2 Cournot Oligopoly

The Cournot Model

As mentioned in Chap. 1, Cournot's oligopoly model was one of the first mathematical models proposed in the field of economics See Cournot (1838). It addresses the functioning of a market with numerous atomistic demanders versus few relatively large suppliers. This implies that all the suppliers influence market price appreciably, and hence, like monopolists, take account of the demand function of the consumers on the market in order to calculate their best moves. As a rule, demand is a decreasing function of price. In equilibrium demand equals supply, and one can also speak of the inverse demand function which states how market price depends on supply. In the case of Cournot, it is most convenient to speak of this inverse demand function,

$$p = f(Q), \tag{2.1}$$

where p denotes price, Q denotes market supply, and $f'(Q) < 0$.

Cournot takes the quantity of supply for each oligopolist as the proper decision variable, so the price somehow results through a not specified market clearing mechanism. This was later the goal for criticism, as a few big firms would rather set supply prices than quantities supplied; and this would open up for the possibility of cutting out competitors in case the commodity marketed is conceived as homogenous by the consumers.

Supposing there are n suppliers whose individual supplies are denoted q_i, market supply becomes,

$$Q = \sum_{i=1}^{i=n} q_i \ . \tag{2.2}$$

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There is a point in defining residual supply,

$$Q_i = Q - q_i, \tag{2.3}$$

the supply of all the other firms, which is not under the control of the ith firm.

The profits, revenue minus cost for the ith firm now become

$$\Pi_i = f(q_i + Q_i)q_i - C_i(q_i), \tag{2.4}$$

where $C_i(q_i)$ is the cost function. Differentiating partially with respect to q_i, results in the first order condition,

$$f(q_i + Q_i) + f'(q_i + Q_i)q_i = C_i'(q_i). \tag{2.5}$$

Unlike the case of monopoly, the optimum condition does not determine the value of q_i; the solution depends on Q_i, the residual supply by the competitors about which the ith firm has no certain knowledge. It can only have more or less sophisticated expectations, based on past experience, and calculate the best decisions under each such expected Q_i.

The outcome is the reaction function,

$$q_i = \phi_i(Q_i), \tag{2.6}$$

which reduces the optimality condition to an identity, i.e.,

$$f(\phi_i(Q_i) + Q_i) + f'(\phi_i(Q_i) + Q_i)\phi_i(Q_i) \equiv C_i'(\phi_i(Q_i)). \tag{2.7}$$

To find such reaction functions $\phi_i(Q_i)$ in nice closed form would be desirable for any modeller. Unfortunately, there are very few demand and cost functions that allow one to do so. Next follow a few examples where this programme can be accomplished, i.e., the traditional case of a linear demand function, and the case of an iso-elastic, or hyperbola shaped demand function.

Example 1: Linear Demand

Suppose we have the inverse demand function

$$p = a - bQ, \tag{2.8}$$

where a, b are two positive constants. Further, assume the cost functions are

$$C_i = c_i q_i,$$

where c_i are constant marginal, equal to average variable costs.

Then the profit of the ith firm becomes,

$$\Pi_i = \left(a - b\left(q_i + Q_i\right)\right)q_i - c_i q_i = \left(a - c_i - bQ_i\right)q_i - bq_i^{\,2}, \tag{2.9}$$

and the maximum condition,

$$a - c_i - bQ_i = 2bq_i. \tag{2.10}$$

Solving, the reaction function is readily obtained,

$$q_i = \frac{a - c_i}{2b} - \frac{1}{2}Q_i. \tag{2.11}$$

These reaction functions are straight lines with the constant slope $-\frac{1}{2}$. Obviously we must have $a > c_i$, i.e., the maximum price obtainable must exceed the unit cost, otherwise the firm could not obtain any profit. Further, the reaction function would return a negative output q_i, unless

$$Q_i < \frac{a - c_i}{b} \tag{2.12}$$

holds. As this is meaningless, any negative q_i, would be replaced by zero, which means that if the competitors supply too much, then the firm drops out.

To get a complete picture it is better to explicitly check for nonnegativity of profits.[1] Substituting the reaction function in the profit expression one obtains

$$\Pi_i = (a - c_i - bQ_i)\left(\frac{a-c_i}{2b} - \frac{1}{2}Q_i\right) - b\left(\frac{a-c_i}{2b} - \frac{1}{2}Q_i\right)^2, \tag{2.13}$$

which equals

$$\Pi_i = \frac{b}{4}\left(\frac{a-c_i}{b} - Q_i\right)^2. \tag{2.14}$$

It may seem that this expression is always nonnegative, but when the reaction q_i is negative, this is due to the fact that negative costs outweigh negative revenues, which has no factual meaning. Accordingly, considerations of profits add nothing new to the constraint for a positive reaction already stated, so the complete reaction function reads

$$q_i = \begin{cases} \dfrac{a-c_i}{2b} - \dfrac{1}{2}Q_i, & Q_i < \dfrac{a-c_i}{b} \\ 0, & Q_i \geq \dfrac{a-c_i}{b} \end{cases}. \tag{2.15}$$

Note that the resulting reaction function is piecewise linear, i.e., nonlinear, even in this simplest case with a linear demand function and constant unit costs. In setting up these reaction functions, all the steps needed to be considered at setting up reaction functions in further examples were already encountered.

