

2. Geometrical Preliminaries

The description of the geometry is essential for the definition of a shell structure. Our objective in this chapter is to survey the main geometrical concepts, to introduce the related notation and to recall some essential results that will be needed in this book.

2.1 Vectors and Tensors in Three-Dimensional Curvilinear Coordinates

In this section, we provide a brief review of tensor analysis and differential geometry. For more details on these concepts, the reader can refer to (Green & Zerna, 1968), see also (Bernadou, 1996; Ciarlet, 1998).

2.1.1 Vectors and tensors

In the following discussion, we distinguish between

- the three-dimensional (3D) *physical space* – also called the *Euclidean space* – that we denote by \mathcal{E} , in which we can define the geometry and the kinematics of all the mechanical objects that we want to consider (3D bodies, plates, shells...);
- the 3D *mathematical space*, denoted by \mathbb{R}^3 , simply defined as the set of all triples of real numbers, i.e. quantities of the type (ξ^1, ξ^2, ξ^3) where ξ^1 , ξ^2 and ξ^3 are reals¹.

We assume that an origin O is given in \mathcal{E} , so that we can identify points and vectors. Then, by choosing a basis in \mathcal{E} , i.e. three independent vectors \vec{v}_1 , \vec{v}_2 and \vec{v}_3 that are attached to O , we obtain a natural (we can say “canonical”) one-to-one mapping from \mathbb{R}^3 to \mathcal{E} defined by

$$(\xi^1, \xi^2, \xi^3) \mapsto \xi^1 \vec{v}_1 + \xi^2 \vec{v}_2 + \xi^3 \vec{v}_3. \quad (2.1)$$

¹ The reason why we use superscripts in this notation will be explained in the sequel, see Remark 2.1.2.

It is then said that ξ^1 , ξ^2 and ξ^3 are the *components* of the vector “ $\xi^1\vec{v}_1 + \xi^2\vec{v}_2 + \xi^3\vec{v}_3$ ” in the $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ basis. If we now consider the point identified with this vector, i.e. M such that

$$O\vec{M} = \xi^1\vec{v}_1 + \xi^2\vec{v}_2 + \xi^3\vec{v}_3, \quad (2.2)$$

then ξ^1 , ξ^2 and ξ^3 are also called the *coordinates* of M in the *coordinate system* defined by O and $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$.

A *tensor* is an object that generalizes vectors to a higher dimension. For example, we can consider the family of couples of base vectors, that we write

$$\vec{v}_m \otimes \vec{v}_n, \quad m, n = 1, 2, 3, \quad (2.3)$$

calling the symbol \otimes the *tensor product*², and we can use these couples as base vectors for a higher-order vector space that we call the space of second-order tensors, denoted by $\mathcal{E} \otimes \mathcal{E}$. Similarly, we can consider tensors of any order. In order to immediately identify the order of a given tensor, we place over the corresponding symbol a number of arrows equal to its order (or a parenthesized number as left superscript for orders higher than 2). For example, ${}^{(4)}S$ denotes a fourth-order tensor, and $\vec{\vec{T}}$ denotes a second-order tensor that decomposes on the basis given in (2.3) as follows

$$\vec{\vec{T}} = \sum_{m=1}^3 \sum_{n=1}^3 T^{mn} \vec{v}_m \otimes \vec{v}_n. \quad (2.4)$$

Note that, of course, a first-order tensor is a vector. By extension, we will say that a scalar is a zero-order tensor. Of course, a zero-order tensor has no components since it does not depend on any basis.

The main operations on tensors that we will consider are

- **Tensor product:** the tensor product of two tensors is the tensor with an order equal to the sum of the two orders, and components equal to the product of the components. For instance, for two second order tensors $\vec{\vec{U}}$ and $\vec{\vec{T}}$, we have

$$\vec{\vec{U}} \otimes \vec{\vec{T}} = \sum_{m=1}^3 \sum_{n=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 U^{mn} T^{kl} \vec{v}_m \otimes \vec{v}_n \otimes \vec{v}_k \otimes \vec{v}_l. \quad (2.5)$$

- **Dot product:** the dot product on tensors generalizes the classical concept on vectors. It takes the last order of the first argument and the first order

² In some texts the tensor product symbol is not used and (2.3) is simply written as $\vec{v}_m \vec{v}_n$.

of the second argument and combines them using the vector dot product. For example, for two second-order tensors $\vec{\vec{U}}$ and $\vec{\vec{T}}$, we have

$$\vec{\vec{U}} \cdot \vec{\vec{T}} = \sum_{m=1}^3 \sum_{n=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 U^{mn} T^{kl} (\vec{t}_n \cdot \vec{t}_k) \vec{t}_m \otimes \vec{t}_l. \quad (2.6)$$

Note that the result is a tensor of order $s - 2$, where s is the sum of the two orders.

- Double-dot product: the double-dot product combines the last order of the first tensor with the first order of the second one like the dot product and, in addition, it also combines the last but one of the first tensor with the second of the second tensor. Namely, for two second-order tensors, we get

$$\vec{\vec{U}} : \vec{\vec{T}} = \sum_{m=1}^3 \sum_{n=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 U^{mn} T^{kl} (\vec{t}_n \cdot \vec{t}_k) (\vec{t}_m \cdot \vec{t}_l) \quad (2.7)$$

This produces a tensor of order $s - 4$, i.e. a scalar in (2.7).

- Transposition of second-order tensors: for a second-order tensor $\vec{\vec{T}}$, we define the *transposed* tensor $\vec{\vec{T}}^T$ by

$$\vec{\vec{T}}^T = \sum_{m=1}^3 \sum_{n=1}^3 T^{nm} \vec{t}_m \otimes \vec{t}_n. \quad (2.8)$$

2.1.2 Covariant and contravariant bases. Metric tensor

The *contravariant basis* $(\vec{t}^1, \vec{t}^2, \vec{t}^3)$ is inferred from the original (also called *covariant*) basis $(\vec{t}_1, \vec{t}_2, \vec{t}_3)$ by the relations

$$\vec{t}_m \cdot \vec{t}^n = \delta_m^n, \quad \forall m, n = 1, 2, 3, \quad (2.9)$$

where δ denotes the Kronecker symbol³. Note that the relations (2.9) uniquely determine the contravariant base vectors. Then the components of any vector \vec{u} in the covariant basis $(\vec{t}_1, \vec{t}_2, \vec{t}_3)$ will be called the *contravariant components*, denoted by (u^1, u^2, u^3) , and these components can be easily calculated by the formula

$$u^m = \vec{u} \cdot \vec{t}^m, \quad m = 1, 2, 3, \quad (2.10)$$

³ $\delta_m^n = 1$ if $m = n$ and 0 otherwise.

i.e. by using the contravariant basis. Of course, similar expressions hold for higher-order tensors.

