, \ldots x_d)$. Clearly the paths from x_0 to x_d are different, and so, as we proceed out from x_0, there is a first instance in which the path q departs from D_1 – say $t_i = x_i$ for $i \leq k$, but $t_{k+1} \neq x_{k+1}$. Then, after that divergence of paths, there is a *next* instance at which the two paths join up – say when $t_j = x_m$, $\min(j, m) > k + 1$. Then

$$T = (t_k, t_{k+1}, \ldots, t_j = x_m, x_{m-1}, x_{m-2}, \ldots, x_k = t_k)$$

is a circular tour of length

$$(j - k) + (m - k) < 2d. \tag{9.26}$$

(Note that, since $e < d$, this equation holds even in the extreme case when $k = 0$ and $m = d$ (or equivalently, $j = e$).) But this contradicts the minimality of d. Thus we have $d(x_0, x_d) = d$ as well as a similar result for all antipodal pairs of vertices of D. Thus D is isometrically embedded.

Now, using Step 4, we can extend D_1 to a geodesic path $(x_0, \ldots, x_d, y_{d+} 1, \ldots, y_n)$. Then $d(x_0, y_{n-1}) = n - 1$, while, $d(x_{2d-1}, y_{n-1}) = d(x_1, y_{n-1}) = n - 2$, against $[\mathbf{A}V](y_{n-1}, x_0)$. Step 4 is proved. □

Now it follows that Γ satisfies axioms (GP1)–(GP3) of p. 334 with n finite, and so is a generalized n-gon geometry. □

9.4.4 Existence of Generalized Polygons

As we have seen, the class of generalized ∞-gons is bijective with the class of trees with no degree-one vertices. So existence is assured here.

Similarly, when $n = 2$, we are just dealing with the class of complete bipartite graphs with at least two vertices in each component part, so existence is assured here also.

Generalized *n*-Gons Having Both Thick and Thin Lines

Recall that any vertex of the incidence graph $\Gamma = (\mathcal{P} \cup \mathcal{L}, *)$ of a rank two geometry is said to be *thick* if it is on at least three edges. The object (point or line) is said to be *thin* if it is on exactly two edges. Of course, if Γ is firm, as in the case of generalized n-gons, there are no vertices on just one edge, so all vertices are either thick or thin.

In a beautiful paper by Arthur Yanushka [147], it is shown that all generalized polygons of finite diameter having both thin and thick objects are obtained from a thick generalized polygon by a certain construction. We describe this theory next.

For any finite integer $n > 1$, we have defined a generalized n-gon as a bipartite graph of diameter n and girth $2n$. The *girth*, one may recall, is the minimal length of a *circular tour* – that is, a circular path with no edge repeated. With that understanding there is no need to exclude bipartite graphs with multiple edges, for a graph has multiple edges if and only if it has girth 2.

With this insight, one may define a *generalized* 1*-gon* to be a bipartite graph of diameter 1 and girth 2 – that is, exactly two vertices connected by at least two (possibly infinitely many) edges. (Note the 1-gon is thick, if the two vertices are connected by at least three edges.)

Now we construct a generalized polygon with possible thin objects from a thick polygon. Suppose $\Gamma = (\mathcal{P} \cup \mathcal{L}, *)$ is a thick generalized n-gon. Fix a positive integer k. We construct a generalized kn-gon from the following recipe:

1. Each edge e of the bipartite graph Γ is replaced by a path $p(e)$ of length k whose extremal vertices are those of the original edge e.
2. If e_1 and e_2 are distinct edges of Γ, the two paths $p(e_1)$ and $p(e_2)$ shall share no vertices except possibly the extremal vertices – that is, the original thick vertices of e_1 or e_2.

The result of this replacement is a new bipartite graph $k\Gamma$ whose vertices are the old (thick) vertices of Γ and the new (thin) vertices that are non-extremal vertices of one of the paths $p(e)$, as e ranges over all edges of Γ. Each tour of length ℓ in Γ is then converted to a tour of length $k\ell$ in $k\Gamma$, and any pointed circular tour of Γ becomes a circular tour of length $2k\ell$, pointed at a thick vertex. It follows that the graph $k\Gamma$ is bipartite of girth $2kn$ and diameter kn, and so is a generalized kn-gon.

Of course, if $k = 1$, the edges of Γ remain unadulterated, and the graphs $k\Gamma$ and Γ coincide.

Let us examine 3Γ when Γ is the generalized 1-gon consisting of two vertices connected by three edges. The result is a generalized 3-gon with two thick objects and six thin ones. Now (unlike the original 1-gon) 3Γ possesses an interpretation as a point-line geometry: namely the (self-dual) projective plane consisting of one thick point incident with three thin points, and one thick point not on that thick line.

This brings us to the remarkable theorem of A. Yanushka [147].[12]

Theorem 9.4.15 (A. Yanushka.) *Suppose* $\Gamma = (\mathcal{P} \cup \mathcal{L}, *)$ *is a generalized* n*-gon with* n *finite, having both thin and thick vertices. Then the thick vertices form a generalized* m*-gon* Γ_0 *where* $n = km$ *and* Γ *is isomorphic to* $k\Gamma$.

The argument is that if two thick vertices of Γ are connected by a path of lenth k, all of whose non-extremal vertices are thin, then all such paths connecting two thick vertices have this length. The geometry Γ_0 is then recovered by replacing each such path (whose extremal vertices are thick, and whose non-extremal vertices are thin) by a simple edge uniquely defined by the path. In this way all thin vertices are erased.

Theorem 9.4.15 reduces the existence question to thick polygons.

Thick Generalized Polygons

For $2 < n < \infty$ we divide the existence question into two parts: (1) the case that $|\mathcal{P}|$ is finite and (2) the case of infinitely many points. We consider the finite case first.

We say that a finite generalized n-gon geometry $(2 < n)$ has *order* (s, t) if and only if each point is incident with $t + 1$ lines and each line is incident with $s + 1$ points. The order is forced when all points and lines are thick (that is, they are incident with at least three other objects). When thin objects are allowed, as discussed in the previous section, there may be no order. For example:

1. A generalized 3-gon is a generalized projective plane which may have both thick and thin lines. If both types of lines occur, the 3-gon has no order.
2. We have seen that there are cases where a finite generalized quadrangle does not possess an order. For example, if every point is on just two lines, the quadrangle may be a grid with two distinct line sizes.
3. A similar absence of an order may occur in hexagons and octagons which have thin lines or thin points.

Generalized Polygon Geometries of Finite Order (s, t)

This much is known: if all lines are thick and all points are thick (that is, they lie on at least three distinct lines) then the finite generalized polygon does possess an order (s, t). Of course, the converse fails: there *are* polygons of order (s, t) with $s = 1$ or

[12] Although the context of the paper seems to be finite polygons, the proof of Theorem 9.4.15, which is confined to the structural relation between thick and thin vertices, seems to proceed without using finiteness.

with $t = 1$. If $s = 1 = t$, the generalized polygon is just the vertices and edges of a regular n-gon (or any n-circuit that is a circular path for that matter).

It is easy to show that in a generalized n-gon geometry $(\mathcal{P}, \mathcal{L})$ of order (s, t) $(n > 2)$, the number of points and lines are exactly determined. Either

1. n is odd, one must have $s = t$, and

$$|\mathcal{P}| = |\mathcal{L}| = 1 + s + s^2 + \cdots + s^{n-1},$$

or

2. n is even and

$$|\mathcal{P}| = (1 + s)(1 + (st) + \cdots + (st)^{n-1}),$$
$$|\mathcal{L}| = (1 + t)(1 + (st) + \cdots + (st)^{n-1}).$$

A key result is the Feit-Higman theorem.

Theorem 9.4.16 (Feit and Higman [62].) *Suppose $\Gamma = (\mathcal{P}, \mathcal{L})$ is a generalized n-gon geometry of finite order (s, t). Then exactly one of the following occurs:*

1. $s = t = 1$ *(the ordinary n-gon).*
2. $n = 2$ *(the generalized digon).*
3. $n = 3$ *and* $s = t > 1$ *(the projective plane).*
4. $n = 4$ *(a generalized quadrangle with more than four points).*
5. $n = 6$ *and at most one of s and t is* 1 *or* $min(s, t) > 1$ *and st is a square (generalized hexagons with an order).*
6. $n = 8$ *and at most one of s and t is* 1 *or* $min(s, t) > 1$ *and $2st$ is a square (generalized octagon of order (s, t)).*
7. $n = 12$ *and exactly one of s and t is equal to* 1.

For the cases with $n = 8$ or 12 in which exactly one of the parameters is equal to 1, the generalized n-gon geometry Γ or its dual must arise by the construction of Theorem 9.4.15 given for $2\Gamma^*$.

We have discussed the existence of projective planes of order s at some length in the Appendix to Chap. 5. No planes are known when n is not a prime power, but otherwise they seem to be so plentiful as to discourage any idea of classification.

Finite generalized quadrangles appear in six classical varieties: $Sp(4, q)$, $U(4, q^2)$, $U(5, q^2)$, and their duals $O(5, q)$, $O^-(6, q)$, and the dual of $U(5, q^2)$, which seems not to have a special name. In addition, there are non-classical quadrangles of various sorts:

1. $T_2(\mathcal{O})$, of order (q, q), q even. $\mathcal{O}$ is an oval of $PG(2, q)$ which is not a conic.
2. Generalized quadrangles of order $(s, s + 2)$, with these constructions: (a) $T_2^*(\mathcal{O})$ where $\mathcal{O}$ is a hyperoval of $PG(2, q)$, q even, and (b) the Payne derivative about a regular point of quadrangle of order (s, s).
3. Generalized quadrangles of order (q, q^2): These are (a) the quadrangles arising from Tits' construction $T_3(\mathcal{O})$, where $\mathcal{O}$ is an ovoid of $PG(3, q)$ which is not an elliptic quadric (so q is even) and (b) more than a dozen families of quadrangles

of order $(s, t) = (q, q^2)$ derived from ovoids of $PG(3, q)$ using flocks, Kantor families and q-clans.

These were discussed in the Appendix to Chap. 7. A great many characterizations of the classical quadrangles are surveyed in the excellent article of J. Thas [130] in the *Handbook for Incidence Geometry*. But much more is known and the reader is encouraged peruse the masterful book *Translation Generalized Quadrangles* by J. Thas, K. Thas, and H. Van Maldeghem.

There are only two known infinite families of generalized hexagons with an order (s, t), each parameterized by a finite field. There are the *split Cayley hexagons* of order (q, q) which are related to the group $G_2(q)$ and the *twisted triality hexagons* of order (q, q^3) which are related to the group ${}^3D_4(q)$, and the duals of each type. They are rank two coset geometries defined by the two classes of maximal parabolic subgroups of the indicated group of Lie type.

Finally, the only known octagons of order (s, t), $\min(s, t) > 1$ occur as a coset geometry for the groups ${}^2F_4(q)$ of Ree type. Here q is an odd power of 2. These octagons are called the *Ree–Tits octagons* and have order (q, q^2) (or (q^2, q) in the case of their duals). No other thick octagons are known.

9.5 Diagrams

9.5.1 Introduction

A rank-two diagram is no more or no less than an isomorphism-closed class of rank-two geometries. The word "diagram" comes from our habit of representing nice classes of firm rank two geometries by a labeled edge connecting an ordered pair of vertices, each vertex representing one of the two classes of objects of the geometry. These edges are then incorporated into a larger graph whose vertices (called nodes) are labelled by a set I which we call a "diagram." Geometries over a typeset I are said *to belong to a diagram D* if each of their rank two residues of type $\{i, j\}$ is a member of the class of geometries designated by the label of the edge connecting vertex i and vertex j. Note that a geometry over I cannot belong to a diagram D unless its rank two residues are non-empty and firm.

There is another approach to diagrams. One can also consider diagrams with the labelled edges denoting isomorphism-closed classes of firm rank two chamber systems rather than firm rank two geometries. Then, given a diagram D with vertex set I, we are able to say that a chamber system C over *I belongs to the diagram D* if and only if every rank two residue of type $\{i, j\}$ $(i, j \in I)$ is a chamber system belonging to the class of chamber systems determined by the edge-label of the edge connecting node i to node j.

If $M = (m_{ij})$ is a Coxeter matrix, with rows and columns indexed by I, one can define a diagram $D(M)$ by labeling the edge connecting node i with node j by the class of generalized m_{ij}-gons. Thus in a geometry Γ over I belonging to diagram $D(M)$ (called a *geometry of type M*), the residue $\mathrm{Res}_\Gamma(F)$ of any flag of cotype $\{i, j\}$ is a generalized m_{ij}-gon geometry. Similarly, a *chamber system of* *type M* is a chamber system over I belonging to the diagram $D(M)$ – which means that every residue of type $\{i, j\}$ is a generalized m_{ij}-gon (as originally defined as a chamber system). Chamber systems of type M will be the context of our definition of *Building*.

So we have to cover a number of concepts.

9.5.2 Rank Two Diagrams

We begin our glossary of rank two diagrams:

1. (Digons.) These are the rank two geometries whose incidence graph is complete bipartite – that is, *every object of type* 1 *is incident with every object of type* 2.
2. (Generalized trigons = projective planes.) The incidence graph is that of the points and lines of a projective plane.
3. (Generalized quadrangles.) By now the reader should know what these are.
4. (Generalized $2n$-gons.) A class of near polygons $(\mathcal{P}, \mathcal{L})$ with the property that for every line L and point p, there is a unique path from p to its gate (unique nearest point in L).

These cases are indicated by the respective two-vertex diagrams in Fig. 9.7a.

Why are these diagrams left–right symmetric? It is because ultimately a generalized polygon is defined by a hypothesis that reads the same after the words "points" and "lines" are transposed, but "incidence" is left the same. Thus the dual point-line geometry of a digon, projective plane, generalized quadrangle, generalized hexagon, etc., is also one of the same species.

There are other rank two diagrams which are not symmetric. For example, the class of *linear spaces* $(\mathcal{P}, \mathcal{L})$ are those rank two geometries with the property that any two points are incident with a unique line. We denote the class of such geometries by the simple diagram presented in Fig. 9.7b.

As a special case, one might consider the affine planes encountered in Chap. 4. This diagram is depicted in Fig. 9.7 after the label "(c)".

In a dual linear space, any pair of lines meet at a unique point depending on that pair. Since affine planes are missing this property, the class L is not self dual. So it is good that the edge label "L" is not left–right symmetric.

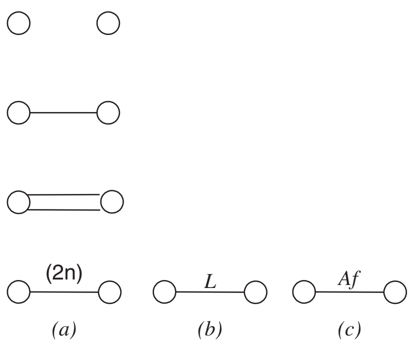

Fig. 9.7 Basic rank two diagrams

9.5.3 Diagram Geometries of Higher Rank

We fix once and for all a type set I, the (global type set). A *diagram (of geometries) over I* is an assignment

$$D : \binom{I}{2} :\to \mathcal{G}_2$$

which assigns to each 2-subset of elements of I an isomorphism-closed class of rank two geometries $D(i, j)$ over $\{i, j\}$. For every subset J of I let $D_{I-J} := D(J)$ denote the restriction of this function to the collection of ordered pairs chosen from J. (We shall always write $D(i, j)$ for $D(\{i, j\})$.) The symbolism above gives us a method of associating with D a graph (also called D) whose nodes are indexed by the elements of I, and for every 2-subset $\{i, j\}$ of I the edge directed from i to j should bear the appropriate rank-two symbol $D(i, j)$ or its graphic representative. The same edge directed from j to i should bear the graphic representative of the class $D(j, i)$. (It is necessary only to assign one of these symbols in the proper orientation since $D(j, i)$ is the class $D(i, j)^*$, the dual point line geometry of $D(i, j)$.) Thus if $D(i, j)$ is the class of linear spaces, any two distinct objects of type i are incident with a unique object of type j and the undirected edge (i, j) is affixed with the symbol "L" written with i to the left and j to the right – or, if it is convenient in drawing the diagram, one can write L^* with j to the left of the symbol.[13]

Next, suppose J is a proper subset of the index set I. Then $D(J)$ is the graph that is obtained when the nodes of $I - J$ and all edges involving at least one of these nodes is removed – that is, it is the labeled graph induced on nodes indexed by J. (Notation is simplified by writing $D(k)$ for $D(\{k\})$. Note that notation has already been arranged so that $D(\{i, j\}) = D(i, j)$.)

In this way we can always describe a graph with $|I|$ nodes associated with the function D : ordered pairs from $I \to$ rank two diagrams.

Conversely, given an edge-labelled graph $D = K_I$, the complete graph over vertex set I, whose edge-labels are prescribed by a function D, directly describes a diagram D.

One says that a geometry Γ over I *belongs to a diagram D* (of geometries) if and only if:

1. For any pair of distinct types $\{i, j\}$, the residue of every flag F of cotype $\{i, j\}$ is a member of the class of rank two geometries $D(i, j)$ over $\{i, j\}$.
2. Every flag F of corank at least three lies in a flag of cotype $\{i, j\}$ for each 2-subset $\{i, j\}$ of the cotype of F.

---

[13] Of course it is not always possible to represent graphically the diagram so that edges are horizontal, but we still expect the assignment of the asymmetric symbols such as "L" to be oriented with respect to the two vertices so as to reflect the diagram D.

Since the diagram assigns a rank-two geometry for each pair of indices $\{i, j\}$ we see that condition 2 of the definition just given is equivalent to:

2' Every flag of corank at least three lies in a chamber flag.

A *diagram geometry* is simply a geometry which belongs to some diagram.[14]

Are Diagram Geometries Residually Connected?

Of course this question must be answered in the negative if some $D(i, j)$ is a class of disconnected geometries or if Γ itself is not connected.

So let us assume that Γ is a connected geometry belonging to a diagram D with all the $D(i, j)$ being classes of connected geometries. One easily sees that if Γ has rank just three, then Γ is residually connected.

However it is possible to produce a connected geometry of rank four which belongs to a diagram D for which all rank two residues are connected – yet there are rank three residues which are not connected. This is developed in Exercise 9.19 (see p. 397).

So far we have been discussing full geometries belonging to a diagram. In the later chapters we shall discuss how point–line geometries are derived from diagram geometries with special emphasis on classical geometries belonging to diagrams (some of infinite rank).