[1] Observe that the non-negativity constraint refers to calculated profits based on the competitors' expected moves. If they turn out wrong, then in a dynamic process the firms may still experience actual negative profits.

Example 2: Isoelastic Demand

Linear demand functions are just simple first approximations. If one assumes individual consumers or groups of consumers to have different linear demand functions, then market demand inevitably results as a broken train of line segments, as proposed by Robinson (1933). Used in oligopoly models it results in the interesting phenomena described by Palander (1936, 1939), but the analysis becomes more messy than is suitable for this simple exemplification, so we defer the analysis of this case to a later chapter.

In stead we now assume that the consumers maximize utility functions of a Cobb–Douglas type, a product of fractional powers of the quantities of different commodities consumed.[2] Without loss of generality such utility functions can be rescaled so that the sum of these power fractions equals unity. Maximizing such utility functions under given budget constraints, then result in income being split in budget shares where the power fractions from the utility function provide the weights. Fixing budget shares of a given income means that quantity demanded and price are reciprocal. See Puu (1991, 2004).

Passing from the individual demand functions to the market, as all individual functions are of the same form results in market demand as well being reciprocal to price, the numerator just being the sum of all the individual budget shares.

As the measurement unit for the commodity is optional, one can choose it in such a way as to normalize the numerator to unity, and thus obtain the inverse demand function

$$p = \frac{1}{Q} \tag{2.16}$$

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[2] Assume a utility function $U = q_1^{\alpha_1} \cdot q_2^{\alpha_2} \cdot \ldots q_n^{\alpha_n}$ for a representative consumer, where the q_i for the moment represent quantities of *different* commodities consumed, *n* is the number of commodities, and α_i are some constants, such that $\alpha_1 + \alpha_2 + \ldots \alpha_n = 1$ holds. The consumer, maximizing utility under the budget constraint $y = p_1 q_1 + p_2 q_2 + \ldots p_n q_n$, then chooses $p_i q_i = \alpha_i y$, which is the same as $q = \alpha y / p$, where we drop the indices as only one commodity is of interest. Summing over all consumers $Q = \sum \alpha y / p$. The consumer identification indices for the α :s and y:s have been suppressed in order not to overload notation, but they are implicit, and can be different for all the consumers. Market price *p*, on the contrary, is the same for all consumers.

The profits of a representative firm now become

$$\Pi_i = \frac{q_i}{q_i + Q_i} - c_i q_i, \tag{2.17}$$

again assuming constant unit costs, and the profit maximum condition reads,

$$\frac{Q_i}{(q_i + Q_i)^2} = c_i . \tag{2.18}$$

As the numerator, residual demand, as well the right hand side, unit cost, are positive, one can take roots, and solve for the simple reaction function,

$$q_i = \sqrt{\frac{Q_i}{c_i}} - Q_i \tag{2.19}$$

This function has a leading square root term of Q_i, and an eventually dominant negative linear term, so its shape is unimodal, starting in the origin, increasing to a maximum, and then decreasing to zero.

The reaction q_i remains positive as long as

$$Q_i < \frac{1}{c_i} \tag{2.20}$$

holds. If not, then again the negative outcome will have to be replaced by zero.

Accordingly, the complete reaction function reads

$$q_i = \begin{cases} \sqrt{\dfrac{Q_i}{c_i}} - Q_i, & Q_i < \dfrac{1}{c_i} \\ 0, & Q_i \geq \dfrac{1}{c_i} \end{cases} . \tag{2.21}$$

One should check profits for nonnegativity. Substituting the nonzero branch of the reaction function in the profit expression yields

$$\Pi_i = \frac{\sqrt{\frac{Q_i}{c_i}} - Q_i}{\sqrt{\frac{Q_i}{c_i}}} - c_i\left(\sqrt{\frac{Q_i}{c_i}} - Q_i\right) = \left(1 - \sqrt{c_i Q_i}\right)^2 . \tag{2.22}$$

Again profit is nonnegative, but again, when the reaction is negative, it is due to negative costs outweighing negative revenues, which is nonsense. Even this time we, however, do not need to introduce any more constraints than already represented by the zero branch condition.