In addition, the contravariant basis can be used as an alternative to the covariant basis to express components of vectors and tensors. For example, the components of a vector \vec{u} in the contravariant basis are called the *co-variant components*. They are denoted by (u_1, u_2, u_3) and can be computed by

$$u_m = \vec{u} \cdot \vec{i}_m, \quad m = 1, 2, 3. \quad (2.11)$$

For a higher-order tensor, mixed forms combining covariant and contravariant orders can be used. For example the expression

$$\vec{\vec{T}} = \sum_{m=1}^3 \sum_{n=1}^3 T_{\cdot n}^m \vec{i}_m \otimes \vec{i}^n \quad (2.12)$$

involves the contravariant-covariant components of the second-order tensor $\vec{\vec{T}}$. Note that the “ \cdot ” symbol in $T_{\cdot n}^m$ is used to identify the position of the contravariant and covariant vectors in the decomposition considered (in particular, in general $T_{\cdot n}^m \neq T_n^{\cdot m}$).

From now on, we will use the *Einstein summation convention*, i.e. we will not write summation signs for all indices that appear once as a subscript and once as a superscript in an expression. For example, instead of (2.12) we will simply write

$$\vec{\vec{T}} = T_{\cdot n}^m \vec{i}_m \otimes \vec{i}^n. \quad (2.13)$$

In this case, the indices m and n can be replaced by any other letters. For this reason, they are called dummy indices.

Combinations of covariant and contravariant indices can be very effectively used to compute the results of dot and double-dot products. For example, for two vectors \vec{u} and \vec{v} we have

$$\vec{u} \cdot \vec{v} = (u^m \vec{i}_m) \cdot (v_n \vec{i}^n) = u^m v_n (\vec{i}_m \cdot \vec{i}^n) = u^m v_n \delta_n^m = u^m v_m. \quad (2.14)$$

Similarly, for second-order tensors we obtain

$$\vec{\vec{U}} \cdot \vec{\vec{T}} = U^{mk} T_{kn} \vec{i}_m \otimes \vec{i}^n = U_{\cdot k}^m T_{\cdot n}^k \vec{i}_m \otimes \vec{i}^n = U_m^{\cdot k} T_{kn} \vec{i}^m \otimes \vec{i}^n = \dots \quad (2.15)$$

and

$$\vec{\vec{U}} : \vec{\vec{T}} = U^{mk} T_{km} = U_{\cdot k}^m T_{\cdot m}^k = U_m^{\cdot k} T_k^{\cdot m} = U_{mk} T^{km}. \quad (2.16)$$

Note how the Einstein convention makes all these expressions natural.

We now introduce the *metric tensor*, $\vec{\vec{g}}$, with covariant-covariant components defined by

$$g_{mn} = \vec{i}_m \cdot \vec{i}_n, \quad m, n = 1, 2, 3. \quad (2.17)$$

For any two vectors \vec{u} and \vec{v} ,

$$\vec{u} \cdot \vec{\vec{g}} \cdot \vec{v} = u^m g_{mn} v^n = (u^m \vec{i}_m) \cdot (v^n \vec{i}_n) = \vec{u} \cdot \vec{v}. \quad (2.18)$$

This justifies the term “metric tensor”, since the dot product allows to compute lengths of vectors, hence distances. The other components of the metric tensor are easily obtained. We have

$$g^{mn} = \vec{i}^m \cdot \vec{i}^n, \quad (2.19)$$

and

$$g^m_{\cdot n} = g_n^{\cdot m} = \delta_n^m. \quad (2.20)$$

The components of the metric tensor can be used to obtain contravariant components from covariant components and vice-versa. Indeed, for a vector \vec{u} ,

$$u^m = \vec{u} \cdot \vec{i}^m = (u_n \vec{i}^n) \cdot \vec{i}^m = g^{mn} u_n, \quad (2.21)$$

and similarly

$$u_m = g_{mn} u^n. \quad (2.22)$$

For tensors, similar conversions can be performed by using appropriate components of the metric tensor as many times as necessary. For instance,

$$T_{mn} = g_{mk} T^k_{\cdot n} = g_{nk} T_m^{\cdot k} = g_{mk} g_{nl} T^{kl}. \quad (2.23)$$

Remark 2.1.1. In the special case where the (covariant) basis $(\vec{i}_1, \vec{i}_2, \vec{i}_3)$ is orthonormal, the contravariant basis is identical to the covariant one and the metric tensor satisfies the properties

$$g_{mn} = g^{mn} = \delta_m^n. \quad (2.24)$$

As a consequence, covariant and contravariant components of vectors and tensors are all equal in this case. ■

Example 2.1.1

Consider the three vectors $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ defined by their coordinates in some given orthonormal basis as follows

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \quad (2.25)$$

Clearly, these three vectors are linearly independent, hence they can be used as a covariant basis. With their components in the orthonormal basis we can directly compute the covariant-covariant components of the metric tensor (shown in matrix form)

$$(g_{mn}) = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 3 \\ 2 & 3 & 14 \end{pmatrix}. \quad (2.26)$$

Then, noting that

$$g_{mn}g^{np} = g_m^{\cdot p} = \delta_m^p \quad (2.27)$$

implies that the matrices of coefficients (g_{mn}) and (g^{mn}) are the inverses of each other, we have

$$(g^{mn}) = \begin{pmatrix} \frac{19}{36} & -\frac{11}{18} & \frac{1}{18} \\ -\frac{11}{18} & \frac{13}{9} & -\frac{2}{9} \\ \frac{1}{18} & -\frac{2}{9} & \frac{1}{9} \end{pmatrix}. \quad (2.28)$$

In order to derive the contravariant base vectors, we note that (2.10) implies the identity

$$\vec{u} = (\vec{u} \cdot \vec{v}^n) \vec{v}_n, \quad (2.29)$$

which, used with $\vec{u} = \vec{v}^m$, gives

$$\vec{i}^m = (\vec{i}^m \cdot \vec{i}^n) \vec{i}_n = g^{mn} \vec{i}_n. \quad (2.30)$$