9.5.4 Chamber Systems Belonging to a Diagram

We may also attach diagrams to chamber systems of higher rank. Here we define a *diagram of chamber systems* as a function

$$D : \binom{I}{2} :\to \mathcal{CH}_2$$

which assigns to each 2-subset $\{i, j\}$ of I a collection $D(i, j)$ of isomorphism classes of chamber systems over $\{i, j\}$.

We say that a chamber system C over I *belongs to the diagram of chamber systems* D if and only if every rank two residue of type $\{i, j\}$, is a rank two chamber system whose isomorphism type is in the class $D(i, j)$.

We allow ourselves to depict the diagram D in the same graphic way as we did for geometries with the understanding that if nodes "i" and "j" are connected by an

[14] Normally, many authors intend only condition 1 for the definition of a geometry Γ belonging to a diagram D. But since a diagram merely expresses the nature of the rank two residues of a geometry or chamber system, this datum may miss anomalies about what is going on at flags of higher corank. Could some of these flags be maximal – that is, have empty residues?

Flags that do not even lie in a flag of corank two are beyond any proscription that a diagram could impose. So it makes sense to propose a more intimate relation between a geometry Γ and a diagram D. That is the reason for condition 2 or $2'$.

edge labelled "(n)," then any residue of type $\{i, j\}$ in a chamber system C belonging to diagram D is a generalized n-gon (recall this was defined as a chamber system in the previous section).

Of course edges labelled "L" would now have to denote chamber systems of linear spaces – that is the system of point-line flags with adjacencies determined by their sharing of a common point or line. This may seem a little akward, but we shall not really encounter it. Virtually all chamber systems in this book are "type M" – which means each rank two residue of type $\{i, j\}$, is an m_{ij}-gon, where m_{ij} is an integer greater than 2 or is the symbol ∞. These are studied more deeply in the next major section of this chapter.

9.5.5 Diagrams and the Functors Connecting Chamber Systems and Geometries

Suppose Γ is a residually connected geometry belonging to the diagram D. Then, of course, the associated chamber system $\mathbf{C}(\Gamma)$ is residually connected. But does it belong to the same diagram D? Of course we are now viewing D as a diagram of chamber systems – strictly speaking a $\mathbf{C}(D)$ with the rank two edges being the chamber systems of the geometries $D(i, j)$ – precisely, for each ordered pair (i, j) chosen from I, we have

$$(\mathbf{C}(D))(i, j) = \mathbf{C}(D(i, j)),$$

as classes of chamber systems.

Now we have the following.

Theorem 9.5.1 *Suppose Γ and C are, respectively, a geometry and a chamber system over the same type set I. Assume both are residually connected and either (1) $\Gamma = \mathbf{\Gamma}(C)$ or (2) that I is finite and $C = \mathbf{C}(\Gamma)$.*

1. *If the geometry Γ belongs to the diagram D then so does C.*
2. *If the chamber system C belongs to the diagram D, then so does Γ.*

Proof This is a direct consequence of Corollary 9.2.8. □

9.5.6 Some Examples Concerning Diagram Geometries and Chamber Systems

Example 9 Let us take a simple example. The symbol $D = A_3$ is the name of the following diagram (Fig. 9.8).

Here the type set I is the set $\{1, 2, 3\}$. The diagram tells us that $D_k := D(\{I - k\})$ is a projective plane when $k = 1$ or 3, and that it is a digon D_2 when $k = 2$.

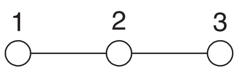

Fig. 9.8 The A_3 diagram

If a geometry Γ belongs to a diagram D, then, according to the definition, for any flag F of cotype $\{i, j\}$ $(i \neq j)$, the residue $\mathrm{Res}_\Gamma(F)$ is a rank two geometry belonging to the class $D(i, j)$. Now consider the geometry $PG(3, K)$ of one-, two-, and three-dimensional vector subspaces of a four-dimensional right vector space V over the division ring K. These are respectively the objects of types 1, 2, and 3 of a geometry in which incidence is vector subspace containment in some order. We have defined and characterized these in Chaps. 3 and 6. The point here is that $PG(3, K)$ *belongs to the diagram* A_3.

Taking the i-dimensional vector subspaces of V to be the objects of type i, saying that $PG(3, K)$ belongs to the diagram A_3, simply means verifying the following three statements:

1. The full collection of objects of types 1 and 2 which are incident with a fixed object of type 3 form a projective plane.
2. Let x, y, and z be objects of types 1, 2, and 3 respectively. If x and z are both incident with y, then they are incident with each other.
3. The collection of all objects of types 2 and 3 which are incident with a given object x of type 1 themselves possess the incidence structure of a projective plane.

This is almost trivial. The last statement holds since the proper subspaces of a three-dimensional vector space over K is a classical projective plane. The first statement it true, since the two- and three-dimensional vector subspaces of V containing a given one-dimensional subspace U are bijective with the one- and two-dimensional subspaces of the three-dimensional factor space V/U (the bijection preserves the incidence relation). The second statement is true simply because incidence is containment.

One might ask whether any geometry belonging to the diagram A_3 is in fact a $PG(3, K)$ for some division ring K? The answer is "yes." One need only show that the truncation to $J = \{1, 2\}$ is a projective space $(\mathcal{P}, \mathcal{L})$ and then use the the Veblen Young theorem. We leave this as an exercise.

9.6 Chamber Systems with a Coxeter Diagram

9.6.1 Coxeter Groups and Coxeter Systems

Coxeter Matrices

Fix an index set I. A *Coxeter matrix over I* is a matrix M whose rows and columns are indexed by I, and whose (i, j)-th entry, m_{ij}, satisfies these properties:

1. m_{ij} is always a positive integer or a formal symbol "∞".
2. $m_{ii} = 1$ for all $i \in I$.
3. $m_{ij} = m_{ji}$ for all $i, j \in I$.

For the moment, the matrix M has no particular linear algebra interpretation. It should simply be regarded as a bank of uncommitted organized data.

The Coxeter Groups $W(M)$

One use of the datum contained in a Coxeter matrix M is that of defining a Coxeter group $W(M)$. Given Coxeter matrix M, the *Coxeter group* $W(M)$ is the factor group

$$F/\langle R^F\rangle,$$

where

1. F is the free group generated by a set $\{x_i\}_{i\in I}$ of generators indexed by I.
2. $\langle R_w^F\rangle$ is the normal closure of the subset of words

$$R_w := \{w_{ij} = (x_i x_j)^{m_{ij}} | i, j \in I, m_{ij} < \infty\}.$$ [15]

Thus, when $m_{ij} = \infty$, $x_i x_j$ is just an element of infinite order in $W(M)$. In effect, $W(M)$ is generated by a collection of elements with trivial square $\bar{R} = \{r_i = x_i\langle R_w^F\rangle\}$, with the property that $\langle r_i, r_j\rangle$ is a dihedral group of order $2m_{ij}$ when m_{ij} is finite, and is the infinite dihedral group when $m_{ij} = \infty$. Moreover, it is the universal group with this property: that is, any other group generated by a class of elements $\{r_i\}_I$ indexed by I with the relations that the product $r_i r_j$ has order m_{ij}, for all $i, j \in I$, is a homomorphic image of the Coxeter group $W(M)$.

Let

$$\rho : F \to F/\langle R_w^F\rangle = W(M)$$

be the natural homomorphism which sends x_i to the element $x_i\langle R_w^G\rangle = x_i^{-1}\langle R_w^G\rangle = r_i$ in $W(M)$. Since each r_i is either an involution or the identity element, and since each element of F is a product of the x_i or their inverses, we can always write an element r of $W(M)$ as a finite product of elements of $\{r_i\}$.

Suppose M is a Coxeter matrix over I and, as above, let $W := W(M)$ be the Coxeter group, and let $R := \{r_i | i \in I\}$, the canonical set of generating elements introduced in the previous two paragraphs.[16] The triple (W, R, M) is called a *Coxeter system*.[17]

[15] Recall that the *normal closure* $\langle X^G\rangle$ of a subset X of a group G is the intersection of all normal subgroups of G which contain X.

[16] Not to be confused with R_w, a collection of words in F.

[17] Usually, in the literature the Coxeter matrix M is suppressed, and one simply writes (W, R) for a Coxeter system. I do not understand the reason for this, but the reader should be forewarned of any differences in notation with the standard literature.

Let Z_2 denote the multiplicative group of integers $\{\pm 1\}$. By the fundamental property of a free group, there is a surjection ϕ from the free group on a set of generators X indexed by I onto Z_2 which takes each element of X to the integer -1. If $x, y \in X$, $\phi(xy) = 1$, whether or not x is distinct from y. In particular, if $M = (m_{ij})$ is a Coxeter matrix over I, then

$$1 = 1^{m_{ij}} = \phi(x_i x_j)^{m_{ij}} = (\phi(x_i) \cdot \phi(x_j))^{m_{ij}}.$$

This means the homomorphism ϕ factors through a surjective homomorphism

$$\mathrm{sgn} : W(M) \to Z_2.$$

Clearly if an element r of $W(M)$ is expressible as a product of an odd number of the generators in R, then $\mathrm{sgn}(r) = -1$ and so r cannot be the identity element of $W(M)$. Thus the following applies.

Theorem 9.6.1 *In the Coxeter system (W, R, M), the following assertions are valid:*

1. *There exists a surjective morphism $W(M) \to Z_2$ taking each element of R to -1.*
2. *The elements of the generating set R are involutions.*
3. *Moreover no product of an odd number of elements of R in $W(R)$ can be the identity.*

The Free Monoid Covering the Coxeter Group

We shall need a mechanism to keep track of the ways to express an element of the Coxeter group $W(M)$ as a product of the r_i.

The *free monoid on the alphabet I* is a monoid I^* whose elements are the "words" $w = i_i \cdots i_\ell$ that can be "spelled" with the alphabet I. (Of course each "word" is nothing more than a sequence of elements written as a "string" (the sequence with the intervening commas removed).) The non-negative integer ℓ is call the *length of the word*, and it is intended that ϕ, the empty word (the unique word of length zero), is to be included in I^*. The binary operation on I^* is the concatenation of words. The concatenation of word w_1 with word w_2 is the word $w_1 \circ w_2$ – the word obtained by first writing the word w_1 (from left to right) and then writing w_2 (from left to right) juxtaposed to the right of the first word w_1. Thus if $w_1 = 123$ and $w_2 = 313$ then $w_1 \circ w_2 = 123313$ and $w_2 \circ w_1 = 313123$.[18] One notes that concatenation is an associative binary operation on the set of words, and that the empty word is a two-sided identity with respect to this operation, so the monoid structure is manifest.

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[18] The author apologizes for adopting the western-European bias in reading from left to right in defining words and in defining concatenation. Of course there is an opposite monoid more suited to semitic writing (Phonecian, Hebrew, Arabic) and Arabic-script renderings of some non-semitic languages (Parsi and Urdu). How could cultures like the Hittites who wrote *Bostrophedron* ever invent a free monoid? There is an answer.

Now there is a morphism $\rho : I^* \to W(M)$ which substitutes r_i for the letter i in each word of the free monoid to form a finite product of involutions in the group $W(M)$. For example, if $I \supseteq \{1, 2, 3\}$, then ρ takes the word $w = 123123321$ to the product of involutions $r_1r_2r_3r_1r_2r_3r_3r_2r_1$ in $G(R)$. Note that since the r_i are involutions $\rho(w) = \rho(123)$, ρ is far from injective.

Clearly this mapping is a morphism in the categroy of monoids since

$$\rho(w_i \circ w_2) = \rho(w_1)\rho(w_2),$$

where the juxtaposition of elements on the right hand side indicates multiplication in the group $W(M)$. The morphism ρ is surjective by the earlier observation that every element of $W(M)$ is expressible as a product of finitely many r_i.

Now, given an element r of the Coxeter group $W(M)$, the fiber $\rho^{-1}(r)$ lists for us all the possible ways of writing the element r as a product of the r_i. Exactly the library of possibilities we want to keep track of. This way we get to live in two worlds: (1) the world of elements of the group $W(M)$ and (2) the world in the sky of ways to express these elements as a product.

9.6.2 *The Cayley Graph of the Coxeter System* (W, R, M)*, and the Coxeter Chamber Systems*

We are going to begin this discussion with a simple assumption, which is easily proved in Lemma 9.6.4 at the beginning of the next section.

(*) The elements of R are pairwise distinct, so there is a bijection

$$R = \{r_i\}_{i \in I} \to I.$$

As a consequence, for every element $s \in W(M)$ and generators r_i and r_j in R, the equation

$$sr_i = sr_j$$

implies $i = j$.

In the previous section we produced a surjective monoid morphism $\rho : I^* \to W(M)$, remarking that the set of all preimages of the group element s (that is, the fiber $\rho^{-1}(s)$) is a library of all possible ways to write the element s as a finite product of the generating involutions $\{r_i\}$.

Actually the beginning student has probably met a structure like this in the guise of *Cayley graphs*.[19] The *Caley graph for the Coxeter group* $W(M)$ *with respect to a*

[19] Indeed the whole idea of representing elements of a group by vertices and words in a set of generators by walks became the basis of both combinatorial topology and combinatorial group theory.

set of generators $R = \{r_i\}_{i \in I}$ is a graph $C(M)$ whose vertex set is the set $W(M)$ of elements of the Coxeter group $W(M)$. We say that (s, t) is an edge *e carrying the label* $j \in I$ if and only if $t = sr_j$ (or equivalently, $s = tr_j$). The result is a graph $C(M)$ (the *Cayley graph of the Coxeter group* $W(M)$), which has edges $e = (s, t)$ which are undirected and which carry with them a set of labels $\lambda(e)$ defined to be the set of indices j in I such that $sr_j = t$.

Since the elements of R are involutions, the Cayley graph $C(M)$ of the Coxeter system (W, R, M) is undirected and has no loops. By the assumption (*) presented at the beginning of this section, the Cayley graph $C(M)$ has no multiple edges. Thus every visible edge defined above carries a unique label, and for every vertex s and label i there is exactly one edge leaving s bearing the label "i."

If 1 denotes the identity element of the Coxeter group $W(M)$, then any walk $p = (1, s_1, s_2, \ldots s_n = t)$ from 1 to t in the Cayley graph $C(M)$ reproduces a unique word $typ(p) = i_1 i_2 \cdots i_n$ $(i = 1, \ldots n)$ of the free monoid I^*, where i_j is an element of $\lambda(s_{j-1}, s_j)$. In that case the element s_n of the Coxeter group is expressible as the product $\prod_{j=1}^{j=n} r_{i_j}$.

We now have a very nice paradigm:

1. *Words in* I^* *correspond one-to-one to walks from* 1 *to any other vertex in the Cayley graph.*
2. *Any relation in* $W(M)$ *among words in its generators corresponds to a circular walk in the Cayley graph* $\mathcal{C}(M)$.
3. *Since* $W(M) = F/\langle R_w^F \rangle$ *is a quotient of a free group by the relation group* $\langle R_w^F \rangle$, *any visible relation (that is, some product over a sequence of the* r_i *is equal to* 1*) must be a logical consequence of basic relations* $(r_i r_j)^{m_{ij}} = 1$ *– that is, every circuit of the Cayley graph is* $\mathcal{C}_2$*-contractible where* $\mathcal{C}_2$ *is the collection of* $2m_{ij}$*-gons defined by taking the orbits of* $\langle r_i, r_j \rangle$ *when* m_{ij} *is finite.*

So it is basically all graphical.

Now it is time to thrust this elementary discussion into a different context. We still retain the assumption (*) maintained at the beginning of this section. Then, as defined, the Cayley graph $C(M)$ of a Coxeter system is a undirected simple graph each of whose edges are labeled by a single element of I.

Lemma 9.6.2 *Under the non-degeneracy condition (*), the Cayley graph* $C(M)$, *with its natural edge-labeling,* λ, *is a connected chamber system over* I. *This chamber system* $C(M)$ *is fully thin (that is, all panels have size exactly two) and satisfies condition (typ) (that the edge labeling assumes a single value on each undirected edge of this simple graph).*

Proof This doesn't really require a bothersome proof. The graph is simple, undirected, and (typ) holds for the edge-labeling. Moreover the "full thinness" condition introduced just above holds. The basic condition defining a chamber system holds automatically – that is, for each $i \in I$, the edges of type i form a "matching" (sometimes called a "1-factor") of the Cayley graph $C(M)$.

We denote the chamber system exhibited by the Cayley graph $C(M)$ by the same symbol $C(M)$.[20]

One observes the following.

Corollary 9.6.3 *1. The Cayley graph $C(M)$ is bipartite. It follows that all its panels (that is its edges) are strongly gated.*
2. The Coxeter group $W(M)$ acts as a group of automorphisms which regularly permutes the chambers.

Proof This is a completely elementary result.

1. According to our paradigm, a circuit in the chamber system $C(M)$ can have type w (a word in the monoid I^*) if and only if $\rho(w)$ is the identity element of $W(M)$. As a consequence, Theorem 9.6.1 implies that $C(M)$ has no circuits of odd length, and so is bipartite.

2. For each element $r \in W(M)$, let r act as a permutation of the chambers by taking chamber c to chamber rc – that is, it acts by *left* multiplication of the chambers. A typical i-adjacency in the chamber system $C(M)$ is an edge $e = (s, sr_i)$ for some $s \in W(M)$. Then r takes e to $re := (rs, rsr_i)$, another i-adjacent pair. This is clearly a group action, and for a chamber c, one has $rc = c$ if and only if $r = 1$, so the action is regular on chambers.

Non-degeneracy of the Coxeter Chamber Systems

The entire subject of groups G *defined by a set X of generators and relations R* (that is, $R \subseteq F(X)$, the free group on generators X, and $G = F(X)/\langle X^{F(X)}\rangle$ is (in the words of J. Humphreys) "notorious" for unexpected results. It could well be that a group G defined in this way by generators and relations is actually trivial. Then of course, the Cayley graph for (G, R) has only one vertex, and all edges would be loops. Even if $G \neq 1$, it is possible that a general Cayley graph has multiple edges.

Because of the *sgn*-epimorphism we know that the first of these pathologies does not occur for the Cayley graph $C(M)$ of a Coxeter system (W, R, M). We know that $C(M)$ has no loops and is bipartite. But what about the second? Can there be multiple edges – or if one prefers, can edges of the chamber system $C(M)$ bear multiple labels? Condition (*) of the previous section asserted that this does not happen. Here we demonstrate (*) by exhibiting a homomorphic image of $W(M)$ in which the elements of R are represented by pairwise distinct elements.