This case with isoelastic demand is quite useful for displaying some of the interesting dynamics possibilities of oligopoly theory, but it has its limitations: In particular, it is not suitable for the discussion of monopoly or collusion. This is because if there is just one firm, the monopolist, then it receives a constant (unit) revenue, $pQ = 1$. On the other hand, production costs increase with output, and all variable costs may be avoided if the monopolist produces nothing. It then sells this nothing at an infinite price, as may be seen from the demand function. In terms of substance this is nonsense, so one just cannot use the model for the monopoly case.

In dynamic oligopoly, the presence of other competitors as a rule keeps the system from landing in zero output for all firms, a case of implicit collusion, which has the same characteristics as monopoly. Further the derived reaction functions all have infinite slope at the origin, so under any perturbation, the system will never stick there. But, in a complete characterization of possible orbits, there remain degenerate orbits which even have their basins of attraction and are stable in a weak Milnor sense. See Tramontana et al. (2010).

Cournot Equilibria

Next consider the equilibria of the general Cournot model. The optimum conditions

$$f(Q) - f'(Q)q_i = C'(q_i) \tag{2.23}$$

with the definition $Q=\sum_{i=1}^{i=n} q_i$ added provide a set of $(n + 1)$ equations in the same number of unknowns, q_i and Q. The solution(s) provide the coordinates of the Cournot equilibria. This also means that all the reaction functions $q_i = \phi_i(Q_i)$ with definition $Q_i = \sum_{j \neq i} q_j$ are satisfied. When the competitors are so few that one can display the phase space graphically, for instance two, the duopoly, then the Cournot equilibria are intersections of the reaction curves. With three competitors, already difficult to visualize, they are intersections in three-space of three surfaces.

Example 1: Linear Demand

In the two exemplifying cases it is possible to calculate the coordinates for the Cournot point. Starting with the linear, the optimality conditions read

$$q_i = \frac{a - c_i}{2b} - \frac{1}{2} Q_i . \tag{2.24}$$

Subtracting $\frac{1}{2} q_i$ from both sides, and multiplying with 2,

$$q_i = \frac{a - c_i}{b} - Q, \tag{2.25}$$

Then, taking the sum over index i,

$$Q = \frac{na - \sum\limits_{i=1}^{i=n} c_i}{b} - nQ \tag{2.26}$$

is obtained, which solves for

$$Q = \frac{n}{n+1} \frac{a-c}{b}, \tag{2.27}$$

where

$$c = \frac{1}{n} \sum_{i=1}^{i=n} c_i \tag{2.28}$$

denotes average unit cost for the competitors. Substituting for total supply in the Cournot equilibrium point, one gets

$$q_i = \frac{a - c_i}{b} - \frac{n}{n+1} \frac{a - c}{b} \tag{2.29}$$

for the supplies of the individual firms.

It is also easy to calculate the equilibrium market price, through substituting for market supply Q in the inverse demand function. Hence

$$p = a - bQ = \frac{1}{n+1} a + \frac{n}{n+1} c \,. \tag{2.30}$$

This is a well known result from work with linear demand functions: Equilibrium price equals a weighted average of maximum price a, and unit production cost c. The more firms that stay active in the oligopoly, the more weight has the second term and the less has the first. In monopoly, price lands halfway between maximum price and unit cost, in duopoly the weight for the cost term is two thirds and one third for maximum price. With an increasing number of competitors the maximum price term tends to lose all importance, and one approaches marginal cost pricing, as prescribed in perfect competition.

To consider which firms will stay in the oligopoly market, one could either check the Cournot equilibrium supply coordinate for positivity, or check that unit cost be lower than market price, i.e.,

$$c_i < \frac{1}{n+1} a + \frac{n}{n+1} c \tag{2.31}$$

It must be kept in mind that the definition of c contains also the value of c_i. See Canovas et al. (2009). To squeeze out some more information, suppose the firms are numbered after increasing unit costs, i.e.,

$$c_1 < c_2 < \ldots c_n, \tag{2.32}$$

which can be done without loss of generality. Then the conditions

$$c_1 < a, \quad c_2 < \frac{a + c_1}{2}, \quad c_3 < \frac{a + c_1 + c_2}{3}, \quad \& \tag{2.33}$$

or, in general,

$$c_i < \frac{1}{i}\left(a + \sum_{j=1}^{j=i-1} c_j\right), \tag{2.34}$$

are obtained from (2.31).

Hence, how many competitors that may be accommodated in a Cournot equilibrium depends on the precise cost structure in that oligopoly branch. In the special case where all the firms are identical, i.e., $c_i = c$, it is sufficient that $c_i = c < a$. If, on the other hand, the cost situation is very unequal, for instance, $c_2 > (a + c_1)/2$, then only a monopoly of the first firm can make positive profits, if $c_3 > (a + c_1 + c_2)/3$, then a duopoly can persist, etc.