Using this formula to compute the contravariant base vectors, we obtain

$$\vec{i}^1 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{6} \end{pmatrix}, \quad \vec{i}^2 = \begin{pmatrix} 0 \\ 1 \\ -\frac{2}{3} \end{pmatrix}, \quad \vec{i}^3 = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{3} \end{pmatrix}, \quad (2.31)$$

and we can easily check that the relations (2.9) are satisfied. ■

We denote the Euclidean norm of a vector \vec{v} by $\|\vec{v}\|_{\mathcal{E}}$. According to the above discussion we have

$$\|\vec{v}\|_{\mathcal{E}}^2 = \vec{v} \cdot \vec{v} = \vec{v} \cdot \vec{g} \cdot \vec{v} = v_m g^{mn} v_n, \quad (2.32)$$

among other possible expressions. We also have the corresponding inner-product

$$\langle \vec{u}, \vec{v} \rangle_{\mathcal{E}} = \vec{u} \cdot \vec{v} = u_m g^{mn} v_n. \quad (2.33)$$

We can extend these definitions to higher-order tensors. For example, for second-order tensors, we set

$$\|\vec{T}\|_{\mathcal{E}}^2 = T_{mn} g^{mk} g^{nl} T_{kl}, \quad (2.34)$$

with the corresponding inner-product

$$\langle \vec{U}, \vec{T} \rangle_{\mathcal{E}} = U_{mn} g^{mk} g^{nl} T_{kl}. \quad (2.35)$$

Note that we use the same notation as for the norm and the inner-product of first-order tensors, since no confusion is possible. We can also easily see that

$$\langle \vec{U}, \vec{T} \rangle_{\mathcal{E}} = \vec{U}^T : \vec{T}. \quad (2.36)$$

However, this formula is restricted to second-order tensors, whereas (2.35) can be extended to any order.

We now define the *invariants* of a second-order tensor \vec{T} . From the equation

$$\vec{T} \cdot \vec{u} = T_{\cdot n}^m u^n \vec{t}_m, \quad (2.37)$$

we infer that the quantities $T_{\cdot n}^m$ are the coefficients of the matrix corresponding to the linear mapping

$$\vec{u} \mapsto \vec{T} \cdot \vec{u} \quad (2.38)$$

in the covariant basis. We then call the trace and the determinant of this matrix the invariants of the tensor, because they do not depend on the particular covariant basis chosen. Namely we have

$$\boxed{\text{tr} \vec{T} = T_{\cdot m}^m,} \quad (2.39)$$

$$\boxed{\det \vec{T} = \begin{vmatrix} T_{\cdot 1}^1 & T_{\cdot 2}^1 & T_{\cdot 3}^1 \\ T_{\cdot 1}^2 & T_{\cdot 2}^2 & T_{\cdot 3}^2 \\ T_{\cdot 1}^3 & T_{\cdot 2}^3 & T_{\cdot 3}^3 \end{vmatrix}.} \quad (2.40)$$

We will say that the second-order tensor \vec{T} is *symmetric* if it holds that $\vec{T}^T = \vec{T}$. Recalling (2.8), this means that

$$T^{mn} = T^{nm}, \quad m, n = 1, 2, 3, \quad (2.41)$$

which is equivalent to

$$T_{mn} = T_{nm}, \quad m, n = 1, 2, 3. \quad (2.42)$$

and also equivalent to

$$T_{\cdot n}^m = T_n^{\cdot m}, \quad m, n = 1, 2, 3 \quad (2.43)$$

since $T_{\cdot n}^m = g^{mk} T_{kn}$ and $T_n^{\cdot m} = g^{mk} T_{nk}$, so that we can simply denote the mixed components by T_n^m ⁴. Note that the metric tensor is obviously symmetric.

⁴ Note that this does not imply that $T_n^m = T_m^n$, i.e. symmetry of the mixed components.

2.1.3 Curvilinear coordinate systems

It is sometimes useful, natural, or even necessary like in shell analysis, to express relevant quantities using a *curvilinear coordinate system*. Such a coordinate system is defined by the datum of

- a bounded open⁵ subset of \mathbb{R}^3 , denoted by Ω , that we call the *reference domain*;
- a smooth⁶ injective⁷ mapping $\vec{\Phi}$ from $\bar{\Omega}$, the closure⁸ of Ω , into \mathcal{E} . We call this mapping the *chart*.

Any point M in $\vec{\Phi}(\bar{\Omega})$ is then uniquely defined by its coordinates in this curvilinear coordinate system, i.e. the three real numbers (ξ^1, ξ^2, ξ^3) such that

$$OM = \vec{\Phi}(\xi^1, \xi^2, \xi^3). \quad (2.44)$$

A *coordinate curve* is defined by freezing any two of the coordinates and varying the third one. For example, the curve defined by

$$\xi^1 \mapsto \vec{\Phi}(\xi^1, \tilde{\xi}^2, \tilde{\xi}^3), \quad (2.45)$$

for a given choice of $(\tilde{\xi}^2, \tilde{\xi}^3)$, is a ξ^1 -coordinate curve. Of course, in general, these coordinate curves are really curved. Particular examples of curvilinear coordinate systems are provided by spherical and cylindrical coordinates.

At any point M of coordinates (ξ^1, ξ^2, ξ^3) in $\vec{\Phi}(\Omega)$, we can consider the vectors \vec{g}_m , defined by

$$\vec{g}_m = \frac{\partial \vec{\Phi}(\xi^1, \xi^2, \xi^3)}{\partial \xi^m}, \quad m = 1, 2, 3. \quad (2.46)$$

We henceforth assume that the vectors $(\vec{g}_1, \vec{g}_2, \vec{g}_3)$ are linearly independent at all points M in $\vec{\Phi}(\bar{\Omega})$. Then we call $(\vec{g}_1, \vec{g}_2, \vec{g}_3)$ the *covariant basis* at M . We emphasize that, unlike the global basis $(\vec{t}_1, \vec{t}_2, \vec{t}_3)$ considered in the previous sections, the basis $(\vec{g}_1, \vec{g}_2, \vec{g}_3)$ is a *local basis*; i.e., it varies with the point considered. This is an obvious consequence of the curvedness of the coordinate system since, by the definition (2.46), the vectors of the covariant basis at M are tangent to the three coordinate curves passing through M . Figure 2.1 shows an example of a curvilinear coordinate system.