A linear transformation T of a vector space V is *finitary* if and only if the subspace $\mathbf{C}_V(T) : \{v \in V | v^T = v\}$, has finite codimension in V. Working with finitary transformations has the advantage that such transformations are invertible if

[20] This is not an abuse of notation. Rather it is a *use* of definitions. By some extraordinary serendipity which sometimes blesses fumbling authors like myself, the phrases "Cayley" and "Chamber system" begin with the same letter "C." So the mnemonics are preserved when we denote (1) the *Cayley graph* defined by Coxeter matrix M and (2) the *Chamber system* defined by the same Coxeter matrix M, by the very same symbol $C(M)$.

and only if they are injective. We let $GL(V)$ denote the full group of all invertible finitary transformations of V.

Suppose a group G acts as a group of finitary linear transformations on a vector space V. (One normally calls this a *finitary representation of the group* G.) In effect, a finitary representation is simply a group morphism

$$\phi : G \to GL(V).$$

It is said to be a *faithful representation* if and only if $\ker\phi = \{1_G\}$, the identity subgroup of G. Of course that means there is an embedding $G \to GL(V)$.

Let $V = \bigoplus_{i \in I} \mathbf{R}v_i$ be the vector space over the field of real numbers $\mathbf{R}$ whose basis is the set $X := \{v_i\}_{i \in I}$. Let $M = (m_{ij})$ be a Coxeter matrix over I.

Then there exists a symmetric inner product, $B_V : V \times V \to \mathbf{R}$, uniquely defined by the equations

$$B_V(v_i, v_i) = 1, \text{ for all } i \in I, \tag{9.27}$$

$$B_V(v_i, v_j) = B_V(v_j, v_i) = -\cos\left(\frac{\pi}{m_{ij}}\right), \quad i, j \in I, i \neq j, \tag{9.28}$$

$$B_V(v_i, v_j) = -1, \text{ when } m_{ij} = \infty. \tag{9.29}$$

This form B_V is called the *Coxeter form* and is completely determined by the Coxeter matrix M.

As an example, for the diagram of type A_3 considered above,

$$M = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

while the Gramm matrix is

$$\begin{pmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{pmatrix}.$$

For each $i \in I$, $\langle v_i \rangle$ is a non-degenerate subspace of V, and so we have a decomposition

$$V = \langle v_i \rangle \oplus v_i^{\perp},$$

where $v_i^{\perp} := \{v \in V | B_V(v, v_i) = 0\}$.

Let σ_i be the reflection on the space V which "inverts" each vector $y \in \langle v_i \rangle$ and which pointwise stabilizes the complementing "perp-space" $v_i^{\perp}$. Since we are over the field of real numbers $\mathbf{R}$ of characteristic not 2, this construction makes sense. Clearly each such reflection σ_i is an involution.

But the form B_V is defined so that the product of the two reflections $\sigma_i\sigma_j$ has order m_{ij} when the latter is a positive integer, or has infinite order when $m_{ij} = \infty$.

Applying the fundamental property of a free group, we see that there is a surjective homomorphism

$$f : W(M) \to S := \langle \sigma_i, i \in I \rangle$$

taking the generator r_i of $W(M)$ to the reflection σ_i of (V, B).

Note that the Coxeter form (V, B_V), the collection $\Sigma := \{\sigma_i | i \in I\}$ of reflections, and the finitary representation $f : W(M) \to S = \langle \Sigma \rangle$ are all uniquely determined (up to isomorphisms) by the Coxeter matrix M alone.

Now we have condition (*) alluded to above.

Lemma 9.6.4 *Let (W, R, M) be a Coxeter system. Then the generators r_i of R are a collection of pairwise distinct involutions.*

Actually, there is quite a bit more.

Theorem 9.6.5 (Bourbaki, Chap V, n^o 4.3) [9].) *The homomorphism $f : W(M) \to S$ defined by the Coxeter form is an isomorphism .*

A very nice proof of this theorem appears in the excellent book of Garret [63, pp. 7–9].

It has a very important consequence.

Corollary 9.6.6 (The Parabolic Subgroup theorem.) *Suppose $(W(M), R, M)$ is a Coxeter system. We suppose I indexes the set of generators R, so that $C(M)$ is a chamber system over I. Let J be any subset of I. Let $R_J = \{r_i \in R | i \in J\}$ and let M_J be the minor of the Coxeter matrix M obtained by restriction to the rows and columns indexed by the elements of J. Let $(W(M_J), S, M_J)$ be the Coxeter system defined by the submatrix M_J, where $S := \{s_j | j \in J\}$ denotes the fundamental set of involutatory generators of the Coxeter group $W(M_J)$.*

Then there is an isomorphism

$$W(M_J) \to \langle R_J \rangle$$

which sends s_j to r_j, for all $j \in J$.

In particular, if $r_{i_1} \cdots r_{i_k} = 1$ is an identical relation in the Coxeter group $W(M)$, and if all indices i_j in this expression belong to J, then this relation is entirely the consequence of the relations

$$(r_I r_j)^{m_{ij}} = 1, \textit{ for } i, j \textit{ restricted to } J \textit{ alone}.$$

Put another way, $(\langle R_J \rangle, R_J, M_J)$ is itself a Coxeter system.

Proof As before, let V be the real vector space with basis $X = \{v_i\}_{i \in I}$, let B be the symmetric bilinear Coxeter form on V defined by Eqs. (9.27), (9.28) and (9.29), and let $\Sigma := \{\sigma_i | i \in I\}$ be the system of reflections which invert the v_i.

For a subset J of I, let

$$V_J := \langle v_j | j \in J \rangle,$$
$$\Sigma_J := \{\sigma_j | j \in J\},$$

let B_J be the restriction of the form B to V_J, and let the glossary for M_J, R_J, and the Coxeter system $(W(M_J), S, M_J)$ be as in the statement of the corollary.

Clearly the reflections in Σ_J induce reflections on the space V_J, each pointwise fixing a subspace of V which complements V_J. For all $j \in J$, let $\bar{\sigma_j} = \sigma_j|_{V_J}$ be the restriction of reflection σ_j to the subspace V_J, and let $\bar{\Sigma}_J$ be the collection of these restricted reflections $\bar{\sigma}_j$, $j \in J$. Then there is an isomorphism

$$\delta : \langle \Sigma_J \rangle \to \langle \bar{\Sigma}_J \rangle,$$

the right side being a subgroup of $GL(V_J)$.

The finitary representation $W(M) \to GL(V)$ restricts to its subgroup $\langle R_J \rangle$ to give a surjection of groups

$$\beta : \langle R_J \rangle \to \langle \Sigma_J \rangle,$$

taking r_j to σ_j, for $j \in J$.

Also, since the R_J and Σ_J satisfy the relations prescribed for R (simply a consequence of the fact that M_J is a submatrix of M), the fundamental property of the free group produces group surjections

$$\alpha : W(M_J) \to \langle R_J \rangle, \text{ and}$$
$$\gamma : W(M_J) \to \langle \Sigma_J \rangle.$$

Now $\beta \circ \alpha = \gamma$ since both sides take the generator s_j to the reflection σ_j, $j \in J$.

Now $W(M_J)$ acts on (V_J, B_J) via

$$\delta \circ \gamma : W(M_J) \to \langle \bar{\Sigma}_J \rangle$$

exactly as in the discussion of the preceding Theorem 9.6.5. An application of that theorem (with M_J and (V_J, B_J) replacing M and (V, B) throughout) implies that $\delta \circ \gamma$ is a group isomorphism. It follows that $\gamma = \beta \circ \alpha$ is a group isomorphism, and so each of the surjective factors α and β are isomorphisms. Now α is an isomorphism sending s_j to r_j, so no relations exist among the elements of R_J other than those dictated by $W(M_J)$. The proof is complete. □

Length Functions in I^* and $W(M)$

We have already discussed the length of a word w in the free monoid I^* over the alphabet I. The length of w is just the number of letters used to "spell" the word w – the empty word being length zero. Given a Coxeter system (W, R, M) defined by a Coxeter matrix M, we have produced a surjective morphism of monoids

$$\rho : I^* \to W(M)$$

in which letter $i \in I$ is mapped to generator $r_i \in R$. We have also seen that ρ induces a bijection $\hat{\rho}$ between I^* and the set of all walks (galleries) in the chamber system $C(M)$ beginning at 1, the identity element of $W(M)$. The inverse of the bijection $\hat{\rho}$ is simply the type function that records the type, typ(G), of a gallery G of $C(M)$ beginning at 1. Note that the terminal chamber of the gallery G is then $\rho(\text{typ}(G))$ and that the length of the gallery $\ell(G)$ is in fact the length $\ell(\text{typ}(G))$ of the word typ(G).

Now there is another notion of length, this time on elements of the Coxeter group $W(M)$. Actually it is a distance function: we define the *length* $\ell(r)$ *of an element* r *of the Coxeter group* $W(M)$ to be the *distance* $d(1, r)$ in the Cayley graph $C(M)$ of the element r from the element 1. That means $\ell(r)$ is the *shortest* length of a word $w \in I^*$ such that $\rho(w) = r$.

We say that a word $w \in I^*$ is a *reduced word* if and only if $\ell(w) = \ell(\rho(w))$. Note that when we use the term "reduced word," there is a Coxeter matrix in the background. It depends on nothing more than that.

Recall from Chap. 1 that a geodesic path in a graph is just a walk of shortest possible length connecting its extremeties. In our case, if w is a reduced word in I^* (with respect to M) there actually is a geodesic from 1 to $\rho(w)$ whose type is w.

One now notes the following.

Lemma 9.6.7 *If r is any element of $W(M)$, and r_i is one of the generators in R, then $\ell(rr_i)$ is either $\ell(r) - 1$ or $\ell(r) + 1$.*

Proof Since $C(M)$ is bipartite, each edge (r, rr_i) of $C(M)$ is gated with respect to the vertex 1. So the two distances $d(1, r)$ and $d(1, rr_i)$ differ by one. The proof is complete. □

Homotopy in the Coxeter Chamber System

Remark We have noted that $\mathcal{C}_2$-homotopy in the Cayley graph $C(M)$ of the Coxeter system (W, R, M) provides a way of asserting (rather than "determining") that, for two words w_1 and w_2 in I^*, one has $\rho(w_1) = \rho(w_2)$. In other words, it provides a "playing board" on which one can graphically restate the fact that two products of elements of R represent the same element of the Coxeter group.[21]

At least $\mathcal{C}_2$-homotopy is the relevent concept.

> *Two walks (or galleries) in $C(M)$ beginning at* 1 *have the same terminus if and only if they are $\mathcal{C}_2$-homotopic.*

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[21] By scrambling the language of the logicians and the topologists, one might say that $\mathcal{C}_2$-homotopy encompasses the "word problem" for the generators and relations. One might say this at a cocktail party, but it would be a bit crude to assert it seriously. Two walks in $C(M)$ might be $\mathcal{C}_2$-homotopic, (that is, they are connected by a finite string of elementary homotopies, each giving a "deduction" of the ρ-equivalence of two words), but it could conceivably be true that there is no *general recipe* to discover this string of elementary homotopies, as would be required by the logician studying word problems in groups.

Referring to our discussion of homotopy in Chap. 1, we may take $\mathcal{C}_2$ to be the collection of all circular walks in the Coxeter chamber system $C(M)$ whose type has the form $(ij)^{m_{ij}}$ for some pair of indices i, j for which $m_{ij} < \infty$.

An *elementary contraction* of a word in I^* is the deletion of a subword of the form ii – thus the concatenation $f = w_1 \circ ii \circ w_2$ is changed to $f' = w_1 w_2$ by an elementary contraction. Similarly, an *elementary expansion* is the insertion of any expression ii (any i in I) at any position in a word w – that is, an elementary contraction performed with the movie film running backwards. These operations *do* change the length of a word by ± 2 – so they do not disturb the parity of the length (its value mod 2).

As with homotopies, we can use the bijection between words of I^* and galleries beginning at chamber c to apply this notion to galleries of C having c as an initial chamber. Thus an elementary contraction of a gallery simply "snips off" a segment a-b-a from the gallery. A sequence of such deletions can be used to trim off all "spurs," as in Fig. 9.9. Given the Coxeter matrix M, an *elementary M-homotopy of words* replaces a subword of type

$$p(i, j; m_{ij}) := ijijij \cdots \text{ of length } m_{ij}$$

by

$$p(j, i; m_{ij}) := jiji \cdots \text{ of length } m_{ij},$$

when m_{ij} is finite.

More generally, an *elementary $\mathcal{C}_2$-homotopy of words* replaces a subword

$$p(i, j; k) := iji \cdots \text{ of length } k$$

by

$$p(j, i; m_{ij} - k) := jiji \cdots \text{ of length } m_{ij} - k.$$

An important observation is as follows.

An elementary M-homotopy does not change the length of the word. Although an elementary $\mathcal{C}_2$-homotopy of words may change the length of a word, it does not change the parity of that length.

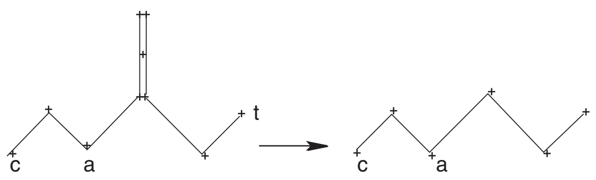

Fig. 9.9 Trimming of spurs

We say two words are *equivalent* if one can be obtained from the other by a finite sequence of expansions, contractions, and elementary $\mathcal{C}_2$-homotopies. The student may easily verify that "equivalent" as just defined is indeed an equivalence relation (the empty sequence of elementary operations is allowed). We write $[w]$ for the equivalence class of all words equivalent to word w.

Clearly if w_i is equivalent to $w'_i, i = 1, 2$, then the concatenation $w_1 \circ w_2$ is equivalent to $w'_1 \circ w'_2$ (just carry out the elementary operations of contraction, expansion, and $\mathcal{C}_2$-homotopy separately in each factor w_i of $w_1 \circ w_2$). Thus there is a well-defined class containing the concatenations of all words in $[w_1]$ with all words in $[w_2]$ and we may write

$$[w_1] \circ [w_2] = [w_1 \circ w_2]$$

as a well-defined operation on the class of equivalent words. Thus the collection of all word-equivalence classes forms a semigroup $I^*/[\,]$ and there is an epimorphism $e : I^* \to I^*/[\,]$.

Lemma 9.6.8 *Two words w_1 and w_2 are equivalent if and only if they have the same image in $W(M)$ under the homomorphism $\rho : I^* \to W(M)$. Thus $I^*/[\,]$ is isomorphic to the Coxeter group $W(M)$.*

Proof Note that ρ is the restriction to the monoid I^* of the canonical surjection $\hat{\rho} : F(S) \to W(M)$ by which $W(M)$, as a presented group, is represented as a homomorphic image of the free group $F(S)$. But ρ factors as $f \circ [\,]$ where $f([w])$ is the constant value of ρ on the class $[w]$. We have a surjection $f : I^*/[\,] \to W(M)$.

It is now clear that $I^*/[\,]$ is a group. But since all of the relators $R_w = \{(s_i s_j)^{m_{ij}}, (s_i)^2\ i, j \in I\}$ lie in $[\emptyset]$, the group $I^*/[\,]$ has its subset $S' := \{s_i[\emptyset] \in I\}$ satisfying the same set of relations. Thus $f(s_i[\emptyset]) = s_i$ and the universality of the presented group $W(M)$ shows that f is an isomorphism taking cosets $s_i[\emptyset]$ (which are elements of $I^*/[\,]$) to $s_i, i \in I$. Thus for any w, w' in $I^*(I)$, $f[w] = e(w')$ if and only if $[w] = [w']$.

9.6.3 Other Properties of Coxeter Chamber Systems

The first property of $C(M)$ that concerns us is the following.

(G_c) If G is a gallery of $C(M)$, then its type $\mathrm{typ}(G)$ is a reduced word of I^* if and only if G is a geodesic path.

To show that $C(M)$ possesses this property, we may apply an automorphism of the chamber system to assure that the initial vertex of G is the identity element 1. In this case G is a gallery of type $w = \mathrm{typ}(G)$ beginning at 1 and ending at $t := \rho(w)$. In fact, by our paradigm, G is the *only* gallery of that type starting at 1. But now, by our definition, w is a reduced word if and only if $\ell(w) = \ell(t)$ — that is, if and only if $\ell(G) = d(1, t)$. The latter equation holds if and only if G is a geodesic path in $C(M)$.

Remark If we seemed to be "proving" that "every Coxeter chamber system possesses property (G_c)" – one will surely notice that the "proof" is really just a tautology resulting from the way we defined "length" on $W(M)$, and the way we defined "reduced". It should then seem odd if I told you that this property (G_c), applied to arbitrary chamber systems C of type M (of course "C" replaces $C(M)$ in the definition of condition (G_c) given above), becomes one of the equivalent conditions for a building. How can possession of a property be a tautology in the context of Coxeter chamber systems, yet be an important concept in the more general context of chamber systems of type M? Being tautological, it can point to no special property of Coxeter chamber systems to bring to the larger context! But it does, for it suggests that Coxter chamber system-like structures are involved in chamber systems of type M. One can almost "smell" apartments in this property (G_c). But perhaps we are getting ahead of ourselves.

M-Homotopy in a Coxeter Chamber System

Our aim in the next sections is to show that in a Coxeter chamber system $C(M)$ one has condition (RG) – the assertion that all residues are strongly gated. This is proved in Theorem 9.6.14 below.

Remark Most accounts of Coxeter groups in the literature focus on quite different defining properties, such as the "Strong Exchange condition," and "the Deletion condition," which, though they characterize Coxeter groups, do not produce efficient proofs of the (RG)-result just quoted. In fact (excepting Tits' "local approach paper" [139], pp 588–621) it is difficult to find in the literature any mention of Theorem 9.6.14 as a direct consequence that one is dealing with a Coxeter group.

Note that $\mathcal{C}_2$-homotopy of galleries of $C(M)$ can drastically change the length of a gallery in $C(M)$. For example, if $m_{ij} = 3$, a gallery G starting at 1 and type ij can be replaced by one of type *jiji* – a longer gallery.

So this raises a question.