Example 2: Isoelastic Demand

Also for the second example it is possible to obtain the closed form expressions for the Cournot point coordinates and the rest of the results just discussed. Recall the main branches (2.19) of the reaction functions,

$$q_i = \sqrt{\frac{Q_i}{c_i}} - Q_i. \tag{2.35}$$

Adding Q_i to both sides,

$$Q = \sqrt{\frac{Q_i}{c_i}}, \tag{2.36}$$

or, taking squares,

$$Q^2 = \frac{Q_i}{c_i}, \tag{2.37}$$

is obtained.

Multiplying by c_i, recalling that $Q_i = Q - q_i$, and reorganizing slightly,

$$q_i = Q - c_i Q^2, \tag{2.38}$$

Taking the sum over i, one gets

$$Q = nQ - ncQ^2, \tag{2.39}$$

where, like in the previous case,

$$c = \frac{1}{n}\sum_{i=1}^{i=n} c_i \tag{2.40}$$

denotes average unit cost.

As $Q > 0$, it is permitted to cancel one power in equation (2.39), so

$$Q = \frac{n-1}{n}\frac{1}{c}, \tag{2.41}$$

and, consequently,

$$q_i = \frac{n-1}{n}\frac{1}{c}\left(1 - \frac{n-1}{n}\frac{c_i}{c}\right) = \frac{n(n-1)c - (n-1)^2 c_i}{(nc)^2}. \tag{2.42}$$

Several facts are worth being noted. As indicated in the introduction, the model is not suitable for discussing monopoly. The conditions (2.41) and (2.42) result in zero output when $n = 1$. One can also easily calculate oligopoly price in the Cournot equilibrium:

$$p = \frac{n}{n-1} c . \tag{2.43}$$

Disregarding again the degenerate monopoly case where price is infinite, equilibrium price is twice the unit cost for a duopoly, one and a half times cost in a triopoly, and so forth. Again, as the number of competitors increases, one approaches marginal cost pricing and the case of perfect competition.

As for the question which firms may obtain positive profits in the Cournot equilibrium, one can again either consider positivity of the reaction, or that unit cost be lower than market price, i.e.,

$$c_i < \frac{n}{n-1} c . \tag{2.44}$$

Again, it should be noted that c is the average of all the unit costs, including that on the left, and again, one can assume the firms to be numbered in order of increasing unit costs. In this way, ignoring the degenerate case of monopoly, one has $c_2 < c_1 + c_2$ for duopoly, which is always fulfilled with positive unit costs, $2c_3 < c_1 + c_2 + c_3$ for triopoly, which is the same as $c_3 < c_1 + c_2$, or, in general.

$$c_i < \frac{1}{i-2} \sum_{j=1}^{j=i-1} c_j \tag{2.45}$$

Hence, for two competitors in a Cournot oligopoly, there are no constraints for the costs; for three, the unit cost of the third must, however, not exceed the sum of the costs of the two firms with the lowest costs. A fourth firm may be added provided its unit cost does not exceed one half of the sum of the costs for the three with the lowest, and so forth. Again, in the special case $c_i = c$, nothing is constrained.

The Cournot Iterative Map

It is now time to return to the general case, and consider the dynamics of Cournot action. To this end it is necessary to state how expectations for the residual supplies Q_i are formed in an evolving system in order that each competitor be able to calculate the proper response according to $q_i = \phi_i(Q_i)$ as stated in (2.6) end exemplified in (2.11) and (2.19). Somehow real competitors and modellers alike have to form such expectations, and, as already stated, the assumption closest at hand is that each competitor assumes the others to maintain their previous moves, even if the assumption constantly shows up as being wrong.

This links future to past, so one can write

$$q_i(t+1) = \phi_i(Q_i(t)) \tag{2.46}$$

with time period identifications for the variables. Of course

$$Q(t) = \sum_{i=1}^{i=n} q_i(t), \quad Q_i(t) = Q(t) - q_i(t), \tag{2.47}$$

and likewise for time period $t+1$. These rules indicate how all the quantity variables, individual supply, total supply, and residual supply for each following period are obtained from those of the current. Hence the whole following orbit of the system could in principle be calculated.

For any such system only four possible orbit types which are approached asymptotically after a transient exist. Fixed points (or equilibria), periodic orbits (which always return to the same state after a finite number of steps), quasiperiodic orbits (those that have an infinite period, coming close to previous states periodically but never quite so in finite time), and chaotic orbits (such that are unpredictable, as nearby orbits are separated at an exponential rate, though bent back and kept in a finite area of the phase space).

