2 Eta Products

2.1 Level, Weight, Nominator and Denominator of an Eta Product

By an *eta product* we understand any finite product of functions

$$f(z) = \prod_m \eta(mz)^{a_m}$$

where m runs through a finite set of positive integers and the exponents a_m may take any values from $\mathbb{Z}$, positive or negative or 0. (Of course, an exponent 0 contributes a trivial factor 1 to the product, and therefore we may as well assume that $a_m \neq 0$ for all m.) Since the product is finite, the lowest common multiple $N = \text{lcm}\{m\}$ exists, and every m divides N. We write

$$f(z) = \prod_{m|N} \eta(mz)^{a_m}, \tag{2.1}$$

and we call f an eta product of *level* N. Here, formally, m runs through all positive divisors of the positive integer N, and some of the exponents a_m might be 0. We will use this notation also in cases when N is bigger than $\text{lcm}\{m\}$; then N is a multiple of the level of the eta product.

Some authors use the term *eta quotient* for functions as in (2.1), and they reserve the term *eta product* for the case when $a_m \geq 0$ for all m.

Often we will use the notation

$$[1^{a_1}, 2^{a_2}, 3^{a_3}, \ldots]$$

as an abbreviation for the eta product $\eta(z)^{a_1}\eta(2z)^{a_2}\eta(3z)^{a_3}\ldots$. This notation is adopted from [42]. The term in square brackets will often be written as a fraction with positive exponents in its numerator and denominator.

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An eta product (2.1) transforms like a modular form of weight

$$k = \frac{1}{2}\sum_m a_m$$

with some multiplier system on the congruence group $\Gamma_0(N)$. This means that for every $L = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ we have

$$f(Lz) = f\left(\frac{az+b}{cz+d}\right) = v_f(L)(cz+d)^k f(z)$$

where $v_f(L)$ is some 24th root of unity which can be computed from the multiplier system v_η of the eta function. We will rarely need to know the values $v_f(L)$ of the multiplier system of f explicitly. We have

$$v_f(L) = v_f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \prod_{m|N}\left(v_\eta\begin{pmatrix} a & mb \\ c/m & d \end{pmatrix}\right)^{a_m}$$

where the values of v_η are given explicitly in Theorem 1.7. Highly important for us, however, is the value $v_f(T)$ for the translation $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. We write

$$\frac{1}{24}\sum_{m|N} m a_m = \frac{s}{t} \tag{2.2}$$

in lowest terms, i.e., with $\gcd(s,t) = 1$. Then it is a trivial consequence from $\eta(z+1) = e\left(\frac{1}{24}\right)\eta(z)$ that we have $v_f(T) = e\left(\frac{s}{t}\right)$,

$$f(Tz) = f(z+1) = e\left(\frac{s}{t}\right) f(z).$$

It follows that f has a Fourier expansion of the form

$$f(z) = \sum_{n\equiv s(\bmod t),\ n\geq s} c_n e\left(\frac{nz}{t}\right) \tag{2.3}$$

with coefficients $c_n \in \mathbb{Z}$, $c_s = 1$. In particular, $\frac{s}{t}$ is the order of f at the cusp ∞. We call s the *numerator* and t the *denominator* of the eta product (2.1). The denominator t is a divisor of 24.

An explicit formula for $v_f(L)$ is given in [105], Theorem 1.64 in the case when the weight k and the number (2.2) are integers (whence $t = 1$) and when also $\frac{1}{24}\sum_{m|N} m a_{N/m}$ is an integer; in this case $v_f(L)$ is a function of d only.

For a Fourier series (2.3), the sign transform is

$$f\left(z+\tfrac{1}{2}\right) = e\left(\tfrac{s}{2t}\right) \sum_{n\equiv s(\bmod t),\, n\geq s} (-1)^{(n-s)/t} c_n e\left(\frac{nz}{t}\right).$$

Modifying our concept from Sect. 1.2, we will also call the series for $e(-\frac{s}{2t}) \times f(z+\frac{1}{2})$ the *sign transform* of the series for $f(z)$.