If G_1 and G_2 are two galleries from 1 to t in $C(M)$ of types w_1 and w_2, respectively, then, of course, the gallery $G_1 \circ G_2^{-1}$ is a circuit pointed at 1, and by the universality of $W(M)$, this circuit is $\mathcal{C}_2$-contractible. By Chap. 1, this means G_1 is $\mathcal{C}_2$-homotopic to G_2.

Now suppose the two words w_1 and w_2 are ($\mathcal{C}_2$-homotopic) *reduced* words – i.e., the two galleries G_1 and G_2 are geodesic paths from 1 to t. Is it really necessary to *lengthen* the galleries – going from a shortest gallery, to longer galleries, and finally back to a final shortest gallery – in order to achieve the $\mathcal{C}_2$-homotopy?

Put another way, is it possible to pass directly from geodesic gallery G_1 to G_2 without ever passing to a longer gallery? One must ask what the elementary homotopies would be in this case? At best, one is replacing some subsegment of type $p(ij) := iji\cdots$ (of length m_{ij}) by one of type $p(ji) := jiji\cdots$ (also of length m_{ij})

and then only in the case that m_{ij} is finite. We call this an *elementary M-homotopy* of $C(M)$. We say two galleries G_1 and G_2 of $C(M)$ are *M-homotopic* if and only if one can be transformed into the other by a chain of elementary M-homotopies. The property we desire is as follows.

(P_c) Suppose G_1 and G_2 are two $\mathcal{C}_2$-homotopic galleries of $C(M)$ which have the same initial and terminal vertices. If both are geodesics, then they are in fact M-homotopic.

Now we assert that this property is true of Coxeter chamber systems $C(M)$.[22] In the next section we shall see that this property (P_c) makes sense in the context of all chamber systems C of type M (of course with C replacing $C(M)$ in the description above) and that in fact this property is equivalent to (G_c), though that is hardly obvious at this stage. We prove that in the next section. It will follow that $C(M)$ possesses the property (P_c), but it is better to wait for a general proof, rather than one geared specifically to Coxeter chamber systems.

9.6.4 Walls, Roots, and Distance in a Coxeter Chamber System

Let (W, R, M) be a Coxeter system. Now the group W, of course, acts from the left on the coset chamber system $C = C(W, 1; R)$ and any element $r \in R^W = \{g^{-1}r_i g | g \in W, r_i \in R\}$ is called a *reflection*.

Given a reflection r, the collection E_r of all edges (rank one residues) of C which are stabilized by r is called a *wall*. Note that because of the free action of W on the chambers of C, any edge e in E_r necessarily has its two vertices transposed by the involution r. (This means, of course, that r transposes the two opposite roots $D^-(e)$ and $D^+(e)$ defined by the edge e. See Exercise 9.22, part 3.)

Suppose, next, that an edge e belonging to the wall E_r, lies in a rank two residue S. Then as S is a connected $\{i, j\}$-component of C, r stabilizes S. If m_{ij} is finite, S is just a $2m_{ij}$-gon (as a graph), r must stabilize the unique edge e' opposite e in this polygon. This means the following.

(9.6.1) *If e is in the wall E_r and e' is the edge opposite e in some finite rank two residue S containing e, then e' is also in the wall E_r.*

We say that a gallery $G = (c_0, c_1, c_2, \ldots, c_m)$ *crosses a wall E_r k times* if exactly k of the edges $e_1 = (c_0, c_1)$, $e_2 = (c_1, c_2)$, ..., $e_n = (c_{n-1}, c_n)$ belong to the wall E_r. We say G *crosses the wall E_r* if and only if it crosses E_r at least once – more precisely: it crosses E_r k times, where $k \geq 1$. We observe the following elementary but far-reaching result.

[22] The beginning reader is invited to try it out on the Coxeter chamber system of type A_3.

Lemma 9.6.9 *Let $E = E_r$ be any wall.*

1. *Any minimal gallery crosses the wall E at most once.*
2. *Given any two chambers x and y, either (a) all galleries from x to y cross E an even number of times, or (b) all such galleries cross E an odd number of times.*

Proof 1. Let $G = (c_0, c_1, \ldots, c_m)$ be a minimal gallery. Then any interior interval $(c_i, \ldots, c_j)$ must also be a minimal gallery. Now suppose G crossed the wall E k times, where $k \geq 2$. Then there is at least a first time – i.e., a smallest i with $(c_{i-1}, c_i) = e_i$ in the wall E – and also a second instance $e_j = (c_{j-1}, c_j) \in E$, with j the second smallest index with this property. Then $A := (c_{i-1}, \ldots, c_j)$ is a minimal gallery, as observed, so $d(c_{i-1}, c_j) = j - i + 1$. But the reflection r takes A to a gallery A^r from c_i to c_{j-1}. But the distance of c_i to c_{j-1} is only $j - i - 1$, since $(c_i, c_{i+1}, \ldots, c_{j-1})$ is a minimal gallery of this length. Thus r maps a minimal gallery A to a gallery A^r which is not minimal. But this is impossible as r is an automorphism of the chamber system C and so must preserve distances in its graph.

2. Let A and B be two galleries from chamber x to chamber y. Then, regarding x and y as elements of $W(M)$ we see that since A and B have the same terminal chamber y and initial chamber x, $\rho(typ(A)) = \rho(typ(B)) = x^{-1}y$ where $typ(A)$ and $typ(B)$ are the *types* (as words in I^*) of the galleries A and B respectively. Thus gallery A can be deformed into gallery B by a sequence of cutting off or adding on spurs (segments that are backtracks) and elementary $\mathcal{C}_2$-homotopies. It thus suffices to show that neither of these elementary processes disturbs the parity of the number of times that the gallery crosses the wall E.

Consider first the cutting off of a spur $(b_i, b_{i+1}, \ldots, b_{i+t}, b_{i+t-1}, \ldots, b_{i+1}, b_i)$. For each edge $((b_j, b_{j+1})$ in the first part of the spur which belongs to E there is a second occurrence of it (b_{j+1}, b_j) encountered on the return trip to b_i. Thus snipping off a spur costs one an even number of edges of E.

Since adjoining a spur is the above process with the direction of time reversed, an even number of edges of E would be acquired by this process.

Finally we consider an elementary $\mathcal{C}_2$-homotopy $G_1 \circ A \circ G_2 \to G_1 \circ B \circ G_2$ where A has type $p(i, j; k)$, and B has type $p(j, i; m_{ij} - k)$ when m_{ij} is finite. Clearly the transformation $A \to B$ is occurring inside a residue R of type $\{i, j\}$ which is a $2m_{ij}$-circuit, and A and B comprise complementary segments of that circuit. By our observation (5.1), for each edge e of E in the segment A, there is a corresponging opposite edge e' in the circuit R, which may or may not be in A. If it is in A, both edges e and e' are lost by the exchange of B for A. But if e' is not in A, the homotopy transformation just exchanges e' for e. The same remarks hold with the terms "A" and "B" transposed.

Thus in all cases a $\mathcal{C}_2$-homotopy of galleries only alters the number of edges in E by an even number. The proof is complete.

Part 2 of the above lemma can be used to define, from a wall E_r, a partition of the chambers into two parts $D^+(E_r)$ and $D^-(E_r)$ which we call – for the moment – *half-apartments*. We describe this.

Once given a wall $E_r = E$, galleries are of two types: those crossing E an odd number of times and those crossing E an even number of times. Call them *odd* and *even galleries*, respectively. Then from this definition the following arises.

(9.6.2) *The concatenation $\alpha \circ \beta$ of two galleries is even if and only if α and β are both even or both odd.*

Now Lemma 9.6.9, part 2 says that two chambers are either connected only by odd galleries or only by even galleries – yielding in this way two symmetric relations on the set of chambers. But now (5.2) shows that the relation of being connected only by even galleries is not only reflexive, it is transitive as well, and that there are exactly two equivalence classes $D^+(E)$ and $D^-(E)$ partitioning C. Two chambers x and y are in opposite classes if and only if they are connected only by odd galleries.

We have seen (from part 2 of Exercise 1.1 of Chap. 1 and elsewhere) that as C is a connected bipartite graph, any edge e determines two sets $D^+(e)$ and $D^-(e)$, which we have called "roots" (the sets closest to one of the vertices of e) and that these sets also partition the vertices of C. Now the wall E is a collection of edges. If e is an edge of the wall E, what is the relationship of the roots $D^{\pm}(e)$ and the half-apartments $D^{\pm}(E)$? In the next theorem it will be shown that these two partitions of C are in fact the same. From this it will follow that if $e, f \in E$ then $\{D^{\pm}(e)\} = \{D^{\pm}(f)\}$ – and more.

Theorem 9.6.10 *Let E be a wall and let $C = D^+(E) + D^-(E)$ be the partition of chambers into half-apartments determined by E.*

1. *The sets $D^+(E)$ and $D^-(E)$ are convex.*
2. *For each edge $e = (x, y)$ bridging $D^+(E)$ and $D^-(E)$ (that is, $x \in D^+(E)$, $y \in D^-(E)$), then*

   (a) $D^+(E)$ and $D^-(E)$ are the opposite roots

$$D_x(e) := \{z \in C | d_C(z, x) < d_C(z, y)\} \text{ and}$$
$$D_y(e) := \{z \in C | d_C(z, y) < d_C(z, x)\},$$

   respectively, and

   (b) e belongs to E.

Proof 1. Let (x, y) be an "even" pair of chambers with respect to the wall E. This means $x \neq y$, and $\{x, y\} \subseteq D^+(E)$ or $\{x, y\} \subseteq D^-(E)$. Let G be a minimal gallery from x to y. On the one hand G must cross E an even number of times while on the other hand, by Lemma 9.6.9 it crosses E at most once. Thus each chamber c of the gallery G also forms an even pair with x and lives in the same half-apartment $D^+(E)$ or $D^-(E)$ as x and y. So half-apartments are convex.

2. Let e be a bridging edge as in part 2. Then e itself is a minimal gallery (x, y) connecting x and y. Since, by definition, (x, y) is an odd pair, this gallery crosses the wall $E = E_r$ an odd number of times. Thus the edge e is r-invariant, proving (b).

Now suppose z is any chamber in $D^+(E)$. We must show that $d(z, x) < d(z, y)$. Let us suppose, on the contrary, that z is closer to y than to x, so that by Exercise 9.22, part 3, $d(z, x) = d(z, y) + 1$. Then if G is a minimal gallery from z to y, the augmented gallery $G' := G \circ (x, y)$ is minimal. Then G' crosses E an even number of times. But as it contains the edge $e = (x, y)$ it crosses E at least once, and so must cross E at least twice. But by Lemma 9.6.9, that is impossible for a minimal gallery G'. Thus we must have $d(z, x) < d(z, y)$ after all.

We have shown, then, that

$$D^+(E) \subseteq D_x(e) := \{z|d(z, x) < d(z, y)\}.$$

Similarly, $D^-(E) \subseteq D_y(e)$ and since $D_x(e) \cap D_y(e) = \emptyset$, all containments here are equalities. Thus (a) holds, and the proof is complete. □

Corollary 9.6.11 *For every pair of distinct chambers x and y, there exist exactly $d(x, y)$ roots containing x but not y. In particular if x is adjacent to y, there is exactly one such root.*

Proof First assume x is i-adjacent to y in C. This means C can be coordinatized so that $x = 1$ and $y = s_i$. Then left multiplication by s_i transposes x and y. Since this was just one of many possible coordinatizations, it is correct to say that some conjugate r of s_i transposes x and y. Thus $e := (x, y)$ belongs to the wall E_r, and by the theorem $D_x(e)$ and $D_y(e)$ are the two roots of E_r, with $D_x(e)$ containing x but not y. Suppose now that D was a root containing x but not y. Then D and its opposite root $-D := C - D$ are formed by a wall E_t containing $e = (x, y)$. But then t and r both transpose x and y to tr fixes x. Since G acts regularly on C, $tr = 1$ so $r = t$. Thus each edge (x, y) belongs to a unique wall – and there is a unique root containing x but not y.

Now let $\gamma = (x = c_0, c_1, \ldots, c_m = y)$ be a minimal gallery from x to y. Let D_i be the unique root containing c_{i-1} but not c_i. Then as x is closer to c_{i-1} than c_i, x belongs to D_i since D_i is a root. But similarly D_i contains y but not x, so y is not in D_i. Clearly, then, the D_i, $i = 1, \ldots, m$ comprise a collection of distinct roots which contain x but not y. We claim they comprise *all* such roots. For suppose D' were any root containing x but not y. Then the gallery γ must pass at some point from the set D' to its opposite root $(C - D')$, at some bridging edge – say (c_{j-1}, c_j). Then D' contains c_{j-1} but not c_j. From the uniqueness of roots separating adjacent chambers, we see $D' = D_j$. □

Corollary 9.6.11 also makes it easy to identify those chambers one might encounter along a minimal gallery stretched between two given chambers.

Lemma 9.6.12 *Fix two chambers x and y. The following conditions on chamber z are equivalent:*

(1) z lies on a minimal gallery from x to y.
(2) $d(x, z) + d(z, y) = d(x, y)$.
(3) z lies in every root containing both x and y.

Proof (2) implies (1). If g and h are minimal galleries from x to z, and from z to y, respectively, then (2) shows that the concatenation $g \circ h$ of these galleries is minimal. Since it contains z, (1) holds.

(1) implies (3). If D is a root containing x and y, it must, because of its convexity, contain any minimal gallery between them and in particular must contain z.

(3) implies (2). Suppose z lies in every root containing both x and y. Now if D is a root containing z but not y, it must contain x. For if not, x, like y, must belong to the opposite root $-D$, contrary to our hypothesis on z. Thus we have

$$\begin{aligned} d(z, y) &= \text{no. of roots on } z \text{ but not } y \\ &= \text{no. of roots on } x \text{ but not } y \text{ which contain } z. \end{aligned}$$

Also if D is a root on x but not z, then by hypothesis D does not contain y. Thus

$$\begin{aligned} d(x, z) &= \text{no. of roots on } x \text{ but not } z \\ &= \text{no. of roots on } x \text{ but not } y \text{ which do not contain } z. \end{aligned}$$

But now

$$\begin{aligned} d(x, y) &= \text{no. of roots on } x \text{ but not } y \\ &= \text{no. of roots on } x \text{ but not } y \text{ which contain } z \\ &\quad + \text{no. of roots on } x \text{ but not } y \text{ which do not contain } z \\ &= d(z, y) + d(x, y) \end{aligned}$$

by the preceding two equations. Thus (2) holds and the proof is complete. □

9.6.5 Gatedness and Convexity of Residues

There are nice consequences of Corollary 9.6.11 and Lemma 9.6.12, of the previous section.

Lemma 9.6.13 *All residues of the Coxeter chamber system $C(M)$ are isometrically embedded.*

Proof Let (W, R, M) be the Coxeter system giving rise to the chamber system $C(M)$. Let x and y be two distinct chambers of a residue S of $C(M)$ of type J. Let d_S denote the internal metric of S. It follows from Corollary 9.6.6 that S is isomorphic to $C(M_J)$, the Cayley graph of the Coxeter system (W_J, R_J, M_J). Then by Corollary 9.6.11, $d_R(x, y)$ is the number of roots of (W_J, R_J, M_J) which contain x but not y. Now suppose $E_J(r) := (D_J^+(r), D_J^-(r))$ is a wall of $S = C(M_J)$ defined by involution $r \in R_J$ (here expressed as a pair of opposite roots). Since $R_J \subseteq R$, we have $D_J^+(r) = D^+(r) \cap S$ and $D_J^-(r) = D^-(r) \cap S$ where $(D^+(r), D^-(r))$ is the partition of $C(M)$ into opposite roots defined by the involution r. The correspondence $(D_J^+(r), D_J^-(r)) \to (D^+(r), D^-(r))$ is therefore an injection. A second application of Corollary 9.6.11 shows that the global distance $d(x, y)$ is *at least as large* as $d_R(x, y)$. But since distance measures the lengths of minimal galleries the inquality also goes the other way. Thus we have $d(x, y) = d_R(x, y)$. Thus S is isometrically embedded. □

Theorem 9.6.14 *In the chamber system* C *of the Coxeter system* (W, R, M)*, all residues are strongly gated and hence convex.*

Proof Suppose (c, g, x, S) is a quartet where S is a residue of $C(M)$, and c, g, and x are chambers such that

1. $x \in S$, $g \in S$ such that $d(c, S) = d(c, g)$, and
2. $d(c, x) < d(c, g) + d_S(g, x)$.

Clearly c is not in S. Lemma 9.6.13 now tells us that we may drop the subscript S in the right-most term of 2. Then 2 informs us that g lies in no geodesic gallery from c to x. But in that case, Lemma 9.6.12 asserts that there exists a root $D^+(r)$ containing c and x but not g. Thus g is in the opposite root $D^-(r)$.

Now a minimal gallery $G = (c = c_0, \ldots, c_m = g)$ from c to g connects a chamber of $D^+(r)$ to a chamber g of $D^-(r)$, and so there is a first edge $e := (c_{i-1}, c_i)$ which bridges the partition $C(M) = D^+(r) + D^-(r)$. Similarly, in a minimal gallery H of S from g to x, there is a first edge f connecting a chamber of $D^-(r)$ to one in $D^+(r)$ (see Fig. 9.10). By part 2 of Theorem 9.6.10, these bridging

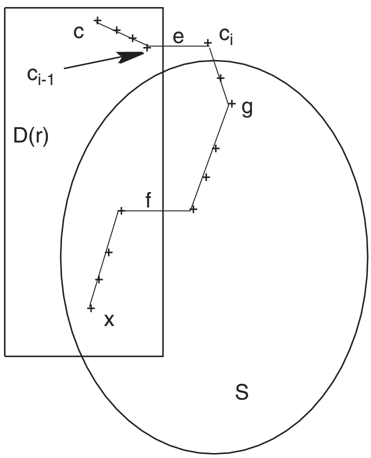

Fig. 9.10 The configuration of a root and a residue in proving strong-gatedness of the residue. The root marked $D(r)$ is $D^+(r)$

edges belong to the wall $E(r)$ and so are r-invariant. Since S is the unique residue of its type containing the edge f, we must have $S^r = S$. In particular $g^r \in S$. But now

$$(c_0, \ldots, c_{i-1}, c_{i+1}^r, c_{i+2}^r, \ldots, c_m^r = g^r)$$

is a gallery from c to g^r whose length is one less than the length of gallery G which is $d(c, S)$ by 1. Thus $g^r \in S$ and yet $d(c, g^r) < d(c, S)$, an absurdity.

