

Chapter 2

Wave Turbulence as a Part of General Turbulence Theory

2.1 Basic Facts about Hydrodynamic Turbulence

The name “Wave Turbulence” may appear paradoxical to people unfamiliar with the subject, because traditionally turbulence is associated mostly with vortices, and the waves are only secondary. We will try to explain that, on the contrary, a system of weakly nonlinear waves behaves very similar to the classical turbulence system, even in absence of vortices. For this, let us first briefly review the classical results in turbulence. For a more detailed discussion of the hydrodynamic turbulence see book [1].

Even though the fluid equations are deterministic, turbulence is a chaotic motion, and it is difficult to obtain a complete detailed information about the fluid flow. Indeed, individual paths of the fluid particles in a turbulence are sensitive to slight changes in initial conditions and imperfections in repeated experiments. However, averaged quantities in turbulence are well defined and can be studied, which is the subject of the *statistical theory* of Turbulence.

As with any statistical system, one might be tempted to approach turbulence using thermodynamics. Unfortunately, a simple-minded straightforward application of the classic thermodynamic theory would predict that turbulent systems have extremely high temperatures (energy per degree of freedom), which is absurd. Instead, rather than through temperature or chemical potential, turbulence is better described by the *energy flux through scales*. If such a flux is *local*, i.e. takes place as a sequence of transfers between eddies of similar sizes, it is called the *energy cascade*. However, as we will see later, there is some space for the thermodynamical solutions too.

The energy spectrum of turbulence is a mean quantity and is one of the main objects studied in turbulence theory. We write this spectrum as follows

$$E^{(3D)}(\mathbf{k}) = \frac{1}{2} \int_{\mathbb{R}^3} \langle \mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x} + \mathbf{r}) \rangle e^{-i\mathbf{k} \cdot \mathbf{r}} \frac{d\mathbf{r}}{(2\pi)^3}, \quad (2.1)$$

and we define *homogeneous turbulence* as turbulence for which the energy spectrum is independent of \mathbf{x} . We also define *isotropic turbulence* as a system where the energy spectrum is independent of the direction of the wave number \mathbf{k} , i.e. $E^{(3D)}(\mathbf{k}) = E^{(3D)}(k)$ with $k = |\mathbf{k}|$. The angular bracket denotes a suitable average, ensemble, volume or space (here we will omit discussion of relevance of different averaging procedures).

The super-script (3D) refers to the fact that $E^{(3D)}$ represents the kinetic energy density in the 3D \mathbf{k} -space, i.e.

$$\frac{1}{2}\langle u^2 \rangle = \int_{\mathbb{R}^3} E^{(3D)}(\mathbf{k}) d\mathbf{k}. \quad (2.2)$$

On the other hand, for isotropic spectra the same information is contained in a 1D spectrum which is obtained from $E^{(3D)}$ by integration over the unit sphere in the 3D \mathbf{k} -space. This gives

$$E^{(1D)}(k) = 4\pi k^2 E^{(3D)}(k)$$

so that $E^{(1D)}(k)$ represents the energy density over $k = |\mathbf{k}|$,

$$\frac{1}{2}\langle u^2 \rangle = \int_0^{+\infty} E^{(1D)}(k) dk. \quad (2.3)$$

2.1.1 Richardson Cascade

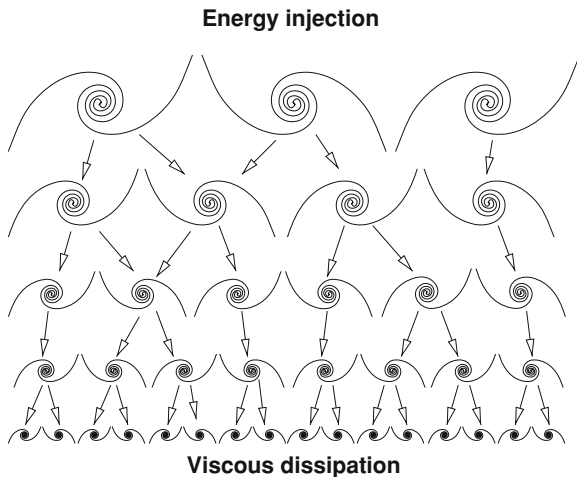
*So, nat'ralists observe, a flea
Hath smaller fleas that on him prey;
And these have smaller yet to bite 'em,
And so proceed ad infinitum.*