⁵ i.e. which does not contain its boundary

⁶ i.e. which can be differentiated as many times as needed

⁷ i.e. two different elements necessarily have different images through $\vec{\Phi}$

⁸ i.e. Ω itself, together with its boundary $\partial\Omega$

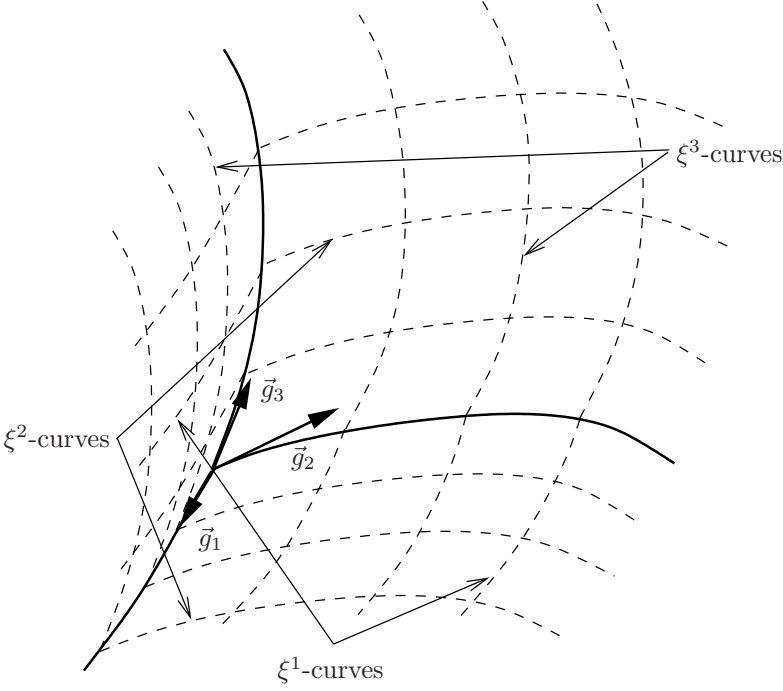


Fig. 2.1. Curvilinear coordinate system

Remark 2.1.2. Using superscripts in the notation of the coordinate symbols (ξ^1, ξ^2, ξ^3) is consistent with the definition of the covariant basis, since we can write

$$d\vec{\Phi} = \vec{g}_i d\xi^i, \quad (2.47)$$

where the symbol “ d ” denotes the differential. ■

Remark 2.1.3. Consider the particular chart given by

$$\vec{\Phi}(\xi^1, \xi^2, \xi^3) = \xi^1 \vec{t}_1 + \xi^2 \vec{t}_2 + \xi^3 \vec{t}_3. \quad (2.48)$$

Then the local covariant basis corresponds to $(\vec{t}_1, \vec{t}_2, \vec{t}_3)$ everywhere, and the coordinate system is in fact rectilinear in this case. ■

Using the local basis, we can define components of vectors and tensors as before, but these components are now defined locally. For instance, the twice-covariant components of the metric tensor are given by

$$g_{mn} = \vec{g}_m \cdot \vec{g}_n, \quad m, n = 1, 2, 3, \quad (2.49)$$

and all other formulae from (2.19) to (2.23) still hold (by substituting \vec{g} for \vec{i} when needed) with, in particular, the contravariant basis $(\vec{g}^1, \vec{g}^2, \vec{g}^3)$ uniquely defined by

$$\vec{g}_m \cdot \vec{g}^n = \delta_m^n, \quad \forall m, n = 1, 2, 3. \quad (2.50)$$

In order to perform integrations using a curvilinear coordinate system, we need to express the infinitesimal volume corresponding to the coordinate differentials $(d\xi^1, d\xi^2, d\xi^3)$. This infinitesimal volume, denoted by dV , is given by

$$dV = |[\vec{g}_1, \vec{g}_2, \vec{g}_3]| d\xi^1 d\xi^2 d\xi^3, \quad (2.51)$$

where $[\vec{g}_1, \vec{g}_2, \vec{g}_3]$ is the mixed product of the three vectors, namely

$$[\vec{g}_1, \vec{g}_2, \vec{g}_3] = \vec{g}_1 \cdot (\vec{g}_2 \wedge \vec{g}_3), \quad (2.52)$$

with the symbol “ \wedge ” denoting the vector cross product. We can show that

$$|[\vec{g}_1, \vec{g}_2, \vec{g}_3]| = \sqrt{g}, \quad (2.53)$$

where g denotes the determinant of the matrix of coefficients $(g_{mn})_{m,n=1,2,3}$.

Proof of (2.53). Let Φ_m denote the components of $\vec{\Phi}$ in an orthonormal basis. Then we have

$$g_{mn} = \vec{g}_m \cdot \vec{g}_n = \frac{\partial \Phi_p}{\partial \xi^m} \frac{\partial \Phi_p}{\partial \xi^n}. \quad (2.54)$$

Hence

$$\det(g_{mn}) = \det(G^T G) = (\det G)^2, \quad (2.55)$$

where G denotes the (3×3) matrix of coefficients

$$G_{mn} = \frac{\partial \Phi_m}{\partial \xi^n}, \quad (2.56)$$

i.e. the matrix of the components of the covariant basis in the orthonormal system, and G^T denotes the transpose of G . Hence

$$\det G = [\vec{g}_1, \vec{g}_2, \vec{g}_3], \quad (2.57)$$

and (2.53) immediately follows. ■

Therefore, the integral of a function f in curvilinear coordinates will take the form

$$\int_{\xi^1, \xi^2, \xi^3} f(\xi^1, \xi^2, \xi^3) \sqrt{g} d\xi^1 d\xi^2 d\xi^3. \quad (2.58)$$

2.1.4 Covariant differentiation

In our subsequent discussions, we will often be concerned with derivatives of *invariant quantities*. We call invariant a quantity that does not depend on a particular choice of coordinate system. For example, a scalar field defined in a region of the physical space (this field may represent a temperature, a pressure, . . .) is invariant. Likewise, a vector field (e.g. a velocity or displacement field) or, more generally speaking, a tensor field are invariant. Also, the invariants of a tensor are, of course, invariant. By contrast, the components of a tensor field (except for a zero-order tensor) depend on the specific coordinate system used and are hence obviously not invariant.