Thus no such quartets (c, g, x, S) satisfying 1 and 2 above can exist. This means that for fixed choice of c and residue S, and then for any g for which $d(c, g) = d(c, S)$, we have

$$d(c, x) = d(c, g) + d_R(g, x) \text{ for all } x \in S.$$

Thus S is strongly gated in $C(M)$. □

Corollary 9.6.15 *Let (W, R, M) be a Coxeter system.*

1. (P) *Any two minimal galleries connecting the same two chambers of $C(M)$ are M-homotopic.*
2. *In the monoid I^*, any two reduced words w_1 and w_2 with $\rho(w_1) = \rho(w_2) \in W$ are M-homotopic words.*

Proof 1. We have just proved that every residue of the Coxeter chamber system $C(M)$ is strongly gated – i.e., (RG) holds. Hence (RG_2) holds, and so by Theorem 9.3.13 condition (Λ-min) holds, that is, any two minimal galleries connecting the same two chambers are Λ-homotopic. But in $C(M)$ an elementary Λ-homotopy is an elementary M-homotopy. Thus (Λ-min) becomes condition (P).

2. This M-homotopy among minimal galleries with the same initial and terminal chambers of $C(M)$ induces (via the type-mapping *typ*) the M-homotopy of words described in part 2. □

Theorem 9.6.16 *Suppose (W, R, M) is a Coxeter system over I. Suppose w is a reduced word of I^* and that for some letter $i \in I$,*

$$\ell(wi) < \ell(w).$$

Then w is M-homotopic to a word w' ending in i.

Proof This argument can be played out in the Cayley graph $C(M)$. Suppose G is a minimal gallery of $C(M)$ of type w, beginning at 1 (the identity element of $W(M)$). (It then necessarily terminates at $t := \rho(w)$.) Now on t there is a unique edge (t, t') of type i, and the hypothesis $\ell(wi) < \ell(w)$ tells us that t' is one unit closer to 1 than is t. Thus there is a minimal gallery H of length $\ell(w) - 1 = \ell(wi)$ from 1 to t' and now G and $H \circ (t', t)$ are two minimal galleries running from 1 to t.

By Corollary 9.6.15, part 1, these two galleries are M-homotopic, and so, taking types of the galleries, there is an M-homotopy of words:

$$typ(G) = w \to w' = typ(H \circ (t', t)),$$

where w' ends in i. □

9.6.6 When Is a Coxeter Group Finite?

The title of this section can be recast in this way: *For which matrices M is $W(M)$ a finite group?* This question was completely answered by H.S.M. Coxeter in 1934 [43]. First, it is immediate that if $W(M)$ is to be a finite group, the matrix M can have no entry equal to "∞." Second, M itself can have only finitely many rows and columns, for otherwise, using the Tits' form of Theorem 9.6.5, there would be a chain of subgroups of unbounded orders. Third, if there is a partition $I = A + B$ for which every generating involution in R_A commutes with every generating involution in R_B (so that $m_{ij} = 2$ for all $(i, j) \in A \times B$), then the Coxeter group $W(M)$ is just the direct product

$$W(M) = W_A \times W_B,$$

and so is finite if and only if each direct factor is finite. Thus one need only consider Coxeter systems in which there is no such partition. This is equivalent to saying that in the Dynkin notation, the diagram $D(M)$ is connected. We call such Coxeter groups $W(M)$, *irreducible Coxeter systems*.

Theorem 9.6.17 (Coxeter) *The Coxeter group $W(M)$ is finite if and only if the connected components of its diagram $D(M)$ are among those listed in Fig. 9.11.*

The diagrams listed in Fig. 9.11 are called the *Dynkin diagrams*.

In the next major section of this chapter we are going to define buildings as a class of chamber systems of type M with certain properties which we would like to keep as simple as possible. Our intention is to derive everything from (the seemingly tautological condition) (G_c) without ever once mentioning a simplicial complex. Everything is in terms of graphs.

But our ultimate objective in this book is to characterize point-line geometries. Merely to show that the most interesting geometries of this type are truncations of homomorphic images of geometries whose chamber systems are of type M and satisfy conditions like (G_c) would have no point if one could not classify the latter.

That is why Tits' theorem classifying all buildings of rank at least three and type M, where $W(M)$ happens to be a finite group (the buildings of "spherical type") is such a beacon. Obviously one cannot mention the latter without mentioning Coxeter's classification of finite Coxeter groups mentioned just above. In the eyes of the author, Tits' classification (not to mention his classification of buildings of affine type of rank at least four) is the "theorem of the century" (to borrow an appellation of Shreeram Abhyankar). We can now prove that point-line geometries

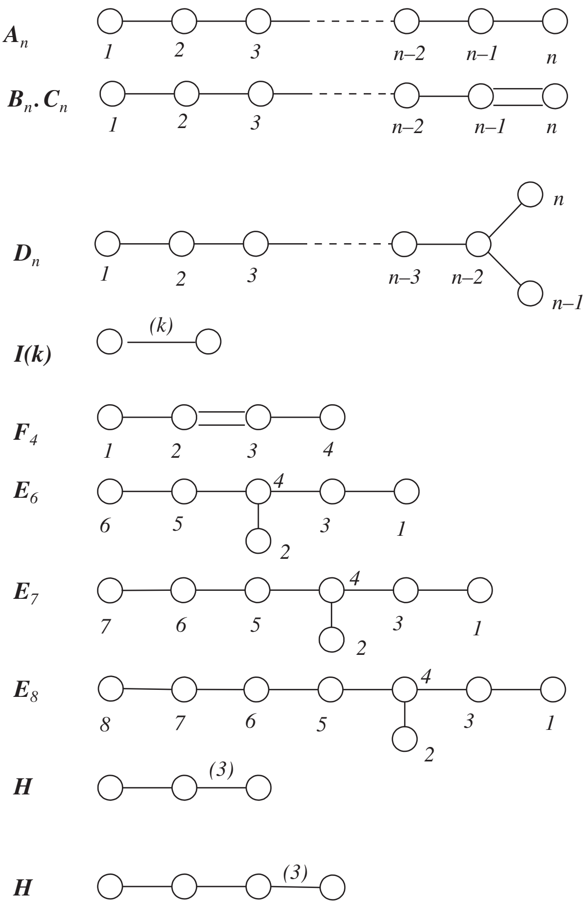

Fig. 9.11 The Dynkin diagrams. These are the diagrams of the irreducible Coxeter systems for which the Coxeter group is finite. (The node labeling is not altogether standardized in the literature. Following Cohen's article in the *Handbook* [35], we have used the numbering adopted by Bourbaki [9]

have real classifiable conclusions because of Tits' theorem. Otherwise one would be condemned to proving that certain axioms imply miscellaneous properties. One could really classify nothing.

9.7 Chamber Systems of Type *M*

9.7.1 Introduction

Let M be a *Coxeter matrix* $M = (m_{ij})$, that is, a symmetric matrix M with all diagonal entries equal to 1, and every other entry equal to either a positive integer greater than one, or the symbol "∞".

Recall that a chamber system $C = (V, E, \lambda)$ over I is said to be a *chamber system of type* M if and only if M is a Coxeter matrix with rows and columns indexed by I, and for every 2-subset $\{i, j\}$ of I, every residue of type $\{i, j\}$ is a generalized m_{ij}-gon.

Note that in a chamber system $C = (V, E, \lambda)$ of type M, each edge of the chamber system must bear a unique label– that is condition (typ) holds. For suppose some edge $e = (c, c')$ carried two or more labels $\lambda(e) = \{i, j. \ldots\}$. Then setting $J = \{i, j\}$ we see that e and its vertices c and c' belong to a connected component R of the graph (V, E_J), which is a residue of type $\{i, j\}$. But since C is type M, R is the chamber system of a generalized m_{ij}-gon, and so cannot have repeated labels.

The *rank of a chamber system of type* M is the cardinality of the set I indexing the rows and columns of the matrix M.

Along with every chamber system of type M we inherit an entire Coxeter system $(W = W(M), R, M)$ and its associated chamber system $C(M)$ – and in fact, *all* of the paraphenalia of the previous section on Coxeter groups and their Cayley graphs. This means we have the following:

1. A monoid epimorphism $\rho : I^* \to W$ from the free monoid I^* over I onto the Coxeter group $W(M)$ defined by sending letter i to the involution $r_i \in R$.
2. A “coordinatization" μ_c of the chambers of $C(M)$ by the elements of W which assigns the identity element $1 \in W(M)$ to chamber c.
3. A bijection ϕ_c:

$$\text{words in } I^* \longrightarrow \text{walks in } C(M) \text{ beginning at chamber } c.$$

4. The notion of a *reduced* word in W, that is, a word w in the monoid I^* whose length is the distance from 1 to $\rho(w)$ in the Cayley graph $C(M)$.

All these things are in place once a chamber system C of type M is even mentioned. One immediately notices that not one of the concepts listed just above involves the Earthly chamber system C. This is all going on in some sort of Chinese Celestial Heaven above C involving the free monoid and the “Coxeter world”!

However, because of condition (typ), there is associated with each gallery $G = (c_0, c_1, \ldots, c_k)$ (that is, a walk in C) a unique word $\mathrm{typ}(G) = i_0 i_1 \cdots i_{k-1}$ in the monoid I^*, called the *type of the gallery* G, where $i_j := \mathrm{typ}((c_j, c_{j+1}))$ for $j = 0, \ldots, k-1$. (We defined this before for the Coxeter chamber systems; here we are simply observing that the definition makes sense for *any* chamber system of type M.)

Noting that each chamber of C lies on an edge with each possible label, we see that for any chamber c of C, restriction of the type function produces a *surjection*

$$\mathrm{typ}_c : \{\text{walks of } C \text{ beginning at } c\} \to I^*.$$

Unlike the case for $C(M)$, this is normally not an injection. Many galleries of C of the same type can begin at chamber c.

9.7.2 The Three Levels of Homotopy

Homotopy theories for a chamber system of type M simply amount to citing a set of rules (called "elementary homotopies") for replacing a gallery by another without disturbing the initial and terminal chambers of the gallery. The reverse replacement "undoing" an elementary homotopy is also required to be an "elementary homotopy." Then two galleries become "homotopic" (so the theory runs) if and only if they are connected by a chain of elementary homotopies. (The chain may have length zero, so the identity relation is a homotopy.) Then the relation of being "homotopic" is an equivalence relation on galleries and so "homotopy classes of galleries" are naturally defined. (Since we did not even specify the particular definition of homotopy, one can see that this is a rather general concept.)

In Chap. 1, we studied homotopies in arbitrary graphs which were defined by a family of circuits. We proved that for any graph $G = (V, E)$, and any family $\mathcal{C}$ of its circuits, there exists a universal $\mathcal{C}$-cover of G. In this sort of homotopy, backtracks $P \circ P^{-1}$ are always $\mathcal{C}$-contractible so we cannot expect this sort of homotopy to preserve the length of a gallery. We shall be interested in this for $\mathcal{C}_2$-homotopy.

A second homotopy theory, M-homotopy, is a special case of the first – that is M-homotopic strings or galleries will already be $\mathcal{C}_2$-homotopic. So the former is a finer theory. Both theories have already been discussed for the Coxeter chamber systems $C(M)$. We want to establish how they work for general chamber systems of type M, as well as the free monoids.

So our homotopies will be discussed at three levels:

- Homotopy of words in the free monoid I^*.
- Homotopy of walks beginning at the identity element 1 in the Cayley graph of the Coxeter groups $W(M)$ (in other words, the homotopy of galleries beginning at 1 in the Coxeter chamber system $C(M)$).
- Homotopy of galleries of C.

The relation between the first two levels is basically a bijection, as has been discussed in the guise of an earlier paradigm. The relation between the third and the first two is not so tight.

For our two relevent homotopy theories, we shall show two versions of our earlier paradigm:

1. (Going up in levels.) Any homotopy of galleries at the level of the chamber system C of type M determines a corresponding unique homotopy of words in I^* and of galleries beginning at 1 in $C(M)$.
2. (Going down in levels.) A partial converse in the form of an existence theorem will assert that if $w_1 \to w_2$ is a suitable homotopy of words, and G is a gallery of C of type w_1, then there *exists* a corresponding homotopy of galleries of C taking G to a gallery H of type w_2 with the same initial and terminal vertices as G.[23]

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[23] Of course now the H is not necessarily unique, as it was in $C(M)$, but this enough to prove equivalence of several properties of C.

Some Notation for Words of I^*

We require a little notation concerning words in the monoid I^*. If $w = a_1 \cdots a_n$ is a word in the free monoid I^* spelled out by the letters $a_i \in I$, then w^* will always denote the *reverse word* $a_n a_{n-1} \cdots a_2 a_1$. This is a useful notation, for if the gallery G of a chamber system C of type M has type $w = \mathrm{typ}(G)$, then the reverse gallery G^{-1} must have type w^*.

If the coefficient m_{ij} of the Coxeter matrix M is finite, and $0 < k < m_{ij}$, define the following words in the monoid I^*:

$$p(i,j) := ijij \cdots (\text{length } m_{ij}), \tag{9.30}$$

$$p(i,j;k) := ijij \cdots (\text{length } k). \tag{9.31}$$

Note that if m_{ij} or k is even, then $p(i,j)$ and $p(i,j;k)$ are respectively words beginning in i and ending in j. Similarly, if m_{ij} or k is odd, these respective words begin and end in i. Thus we have these factorizations in I^*:

$$(ij)^{m_{ij}} = p(ij)^2, \text{ if } m_{ij} \text{ is even}, \tag{9.32}$$

$$(ij)^{m_{ij}} = p(i,j:k)p(i,j;m_{ij}-k) \text{ if } k \text{ is even}, \tag{9.33}$$

$$(ij)^{m_{ij}} = p(i,j)p(j,i) \text{ if } m_{ij} \text{ is odd}, \tag{9.34}$$

$$(ij)^{m_{ij}} = p(i,j;k)p(j,i;m_{ij}-k) \text{ if } k \text{ is odd}. \tag{9.35}$$

In order to avoid constant case divisions according to the parity of m_{ij} or k, it will be convenient to have a common notation for the right-most factors of the equations above. This factor is written $p(i,j)^T$ in the first and third equations, and written $p(i,j:k)^T$ in the second and fourth. Then, whenever m_{ij} is finite, we always have $(ij)^{m_{ij}} = p(ij)p(ij)^T = p(i,j;k)p(i,j;k)^T$.

$\mathcal{C}_2$-Homotopy of C

For any chamber system C of type M, $\mathcal{C}_2$ will denote the collection of all circuits in C that lie within some rank two residue of C. Then along the lines of Chap. 1, $\mathcal{C}_2$-homotopy is defined, and universal $\mathcal{C}_2$-covers always exist.

Now in a chamber system C of type M, any circular gallery G expressible as a concatenation $P_1 \circ P_2$, where P_1 has type $p(i,j:k)$ and P_2^{-1} is a gallery of type $p(i,j;k)^T$ – that is $p(i,j;m_{ij}-k)$ or $p(j,i;m_{ij}-k)$ according to whether k is even or odd – is a circuit of (panel-reduced) type $(ij)^{m_{ij}}$ residing within some rank two residue of C of type $\{i,j\}$ – a residue which is a generalized m_{ij}-gon.[24]

Then an *elementary $\mathcal{C}_2$-homotopy of a gallery of C* is a transformation

$$A \circ P_1 \circ B \to A \circ P_2 \circ B,$$

[24] Note that the assumption that G is circular is necessary. Unlike the Coxeter chamber system $C(M)$, for a general chamber system of type M a gallery of type $(ij)^{m_{ij}}$ need not be circular.

where either:

1. one of $\{P_1, P_2\}$ is a spur (x, y, x) and the other is the gallery of length 0 at x, or,
2. for some $i, j \in I$ and appropriate integer k, P_1 is a gallery of type $p(i, j : k)$ and P_2^{-1} is a gallery of type $p(i, j; k)^T$ — that is $P_1 \circ P_2^{-1}$ is a circuit of type $(ij)^{m_{ij}}$. (Note that P_1 and P_2 have lengths k and $2m_{ij} - k$, respectively.)

But these two transformations determine similar transformations of words of I^*:

$$u \circ ii \circ w \to u \circ w$$

for some i or its reverse, or

$$u \circ p(i, j; k) \circ w \to u \circ (p(i, j; k)^T)^* \circ w$$

either of which we call an *elementary $\mathcal{C}_2$-homotopy of words*.

But now, applying the monoid morphism $\rho : I^* \to C(M)$, this same elementary homotopy of words induces a $\mathcal{C}_2$-homotopy of galleries of the Coxeter chamber system $C(M)$, as explained in an earlier section. So we have $\mathcal{C}_2$-homotopies occurring at three levels: (1) in words of I^*, (2) in galleries of the Coxeter chamber system $C(M)$ which begin at the identity element $1 \in W(M)$, and (3) in galleries of C. The latter determines the two former.

Before leaving $\mathcal{C}_2$-homotopy theory, there is an important fact to observe.

Lemma 9.7.1 (Existence of $\mathcal{C}_2$-homotopies in C.) *Suppose C is a chamber system of type M. Suppose $w_1 \to w_2$ is a $\mathcal{C}_2$-homotopy of words. Then, for any gallery G_1 of C of type w_1, there exists a gallery G_2 of type w_2 $\mathcal{C}_2$-homotopic to G_1.*

Proof One merely verifies the assertion at an elementary homotopy. □

M-Homotopy in *C*

We have already met *special M-homotopy* in the context of Coxeter chamber systems.

An *elementary M-homotopy of words* of I^* is a replacement of words of the form

$$w \circ p(i, j) \circ v \longrightarrow w \circ (p(i, j)^T)^* \circ v$$

where $p(i, j)$ is the word of type of type $ijij\cdots$ of length m_{ij}, $p(i, j)^T = p(i, j)$ or $p(j, i)$ according to whether m_{ij} is even or odd, and the "star" operator denotes reversal of a word.[25] Then two words of I^* are M-homotopic if and only if they are connected by a series of elementary M-homotopies. Clearly M-homotopic words have the same length.

[25] A memnonic device is this: $p(i, j)^T$ must always end in "j." Thus $(p(i, j)^T)^* = p(j, i)$ since it is spelled in $\{i, j\}$ without "double letters," has the right length, and begins with letter j.