This is a verse from the poem “On Poetry: a Rhapsody” written by Jonathan Swift in 1733. It was not about insects but, rather, it was meant to be a metaphor with a rather pessimistic conclusion in the lines that immediately follow:

*Thus every poet, in his kind,
Is bit by him that comes behind:
Who, though too little to be seen,
Can teaze, and gall, and give the spleen.*

Swift liked “self-similar” examples, suffices to recall the “Gulliver Travels” with its little people and big people, and it is precisely the self-similarity of the “flea” verse that fascinated Lewis Richardson who in 1922 came up with his own version for turbulence:

Fig. 2.1 Richardson's energy cascade



*Big whorls have little whorls,
Which feed on their velocity,
And little whorls have lesser whorls,
And so on to viscosity.*

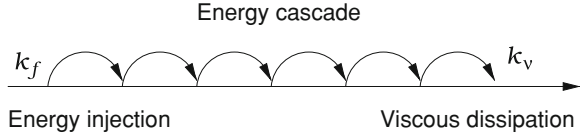
Richardson pictured that small vortices obtain their energy from break-ups of larger ones, only to find themselves breaking to even smaller ones, and so on in a self-similar way, see Fig. 2.1. The largest vortices in the Richardson cascade picture obtain their energy from an external forcing (e.g. a mechanical forcing or an instability mechanism to release the internal energy into the kinetic one) and the smallest vortices are dissipated by viscosity [2]. The total rate of the energy injection at large scales is equal on average to the energy dissipation rate at small scales, so that a statistically steady turbulent state forms.

The Richardson cascade is best represented in Fourier \mathbf{k} -space, see Fig. 2.2. Here, the lengthscale is $1/k$ (where $k = |\mathbf{k}|$), so that we have our turbulence source at small k_f and the energy cascade is in the positive k -direction towards the dissipation scale at large k_v .

2.1.2 Kolmogorov–Obukhov Theory

In 1941, Kolmogorov [3, 4] and Obukhov [5] introduced the universality hypothesis for the inertial range, i.e. for $k_f \ll k \ll k_v$. The idea is that far away from the source and the sink turbulence properties only depend on the energy cascade rate (equal to the energy dissipation rate in the steady state), and not on details of the forcing or the dissipation of energy. This is because the Richardson

Fig. 2.2 Energy cascade in the k -space



cascade is assumed to be *local*, i.e. transferring energy in many steps each involving transfers among eddies with similar sizes only.

The Kolmogorov–Obukhov idea leads to a dimensional argument, where the energy dissipation rate ε and wavenumber k are assumed to be the only relevant dimensional quantities in the inertial range $k_f \ll k \ll k_v$. In particular, the viscosity ν is irrelevant.

We find for the dimensions

$$\begin{aligned} [\varepsilon] &= \left[\frac{\mathbf{u}^2}{t} \right] = \frac{l^2}{t^3}, \\ [E^{(1D)}] &= \left[\frac{\mathbf{u}^2}{k} \right] = \frac{l^3}{t^2}. \end{aligned} \quad (2.4)$$

With these in mind, we see that the only combination of ε and k that results in the correct dimension of $E^{(1D)}$ is

$$E^{(1D)} = C\varepsilon^{2/3}k^{-5/3}, \quad (2.5)$$

which is known as the Kolmogorov–Obukhov spectrum, often also referred to as Kolmogorov spectrum, or K41 [3–5]. Here constant $C \sim 1.6$ is called the Kolmogorov constant. In spite of such a simple dimensional derivation, this spectrum represents the strongest result in turbulence which is supported by numerous experimental confirmations in laboratory and in field observations. However, there have also been observed some deviations from the Kolmogorov scaling, which are small for the spectrum (and other second-order moments) but become significant for the high-order moments of the velocity field. These deviations are associated with the turbulence intermittency phenomenon, which has been attracting a considerable attention in the modern turbulence research [1].

The Richardson cascade picture and the Kolmogorov–Obukhov spectrum are of fundamental importance for the WT theory, because similar cascade states are typical for WT, as we will see later. Moreover, deviations from the Kolmogorov-type scalings associated with WT intermittency are also interesting and they will be discussed in the present book.