Clearly, the gradient of an invariant quantity must also be an invariant quantity, hence *the gradient of a tensor is also a tensor*. Note that in this statement we omitted the word “field” which will henceforth be implicit when considering derivatives (or gradients) of tensors. We now want to express the components (in a given basis) of the gradient of a tensor using the components of the tensor itself.

Let us start with a zero-order tensor (scalar field) T . By definition, the gradient of T is the first-order tensor, denoted by $\vec{\nabla}T$, such that, for any vector \vec{u} , the quantity $\vec{\nabla}T \cdot \vec{u}$ gives the “variation of T along \vec{u} ”. Namely

$$\vec{\nabla}T \cdot \vec{u} = \frac{d}{dx} [T(\xi^1(x), \xi^2(x), \xi^3(x))], \quad (2.59)$$

for any parametrization $x \mapsto (\xi^1(x), \xi^2(x), \xi^3(x))$ chosen so that \vec{u} is the tangent vector of the curve given by $\vec{\Phi}(\xi^1(x), \xi^2(x), \xi^3(x))$. By the chain rule, we have

$$\frac{d}{dx} \vec{\Phi}(\xi^1(x), \xi^2(x), \xi^3(x)) = (\xi^1)'(x) \vec{g}_1 + (\xi^2)'(x) \vec{g}_2 + (\xi^3)'(x) \vec{g}_3 = \vec{u}. \quad (2.60)$$

Hence an appropriate parametrization must satisfy

$$(\xi^m)' = u^m, \quad m = 1, 2, 3. \quad (2.61)$$

Using the chain rule in (2.59) we infer

$$\vec{\nabla} T \cdot \vec{u} = T_{,1}u^1 + T_{,2}u^2 + T_{,3}u^3, \quad (2.62)$$

where we adopt the classical notation

$$T_{,m} = \frac{\partial T}{\partial \xi^m}, \quad m = 1, 2, 3. \quad (2.63)$$

Hence the covariant components of $\vec{\nabla} T$ are simply the ordinary derivatives of T along each coordinate curve, and of course

$$\vec{\nabla} T = T_{,m} \vec{g}^m. \quad (2.64)$$

For a first-order tensor (namely, a vector), the mathematics is slightly more complicated. For a vector \vec{u} , by definition, the gradient of \vec{u} is a second-order tensor, denoted by $\vec{\vec{\nabla}} \vec{u}$, such that

$$\vec{\vec{\nabla}} \vec{u} \cdot \vec{v} = \frac{d}{dx} [\vec{u}(\xi^1(x), \xi^2(x), \xi^3(x))], \quad (2.65)$$

for a parametrization $x \mapsto (\xi^1(x), \xi^2(x), \xi^3(x))$ such that $(\xi^m)' = v^m$ ($m = 1, 2, 3$) (i.e. the corresponding tangent vector is \vec{v}). By applying the chain rule, we obtain

$$\begin{aligned} \vec{\vec{\nabla}} \vec{u} \cdot \vec{v} &= (\vec{u}_{,m} v^m) \\ &= (u_{n,m} \vec{g}^n + u_n \vec{g}^n_{,m}) v^m \\ &= (u_{n,m} \vec{g}^n + u_k \vec{g}^k_{,m}) v^m \\ &= (u_{n,m} + u_k \vec{g}^k_{,m} \cdot \vec{g}_n) \vec{g}^n v^m, \end{aligned}$$

where we used the identity

$$\vec{w} = (\vec{w} \cdot \vec{g}_n) \vec{g}^n \quad (2.66)$$

with $\vec{w} = \vec{g}^k_{,m}$ in order to obtain the decomposition of this vector on the contravariant basis. Then, since $\vec{g}^k \cdot \vec{g}_n = \delta_n^k$ implies $\vec{g}^k_{,m} \cdot \vec{g}_n = -\vec{g}^k \cdot \vec{g}_{n,m}$, we obtain

$$\vec{\vec{\nabla}} \vec{u} \cdot \vec{v} = (u_{n,m} - u_k \vec{g}_{n,m} \cdot \vec{g}^k) \vec{g}^n v^m \quad (2.67)$$

We classically define the 3D *Christoffel symbols*

$$\bar{\Gamma}_{nm}^k = \vec{g}_{n,m} \cdot \vec{g}^k.$$

(2.68)

Hence, the (n, m) covariant-covariant component of $\vec{\nabla} \vec{u}$, that we denote by $u_{n||m}$, is

$$\boxed{u_{n||m} = u_{n,m} - \bar{\Gamma}_{nm}^k u_k,} \quad (2.69)$$

and then, of course,

$$\vec{\nabla} \vec{u} = u_{n||m} \vec{g}^n \otimes \vec{g}^m \quad (2.70)$$

We call $u_{n||m}$ the *covariant derivative* of u_n . If we want to use the contravariant components of \vec{u} , it can easily be checked that the (n, m) contravariant-covariant component of $\underline{\nabla} \vec{u}$, denoted by $u^n_{\cdot||m}$ and called the covariant derivative of u^n , is given by

$$\boxed{u^n_{\cdot||m} = u^n_{\cdot,m} + \bar{\Gamma}_{mk}^n u^k,} \quad (2.71)$$

and therefore also

$$\vec{\nabla} \vec{u} = u^n_{\cdot||m} \vec{g}_n \otimes \vec{g}^m. \quad (2.72)$$

There is no simple expression of the components of the gradient for which the second index is in contravariant position.