Applying the monoid morphism ρ, any M-homotopy of words is converted into an M-homotopy of galleries of the Coxeter chamber system $C(M)$.

Meanwhile back on Earth we have the 'mundane' chamber system C of type M. Even here, Heaven can be "mirrored" by replacing a gallery G_1 of C from chamber s to t expressible as a concatenation $A \circ P_1 \circ B$ by another gallery $G_2 := A \circ P_2 \circ B$ from s to t where $\text{typ}(P_1) = p(i, j)$ and $\text{typ}(P_2) = (p(i, j)^T)^*)$. This distortion defines an *elementary M-homotopy of galleries in C* which does not change the length of the gallery. As usual this extends to a definition of *(special) M-homotopy* among the walks of graph C – that is, among the galleries of the chamber system C.

With these notions in place, we may easily deduce the second basic property of chamber systems of type M.

Lemma 9.7.2 *Suppose C is a chamber system of type M.*

(1) (Existence of M-homotopies in C.) *Suppose G is a gallery in C of type $w \in I^*$ connecting chamber x to chamber y. If the word w is M-homotopic to the word v in I^*, then gallery G is M-homotopic in C to a gallery H having type v.*
(2) *Conversely, if two galleries of C are M-homotopic, then there is a corresponding M-homotopy of words in $M(I)$ connecting their types.*

In other words, any special M-homotopy that can be formed in Heaven can be modeled on Earth (though not necessarily in a unique way). Similarly, any M-homotopy down in C is forever inscribed in Heaven as an M-homotopy of words in I^*.

Proof For any elementary M-homotopy in C, the segment P_1 is a gallery of type $p(i, j) = ijij \cdots$ in a residue which is a generalized m_{ij}-gon. By the standard generalized polygon conditions, there exists a gallery P_2 of type $p(j, i)$ beginning and ending at the same chambers.

The second part is just a consequence of our definitions. □

Geodesics in C have Reduced Type

We are now ready for one more basic property of chamber systems of type M. Recall that a gallery G is a *geodesic* in the chamber system C if and only if it is a shortest possible path connecting its extremeties.

Lemma 9.7.3 *Suppose C is a chamber system of type M. If G is a geodesic in C, then the type of G is a reduced word.*

Proof If the type of G were a word which was *not* reduced then there would be a series of elementary $\mathcal{C}_2$-homotopies of words in I^* which changed w to a shorter word v. Now by the second part of Lemma 9.7.1 there must exist a gallery H of type v which is $\mathcal{C}_2$-homotopic to G. But in that case the length of v, which is the length of H, is shorter than the length of G, but still connects the initial and terminal chambers of G. This contradicts the fact that G was a geodesic. □

9.8 Buildings

9.8.1 Introduction

Buildings are a species of connected chamber systems of type M – possibly of infinite rank. There are many properties which become equivalent in the context of connected chamber systems of type M. We have already shown the equivalence of the three strong-gatedness properties (RG^1), (RG), and (RG_2) for arbitrary chamber systems with condition (typ). In the context of chamber systems of type M any of these conditions are equivalent to the following:

1. Condition (G_c). Any gallery of reduced type beginning at chamber c is a geodesic (or minimal) gallery.
2. Condition (P_c) (Tits). Any two galleries of reduced type connecting chamber c to another chamber are M-homotopic.
3. Condition (G_x). By this we intend to assert condition (G_x) for every chamber x.
4. Condition (P_x). Similarly, the assertion that (P_x) holds for every chamber x.

In the next section we shall demonstrate the equivalence of these conditions for connected chamber systems of type M by means of the following chain of proved results:

1. (Theorem 9.8.1) The conditions (P_c) and (G_c) are equivalent.
2. (Theorem 9.8.2) The conditions (G_c) and (G_x) are equivalent.
3. (Theorem 9.8.4) The condition (G_c) implies (RG).
4. (Theorem 9.8.5) The condition (P_c-min) together with (RG) implies (G_c).

How does this imply the equivalence? We are missing only (RG) implies (G_c). But from the section on chamber systems with strongly gated residues, we proved that (RG) trivially implies (RG_2) which in turn implies condition (Λ-min) (Theorem 9.3.13) . In the context of chamber systems C of type M, the condition (Λ-min) is manifestly as follows.

(P_c-min) Any two minimal galleries connecting the same two chambers of the chamber system C are M-homotopic.

So the assumption (RG) gives us (P_c-min) as well, and then by the fourth result enumerated above, one has (G_c).

We consider a connected chamber system of type M satisfying any one of the properties listed above, as well as any one of the three strongly gated conditions (all seven of which conditions are equivalent) to be a *building*.

In a subsequent section, we show that these conditions are also equivalent to the formulation of Ronan and Tits in which there is a Coxeter-group-valued metric with relatively simple properties[26] and then show (as in Ronan's book) that all of this is

[26] See Ronan [103], Chap. 3.

equivalent to the traditional formulation of the definition of “building” involving a Tits’ system of apartments.

Of course this means that now there are very simple ways to define a building in the context of connected chamber systems of type M. What could be simpler than saying that all corank one residues are gated (RG^1)? Or to assert condition (G_x) that all galleries of reduced type are minimal galleries? To this author, simple definitions have an appeal well beyond the fact that such definitions are easier to teach: rather, it is that the more simple a concept, the more one acquires the esthetic feeling that it is forced on us by the nature of the universe – or at least our mental universe. But there is a down side: simple definitions can give a false illusion of understanding, making invisible all the structural complexity underneath. That is in fact the way it is with buildings. We have merely caressed the upper surface.

9.8.2 The Conditions (G_c) and (P_c)

In the previous section we noted that in any chamber system of type M, the type of any geodesic gallery is a word of reduced type (Lemma 9.7.3). The relevant property here is in fact the converse of this assertion. Suppose C is a chamber system of type M and fix a chamber c in C. Then we define this property.

(G_c) *Every gallery of reduced type beginning at chamber c is a geodesic.*

Tits introduced the following (equivalent) condition in [139].

(P_c) *If two galleries G and H have reduced type, and have the same extremities, then these two galleries are M-homotopic.*

The Equivalence of (G_c) and (P_c)

Theorem 9.8.1 *The properties* (P_c) *and* (G_c) *are equivalent.*

Proof (G_c) implies (P_c). Let G and H be galleries of reduced types $w = \mathrm{typ}(G)$ and $u := \mathrm{typ}(H)$ beginning at chamber c and terminating at chamber t. Let s be the next-to-last chamber of gallery H, so s is i-adjacent to t. Now by (G_c), $d = d(c, t)$ is the length of words $\mathrm{typ}(H)$ and $\mathrm{typ}(G)$; but as $d(c, s) = d - 1$, by (G_c) the gallery $G \circ (t, s)$ cannot be of reduced type. Thus the concatenation $\mathrm{typ}(G) \circ i$ is not reduced and so its length is less than $\mathrm{typ}(G)$. Then by Theorem 9.6.16, $\mathrm{typ}(G)$ is M-homotopic in I^* to a word $w' \circ i$ which is reduced. Then by Lemma 9.7.2 above, G is homotopic to a gallery $G' \circ (r, t)$ where r is i-adjacent to t and G' is of reduced type $w' = \mathrm{typ}(G')$.

If $r = s$, then by induction on the length, G' is homotopic to H', the subgallery of H from c to s. Thus

$$G \sim G' \circ (s, t) \sim H' \circ (s, t) = H$$

and we are done.

If $r \neq s$, $G' \circ (r, s)$ is the same reduced type as $G' \circ (r, t)$. But the former is not a geodesic as $d(c, s) = d - 1$, contradicting (G_c).

(P_c) implies (G_c). Now let $G = G' \circ (y, t)$ be a gallery of reduced type $w' \circ i$ (that is, G' has type w' and the final edge (y, t) is labelled i) chosen of minimal length so $w' \circ i$ is reduced, but the distance of its terminus t from c is less than its length. Then, as w' is reduced, G' is a geodesic from c to y of length $d = d(c, y)$, and $d(c, t) = d$ or $d - 1$. Let H be any geodesic from c to t. Then H has reduced type typ(H) by Lemma 9.7.3. By (P_c), H and G are M-homotopic galleries. But that is impossible as g has length $d+1$ while H has length d or $d-1$. Thus no such gallery G can exist – i.e., (G_c) holds. □

Transporting (G_c) to Other Chambers

Theorem 9.8.2 *A chamber system C of type M satisfies condition* (G_c) *if and only if it satisfies condition* (G_x) *for all its chambers x.*

Proof We need only prove the forward implication, and, as the chamber system C is connected, we need only show that (G_c) implies ($\mathrm{G}_{c'}$) where c' is any chamber j-adjacent to c. Let G be a gallery of reduced type w beginning at c'. We must prove that G is a geodesic, and as the result is obvious if G has length 1, we may assume all shorter galleries of reduced type starting at c' are geodesics.

Case 1: $\ell(jw) > \ell(w)$ – i.e., jw is a reduced word. Then the augmented gallery $(c, c') \circ G$ has reduced type jw and so is a geodesic by (G_c). Then its long "tail," G, is also a geodesic.

Case 2: $\ell(jw) < \ell(w)$. Then jw is not a reduced word. It then follows from Theorem 9.6.16 that the word w is homotopic to a word beginning with j, and so by Lemma 9.7.2 (2), G is M-homotopic to a gallery beginning with a j-adjacency. Since the latter gallery is a geodesic if and only if the former gallery is a geodesic, we may, without loss, assume G *is* this latter gallery. Thus we write

$$G = (c' = c_0, c_1, \ldots, c_d)$$

of reduced type jw', so (c_0, c_1) is a j-adjacency.

Subcase 2.1. $c_1 \neq c$. Then letting G' be the subgallery of G running from c_1 to c_d, we see that $(c, c_1) \circ G'$ is also of reduced type jw' and so is a geodesic. Thus $d(c, c_d) = d$. If then $d(c', c_d) = d$ we are done as G is then a geodesic. Thus we must assume $d(c', c_d) = d - 1$. There is thus a geodesic gallery H' of length $d - 1$ running from c' to c_d, say of reduced type w''. Then as $d(c, c_d) = d$, $(c, c') \circ H'$ is a minimal gallery, so its type jw'' is reduced. Now $(c, c') \circ H'$ and $(c, c_1) \circ G'$ are two galleries of reduced types jw'' and jw' terminating at c_d, and so by condition (P_c), the galleries are M-homotopic. By Lemma 9.7.2 (2), their respective types jw'' and jw' are homotopic words. It follows that w'' and w' are also homotopic words. Thus by Lemma 9.7.2 (1), there exists a gallery H from c' to c_d of reduced type w'. We have now the configuration of Fig. 9.12.

Now let H be the gallery $(c' = h_1, \ldots, h_d = c_d)$ of length $d - 1$. Then h_{d-1} and c_{d-1} are both k-adjacent to c_d, since h and g' are *both* type w'.

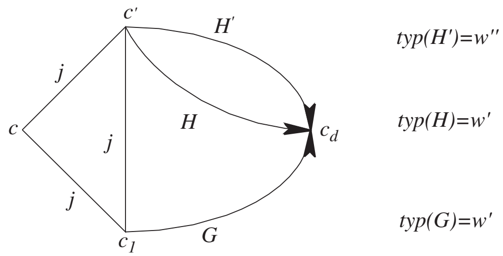

Fig. 9.12 The configuration of Subcase 2.1

Assume now $c_{d-1} \neq h_{d-1}$. Then the gallery

$$(c, c', h_2, \dots h_{d-1}, c_{d-1})$$

(marked by the dotted path in Fig. 9.13a) is reduced type jw' but travels only a distance $d(c, c_{d-1}) = d - 1$. This contradicts (G_c).

On the other hand, if $h_{d-1} = c_{d-1}$, the gallery

$$(c', c_1, \dots, c_{d-1})$$

(marked by the dotted path in Fig. 9.13b) has reduced type (a factor of jw') and length $d - 1 > d - 2 = d(c', c_{d-1})$ against the minimality of G.

Subcase 2.2. $c = c_1$. We have $d(c', c_d) \leq d(c, c_d) = d - 1$ despite the fact that $G = (c', c) \circ G'$ has reduced type jw'. We may then supply a minimal gallery K of reduced type v from c' to c_d so $|K| \leq d - 1$. We then have the configuration of Fig. 9.14.

Suppose first that jv is a reduced word. Then by (P_c), jv is a word homotopic to w'. But in that case jw' would be a word M-homotopic to *jjv* and so could not be a reduced word.

Thus we may assume that jv is not reduced. Then, as v is reduced and jv is not, Theorem 9.6.16 tells us that v is M-homotopic to a word jv', and so by Lemma 9.7.2 (1), K is M-homotopic to another minimal gallery $(c', a) \circ K'$ of type jv'. The picture is now that of Fig. 9.15, since $d(c, c_d) = d - 1 \neq d(a, c_d)$ forced $a \neq c$.

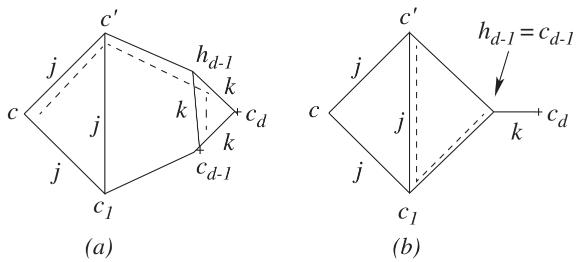

Fig. 9.13 Two impossible galleries

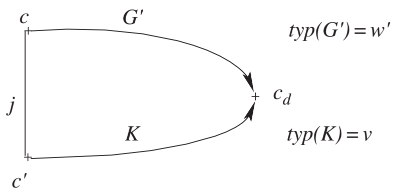

Fig. 9.14 The configuration of Subcase 2.2

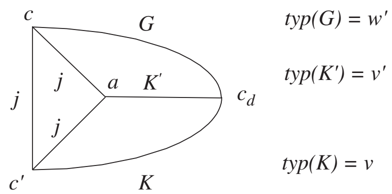

Fig. 9.15 Refinement of Subcase 2.2

Then $(c, a) \circ K'$ is a gallery of reduced type jv' and so by (P_c) is M-homotopic to G', whence by Lemma 9.7.2 (2), w' is homotopic to jv'. Then jw' is homotopic to jjv' and this contradicts the case assumption that jw' is a reduced word. The proof is complete. □

The Condition (G_c) Implies Condition (RG)

We begin with a lemma.

Lemma 9.8.3 *The following statements hold:*

1. *Let C satisfy condition (G_c). Then all residues of C are isometrically embedded induced subgraphs of C. Moreover, they are convex.*
2. *As a chamber system, any residue R also satisfies (G_c).*

Remark One may note that if our definition of building was (G_c) then part 2 asserts that every residue of a building is also a building. This can be seen as an extension of the so-called "parabolic subgroup theorem" to chamber systems of type M. Part 1 of this theorem is the most important. It is in fact the perfect analog of Lemma 9.6.13 for Coxeter chamber systems moved into the more general realm of chamber systems of type M. But now we cannot talk about walls and roots and methods of measuring distance by the number of roots contining one chamber but not the other. We need an entirely new proof.

Proof 1. Suppose x and y are two chambers in a residue R of type J in C. Then at least R is a chamber system of type M_J, the Coxeter matrix restricted to the rows and columns indexed by J. Let G be a shortest gallery of the chamber system R connecting x and y. By Lemma 9.7.3 applied to R, the type w of gallery G is a *reduced* word in $M(J)$.

By a well-known property of Coxeter systems (Lemma 9.6.13, for example), w is a reduced word of I^*. Similarly, if H is any geodesic of C connecting x to y, H also has reduced type. By condition (P_x), H is M-homotopic to G, since these two galleries have the same extremeties and are of reduced type. Thus we see two things:

1. G is a geodesic of C.
2. H is already a geodesic of the residue R.

Statement 1 shows that R is isometrically embedded. Statement 2 shows that R is an induced convex subgraph.

2. From 2 it is clear that (G_c) holds relative to the chamber system R.

The proof is complete. □

Theorem 9.8.4 *For a chamber system C of type M, condition (G_c) implies condition (RG).*

Proof In view of Lemma 9.8.3 all residues are convex induced subgraphs of C, and so ordinary gatedness (in the sense of Scharlau and Dress) implies strong gatedness. Thus it suffices to show that any residue is gated in the ordinary sense.

Choose a residue R of type J and a chamber c and let p_1 be a chamber of R nearest c. If we show $d(c, p_1) + d(p_1, r) = d(c, r)$ for all $r \in R$, the uniqueness of p_1 and the gatedness of R will be shown in one stroke. For this purpose, it is clear that c can be assumed not to lie in R (for then the distance equation follows with $c = p_1$).

We prove the distance equation by induction on $d(p_1, r) = d$, it being true when $d = 0$. So, if the equation is false for some r, it is false for some t of minimal distance from p_1. Thus we have a minimal gallery $G' \circ (s, t)$ of length d and reduced type wj, from p_1 to t and a minimal gallery G_1 of reduced type u from c to p_1. Minimality of $d(c, p_1)$ (which follows from the choice of p_1) implies that the word *u is not M-homotopic to a word ending in a letter of J, the type of the residue R.*

Now uw is reduced as $d(c, s) = d(c, p_1) + d(p_1, s)$ holds by the minimality of d. But uwj is *not* reduced, since otherwise, by (G_c), $G_1 \circ G' \circ (s, t)$ would be a geodesic against the choice of t. Thus in the Coxeter system W, $\ell(uwj) < \ell(uw)$. Thus by Theorem 9.6.16 uw is M-homotopic (in the monoid I^*) to a word vj ending in j.