2.1.3 2D Turbulence

It is often possible to find more than one positive conserved quantity in WT systems. This will modify the cascade picture presented above. We shall show that

the behavior of these systems is similar to 2D turbulence, where two invariants are transferred in the opposite directions in the scale space, as discovered by Fjørtoft [6]. Kraichnan found Kolmogorov-like spectra for each of the two cascades [9]. Thus, let us review the results concerning 2D turbulence.

Let us consider a 2D flow. For instance, horizontal motions in the planetary atmosphere stretch over far larger scales than the atmosphere height, and this system can therefore be approximated by a 2D flow. Rapidly rotating fluids provide another example (via Taylor–Proudman two-dimensionalization). Charged particles in a very strong external magnetic field give yet another example [7].

2.1.3.1 Energy and Enstrophy

From a basic Fluid Dynamics course we learn that the 2D incompressible ideal flow conserves two quadratic quantities, energy and enstrophy. Naturally, for infinite homogeneous turbulence these two quantities are infinite, and it only makes sense to talk about their spatial density. We can write both in terms of the energy spectrum $E^{(1D)}(k)$:

$$E = \frac{\text{Energy}}{\text{Area}} = \frac{1}{2} \langle \mathbf{u}^2 \rangle = \int_0^\infty E^{(1D)}(k) dk, \quad (2.6)$$

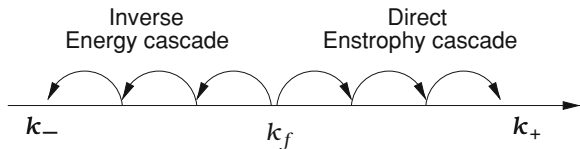
$$\Omega = \frac{\text{Enstrophy}}{\text{Area}} = \frac{1}{2} \langle \omega^2 \rangle = \int_0^\infty k^2 E^{(1D)}(k) dk \quad (2.7)$$

where we have used the relation between the velocity and the vorticity in Fourier space, $\hat{\omega}_k = i\mathbf{k} \times \hat{\mathbf{u}}_k$.

2.1.3.2 Dual Cascade Behavior

Let us introduce the enstrophy production rate η . In statistically stationary turbulence, the dissipation rate is equal to the production rate. Let us consider turbulence excited near wavenumber k_f and dissipated at very small wavenumbers $k_- \ll k_f$ and at very large wavenumbers $k_+ \gg k_f$, and let there be neither forcing nor dissipation at wavenumbers such that $k_- < k < k_f$ or $k_f < k < k_+$, see Fig. 2.3. These intervals are called the *inverse and the direct cascade inertial ranges*

Fig. 2.3 2D turbulence: dual cascade behavior in the k -space



respectively. The relation between η and ε follows from (2.6) and (2.7) which dictate $\eta \sim k_f^2 \varepsilon$.

Let us use an *ad absurdum* argument to find the directions of the energy and the enstrophy cascades. This will constitute so-called Fjørtoft argument [6].

If energy was dissipated at k_+ at the rate comparable to the injection rate ε then enstrophy would be dissipated at a rate $\sim k_+^2 \varepsilon \gg k_f^2 \varepsilon \sim \eta$, which is a contradiction, because in a steady state it is impossible to dissipate enstrophy at a rate which is higher than the injection rate. Therefore, energy must be dissipated at k_- , and we call the energy cascade from k_f to k_- the *inverse energy cascade*. In a picture like Richardson's, we would see vortices merging rather than breaking up, i.e. energy is transferred from small to large vortices.

We can use a similar argument for the enstrophy, assuming *ad absurdum* that it is dissipated at $k_- \gg k_f$ at the rate comparable to the production rate η . But this would imply that the energy is dissipated at k_- at the rate $\sim \eta/k_-^2$ which is much greater than the energy production rate $\varepsilon \sim \eta/k_f^2$ which is impossible in stationary turbulence. Therefore, the enstrophy must dissipate at k_+ . We call this transfer of enstrophy from k_f to k_+ the *direct enstrophy cascade*. In a picture like Richardson's, the enstrophy is transferred into long and thin vorticity filaments during the vortex merging process. Such thin filaments also occur when weaker vortices are stretched by stronger ones.