All these are possible attractors. They can be lonely, or they can coexist. A fixed point can coexist with another fixed point, as demonstrated by Palander (1936, 1939) and Wald (1936), with a periodic orbit (as also shown by Palander), or even with a chaotic attractor. In the case of coexistence, each attractor has its proper basin of attraction, and the separating basin boundaries divide the total phase space in such attraction basins. Basins and basin boundaries can be simple, simply connected areas and curves of finite length, but they can also, even with very simple systems, be very complicated, so called fractals.

These potential attractors, fixed points, periodic, quasiperiodic, and chaotic, can also be unstable, i.e., repellors. They yet remain interesting for describing the dynamics, as they exist and repel all trajectories that come near.

Further, attractors can turn into repellors due to some slight change of a parameter of the model, in which case one speaks of a bifurcation. Modern theory for dynamical systems has got very far in studying and classifying types of bifurcations, as well as of attractors and basins.

Formation of Expectations

In formulating the iterative dynamic system, or map as is the mathematical term, it was assumed that the competitors just assumed all others to retain their previous moves. This is a quite problematic case in an evolving system, except when it is in a stable fixed point.

Observing Chaotic Orbits

So, what can the competitors learn from the actually observed orbit? A chaotic orbit is by definition unpredictable, though deterministic. This is due to the magnification of computational rounding off errors. Determinism just means that the future of the system is totally determined by the initial conditions; but these initial conditions must be known exactly, numerical values with an infinite number of decimals. Such precision is never possible in reality. Now, a slight measurement error can have different consequences. In a system such as focused in traditional dynamics, it just leads to a slight displacement of the orbit. In a chaotic system, orbits starting from different initial conditions, no matter how slight the difference is, need no more than a few moves to end up in completely different parts of the phase space. This is what unpredictability means. For the present context it is obvious that orbits that are never predictable are useless for observation and use in forming expectations. Unfortunately, modern dynamics has shown that this phenomenon tends to be present in even the simplest nonlinear dynamic systems.

Observing Periodic Orbits

Starting at the simplest end, disregarding the obvious case of a stable fixed point, suppose the system actually produces a periodic orbit, such as may be observed and learned by the competitors. Would it then be possible to adapt to this, and produce an orbit of exactly that periodicity? Such an idea is close to the favourite economics idea of rational expectations.

As will now be shown this is always impossible, because such learning and adapting to any periodicity is bound to produce a periodicity *different* from the one learned, unless the orbit is a fixed point. See Puu (2008).

So, suppose there exists a periodic orbit of period T as solution to the dynamic Cournot system. Then, for the factual orbit,

$$q_i(t+T) = q_i(t), \tag{2.48}$$

holds for all time periods t. As the periodicity is assumed to be learned and adjusted to by the competitors, they will form the expectations correctly and respond to observed facts not one period back, but T periods back. Hence,

$$q_i(t+T) = \phi_i(Q_i(t)). \tag{2.49}$$

But, from equations (2.48) and (2.49), then

$$q_i(t) = \phi_i(Q_i(t)), \tag{2.50}$$

which relates the variables in the same time period, and is hence a definition for the Cournot equilibrium fixed point. This relation holds, as assumed, for all time periods t, which shows that the assumed periodic orbit is a fixed point. A fixed point, of course, is a periodic point – of all periodicities. But this simple proof shows that a periodic point which is not a fixed point cannot be learned, adapted to, and yet result in a periodic orbit of the assumed period, unless it is a fixed point.

For the reader who is very fond of the idea of rational expectations, we can add a different argument. Suppose the dynamic system produces an orbit of, say, period 3. This means that the system eventually visits three points here called ***A***, ***B***, ***C*** over and over. Now, suppose the competitors learn this, and adapt. As the system is deterministic, and the same as before, the outcome would be the same sequence, though with two undetermined points in intervening periods. Assuming one again starts in $\boldsymbol{A}$, the system produces

$$\boldsymbol{A}\,\_\,\_\,\boldsymbol{B}\,\_\,\_\,\boldsymbol{C}\,\_\,\_\,.$$

Now, try to make this the original 3-period orbit. In that case $\boldsymbol{B}$ should follow the first entry $\boldsymbol{A}$, and fill out the first blank, but then the third and fifth blanks will be filled out by $\boldsymbol{C}$ and $\boldsymbol{A}$ in that order, producing

$$A\,\boldsymbol{B}\,\_\,B\,\boldsymbol{C}\,\_\,C\,\boldsymbol{A}\,\_\,.$$

Finally, the remaining blanks need to be filled out, by $\boldsymbol{C}$, $\boldsymbol{A}$, $\boldsymbol{B}$, producing

$$A\,B\,\boldsymbol{C}\,B\,C\,\boldsymbol{A}\,C A\,\boldsymbol{B}\,.$$

This starts quite correctly with $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, but then comes $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{A}$, and finally, $\boldsymbol{C}$, $\boldsymbol{A}$, $\boldsymbol{B}$. It is thus impossible to recognize the sequence as anything but a 9-period orbit. After the last entry the sequence starts all over again.