Remark 2.1.4. By definition, the derivative of a vector \vec{u} along the m -th coordinate line satisfies

$$\vec{u}_{\cdot,m} = \vec{\nabla} \vec{u} \cdot \vec{g}_m = u_{n||m} \vec{g}^n \quad (2.73)$$

hence the coefficients $(u_{n||m})_{n=1,2,3}$ (respectively $(u^n_{\cdot||m})_{n=1,2,3}$) are the covariant (respectively contravariant) components of the vector $\vec{u}_{\cdot,m}$. ■

For higher-order tensors, covariant differentiation formulae are more complex. We only give the expressions of covariant derivatives of second-order tensors.

$$T_{mn||p} = T_{mn,p} - \bar{\Gamma}_{mp}^k T_{kn} - \bar{\Gamma}_{np}^k T_{mk}, \quad (2.74)$$

$$T^m_{\cdot n||p} = T^m_{\cdot n,p} + \bar{\Gamma}_{kp}^m T^k_{\cdot n} - \bar{\Gamma}_{np}^k T^m_{\cdot k}, \quad (2.75)$$

$$T^{m \cdot n}_{\cdot||p} = T^{m \cdot n}_{\cdot,p} - \bar{\Gamma}_{mp}^k T^{m \cdot n}_{\cdot k} + \bar{\Gamma}_{kp}^n T^m_{\cdot k}, \quad (2.76)$$

$$T^{\bar{m}\bar{n}}_{\cdot\cdot||p} = T^{mn}_{\cdot p} + \bar{\Gamma}^m_{kp} T^{kn} + \bar{\Gamma}^n_{kp} T^{mk}, \quad (2.77)$$

and we have

$${}^{(3)}\nabla \vec{T} = T_{mn||p} \vec{g}^m \otimes \vec{g}^n \otimes \vec{g}^p, \quad (2.78)$$

and similarly when using the other tensor representations.

Remark 2.1.5. The gradient of the metric tensor is zero. Indeed, in a global orthonormal coordinate system, the covariant and contravariant components of the metric tensor are the same, namely the components of the Kronecker symbol, and covariant differentiation is the same as usual differentiation. Hence, all the components of the gradient in this coordinate system are zero and, since the gradient is a tensor, it is the zero tensor. ■

Another important property of covariant derivatives is that they follow “usual rules” (i.e. the same as ordinary derivatives) for the differentiation of products. For example, for two vectors \vec{u} and \vec{v} , recalling that covariant and ordinary differentiation are the same for a scalar, we have

$$(u_m v^m)_{,n} = (u_m v^m)_{||n} = u_{m||n} v^m + u_m v^m_{||n}. \quad (2.79)$$

Likewise, we have

$$(T^{mn} v_n)_{||p} = T^{\bar{m}\bar{n}}_{\cdot\cdot||p} v_n + T^{mn} v_{n||p}, \quad (2.80)$$

and all other covariant derivatives of products are similarly obtained in this natural fashion.

2.2 The Shell Geometry

The purpose of this section is to introduce the geometric concepts (notations, definitions and basic properties) needed in the analysis of mathematical shell models.

2.2.1 Geometric definition of a shell

We consider a shell to be a solid medium geometrically defined by a *mid-surface* immersed in the physical space \mathcal{E} , and a parameter representing the *thickness* of the medium around this surface. In general, the midsurface of a given shell is defined by a collection of *two-dimensional charts*, i.e. smooth injective mappings from domains of \mathbb{R}^2 (the reference domains associated with each chart) into \mathcal{E} . Note indeed that the mid-surface of a general shell

may consist of the collection of several smooth surfaces assembled along folds, and that even a smooth surface cannot always be defined by a single two-dimensional (2D) chart (e.g. a sphere). However, in complex configurations, the analysis will be decomposed according to each chart and each reference domain (by writing that the global energy is the sum of energies on all sub-parts). Therefore, without loss of generality, we now focus on shells defined using a single chart.

We thus consider a shell with a midsurface (denoted by \mathcal{S}) defined by a 2D chart $\vec{\phi}$ which is an injective mapping from the closure of a bounded open subset of \mathbb{R}^2 , denoted by ω , into \mathcal{E} , hence $\mathcal{S} = \vec{\phi}(\bar{\omega})$. We assume that $\vec{\phi}$ is such that, at each point of the midsurface, the vectors

$$\boxed{\vec{a}_\alpha = \frac{\partial \vec{\phi}(\xi^1, \xi^2)}{\partial \xi^\alpha},} \quad (2.81)$$

are linearly independent, so that they form a basis of the plane tangent to the midsurface at this point. We define the unit normal vector

$$\boxed{\vec{a}_3 = \frac{\vec{a}_1 \wedge \vec{a}_2}{\|\vec{a}_1 \wedge \vec{a}_2\|}.} \quad (2.82)$$

The 3D medium corresponding to the shell is then defined by the 3D chart given by

$$\boxed{\vec{\Phi}(\xi^1, \xi^2, \xi^3) = \vec{\phi}(\xi^1, \xi^2) + \xi^3 \vec{a}_3(\xi^1, \xi^2),} \quad (2.83)$$

for (ξ^1, ξ^2, ξ^3) , in Ω , where Ω is the 3D reference domain defined by

$$\Omega = \left\{ (\xi^1, \xi^2, \xi^3) \in \mathbb{R}^3 \mid \begin{array}{l} (\xi^1, \xi^2) \in \omega, \xi^3 \in \left[-\frac{t(\xi^1, \xi^2)}{2}, +\frac{t(\xi^1, \xi^2)}{2} \right] \end{array} \right\}. \quad (2.84)$$

In this definition, $t(\xi^1, \xi^2)$ represents the *thickness* of the shell at the point of coordinates (ξ^1, ξ^2) . We denote by \mathcal{B} the region of the Euclidean space occupied by the shell body, namely

$$\mathcal{B} = \vec{\Phi}(\bar{\Omega}). \quad (2.85)$$

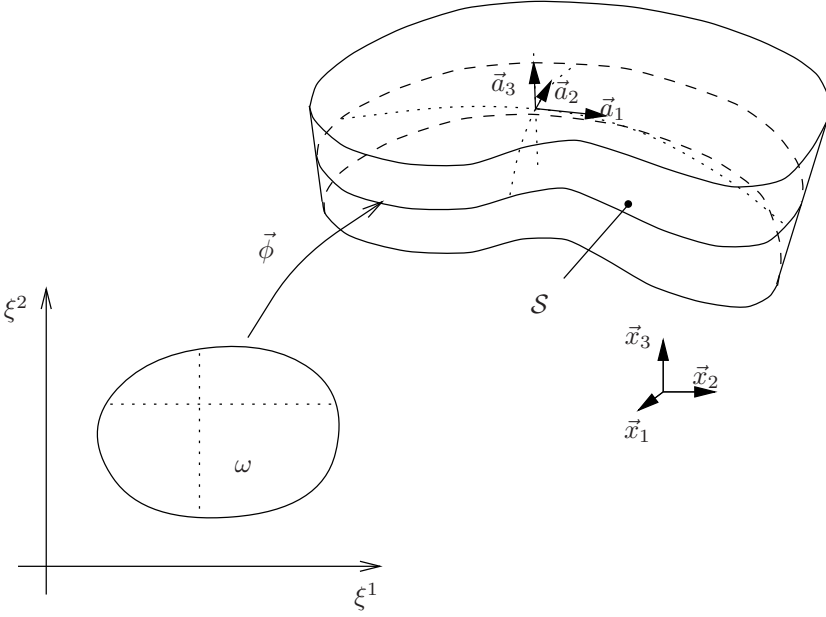


Fig. 2.2. Geometric description of a shell

Figure 2.2 shows an example of shell described by its midsurface and the thickness parameter. The notation $(\vec{x}_1, \vec{x}_2, \vec{x}_3)$ will be used in the sequel, like in this figure, to denote a reference orthonormal basis.