2.1.3.3 Fjørtoft Argument in Terms of Centroids

Note that there exist several versions of the Fjørtoft argument, a couple of which were presented by Fjørtoft himself in his original paper [6], some other variations can be found e.g. in [8]. These different versions vary in degree of rigor from being a mathematical theorem to a less rigorous physical speculation. Here we will present yet another (probably new) way to formulate Fjørtoft in terms of the energy and the enstrophy centroids in the k and the l (the length scale) spaces. This formulation will be rigorous and quite useful for visualizing the directions of transfer of the energy and the enstrophy. In contrast with the above version, this formulation is for a non-dissipative evolving turbulence rather than for a forced/dissipated system. First let us define the centroids.

Definition 2.1 The energy and the enstrophy k -centroids are defined respectively as

$$k_E = \int_0^\infty k E^{(1D)}(k) dk / E, \quad (2.8)$$

$$k_\Omega = \int_0^\infty k^3 E^{(1D)}(k) dk / \Omega, \quad (2.9)$$

and the energy and the enstrophy l -centroids are defined respectively as

$$l_E = \int_0^\infty k^{-1} E^{(1D)}(k) dk / E, \quad (2.10)$$

$$l_\Omega = \int_0^\infty k E^{(1D)}(k) dk / \Omega \equiv k_E E / \Omega, \quad (2.11)$$

Theorem 2.1 *Assuming that the integrals defining $E, \Omega, k_E, k_\Omega, l_E$ and l_Ω converge, the following inequalities hold,*

$$k_E \leq \sqrt{\Omega/E}, \quad (2.12)$$

$$k_\Omega \geq \sqrt{\Omega/E}, \quad (2.13)$$

$$k_E k_\Omega \geq \Omega/E, \quad (2.14)$$

$$l_E \geq \sqrt{E/\Omega}, \quad (2.15)$$

$$l_\Omega \leq \sqrt{E/\Omega}, \quad (2.16)$$

$$l_E l_\Omega \geq E/\Omega. \quad (2.17)$$

Exercise 2.1 *Prove this theorem using Cauchy–Schwartz inequality.*

We see that according to inequalities (2.12) and (2.13), during the system's evolution the energy centroid $k_E(t)$ is bounded from above and the enstrophy centroid $k_\Omega(t)$ is bounded from below (both by the same wavenumber $k = \sqrt{\Omega/E}$), as one would expect from Fjørtoft argument. Further, inequality (2.14) means that if $k_E(t)$ happened to move to small k 's then $k_\Omega(t)$ *must* move to large k 's, that is roughly, *there cannot be inverse cascade of energy without a forward cascade of enstrophy*. Note that there is no complimentary restriction which would oblige $k_E(t)$ to become small when $k_\Omega(t)$ goes large, so the k -centroid part of the Fjørtoft argument is asymmetric, and one has to consider the l -centroids to make it symmetric. Indeed, in addition to conditions (2.15) and (2.16) which are similar to (2.12) and (2.13), we have inequality (2.17) meaning that if $l_\Omega(t)$ happened to move to small l 's then $l_E(t)$ *must* move to large l 's, i.e. *any forward cascade of enstrophy must be accompanied by an inverse cascade of energy*.

Importantly, we do not always have $k_E \sim 1/l_E$ and $k_\Omega \sim 1/l_\Omega$, as illustrated in the following exercise.

Exercise 2.2 *Consider a state with spectrum $E^{(1D)}(k) \sim k^{-5/3}$ for $k_a < k < k_b$ (with $k_b \gg k_a$) and $E^{(1D)}(k) \equiv 0$ outside of this range. Show that for this state $k_E \sim k_b^{1/3} k_a^{2/3}$ and $l_E \sim 1/k_a$ (i.e. $k_E \approx 1/l_E$) and $k_\Omega \sim 1/l_\Omega \sim k_b$.*

2.1.3.4 Spectra of 2D Turbulence

Robert Kraichnan [9] realized that, provided the energy and the enstrophy cascades are local, the energy spectrum in the inverse cascade range will only be determined by the energy flux, whereas in the direct cascade range—by the enstrophy flux. He used a Kolmogorov-style dimensional argument and assumed that ε and k are the only relevant dimensional quantities that can enter the expression for the energy spectrum in the inverse cascade, and for the direct cascade these relevant quantities are respectively η and k .