This shows that if the competitors learn the 3-periodicity they produced before, then they, in fact, produce a 9-period orbit, which, if learned and adapted to, in its turn produces another periodicity. If all the competitors do not learn the periodicity simultaneously, but some are slower in learning and adapting than others, then the result becomes even much more complicated.

The conclusion is that learning and adapting to an observed periodicity by the agents is always bound to change this very periodicity. In a loose way it is similar to the impossibility principle in physics, where the very act of observing and measuring positions and momenta of particles changes the facts to be observed.

It is self-evident, that the change of periodicity does not occur when the periodic orbit is a fixed point with just one point of phase space $\boldsymbol{A}$ visited over and over.

As a general conclusion; some orbits, such as the chaotic, are too complicated to learn and adapt to; some, such as the periodic, are simple to observe, learn, and adapt to, but when the agents do this, they inevitably change the outcome. To adapt to an observed orbit and make the system produce that very orbit seems to be impossible, unless the orbit is a fixed point.

What then can the competitors do to escape from expecting the others to retain their previous moves, sometimes called naive expectations, which always shows up to be wrong? They can observe trends, increases or decreases from two subsequent period observations, or even curvatures in an attempt to identify approaching turning points. However, analysing such systems is bound to become extremely complex and messy as it multiplies up the order of the system.

In the sequel this simple assumption concerning the formation of expectations will be kept, except when discussing conservative moves in terms of adaptive systems, or Stackelberg leadership, which represents another type of learning, not of the actual orbit, but of the reaction functions of the competitors.

Stability

Cournot equilibria, fixed points of the Cournot iterative map, can, as stated above, be stable or unstable. If they are unstable, they can yet be of interest in characterizing the dynamic system, but they are no attractors.

For establishing the stability of a Cournot equilibrium point of the general iterative system,

$$q_i(t+1) = \phi_i(Q_i(t)), \tag{2.51}$$

one first needs to calculate the derivatives of the reaction functions,

$$\frac{dq_i(t+1)}{dQ_i(t)} = \phi_i' . \tag{2.52}$$

Then the Jacobian matrix reads,

$$J = \begin{bmatrix} 0 & \phi_1' & \cdots & \phi_1' \\ \phi_2' & 0 & \cdots & \phi_2' \\ \vdots & \vdots & \ddots & \vdots \\ \phi_n' & \phi_n' & \cdots & 0 \end{bmatrix}. \tag{2.53}$$

Obviously, as the supplies of all the competing firms enter in the sum defined as residual supply, the same entries appear as off-diagonal elements in each row. As the supply of each firm itself is absent in this definition of residual supply, the diagonal elements are zero.

The next move would be to form the matrix

$$J - \lambda I = \begin{bmatrix} -\lambda & \phi_1' & \cdots & \phi_1' \\ \phi_2' & -\lambda & \cdots & \phi_2' \\ \vdots & \vdots & \ddots & \vdots \\ \phi_n' & \phi_n' & \cdots & -\lambda \end{bmatrix}, \tag{2.54}$$

where I denotes the n by n identity matrix, and the characteristic equation,

$$|J - \lambda I| = 0. \tag{2.55}$$

This nth degree polynomial equation in λ determines the eigenvalues, real or complex, and for stability one must ensure that all these $\lambda_1, \lambda_2, \ldots \lambda_n$ are in the unit circle in the complex plane.

This is a formidable programme, and given the general equations for the Cournot system,

$$f\left(q_i(t+1) + Q_i(t)\right) + f'\left(q_i(t+1) + Q_i(t)\right)q_i(t+1) = C_i'\left(q_i(t+1)\right), \tag{2.56}$$

implicit differentiation results in

$$\phi_i' = -\frac{f' + \phi_i f''}{2f' + \phi_i f'' - C_i''}, \tag{2.57}$$

which seems to be exceedingly messy. One would have to calculate the Cournot point coordinates from (2.23), substitute those in the derivatives of the reaction functions (2.57), then those in the Jacobian (2.53), and finally solve the polynomial characteristic equation (2.55). Several of these steps are just impossible in view of the general procedures of mathematical economics.

That much just to determine the stability of a single fixed point! How could one at all deal with the orbits resulting from an unspecified iterative map with just a few qualitative properties? These facts emphasize the absolute need to work with specified global models resulting in credible closed form reaction functions.

Example 1: Linear Demand

It is therefore good to return to the two examples. In the literature Theocharis (1959) is generally still credited for having been the first to show that a Cournot equilibrium in an oligopoly with a linear demand function and constant unit costs for the competitors is destabilised when their number exceeds three, whereas stability becomes neutral with three competitors. Then the (unique) fixed pint is not yet unstable, but any slight deviation from it puts up an endless oscillatory motion; each such motion being neutrally stable as well.