Let ρ denote the canonical monoid homomorphism $\rho : I^* \to W$ which evaluates words in I^* as products of generating involutions of W. As we have seen in the previous section, the Coxeter chamber system $C(W)$ has its residues gated. This means that if J is the type of the residue R of C and $C(W)$ is coordinatized by W, then $\rho(u)$ is the gate of the residue $\rho(u)W_J$ with respect to the chamber coordinatized by the identity element (that is, since u is not M-homotopic to a word ending in a letter from J, u is the word of shortest length representing an element in the coset $\rho(u)W_J$). But v is a word representing an element in this coset $e(u)W_J$. Thus v is M-homotopic to a word uv'. We now have these M-homotopies:

$$uw \sim vj \sim uv'j \text{ so } w \sim v'j.$$

But this contradicts the fact that as $G' \circ (s, t)$ was a geodesic, the word wj must be reduced. Thus no such chamber t exists. □

Theorem 9.8.5 *If condition $(P_c$ min $)$ holds for all chambers c of C, and the gated hypothesis (RG) holds, then (G_x) holds for all chambers $x \in C$.*

Proof We assume (G_c) false, and select a chamber c and gallery $G = (c = c_0, \ldots, c_d)$ of reduced type wij of minimal length so that G is not a geodesic. Then $d(c, c_{d-2}) = d-2 = d(c, c_{d-1}) - 1$ and $d(c, c_d) \le d(c, c_{d-1}) = d-1$. Since wij is reduced, $i \ne j$. If $d(c, c_d) = d(c, c_{d-1}) = d-1$, there exists, via the gatedness of rank one residues, a vertex s which is j-adjacent to c_{d-1} but distance $d-2$ from c. On the other hand, if $d(c, c_d) = d-2$, set $s = c_d$, so s has the same property in either case. Then for any minimal gallery H' from c to s, we see that $H' \circ (s, c_{d-1})$ is a minimal gallery, of type ending in j, stretching from c to c_{d-1}. But $G' := (c_0, \ldots, , c_{d-1})$ is also a minimal gallery from c to c_{d-1}. By $(P_c$ min$)$, $H' \circ (s, c_{d-1})$ is M-homotopic with G'. Thus wi, the type of G', is M-homotopic to a word ending in j and this contradicts the hypothesis that wij, the type of G, is reduced.

This completes the proof. □

9.9 Apartments

Our aim is to show that in any building C of type M, there exists a family $\mathcal{A}$ of isometric embeddings $C(M) \to C$ of the Coxeter chamber system defined by M whose images are convex, and which for every pair of chambers, possesses an image covering them.

9.9.1 The Tits Metric of a Building

Let C be a building of type M. Then (P_c) holds for each chamber c. Thus given two chambers x and y, there is just one M-homotopy class of minimal galleries stretched from x to y, and hence just one M-homotopy class $w(x, y)$ of reduced words representing the types of these minimal galleries. Then $w(x, y)$, being an M-homotopy class of reduced words, can be regarded as an element of the Coxeter group, $W = W(M)$. Thus there is a well-defined function

$$w : C \times C \longrightarrow W$$

satisfying

$$(i)\ \ w(x, x) = 1_W \tag{9.36}$$

$$(ii)\ w(x, y) = (w(y, x))^{-1} \tag{9.37}$$

which we call the *Tits metric on C*.

We next observe:

- *The mapping $\rho_a : x \to w(a, x)$ is a morphism of C onto W as chamber systems.*

This is clear, for if x is j-adjacent to y, then $w(a, y) = w(a, x)$ or $w(a, x)j$ according to whether $d(a, y) \leq d(a, x)$ or not.

Since the morphism cannot increase distances (as with any morphism of chamber systems) we see

$$(iii)\ \ \ell(w(x, y)) \leq d(x, y) \tag{9.38}$$

Let X be any subset of W. A *strong isometry* $\alpha : X \to C$ is a mapping such that for all $x, y \in X$,

$$w(\alpha(x), \alpha(y)) = x^{-1}y.$$

We have the following.

Theorem 9.9.1 *Any strong isometry $\alpha : X \to C$ can be extended to an isometry of W into C.*

Proof (The proof we give here is essentially that appearing in Ronan's book [103], pp. 31–32.) By Zorn's lemma it is sufficient to show that for any proper subset X of W, a strong isometry can be extended to a larger set. We may assume $X \neq \emptyset$, and recoordinatizing W by pre-left-multiplication if necessary, that $1_W = x_0 \in X$. Since X is proper in W, $X \neq Xr$ for some involution r in S, and the coordinatization can be chosen so $x_0 r = r \notin X$. We therefore need only extend the isometry $\alpha : X \to C$ to $X \cup \{r\}$.

Case 1: Suppose $\ell(rx) > \ell(x)$ for all $x \in X$. Then we may choose $y = \alpha(r)$ to be any chamber r-adjacent to $\alpha(1_W)$ in C. This is a strong isometry on $X \cup \{r\}$ since rx reduced implies $w(\alpha(r), x) = rx$.

Case 2: $\ell(rx_1) < \ell(x_1)$ for some $x_1 \in X$. Then x_1 is homotopic to a word rf in W and we have the configuration in the Coxeter chamber system W given in Fig. 9.16

Then in C there is a unique gallery of reduced type rf from $\alpha(1_W)$ to $\alpha(x_1)$ by the fact that condition $(P_{\alpha(1)})$ holds in C (Theorem 9.8.1). Let y be the second

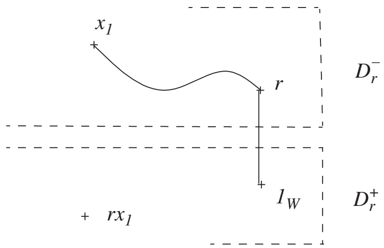

Fig. 9.16 The relation of x_1 to the wall E_r in W

member of this gallery, so $w(\alpha(1_W), y) = r$ and $w(y, \alpha(x_1)) = f$. Now for any $x \in X$, we have

$$\rho_{\alpha(x)}(a) = w(\alpha(x), y) \tag{9.39}$$
$$= w(\alpha(x), \alpha(1_W)) = x \text{ or} \tag{9.40}$$
$$= w(\alpha(x), \alpha(1_W))r = xr \tag{9.41}$$

since y is r-adjacent to $\alpha(1_W)$ and $\rho_{\alpha(x)}$ is a morphism.

Thus, using Eqs. (9.39) and (9.40) or (9.41) above,

$$w(y, \alpha(x)) = x \text{ or } xr \text{ for each } x \in X.$$

Now for each $x \in X$ define

$$\beta(x) := r \circ w(y, \alpha(x)).$$

Then $\beta : X \to W$ is the composition of a strong isometry $\alpha : X \to C$, a morphism $\rho_y : C \to W$, and left multiplication by r on W which is an automorphism of W. Thus β cannot increase distances as none of the factors α, ρ_y, $x \to rx$, do so. Moreover, from the above,

$$\beta(x) = rw(y, \alpha(x)) = r(x \text{ or } rx) = rx \text{ or } x$$

and one calculates that

$$\beta(1_W) = r \circ w(y, \alpha(1_W)) = r \circ r = 1_W \text{ in } D_r^+$$

and

$$\beta(x_1) = r \circ w(y, \alpha(x_1)) = rf = x_1 \text{ in } D_r^-.$$

At this point one can show that β is the inclusion mapping. Suppose otherwise. Then for some $x \in X$, we have $\beta(x) = rx$. Now, if $x \in D_r^+$ (that is, its distance to 1_W in $C(M)$ is shorter than its distance to r) then $\beta(x) = rx$ is further from $\beta(1_W) = 1_W$ than was x – i.e., β has increased distance, a contradiction. On the other hand, if $x \in D_r^-$ (that is, it, like x_1, is closer to r than it is to 1_W) then $\beta(x) = rx$ is further from $\beta(x_1) = x_1$ than is x, and so x has again increased distance. Since β cannot do this, we conclude that no such x exists. Thus for all $x \in X$,

$$\beta(x) = x = r \circ w(y, \alpha(x))$$

so

$$w(y, \alpha(x)) = rx \text{ for each } x.$$

But since $y = \alpha(r)$, this shows that α satisfies

$$w(\alpha(u), \alpha(v)) = u^{-1}v \text{ for } u, v \in X \cup \{r\}$$

– i.e., the "new" α is a strong isometry $X \cup \{r\} \to C$.

This completes the proof. □

9.9.2 Strong Isometries and the Standard Apartment Axioms for a Building

We now let $\mathcal{A}'$ be the class of all strong isometries $W \to C$ obtained by extending possible strong isometries $X \to C$. Next let $\mathcal{A}$ denote the set of *images* of W in C under the various morphisms of $\mathcal{A}'$.[27] We have the following.

(9.9.1) *Any two chambers x and y of C lie in a common member of $\mathcal{A}$.*

Let G be a minimal gallery of type w from x to y. Then let X be the unique gallery of the Coxeter chamber system $W = C(S, M)$ of type w beginning at 1_w. Then there is a clear strong isometry $X \to G$ which extends to a member of $\mathcal{A}'$ whose image in C is an element of $\mathcal{A}$ contained in C . This step is finished. □

(9.9.2) *Any sub-chamber system $A \in \mathcal{A}$ is convex.*

Let x and y be chambers in A and let G be a minimal gallery of reduced type w connecting x and y. By Theorem 9.9.1 there is a morphism $\alpha : W \to C$ such that $A = \alpha(W)$ contains x and y. Since α is a strong isometry,

$$w(x, y) = [w] = (\alpha^{-1}(x))^{-1}(\alpha^{-1}(y)),$$

so $\alpha^{-1}(x)$ is connected to $\alpha^{-1}(y)$ in W by a unique gallery H of reduced type w. Then $\alpha(H)$ is a gallery of type w connecting x and y. By (Q_x), $\alpha(H) = G$ so $G \subseteq A$. This proves that A is convex. □

(9.9.3) *If A_1 and A_2 are two members of $\mathcal{A}$ containing chambers x and y, then there is a chamber system isomorphism $\mu : A_1 \to A_2$ fixing x and y and every vertex on any minimal gallery connecting them.*

First we observe that if H_1 and H_2 are two galleries of reduced type in the Coxeter chamber system W, then there is a unique automorphism of W taking H_1 to H_2.

[27] Note that distinct members of $\mathcal{A}'$ can lead to the same member of $\mathcal{A}$ by composing an isometry with an automorphism of its image.

This is because: (1) (G_c) for all $c \in W$ implies H_1 and H_2 are minimal, (2) on each chamber c there is a unique gallery of type w, and (3) $\mathrm{Aut}(W) \simeq W$ is regular on W.

Now let A_1 and A_2 be two members of $\mathcal{A}$ containing chambers x and y and let G be a minimal gallery of reduced type w connecting x and y and let $\alpha_i : W \to A_i, i = 1, 2$ be the strong isometries. Then by (9.9.3) $G \subseteq A_1 \cap A_2$. Then $H_i := \alpha_i^{-1}(G), i = 1, 2$ are two galleries of reduced type w in W. By the preceding paragraph, there is a unique automorphism $\beta : W \to W$ so that $\beta(H_1) = H_2$ (with the orientation preserved if w is a palindrome). Then the composition of mappings $\alpha_2 \circ \beta \circ \alpha_1^{-1}$ is an isomorphism $A_1 \to A_2$ fixing gallery G chamberwise.

If G' is a second minimal gallery from x to y of type w', then $e(w^{-1}w') = 1_W$ and so, for $i = 1, 2$, the two lifts $H_i' := \alpha_i^{-1}(G')$ are such that $H_i^{-1} \circ H_i'$ are circuits in W. It follows that $\beta(H_1') = H_2'$ and so the the composition $\alpha_2 \circ \beta \circ \alpha_1^{-1} : A_1 \to A_2$ fixes the vertices and edges of G' as well as G. $\square$

In general, if $\mathcal{A}$ is a collection of thin chamber systems of type M isometrically embedded in chamber system C of type M so that (9.9.1), (9.9.2), and (9.9.3) hold, we say that $\mathcal{A}$ *is a system of apartments* for C.

Theorem 9.9.2 *Any building possesses a system of apartments.*

9.10 Appendix to Chapter 9: Spherical Buildings and (B, N)-Pairs

9.10.1 Tits Systems

It had been recognized that many classical groups are generated by two subgroups. The first is the stabilizer of a chamber of C: this is traditionally called the *Borel subgroup* and is denoted B for that reason. The other group is the stabilizer of an apartment A; this group is traditionally denoted N.[28] Of course at the very beginning one didn't have a clear notion of chamber system or what an apartment was. One knew from examples in finite rank that B was a maximal group of upper triangular matrices and that N acted on some sort of spanning frame to be realized as a monimial group. It took a while to recognize the axioms these groups obeyed.[29]

What are they?

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[28] N seems not to have been named after any person, however obscure. Perhaps it was for "normalizer," for in practice N is the normalizer of $B \cap N$.

[29] Of course Tits did it all. But others were thinking about it. The infrequently-quoted paper "*Geometric ABA groups*" by Jack McCaughlin and Don Higman probably anticipated this development to a certain extent, even if only by a few months [75].

9.10.2 (B, N)-Pairs and Tits Systems

A *Tits system* is a quadruple (G, B, N, S) subject to these axioms:

(T1) G is a group generated by two subgroups B and N. Moreover, $B \cap N$ is a normal subgroup of N.
(T2) The factor group $W = N/(B \cap N)$ has a distinguished system S of generating involutions such that:

(a) (W, S) is a Coxeter system, that is S is a generating set of involutions, subject only to the relations given by the Coxeter matrix $M = (m_{ij})$.
(b) Moreover, for every involution $s \in S$ and element $w \in W$,

$$sBw \subseteq BwB \cup BswB.$$

(c) Finally,

$$sBs \neq B \text{ for any } s \in S.$$

Note that if n and n' are two elements of N which are congruent mod $N \cap B$, so $w = (B \cap N)n = (B \cap N)n' \in W$, then $Bn = Bn'$ which can unambiguously be written as Bw. This slight abuse of notation is convenient for expressing the above axiom (T2)(b).

What is the significance of the axiom (T2)(b)? Suppose W_J is the subgroup of the Coxeter group generated by the involutions $\{r_i | I \in J\}$. Let N_J be the preimage of this group in N — that is $W_J = N_J/B \cap N$. The significance is manifest in the following.

Lemma 9.10.1 *BN_JB is a subgroup of G. In particular $G = BNB$. Conversely, if H is any subgroup containing B, and $B \neq 1$, then $H = BN_JB$ for some subset J of the set I indexing the involutions S of W.*

The first statement is an elementary consequence of the relation given in the second part of axiom (T2). A simple proof of the second statement is given in Ronan's book [103], pp 59–60.

These subgroups containing B are called *parabolic subgroups*. Note that when $|S| = |I| = k$ is finite, there are exactly 2^k parabolic subgroups.

Since each parabolic subgroup is determined by a subset J of I, we write $P_J := BN_JB$ and write P_s for $P_{\{i\}}$, when i indexes involution $s \in S$. The poset of parabolic subgroups is easy to describe, for the following reason.

Lemma 9.10.2 *$P_J \cap P_K = P_{(J \cap K)}$ for any subsets J and K of I.*

Of course B is at the bottom on this poset, and the subgroups $P_s = B \cup BsB$, where $s \in S$, are the *minimal parabolic subgroups* (actually minimal among those properly containing B).

Theorem 9.10.3 *The chamber system $C(G, B; \{P_s | s \in S\})$ is a building.*

Proof This is a direct consequence of the fact that the N-orbit BN is an apartment whose conjugates under G form a system of apartments. (There are other proofs. For example, the W-valued metric ∂ can be directly defined from the (B, N)-pair axioms.) □

Theorem 9.10.4 *Suppose C is a building of type M and let $W := W(M)$. Assume the following:*

(B1) *The automorphism group of C transitively permutes chambers, and the stabilizer B of a chamber c is transitive on all chambers d for which $\delta(c, d)$ has a fixed value $r \in W$.*
(B2) *The stabilizer N of an apartment A induces the full Coxeter group $W(M)$ on that apartment.*

Then $G := Aut(C)$ forms a Tits system (G, B, N, S). (S, of course is the canonical set of generating involutions for $W(M)$.)

9.10.3 Sphericity

Recall that a *spherical building* is simply a chamber system of type M with these properties:

1. All corank one residues are strongly gated (among other equivalent conditions).
2. The matrix M produces a finite Coxeter group $W(M)$.

We have also seen that when $W(M)$ is finite, the connected diagrams are as given in Fig. 9.11. Let us suppose from here on in that we have a building whose type matrix M defines a spherical irreducible Coxeter system.

What happens next is a quite remarkable discovery of J. Tits – a transformation of a purely geometric property into a group-theoretic one.

Let C be the chamber system of a spherical building of type M. Clearly the Weyl group is finitely generated and so apartments exist. So the covering properties of apartments show us that the diameter of this chamber system (as a graph) is bounded by the diameter of an apartment. (Apartments cover all distance-paired vertices, and are isometrically embedded.) That means that for every chamber c, there exists a chamber c' which lives as far as possible from c – that is, as opposite chambers of a finite apartment. The opposite relation is quite strong. If c is opposite c' and d is adjacent to c, then there is a unique vertex d' adjacent to c' which is opposite d. If the rank is at least three, it even works for triangles. So that means that there is a bijection from the neighbors of c to the neighbors of c' and it is not difficult to see that this mapping preserves the edge labels.

So one has an isomorphism $N(c) \to N(c')$ of the neighborhood of c to that of its opposite vertex c'. But of course c' possesses many neighbors c'' which are also opposite c (in fact, except for a small rank pathology, all but one of its neighbors has this property). That means we can locally map a neighborhood $N(c')$ to $N(c'')$

when c' is adjacent to c''. It is easy to see that this consistent neighborhood mapping extends to triangles, and so the thing extends globally to automorphisms of C.[30]

I hope I have led the reader to believe that any spherical building of rank at least three automatically possesses a rich group of automorphisms. In fact it is so rich that the two conditions of Theorem 9.10.4 hold, and a (B, N)-pair ensues. Thus the following arises.

Theorem 9.10.5 *Suppose C is a chamber system of spherical type M with thick panels. Call its automorphism group G.*

1. *The stabilizer B of a chamber c and the stabilizer N of an apartment, form a Tits system (G, B, N, S).*
2. *If $B \neq 1$, the diagram $D(M)$ belongs to one of the diagrams listed in Fig. 9.11, excluding H_3 and H_4 and $I(k)$ for $k = 5, 7$, or $k \geq 8$.*
3. *Let $n = |S|$ be the finite rank of the Coxeter matrix M. Then there are exactly 2^n parabolic subgroups P containing B. In particular there is a system of maximal parabolic subgroups $P_1, \ldots, P_n$ where each $P_i = P_{J-\{i\}}$.*
4. *In the chamber system $C = C(G, B; M_1, \ldots, M_n)$, where the M_i are the minimal parabolic subgroups, the double cosets BgM_i and BgP_i are the residues of type i (and rank one), and cotype i (and corank one), respectively.*
5. *The chamber system C is residually connected and so is derived as $\mathbf{C}(\mathbf{\Gamma}(C))$ from the geometry $\Gamma := \mathbf{\Gamma}(C)$.*
6. *The building geometry Γ is the coset geometry: $\Gamma = \Gamma(G; P_1, \ldots, P_n)$, where the P_i are the maximal parabolic subgroups, as above.*

A Final Comment

It is always good to understand what a theory does. As we shall see in the next chapters of this book, the theory of buildings allows point-line characterizations on a level far beyond projective spaces and polar spaces. But local characterizations suffer the occupational hazard that any global universal object can be folded up a little bit by a homomorphism that does not distort things locally. That means one must be prepared to accept the conclusion "X is a homomorphic image of a universal object." If the universal objects have been classified, at least that is the best one can expect for a classification.