An eta product f of level N as in (2.1) will be called *old* if there is an integer $d \geq 1$, a proper divisor N_1 of N and an eta product g of level N_1 such that $f(z) = g(dz)$. Otherwise f will be called a *new* eta product. Since f and g have identical Fourier coefficients, it often suffices to study new eta products. Nevertheless, sometimes it is advantageous to consider old ones. For example, $g(z) = \eta(z)\eta(2z)$ and $f(z) = \eta(8z)\eta(16z)$ both are old eta products of level 16, while g is new of level 2. But f has period 1, and hence its Fourier expansion is a power series in the variable $q = e(z)$, which might be nicer than the expansion of g with fractional powers of q.—We emphasize that our concept of a new eta product has little to do with the concept of a newform in the theory of Hecke operators as explained in Sect. 1.7. Only occasionally it will happen that a new eta product is also a Hecke eigenform. (Incidentally, $\eta(z)\eta(2z)$ is such an example; see Sect. 10.1.)

2.2 Eta Products on the Fricke Group

For the moment, let us put $f_m(z) = \eta(mz)$, where m is a positive integer. From $\eta(-1/z) = \sqrt{-iz}\,\eta(z)$ it follows that

$$f_m(W_N\, z) = f_m\Big(-\frac{1}{Nz}\Big) = \eta\Big(-\frac{1}{(N/m)z}\Big) = \sqrt{-(iN/m)z}\,\eta\Big(\frac{N}{m}z\Big).$$

Thus, for an eta product f of level N as in (2.1), we obtain

$$\begin{aligned} f(W_N\, z) &= \prod_{m|N}\Big((-i(N/m)z)^{1/2}\eta\Big(\frac{N}{m}z\Big)\Big)^{a_m} \\ &= \prod_{m|N}\big((-imz)^{1/2}\eta(mz)\big)^{a_{N/m}} \\ &= (-iz)^k\Big(\prod_{m|N} m^{a_{N/m}}\Big)^{1/2}\prod_{m|N}\eta(mz)^{a_{N/m}}. \end{aligned}$$

The eta product f transforms like a modular form of weight k for the Fricke group $\Gamma^*(N)$ if and only if

$$f(W_N z) = \big(-i\sqrt{N}z\big)^k f(z).$$

We see that this holds if and only if the condition

$$a_{N/m} = a_m \qquad \text{for all} \qquad m|N \tag{2.4}$$

is satisfied. An eta product with this property will be called an *eta product on the Fricke group* of level N.

We observe that an eta product of level N is determined by its system of $\tau(N)$ exponents a_m, whereas roughly half of these parameters—exactly $\lceil \tau(N)/2 \rceil$ of them—suffice to determine an eta product on the Fricke group. Here, $\tau(N) = \sigma_0(N)$ is the number of positive divisors of N, as introduced in Sect. 1.5.

2.3 Expansion and Order at Cusps

The product for $\eta(z)$ tells us that this function is nowhere 0. Therefore, eta products (2.1) are holomorphic on the upper half plane regardless of their system of exponents a_m. However, we will restrict our study to eta products which are holomorphic at all cusps, too. In particular, the order at the cusp ∞ should be non-negative, i.e.,

$$\frac{s}{t} \geq 0.$$

We need conditions for an eta product to be holomorphic at the other cusps $r \in \mathbb{Q}$. For this purpose we give a formula for the order of functions $\eta(mz)$ at an arbitrary cusp and, somewhat more general, for the Fourier expansion of $\eta(mz)$ at cusps. This expansion will eventually be useful when we want to decide whether a linear combination of eta products is a cusp form, where the eta products are holomorphic at all cusps, but not cusp forms themselves.