Palander (1939), whose more important contributions will be dealt with in a later chapter, stated exactly the same 20 years earlier. Unfortunately, the article was written in Swedish, a language even present day Anglo-Saxon oriented Swedish economists neither read nor cite.

The stability issue, which was extremely complicated in the general case, is now equally simple. From the reaction functions,

$$q_i(t+1) = \phi_i(Q_i(t)) = \frac{a - c_i}{2b} - \frac{1}{2} Q_i(t) . \tag{2.58}$$

The derivatives are always

$$\frac{dq_i(t+1)}{dQ_i(t)} = \phi_i' = -\frac{1}{2} , \tag{2.59}$$

i.e., constant, so that one does not even need the particular coordinates of the Cournot equilibrium point to put up the n by n Jacobian matrix

$$J = \begin{bmatrix} 0 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \cdots & -\frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2} & -\frac{1}{2} & \cdots & 0 \end{bmatrix}. \tag{2.60}$$

The characteristic equation is easily obtained as an *n*th degree polynomial that factorizes into

$$|J - \lambda I| = P^n(\lambda) = \left(\lambda - \tfrac{1}{2}\right)^{n-1}\left(\lambda + \tfrac{1}{2}(n-1)\right). \tag{2.61}$$

The eigenvalues are

$$\lambda_1, \ldots \lambda_{n-1} = \tfrac{1}{2}, \lambda_n = -\tfrac{1}{2}(n-1). \tag{2.62}$$

The first $n-1$ are equal to $\frac{1}{2}$ and are well within the unit circle of the complex plane, but the last one $\frac{1}{2}(n-1)$ falls outside it when $n > 3$. For $n = 3$, $\lambda_n = -1$, so this is a boundary case.

The multiple eigenvalues are associated with differences of the variables. They always indicate stability, even when the system diverges. This means that, in any motion, differences between the behaviour of the firms are evened out, so that the actions are asymptotically coordinated. The last eigenvalue is associated with total market supply, and it is it that brings instability to the system when the number of competitors grows.

Theocharis also gave the complete global solution to the system, but there is no need to present it here, because an unstable linear system always just explodes, resulting in infinite deviations, negative supplies, and negative prices. His solution is therefore not factually relevant. The global dynamics of the piecewise linear case, where account is taken of nonnegativity of price, supplies, and profits, was only analysed a few years ago. See Canovas et al. (2008).

Palander too, was only concerned about the instability, but he stated the results completely, as mentioned, 20 years before Theocharis. In 1939 he wrote "*as a condition for equilibrium with a certain number of competitors to be stable to exogenous disturbances, one can stipulate that the* *derivative of the reaction function ϕ' must be such that the condition $|(n-1)\phi' < 1|$ holds. If this criterion is applied to, for instance, the case with a linear demand function and constant marginal costs, the equilibria become unstable as soon as the number of competitors exceeds three. Not even in the case of three competitors will equilibrium be restored, rather there remains an endless oscillation*".

As for the global dynamics of the piecewise linear case, not much exciting occurs. There are just the possibility of a stable monopoly, or a duopoly; otherwise two-period oscillations where all the firms move in phase are the attractors. If the number of competitors exceeds three, all firms drop out every second period. This might seem to present an excellent case for learning the orbit by some competitors who might try to move out of phase, even attempting to become monopolists every second period. But then one encounters the problem stated above that this learning and adaptation alters the periodicity itself.

Which firms will stay on the market depends, as was shown in the section on Cournot equilibria, on the cost structure. It may occur that, even with very numerous firms, the cost structure among them might be such that only one or two firms with cost advantages stay in the Cournot equilibrium; all the other being bound to drop out in the dynamic process.

Example 2: Isoelastic Demand

From (2.35) the main branches of the reaction functions in the case of isoelastic demand were

$$q_i(t+1) = \phi_i(Q_i(t)) = \sqrt{\frac{Q_i(t)}{c_i}} - Q_i(t), \tag{2.63}$$

so the derivatives,

$$\frac{dq_i(t+1)}{dQ_i(t)} = \phi_i' = \frac{1}{2}\frac{1}{\sqrt{c_i Q_i(t)}} - 1 \tag{2.64}$$

are easily obtained. Further, from (2.37), in Cournot equilibrium,

$$Q_i = c_i Q^2 . \tag{2.65}$$

Hence,

$$\phi_i' = \frac{1}{2}\frac{1}{c_i Q} - 1 \tag{2.66}$$

in the Cournot fixed point. Further, it was true that

$$Q = \frac{n-1}{n}\frac{1}{c}, \tag{2.67}$$

where

$$c = \frac{1}{n}\sum_{i=1}^{i=n} c_i \tag{2.68}$$

was average marginal cost. Using (2.67) for Q,

$$\phi_i' = \frac{1}{2}\frac{n}{n-1}\frac{c}{c_i} - 1 \tag{2.69}$$

in Cournot equilibrium.