The 3D chart $\vec{\phi}$ and the reference domain Ω provide a natural parametrization of the shell body, i.e. describe the shell with a natural (since it is based on the midsurface) curvilinear coordinate system. In order to be able to easily express and manipulate tensors in this specific coordinate system, we now introduce some basic concepts of surface differential geometry.

2.2.2 Differential geometry on the midsurface

We can introduce and use *surface tensors* on the midsurface of the shell in a manner very similar to what we did in three dimensions in Section 2.1. At each point, recalling that (\vec{a}_1, \vec{a}_2) is a basis of the tangent plane, we call this basis the *covariant basis* and we define the *contravariant basis* of the tangent plane (\vec{a}^1, \vec{a}^2) by

$$\vec{a}_\alpha \cdot \vec{a}^\beta = \delta_\alpha^\beta, \quad \alpha, \beta = 1, 2. \quad (2.86)$$

First-order surface tensors are vectors of the tangent plane, hence they are uniquely determined by their components in either one of the above-defined

bases. To distinguish surface tensors from 3D tensors, we denote the former by symbols with a number of underbars corresponding to their order (or with a parenthesized number as left *subscript* for orders higher than 2). For example, a first-order surface tensor is denoted such as in “ \underline{v} ”; however, the notation of the bases (\vec{a}_1, \vec{a}_2) and (\vec{a}^1, \vec{a}^2) is an exception to this rule. Note that we use Greek indices for the components of surface tensors, in order to distinguish them from components of 3D tensors denoted with latin indices, hence Greek indices will henceforth implicitly vary in $\{1, 2\}$.

The restriction of the metric tensor to the tangent plane, also called the *first fundamental form* of the surface, is given by its components

$$\boxed{a_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta,} \quad (2.87)$$

or alternatively in contravariant form by

$$\boxed{a^{\alpha\beta} = \vec{a}^\alpha \cdot \vec{a}^\beta.} \quad (2.88)$$

Of course, we also have

$$\boxed{a^\alpha_{\cdot\beta} = a^\alpha_{\beta\cdot} = \delta^\alpha_\beta.} \quad (2.89)$$

The first fundamental form can be used to convert covariant components into contravariant ones, such as in

$$v^\alpha = a^{\alpha\lambda} v_\lambda, \quad (2.90)$$

where we use the Einstein convention (with Greek indices varying from 1 to 2).

We denote the Euclidean norm of surface tensors by $\|\cdot\|_\mathcal{E}$, and the corresponding inner-product by $\langle \cdot, \cdot \rangle_\mathcal{E}$, like for 3D geometry. Of course, in order to evaluate these quantities we can use the first fundamental form. For example we have

$$\langle \underline{u}, \underline{v} \rangle_\mathcal{E} = u_\alpha a^{\alpha\beta} v_\beta, \quad (2.91)$$

$$\|\underline{v}\|_\mathcal{E}^2 = v_\alpha a^{\alpha\beta} v_\beta, \quad (2.92)$$

$$\langle \underline{\underline{T}}, \underline{\underline{U}} \rangle_\mathcal{E} = T_{\alpha\beta} a^{\alpha\lambda} a^{\beta\mu} U_{\lambda\mu}, \quad (2.93)$$

$$\|\underline{\underline{T}}\|_\mathcal{E}^2 = T_{\alpha\beta} a^{\alpha\lambda} a^{\beta\mu} T_{\lambda\mu}. \quad (2.94)$$

The first fundamental form is also useful to express surface integrals. Indeed, the infinitesimal area corresponding to the differentials $(d\xi^1, d\xi^2)$ of the coordinates can be expressed as

$$dS = \sqrt{a} d\xi^1 d\xi^2, \quad (2.95)$$

with

$$a = a_{11}a_{22} - (a_{12})^2. \quad (2.96)$$

Proof of (2.95). The infinitesimal surface area is given by

$$dS = \|\vec{a}_1 \wedge \vec{a}_2\| d\xi^1 d\xi^2. \quad (2.97)$$

and we show in Section 2.2.3 that

$$\|\vec{a}_1 \wedge \vec{a}_2\| = \sqrt{\det(a_{\alpha\beta})} = \sqrt{a}. \quad (2.98)$$

■

Note that second-order surface tensors also have invariants. Namely

$$\text{tr}\underline{\underline{T}} = T^\alpha_{\cdot\alpha}, \quad (2.99)$$

$$\det\underline{\underline{T}} = \begin{vmatrix} T^1_{\cdot 1} & T^1_{\cdot 2} \\ T^2_{\cdot 1} & T^2_{\cdot 2} \end{vmatrix}. \quad (2.100)$$

Another crucial second-order tensor is the *second fundamental form* of the surface, denoted by $\underline{\underline{b}}$. It is defined by

$$b_{\alpha\beta} = \vec{a}_3 \cdot \vec{a}_{\alpha,\beta}. \quad (2.101)$$

The second fundamental form is also called the *curvature tensor*, because it contains all the information on the curvature of the surface. Consider indeed \underline{n} , a unit vector in the tangent plane, and the curve obtained by intersecting