Proposition 2.1 *Let $f_m(z) = \eta(mz)$ with $m \in \mathbb{N}$, and let $r = -\frac{d}{c} \in \mathbb{Q}$ be a reduced fraction with $c \neq 0$. Let a, b be chosen such that $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then we have*:

(1) *The expansion of f_m at the cusp r is*

$$\begin{aligned} f_m(A^{-1}z) &= v_\eta(L)\left(\frac{\gcd(c,m)}{m}(-cz+a)\right)^{1/2} \\ &\quad \times \sum_{n=1}^{\infty}\left(\frac{12}{n}\right) e\left(\frac{n^2}{24m}\left((\gcd(c,m))^2 z + \nu \gcd(c,m)\right)\right) \end{aligned}$$

where $L = \left(\begin{smallmatrix} x & * \\ u & * \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $x = \frac{md}{\gcd(c,m)}$, $u = -\frac{c}{\gcd(c,m)}$, and ν is some integer.

(2) *The order of f_m at the cusp r is*

$$\mathrm{ord}(f_m, r) = \frac{1}{24m}\left(\gcd(c,m)\right)^2.$$

Proof. Since c, d are relatively prime, we can choose $a, b \in \mathbb{Z}$ such that $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. We get $A^{-1}(\infty) = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)(\infty) = -\frac{d}{c} = r$ and

$$f_m(A^{-1}z) = \eta\left(\frac{mdz - mb}{-cz + d}\right) = \eta(\alpha z)$$

where $\alpha = \left(\begin{smallmatrix} md & -mb \\ -c & a \end{smallmatrix}\right)$, $\det(\alpha) = m$. The expansion of f_m at r is given by the expansion of $f_m(A^{-1}z)$ at ∞. In order to find it, we need some matrix $L = \left(\begin{smallmatrix} x & y \\ u & v \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that the lower left entry in $L^{-1}\alpha$ vanishes. We have

$$L^{-1}\alpha = \begin{pmatrix} v & -y \\ -u & x \end{pmatrix} \begin{pmatrix} md & -mb \\ -c & a \end{pmatrix} = \begin{pmatrix} * & * \\ -mdu - cx & * \end{pmatrix}.$$

Therefore we need that $mdu + cx = 0$. Thus for the first column of L we can choose the relatively prime integers

$$x = \frac{md}{\gcd(c, md)} = \frac{md}{g}, \qquad u = -\frac{c}{g}, \quad \text{with } g = \gcd(c, m).$$

From $\det(L^{-1}\alpha) = \det(\alpha) = m$ we infer that

$$L^{-1}\alpha = \begin{pmatrix} * & * \\ 0 & m/g \end{pmatrix} = \begin{pmatrix} g & \nu \\ 0 & m/g \end{pmatrix}$$

with some $\nu \in \mathbb{Z}$. (Observe that we can compute $\nu = -mbv - ya$ explicitly, depending on m and r.) Now we get

$$\begin{aligned} f_m(A^{-1}z) &= \eta(\alpha z) = \eta(LL^{-1}\alpha z) \\ &= v_\eta(L)\left(u\frac{gz + \nu}{m/g} + v\right)^{1/2} \eta(L^{-1}\alpha z) \\ &= v_\eta(L)\left(\frac{-cz - c\nu/g}{m/g} + v\right)^{1/2} \eta\left(\frac{gz + \nu}{m/g}\right) \\ &= v_\eta(L)\left(\frac{g}{m}\left(-cz - \frac{c\nu - vm}{g}\right)\right)^{1/2} \eta\left(\frac{g^2}{m}z + \frac{\nu g}{m}\right) \\ &= v_\eta(L)\left(\frac{g}{m}(-cz + a)\right)^{1/2} \eta\left(\frac{g^2}{m}z + \frac{\nu g}{m}\right) \\ &= v_\eta(L)\left(\frac{g}{m}(-cz + a)\right)^{1/2} \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) e\left(\frac{n^2}{24m}(g^2 z + \nu g)\right). \end{aligned}$$

This proves our first assertion. The first non-vanishing term in $(-cz+a)^{-1/2} \times f_m(A^{-1}z)$ is a constant multiple of $e(g^2 z/24m)$. Thus, by our definition of the order, we obtain $\mathrm{ord}(f_m, r) = g^2/24m$, which is the second assertion. $\square$

We note an immediate consequence of the second assertion:

