

# Preface

In der Theorie der Thetafunctionen ist es leicht, eine beliebig grosse Menge von Relationen aufzustellen, aber die Schwierigkeit beginnt da, wo es sich darum handelt, aus diesem Labyrinth von Formeln einen Ausweg zu finden. Die Beschäftigung mit jenen Formelmassen scheint auf die mathematische Phantasie eine verdorrende Wirkung auszuüben

G. Frobenius, 1893

Theta functions have never ceased to be a source of inspiration for mathematicians. Since their invention by Euler, Gauss, Jacobi, and others, the concept of a theta function was vastly generalized, these functions found applications in physics, theoretical chemistry and engineering sciences, and they play a central role in number theory and other branches of mathematics. In the present monograph only a special type of theta functions will be discussed: Beginning in 1920, Erich Hecke (1887–1947) introduced theta series with characters on algebraic number fields. These series define holomorphic functions on the upper half plane of one complex variable. For quadratic number fields they provide a way to construct modular forms on subgroups of the modular group  $SL_2(\mathbb{Z})$ , notably in the case of smallest integral weight 1, when other methods of construction are troublesome or fail.

My work on the identities in this monograph started some 25 years ago when I first used Eisenstein series and eta products for the construction of Hecke eigenforms on some subgroups of the modular group. The arithmetic of quadratic number fields and the very definition of Hecke theta series imply that these functions are Hecke eigenforms; their Fourier coefficients are multiplicative and satisfy simple recursions at powers of primes. Thus, in order to corroborate that a given combination of Eisenstein series or eta products is in fact a Hecke eigenform, a convenient way would be to identify that function with a Hecke theta series. Of course, this method will only work for the minority of modular forms which are in fact Hecke theta series, that is, in a different terminology, which are of *CM*-type. But it will always work in the case of weight 1.

A few of my results have previously been published in journals. In the course of time the number of examples grew, and apparently it did not make sense any longer to submit them to journals. Finally I decided to pull all the examples out of my desk and to collect them in a research monograph so that they can be used by the community. During my work on this monograph many more new examples emerged. In particular, I would like to draw the attention to some 150 examples where theta series of weight 1 on three distinct quadratic number fields (two of them imaginary, the other one real) coincide. Only four of these examples were previously known to me from the literature.

For a reader who wants to use a book like this there is always a problem to judge whether a specific result might be contained in it, and where to find it. The [Table of Contents](#) at the beginning and the “Directory of Characters” at the end of the book will be helpful in this respect.

Hopefully, neither myself nor anyone of my readers will be a victim to the peril which, according to Georg Frobenius, threatens those who are interested in theta identities.

I am grateful to my home institution, Mathematisches Institut der Universität Würzburg, for providing me with office space and with library and computer resources, several years beyond the time of my retirement. My special thanks are due to Richard Greiner for teaching me how to use the computer resources. I would like to thank Aloys Krieg and Jörn Steuding for reading parts of earlier versions of the manuscript and for helpful criticism. Also, I would like to thank Springer Verlag for publishing this book.

In preparing the manuscript I tried hard to avoid errors. But there are too many chances to commit errors, by mixing up character symbols, confusing signs, and so on, especially when you change notations. I will be grateful to any reader for comments and for communicating errors to my E-mail address, [koehler@mathematik.uni-wuerzburg.de](mailto:koehler@mathematik.uni-wuerzburg.de).

Würzburg, Germany  
August 2010

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<http://www.springer.com/978-3-642-16151-3>

Eta Products and Theta Series Identities

Köhler, G.

2011, XXII, 622 p., Hardcover

ISBN: 978-3-642-16151-3