But Tits' theory of buildings actually introduced far more. If we start over again, one sees that Coxeter's classification of the finite "Coxeter groups" was really a

[30] Some authors (for example Ziechang [150]) call this the "reduction theorem" because a geometric problem is reduced to a group-theoretic one. One might see it in the other direction: a rare priveledge to "go upstairs" to group theory. In fact Professor Ziechang has shown that this phenomenon of producing groups from opposite local isomorphisms has a general life in association schemes from which viewpoint the building arguments are "ad hoc" – that is, they depend on enough special properties of buildings that they are not directly transportable to the world of association schemes. More general arguments must be supplied – and these are described in his great book.

way of saying that certain generators and relations actually determinine the group. Another way of viewing Tits' accomplishment is that it provided an extension of the generator-relations of Coxeter to many classical groups — after all, the (B,N)-pair axioms are nothing less than generator-relation specifications. This discovery received its codification for finite group theorists in the famous Curtis–Tits theorem – a key part of the classification of finite simple groups (see the review of Ron Solomon in a recent issue of the *Bulletin of the American Math Soc* [120]). And then, finally, there is the still unexplored logical reason why these axioms work, and here we enter the realm of mathematical logic.

Aside from Tits' classification of the spherical buildings, few theorems of the twentieth century gave so much focus to so many different fields of mathematics.

9.11 Exercises for Chapter 9

9.11.1 Exercises on Chamber Systems

9.1 Prove both parts of Lemma 9.2.1. (The second part utilizes the fact that any corank one residue of C_K, when viewed as collection of chambers of C, is a corank one residue of C with cotype in K.)

9.2 Suppose C is a set. Let $\{\pi_i | i \in I\}$ be a collection of partitions of C indexed by I. The reader is asked to note the slight ambiguity involved in the previous sentence. It could simply mean (1) that there is a mapping $t : I \to \Pi(C)$, where $\Pi(C)$ denotes the lattice of all partitions of C. (2) On the other hand, we may view the collection $P \subseteq \Pi(C)$ of partitions as already existing, so that the indexing is a bijection $\iota : P \to I$. Under this view the mapping $(\iota)^{-1} := t : I \to \Pi(C)$ given in (1) is injective. So, when we say "let $\{\pi_i | i \in I\}$ be a collection of partitions of C indexed by I" this may allow π_i and π_j to be the same partition even if $i \neq j$ (interpretation (1)) or it might means that π_i and π_j are assumed distinct when i and j are distinct (interpretation (2)).

Under either interpretation, given $\{\pi_i | i \in I\}$, say that two chambers c_1 and c_2 of C form an *edge* if and only if there exists at least one partition π_i in the collection for which c_1 and c_2 belong to the same component of π_i – that is π_i does not separate the chambers. In that case let $\lambda(c_1, c_2)$ be the set of all indices i for which π_i does not separate c_1 and c_2. Let E be the set of all edges, so that λ is now a mapping $E \to 2^C - \emptyset$, taking the edges to non-empty subsets of I.

1. Under interpetation (1), show that (C, E, λ) is an abstract chamber system over I as defined on p. 294. [Check the triangle axiom.]
2. Given an abstract chamber system over I as defined on p. 294, the relation of being equal or i-adjacent is an equivalence relation $\sim_i$, as already remarked. Let π_i be the partition of all chambers C into the $\sim_i$-equivalence classes (or panels of type i). Then $\{\pi_i | i \in I\}$ is a collection of partitions of C. Show by example that it is possible for two relations $\sim_i$ and $\sim_j$ to be the

same equivalence relation, so that $\pi_i = \pi_j$. Conclude that *an equivalent definition of an abstract chamber system is the hypothesis of a collection of partitions of a set C indexed by I, under interpretation (1).*

3. Conclude that a collection of partitions indexed by I under interpretation (2) is an abstract chamber system with this property: (sep) *Given any two distinct types i and j there exists a pair of distinct chambers (x, y) such that either (a) x is i-adjacent but not j-adjacent to y or (b) x is j-adjacent but not i-adjacent to y.*

9.3 Let $A = \langle \sigma, \tau \rangle$, a subgroup of the automorphism group of the chamber system C of Example 5. Show that the canonical homomorphic image C/A is isomorphic to the chamber system of Example 3.

9.4 Display the tripartite 18-vertex graph of $\mathbf{\Gamma}(C)$ where C is the "shaved cube" of Example 4.

9.5 Let $C = (G, B; \mathcal{H})$ be a coset chamber system. As usual, $\mathcal{H}$ is a collection of subgroups of G containing B whose (not necessarily distinct) members are indexed by I. In practice, one often assumes for coset chamber systems that $\mathcal{H}$ is an antichain in the poset of subgroups of G containing B. This condition is not really necessary. It is only there to avoid having i-adjacency imply j-adjacency, for distinct i and j – that is, having partition π_i refine partition π_j in the formulation of Exercise 1. The condition implies (sep). One still gets an abstract chamber system without this condition on $\mathcal{H}$.

1. Now choose any subset J of the set I indexing the subgroups in $\mathcal{H}$ and let H_J be the subgroup of G generated by the subgroups $\{H_j | j \in J\}$. Show that any residue of C of type J is a coset $H_J g$ (regarded as a collection of right cosets of B so that it is a subset of $C = G/B$, the cosets of B). [One must show that this chamber subsystem is connected.]
2. Show that in the coset chamber system C, it is possible for two chambers to be both i- and j-adjacent for distinct i and j. What does this imply about H_i and H_j?
3. Show right multiplication of all cosets by an element $g \in G$ induces an automorphism of C. Thus we have a morphism $G \to \mathrm{Aut}(C)$. Show that this morphism is injective if and only if B contains no non-identity normal subgroups of G.
4. Let $f : G \to K$ be a surjective homomorphism of groups. Show that f induces a morphism of chamber systems

$$c(f) : (G, B; \{H \in \mathcal{H}\}) \to (K, f(B); \{f(H) | H \in \mathcal{H}\}).$$

[Hint: The mapping $c(f)$ takes coset Bg to $f(Bg) = f(B)f(g)$. Show that if $Bg \cup Bh \subseteq H_i h \in \mathcal{H}$ then $f(gh^{-1}) \in f(B)$ (equivalently,

$f(B)f(g) = f(B)f(h))$ or $f(B)f(g) \cup f(B)f(h) \subseteq f(H_i)f(h)$ – that is, $c(f)$ preserves i-adjacency on edges whose image is an edge.]

5. Suppose M is a subgroup of G. Show that there is a chamber-injective chamber system morphism

$$\sigma_M : (M, M \cap B; \{H_i \cap M | H_i \in \mathcal{H}\}) \to (G, B; \mathcal{H}).$$

[Let σ_M send $(M \cap B)m$ to Bm for each $m \in M$. Show that i-adjacency is preserved by σ_M on edges whose images are edges.]

6. As in part 1 of this exercise, let $H_J = \langle H_j | j \in J \rangle$. Suppose K is a proper subset of I and set $J = I - K$. We can form a chamber system over K whose chambers are the cosets G/H_J. Two distinct cosets $H_J g$ and $H_J h$ are k-adjacent, for $k \in K$, if and only if $H_J g \cup H_J h \subseteq H_{J\cup\{k\}} g$. Show that this chamber system is the coset chamber system $(G, H_J; \{H_{J\cup\{k\}} | k \in K\})$. Show that it is the truncation C_K of type K of the chamber system $C = (G, B; \{H_i | i \in I\})$.

9.11.2 Exercises on Residual Connectedness

9.6 Show that the chamber system C of Example 1 is not residually connected, while its associated geometry $\mathbf{\Gamma}(C)$ is residually connected.

9.7 Show that the chamber systems of Examples 2, 4, and 5 are residually connected, while the chamber system of Example 3 is not residually connected.

9.8 Prove Lemma 9.2.3.

9.9 Formalize the argument for Corollary 9.2.8.

9.11.3 A Few Exercises on Gatedness

9.10 Prove Lemma 9.3.1. Also prove Lemma 9.4.4.

9.11 A subgraph (X, E') of a graph (V, E) is said to be *gated* in (V, E) if, for every vertex $v \in V$, there is a vertex $g(v) \in X$ such that for every vertex $x \in X$,

$$d(v, x) = d(v, g(v)) + d(g(v), x).$$

(Note that, unlike strong gatedness, the second summand on the right involved global distance, not the distance in the subgraph (X, E'). This concept was first studied by A. Dress and R. Scharlau [59].) Show that every residue of the rank three chamber system of the following example is gated in the full chamber system.

Example 10 Let m be an odd integer greater than two. Consider this vertex set $V := \{x_0, \dots, x_{2m-1}\}$. Writing $x_{2m} = x_0$ and reading subscripts modulo $2m$ where necessary, we define (x_{2i}, x_{2i+1}) to be an edge labelled "1"; we define (x_{2i-1}, x_{2i}) to be an edge labelled "2"; and we define $(x_i, x_{i+m}$ to be an edge labelled "3" – for all i. Then clearly $(x_0, x_1, \dots, x_{2m-1}, x_{2m} = x_0)$ is a (pointed) circular gallery of type $1212\cdots 2$ of length $2m$. The antipodal pairs of vertices of this circuit form the panels of type "3." Since m is odd, all rank two residues involve all of the chambers.

9.11.4 Exercises on Generalized Polygons

9.12 Prove Theorem 9.4.12.

9.13 Consider the geometry $(\mathcal{P}, \mathcal{L})$ where

$$\mathcal{P} := \{p, q, a_i, b_i | i = 1, \dots, t+1\},$$

and

$$\mathcal{L} = \{(p, a_i), (q, b_i), (a_i, b_i) | i = 1, \dots, t+1\},$$

a collection of thin lines. Show that $(\mathcal{P}, \mathcal{L})$ is a generalized hexagon.

9.14 Suppose $\mathcal{P} = \{p_i, a_i, b_i, q_i | i = 1, \dots, s+1\}$. Set $P := \{p_1, \dots p_{s+1}\}$ and $Q := \{q_1, \dots, q_{s+1}\}$. Let L be the two thick lines P and Q together with the collection of thin lines

$$\{(p_i, a_i), (q_i, b_i), (a_i, b_i) | i = 1, \dots, s+1\}.$$

1. Show that $(\mathcal{P}, \mathcal{L})$ is a generalized octagon with each point on just two lines.
2. Show that $((\mathcal{P}, \mathcal{L}))$ is $2\Gamma^*$ where Γ^* is the 2-by-3 grid.

9.15 The Feit–Higman theorem shows that the only generalized 5-gon of order (s, t) is the ordinary pentagon where $s = t = 1$.

1. If, hypothetically, there were a generalized 5-gon of order (s, s), $s > 1$, show that there are three relations between a point and a line and describe these relations.
2. The student might speculate upon whether there could be a generalized 5-gon with infinitely many points on each line and infinitely many lines on each point.

9.11.5 An Interplay of Examples and Exercises on Diagram Geometries

9.16 Show that any geometry Γ belonging to the finite rank diagram below (Fig. 9.17) is a linear space when truncated to $(\mathcal{P}, \mathcal{L})$ (the first two nodes from the left), and that each remaining node is a subspace with respect this linear space. [Hint: One could first do this for rank three in order to truncate an $L - L$-diagram to a single L and use induction. On the other hand it can also be done directly.]

9.17 Using the previous exercise, show that any geometry belonging to the diagram below (Fig. 9.18) is a projective space when truncated to types $(\mathcal{P}, \mathcal{L})$ represented by the first and second nodes from the left.

This a good time to introduce a few warnings. The main one is not to expect too much from a diagram. The tendency is to use a nice simple diagram, such as A_n (Fig. 9.19) as a means of getting the concept across. But actually this example is quite atypical. For example, if we designate the leftmost two nodes as points and lines, respectively, we obtain the standard projective space $(\mathcal{P}, \mathcal{L})$. It is then a fact that all objects of other types are *subspaces* when realized by their point-line shadows (i.e., their residues truncated to points and lines). The same occurs when we assign some internal node (say the one labeled k) to be the points, and let the flags whose type is represented by adjacent nodes (in this case those of type $(k-1, k+1)$) be the lines. This separation of the point-node from the rest of the diagram by the type of the flags destined to be lines gives us the desirable axiom as follows.

(A) *Any object not a point or line, which is incident with a line L, is in fact incident with every point that is incident with that line L.*

We can hardly hope objects to be subspaces without such an axiom. Indeed, something like that works for diagrams A_n, D_n, E_6, E_7, and E_8. But these are not typical. Consider the following highly pathological example.

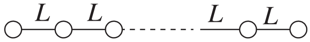

Fig. 9.17 A string diagram with linear-space residues

Fig. 9.18 Another diagram with linear-space residues

Fig. 9.19 The A_n diagram

Example 11 (The flat Neumaier geometry) Consider first the collection $\mathcal{L}$ of all 35 3-subsets of a 7-set $\mathcal{P} = \{0, \ldots, 6\}$. This is acted on by the group $G = \mathrm{Sym}(7)$. Now consider the Fano plane $F = (\mathcal{P}, \mathcal{L}_F)$ whose seven lines are the set $\mathcal{L}_F$ of additive translates mod 7 of the line $\{1, 2, 4\}$. As we know, its stabilizer in G is the simple group H of order 168 and index 30 in G. G acts imprimitively on the cosets G/H in two blocks which are orbits of length 15 under the alternating group $A := \mathrm{Alt}(7)$. Let π_0 and π_1 be these two orbits. If F and F' are planes in different orbits, then these two planes share three lines or no lines at all. On the other hand, if F and F' are planes within a common A-orbit, they always share exactly one line.

Now form the geometry $N := (\mathcal{P}, \mathcal{L}, \pi_0)$, of 7 points, 35 lines, and 15 planes from one class. Incidence has been defined by the definition of π_0. Let us examine the residue of a point – say, the point $p = 0$. The remaining points form a 6-set $\Omega = \{1, \ldots, 6\}$. The lines incident with p are thus identified with the 15 2-subsets of Ω. Any plane of π_0 produces three lines on p which induce a partition of Ω into three 2-sets. There are 15 such partitions. On, the other hand, any two distinct planes of $\mathcal{L}_0$ share just 1 line, and so these 15 planes each induce a different partition of Ω into 2-subsets. So there is a bijection between $2-2-2$-partitions of Ω and planes on p.

This is enough information to see that the residue of a point in N is faithfully modelled by the rank-two geometry of 2-subsets and $2-2-2$-partitions of a 6-set Ω which, as we know, is the generalized quadrangle of order $(2, 2)$. Thus N belongs to the diagram below (Fig. 9.20) which we call the C_3 diagram.

The message here is that the 15 planes of this geometry are *not* subspaces of the truncation to points and lines.

9.18 The reader might consider what happens with the truncation $(\mathcal{P}', \mathcal{L}') = (\pi_0, \mathcal{L})$ to planes and lines. Here we have 15 "points", 35 "lines," and 7 "quadrangles." Are the quadrangles subspaces of $(\mathcal{P}', \mathcal{L}')$?

9.19 Show that the rank three chamber system of Example 10 (p. 394) belongs to a diagram consisting of three nodes with the edge connecting any two of them labelled by "(m)", indicating a generalized m-gon.

Example 12 Let $K_i, i = 1, 2, 3$, be three disjoint copies of the complete tripartite graph $K_{2,2,2}$, each regarded as a geometry over the typeset $\{2, 3, 4\}$. The geometry Γ of this example is over the type-set $\{1, 2, 3, 4\}$. Its truncation to $\{2, 3, 4\}$ is the union $K_1 \cup K_2 \cup K_3$ – a disconnected graph. The set of objects of type 1 is $\{x_1, x_2, x_3\}$. If $\{i, j, k\} = \{1, 2, 3\}$, the object x_i is incident with every object of $K_j \cup K_k$. There are no further incidences.

Fig. 9.20 The C_3 diagram

9.20 In the geometry Γ of Example 12, show the following:

1. Every flag of Γ lies in a chamber flag.
2. Every residue of rank two of Γ is connected.
3. Γ is not residually connected.

9.11.6 Exercises Concerning Chamber Systems of Type M

9.21 Show that in the chamber system $C = C(G, 1 : \langle s_1 \rangle, \ldots, \langle s_n \rangle)$ – which we have been writing as $C(G, 1; S)$ – a gallery is of reduced type if and only if it is a *minimal* gallery – i.e., a shortest gallery (geodesic path) connecting its initial and terminal chambers. [This is not true of general chamber systems, and occurs here because G is a presented group.]

9.22 1. Show that in C, any circuit has even length. [Hint: Any circuit can be deformed to the trivial circuit by some chain of expansions, contractions, and elementary $\mathcal{C}_2$-homotopies, which do not disturb the length parity.]
2. Conclude that C is a bipartite graph.
3. Show that for any bipartite graph and edge $e = (x, y)$, for any vertex v, $d(v, x) = d(v, y) \pm 1$. In particular this must hold for C. So the edge e determines a partition of the vertices of C into two sets: $D^-(e) = \{v | d(v, x) < d(v, y)\}$ and $D^+(e) = \{v | d(v, x) > d(v, y)\}$. These two sets are called *opposite roots* and exist for any bipartite graph.
4. Observe that the partition $C = D^-(e) + D^+(e)$ does not in general match the partition $C = C_1 + C_2$ into two cocliques which defines the bipartness of C.
5. Suppose $e = (x, y)$ is an edge in a bipartite graph C as in the previous two parts of this exercise. Give an example of a bipartite graph C and a"bridging edge" $f = (u, v)$ where $(u, v) \in D^-(e) \times D^+(e)$ for which the partition $D^-(e) + D^+(e)$ does not coincide with the partition $D^-(f) + D^+(f)$. [Compare this with Theorem 9.6.10 which is for bipartite graphs arising from Coxeter chamber systems. In your example, make sure all circuits have even length. This can be done with five vertices.]

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