Corollary 2.2 *Let f be an eta product as in (2.1), and let $r = -\frac{d}{c} \in \mathbb{Q}$,* $\gcd(c,d) = 1$. *Then the order of f at the cusp r is*

$$\operatorname{ord}(f,r) = \frac{1}{24}\sum_{m|N} \frac{(\gcd(c,m))^2}{m} a_m.$$

An eta product f will be called a *holomorphic eta product* if its orders at all cusps are non-negative,

$$\operatorname{ord}(f,r) \geq 0 \qquad \text{for all} \qquad r \in \mathbb{Q} \cup \infty.$$

Holomorphic eta products (2.1) are (entire) modular forms for $\Gamma_0(N)$. They are cusp forms if and only if all the orders are positive,

$$\operatorname{ord}(f,r) > 0 \qquad \text{for all} \qquad r \in \mathbb{Q} \cup \infty.$$

In this case we will call them *cuspidal eta products*, and *non-cuspidal* otherwise.

2.4 Conditions for Holomorphic Eta Products

From Corollary 2.2 we get conditions for an eta product to be holomorphic or a cusp form. These are conditions for infinitely many cusps. Of course, it suffices to check these conditions for a finite system of representatives of inequivalent cusps of $\Gamma_0(N)$, i.e., for the orbits of this group on $\mathbb{Q} \cup \infty$. The number of inequivalent cusps of $\Gamma_0(N)$ is $\sum_{m|N} \varphi(\gcd(m, N/m))$, where φ is the Euler function; this is known from several textbooks; see [125], p. 102, for example. A set of representatives of inequivalent cusps is given in [92], formula (2). Using this, it would be possible to characterize holomorphic and cuspidal eta products by systems of finitely many inequalities. In fact, one can find such a characterization using nothing else but Corollary 2.2:

We observe that the order of f at a cusp does only depend on the denominator c of that cusp. If m is any divisor of N then for all $c \in \mathbb{Z}$ we have

$$\gcd(c,m) = \gcd(\gcd(c,N),m),$$

and $\gcd(c,N)$ is a divisor of N. Therefore the conditions $\operatorname{ord}(f,r) \geq 0$ are satisfied for all $r \in \mathbb{Q} \cup \infty$ if and only if

$$\operatorname{ord}(f,1/c) \geq 0 \qquad \text{for all} \qquad c|N,$$

and similarly for strict inequalities. This proves the following result:

Corollary 2.3 *An eta product f as in (2.1) is holomorphic if and only if the inequalities*

$$\sum_{m|N} \frac{(\gcd(c,m))^2}{m}\, a_m \;\geq\; 0$$

hold for all positive divisors c of N. It is a cuspidal eta product if and only if all these inequalities hold strictly.

2.5 The Cones and Simplices of Holomorphic Eta Products

According to Corollary 2.3, we introduce rational numbers α_{cm}, a matrix A and a column vector X by

$$\alpha_{cm} = \frac{(\gcd(c,m))^2}{m}, \qquad A = A(N) = (\alpha_{cm})_{c,\,m}, \qquad X = (a_m)_m \in \mathbb{R}^{\tau(N)}, \tag{2.5}$$

where the positive divisors m, c of N are taken in some arbitrary, but fixed order. (Usually the divisors will be in their natural order.) Then the condition for holomorphic eta products of level N reads

$$A(N) \cdot X \geq 0, \tag{2.6}$$

and cuspidal eta products are characterized by $A(N)\,X > 0$. The system of linear inequalities in (2.6) defines an intersection of $\tau(N)$ closed halfspaces in $\mathbb{R}^{\tau(N)}$ whose bounding hyperplanes all pass through the origin. So this system defines a closed simplicial cone with its vertex at the origin. We denote this cone by $\mathcal{K}(N)$, i.e.

$$\mathcal{K}(N) = \{X \in \mathbb{R}^{\tau(N)} \mid A(N)X \geq 0\}. \tag{2.7}$$

We can reformulate Corollary 2.3 as follows:

Corollary 2.4 *An eta product (2.1) is holomorphic if and only if its vector of exponents $X = (a_m)_m$ is a lattice point in the cone $\mathcal{K}(N)$. It is cuspidal if and only if X is an interior point of $\mathcal{K}(N)$.*

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