Agiza (1998) and Ahmed et al. (1998) wanted to demonstrate that the Cournot equilibrium is destabilised in this case when the number of competitors exceeds four, and is neutrally stable when the number is exactly four. The frontier of destabilisation is hence just pushed one competitor further as compared to the case of a linear demand function. Their proof could not be completed without changing the demand function to

$$p = \frac{1}{Q} + p_0 , \tag{2.70}$$

through adding a constant. This, however, unlike the original case of the isoelastic function, is not derived from any basic microeconomics of consumers maximizing their utility functions. Further, note that this change has no connection with the problem discussed that the isoelastic demand function is unsuitable for dealing with the case of monopoly; to avoid problems in that connection, one would have to add a constant to supply, rather than to price. But, unfortunately, this is not based on the theory of the consumer either.

However, there is a simple way to arrive at the conclusions of Agiza (1998) and Ahmed et al. (1998) with the original demand function. That is to assume that the competitors are identical, so that all $c_i = c$.

Then

$$\phi_i' = -\frac{1}{2}\frac{n-2}{n-1}, \tag{2.71}$$

which facilitates things a lot. The n by n Jacobian matrix for the Cournot equilibrium point now becomes

$$J = \begin{bmatrix} 0 & -\frac{1}{2}\frac{n-2}{n-1} & \cdots & -\frac{1}{2}\frac{n-2}{n-1} \\ -\frac{1}{2}\frac{n-2}{n-1} & 0 & \cdots & -\frac{1}{2}\frac{n-2}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2}\frac{n-2}{n-1} & -\frac{1}{2}\frac{n-2}{n-1} & \cdots & 0 \end{bmatrix}, \tag{2.72}$$

and the characteristic equation

$$|J - \lambda J| = P^n(\lambda) = \left(\lambda - \frac{1}{2}\frac{n-2}{n-1}\right)^{n-1} \left(\lambda + \frac{1}{2}(n-2)\right) = 0. \tag{2.73}$$

Again, the polynomial factorizes, and among the eigenvalues,

$$\lambda_1, \ldots \lambda_{n-1} = \frac{1}{2}\frac{n-2}{n-1}, \lambda_n = -\frac{1}{2}(n-2), \tag{2.74}$$

the first, of multiplicity $n-1$, are inside the unit circle in the complex plane, whereas the last one comes outside for $n > 4$, and is hence associated with instability.

The present model with isoelastic demand offers better perspectives for interesting dynamics in global analysis than the case with linear demand, but, quite as in the linear case, the Cournot equilibrium is destabilsed when the number of competitors exceeds a very small number, though now four in stead of three.

The significance of the result, attributed to Theocharis (1959), has relevance for the issue of increasing competition transforming the market from monopoly over oligopoly to perfect competition. This means that with an increasing number of competitors, the Cournot equilibrium approaches marginal cost pricing and elimination of profits for the marginal firm (the one with highest unit cost). But, this, which, as was shown above, happens in both models, is of little interest if in this same process equilibrium is destabilised. If the system is no longer attracted to the Cournot equilibrium it does not matter that it transforms into a perfect competition equilibrium.

Yet, intuitively, one would like to keep the possibility of oligopoly seamlessly transforming into perfect competition. The question, which seems, not to have been addressed is; why does this happen with different reasonable demand functions?

The clue lies in the assumptions on the cost side, constant marginal and average variable costs. These emerge from production under constant returns. But a firm producing under constant returns has in principle infinite capacity. In perfect competition, a market price, taken as constant by the individual firms, which exceeds a constant unit cost by the tiniest fraction, makes it possible to blow up the profit to any value just through multiplying up the scale of operation.

Admitting this, one may say that destabilisation due to adding more and more infinite sized firms is not very surprising; nor is it relevant to the issue of transforming oligopoly into perfect competition. It has been implicit that the comparison should be between cases of few large firms versus many small firms. But large and small cannot be modelled with constant unit costs. One would need variable, not constant, returns, preferably including capacity limits for such modelling.

This can, as shown in a following chapter, be modelled in different ways. In the piecewise linear model, one could keep constant unit costs, but at a blunt capacity limit let the cost jump up to infinity. It is also possible to use the CES production function, with capital fixed through an act of investment, to obtain a cost function that asymptotically goes to infinity at a capacity limit due to the fixed capital. This makes it necessary to include assumptions of capital formation, and paves the ground for endogenous modelling of how capital formation and competition may evolve in a branch. See Puu and Panchuk (2009).

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