The dimensional argument for the energy spectrum is basically identical to the one presented above for 3D turbulence, so we have [9]

$$E^{(1D)}(k) = C_\varepsilon \varepsilon^{2/3} k^{-5/3}.$$

Note, however, that the value of the dimensionless constant $C_\varepsilon \sim 6$ is not determined by the dimensional argument and, unsurprisingly, is different from the Kolmogorov constant C for 3D turbulence.

To find the enstrophy cascade spectrum, we note that the dimension for η is

$$[\eta] = [k^2][\varepsilon] = \frac{1}{l^2} \frac{l^2}{t^3} = \frac{1}{t^3}. \quad (2.18)$$

Thus, we find for the spectrum

$$E^{(1D)} = C_\eta \eta^{2/3} k^{-3}, \quad (2.19)$$

which is called the Kraichnan spectrum [9].¹ Here $C_\eta \sim 1.9$ is yet another dimensionless constant. Note a rather large difference ~ 3 between the values for C_ε and C_η . For an explanation of this large difference see paper [11].

The Kraichnan dual-cascade picture presented above was recently confirmed numerically in high-resolution (16384^2) DNS in [12].

2.1.3.5 Yet Another Argument about the Cascade Directions

In Sects. 2.1.3.2 and 2.1.3.3 we already presented two versions of the arguments predicting the energy and the enstrophy cascade directions, and here we will give yet another simple argument.

Let us consider a power-law spectrum $n_k = Ak^{-x}$ (not necessarily a stationary one) and let us construct plausible plots for the energy flux ε and the enstrophy flux η as functions of x , see Fig. 2.4.

We know that $\varepsilon(x)$ will cross zero for both of the exponents corresponding to the thermodynamic states with an energy equipartition, $x = -1$ (point *TE* on

¹ Later [10], Kraichnan found this spectrum to be marginally nonlocal, and remedied it with introducing a log-correction, $E^{(1D)} = C_\eta \eta^{2/3} k^{-3} \ln^{-1/3}(k/k_f)$.

Fig. 2.4 The energy and the enstrophy fluxes as a function of the spectral index x for the 2D turbulence

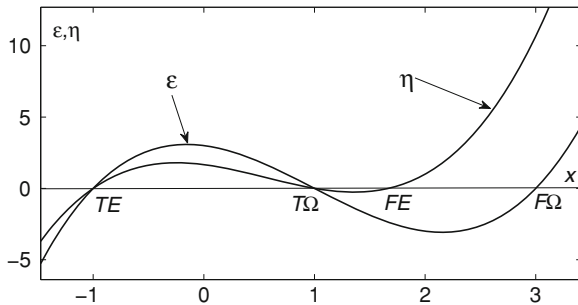


Fig. 2.4), and an enstrophy equipartition, $x = 1$ (point $T\Omega$ on Fig. 2.4). Also, $\varepsilon(x)$ will cross zero at the enstrophy cascade solution, $x = 3$ (point $F\Omega$ on Fig. 2.4).

In turn, $\eta(x)$ will cross zero at both of the thermodynamics solutions (points TE and $T\Omega$ on Fig. 2.4) and on the energy cascade spectrum, $x = 5/3$ (point FE on Fig. 2.4).

It is clear that the fluxes will be both positive if x is very large, see Fig. 2.4. Thus, starting with large x and tracing the plots $\varepsilon(x)$ and $\eta(x)$ toward smaller x and making the necessary zero crossings as described above, we uniquely determine the x -ranges for which these fluxes are positive and negative. In particular, we find that the energy flux ε is negative at point FE (inverse energy cascade), and the enstrophy flux η is positive at point $F\Omega$ (direct enstrophy cascade).

2.2 Placing Wave Turbulence in the Context of General Turbulence

Now we are in the position to discuss similarities and differences between WT and hydrodynamic turbulence and to place WT in the context of the general Turbulence theory. Let us recall the definition of WT which we gave in the beginning of this book:

Wave Turbulence is out-of-equilibrium statistical mechanics of random nonlinear waves.

This definition refers to two features common for all turbulent systems, the *randomness* and the *non-equilibrium character*, as well as to properties specific for WT only, particularly the fact that it is a system of *waves rather than vortices*. Let us discuss these common and the distinct properties in greater detail.

2.2.1 Common Turbulence Properties

Randomness and large deviation from the thermodynamic equilibrium are important ingredients of the above definition which make WT similar to the other

turbulent systems. Randomness is related to (but not completely predetermined by) the fact that there are many wave modes excited in the system, covering a wide range of dynamical length- and time-scales. Thus, the dynamics of these modes is chaotic and should be described by a statistical rather than a deterministic theory.

It is important to understand that the set of waves form an open system, with sources and sinks of energy at different (and often well separated) scales. This statistical system is far from thermodynamical equilibrium. The most typical states for such a system are determined by a flux of energy (or another conserved quantity) through scales rather than temperature or any other thermodynamic potentials. This property is common for all turbulent systems, and the most known example here is Richardson cascade and the Kolmogorov–Obukhov spectrum which corresponds to this statistical state. In WT, the analogs of the Kolmogorov–Obukhov spectrum are Kolmogorov–Zakharov (KZ) spectra. As we will see later, KZ spectra can also be obtained from a dimensional analysis which strengthens their similarity to the Kolmogorov–Obukhov spectrum of hydrodynamic turbulence. It is reasonable to consider the prevalence of the flux/cascade dominated statistical states as a definition of Turbulent systems. In this case, one could mention another non-equilibrium system which qualifies to be called turbulence under this definition: a set of sticky particles coagulating upon collision which also exhibit cascades [13–15]. In this case the cascade takes form of a mass flux in the space of particle sizes.

Finally, some WT systems have several positive conserved quantities and, like 2D turbulence, exhibit a dual cascade behavior. Also like in 2D turbulence, inverse cascades may lead to accumulation of turbulence at the largest available scales, followed by breakdown of the local cascade picture and onset of the nonlocal interaction. We will see later that this happens, in particular, in the nonlinear Schrödinger (NLS) model, where this process corresponds to a non-equilibrium Bose–Einstein condensation (BEC). We will see that during this process WT changes its nature from being a four-wave to a three wave process, with a non-trivial phase transition between these states involving strong turbulence and vortices [16]. Further examples of inverse WT cascades can be found in the water gravity waves (swell effect) and the Rossby/Drift waves (leading to formation of zonal jets). Later in this book, we will discuss these wave systems too.

2.2.2 Distinct Properties of WT

One obvious difference between the hydrodynamic turbulence and WT is that in the former case the basic type of motion is a hydrodynamic vortex whereas in the latter case it is a propagating wave. Note that in many physical examples there are both vortices and waves which co-exist and interact with each other, for example compressible fluids, MHD turbulence, fluids in rotating and stratified media, quantum turbulence. In these cases WT is only a part of the whole turbulent system.

In most of the WT theory, it is further assumed that the waves are weakly nonlinear and dispersive. These assumptions allow a systematic mathematical treatment of WT. Indeed, weakly nonlinear waves can, for short time intervals, be approximated by independent linear waves. Their amplitudes are sufficiently small and time-independent over such scales. At long times however, the wave amplitudes change, but the large difference in scale between this nonlinear evolution and the linear wave period allows one to average over the fast linear times. It should be noted that the WT description is not a perturbation theory, for even slow nonlinear evolution may lead to order-one changes of wave amplitudes.

In WT, the wave dispersion is important, since non-dispersive waves have constant (k -independent) group velocities and the wave packets can strongly affect each other because they “stick together” for an infinite time. However, in exceptional cases it is possible for WT to be applied to non-dispersive systems, e.g. in Alfvén waves where co-propagating waves do not interact, or in 3D sound where divergence of waves propagating in various space directions plays a similar role as the wave dispersion (this is not the case for the sound in lower-dimensional spaces!).

A remarkable distinct feature of weak WT is that the KZ spectra corresponding to the cascade states can be obtained as exact analytical solutions of the WT kinetic equation and not only dimensionally. Further, stability of the solution and locality of interaction can also be examined analytically. This is a luxury which is not available in the theory of hydrodynamic turbulence, and it brings about yet another important role of WT as a testing ground for universal concepts proposed in the context of more complex turbulent systems, and for developing new ideas that could later be applied more broadly in turbulence.

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Wave Turbulence

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