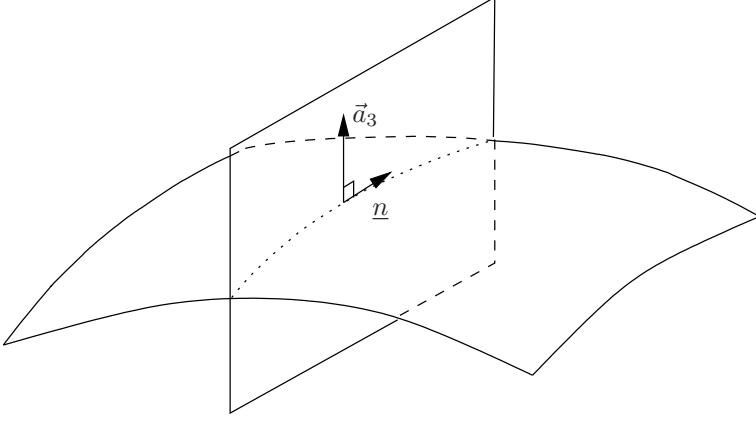


Fig. 2.3. Intersection of the surface with a plane containing \vec{a}_3

the midsurface with the plane defined by \underline{n} (considered as a vector of \mathcal{E}) and \vec{a}_3 , see Figure 2.3. A parametrization of this curve by its arc-length will be of the form

$$x \mapsto \vec{\phi}(\xi^1(x), \xi^2(x)), \quad (2.102)$$

with $(\xi^\alpha)'(x) = n^\alpha$ ($\alpha = 1, 2$) at the point in consideration, as

$$\underline{n} = \frac{\partial \vec{\phi}}{\partial \xi^\alpha} \frac{d\xi^\alpha}{dx} = \frac{d\xi^\alpha}{dx} \vec{a}_\alpha, \quad (2.103)$$

recalling that x represents the arc-length. Then, noting that

$$b_{\alpha\beta} = -\vec{a}_{3,\beta} \cdot \vec{a}_\alpha, \quad (2.104)$$

(since $\vec{a}_3 \cdot \vec{a}_\alpha = 0$), we infer

$$\begin{aligned} \underline{n} \cdot \underline{\underline{b}} \cdot \underline{n} &= b_{\alpha\beta} n^\alpha n^\beta = b_{\alpha\beta} (\xi^\alpha)' (\xi^\beta)' = -(\xi^\beta)' \vec{a}_{3,\beta} \cdot (\xi^\alpha)' \vec{a}_\alpha \\ &= -\frac{d\vec{a}_3}{dx} \cdot \underline{n} = \vec{a}_3 \cdot \frac{d\underline{n}}{dx}, \end{aligned} \quad (2.105)$$

since $\vec{a}_3 \cdot \underline{n} = 0$ along the intersection curve. Hence, noting that \vec{a}_3 is the unit normal vector to this (planar) curve at the point of consideration, this implies that $\underline{n} \cdot \underline{\underline{b}} \cdot \underline{n}$ is the curvature of the curve, counted positively when \vec{a}_3 points towards the center of curvature (for example, it is negative in Fig. 2.3). Therefore, by considering the quantity

$$\frac{b_{\alpha\beta} v^\alpha v^\beta}{a_{\alpha\beta} v^\alpha v^\beta} \quad (2.106)$$

when \underline{v} varies (\underline{v} non-zero but not necessarily of unit length) we obtain all the curvatures of such curves passing through one specific point, since the denominator normalizes the numerator by the square of the norm of \underline{v} and varying \underline{v} in the tangent plane thus amounts to rotating the intersecting plane around \vec{a}_3 in Fig. 2.3. The tensors \underline{a} and \underline{b} are both symmetric (see Remark 2.2.1 below), hence the quantity (2.106) can be seen as a Rayleigh quotient. Therefore there exist two directions corresponding to its minimum and its maximum, and these directions are \underline{a} -orthogonal, i.e. they are orthogonal in the usual sense since \underline{a} is the surface metric tensor. The values of the curvature along these directions are called the *principal curvatures*. The half-sum and the product of the principal curvatures are classically called the *mean curvature* and *Gaussian curvature*, respectively.

The mean and Gaussian curvatures of the surface can be respectively obtained by

$$H = \frac{1}{2}(b_1^1 + b_2^2) = \frac{1}{2}\text{tr}\underline{b} \quad (2.107)$$

and

$$K = b_1^1 b_2^2 - b_2^1 b_1^2 = \det \underline{b}. \quad (2.108)$$

Here we have

$$b_\beta^\alpha = a^{\alpha\lambda} b_{\lambda\beta} = -a^{\alpha\lambda} \vec{a}_{3,\beta} \cdot \vec{a}_\lambda = -\vec{a}_{3,\beta} \cdot \vec{a}^\alpha = \vec{a}_{,\beta}^\alpha \cdot \vec{a}_3, \quad (2.109)$$

since $\vec{a}^\alpha \cdot \vec{a}_3 = 0$. Note that H and K are defined using invariants of the curvature tensor, hence they do not depend on the specific coordinate system considered.

Proof of (2.107) and (2.108). We can change the coordinate system so that, at the point in consideration, the covariant basis coincides with orthonormal vectors tangent to the directions of principal curvatures. In this new coordinate system (and at this specific point), covariant and contravariant components are the same⁹ and the new coefficients of \underline{b} , denoted by $\tilde{b}_{\alpha\beta}$ are such that \tilde{b}_{11} and \tilde{b}_{22} are the principal curvatures, and $\tilde{b}_{12} = \tilde{b}_{21} = 0$. Hence, (2.107) and (2.108) hold in this specific coordinate system and, since

⁹ Note that covariant and contravariant components coincide in *any* orthonormal coordinate system, and this can be used to easily compute the mean and Gaussian curvatures with the above formulae.

these two expressions correspond to the two invariants of $\underline{\underline{b}}$, we infer that they must hold in any coordinate system. ■

We also define the *third fundamental form* by

$$\boxed{c_{\alpha\beta} = b_{\alpha}^{\lambda} b_{\lambda\beta}.} \quad (2.110)$$

Remark 2.2.1. The three fundamental forms are symmetric. This is obvious for $\underline{\underline{a}}$ (as it is for the metric tensor in 3D). For $\underline{\underline{b}}$, it is directly inferred from (2.101), noting that

$$\vec{a}_{\alpha,\beta} = \frac{\partial^2 \phi}{\partial \xi^{\alpha} \partial \xi^{\beta}} = \vec{a}_{\beta,\alpha}. \quad (2.111)$$

As for $\underline{\underline{c}}$, we have

$$c_{\alpha\beta} = b_{\alpha}^{\lambda} b_{\lambda\beta} = b_{\alpha\mu} a^{\mu\lambda} b_{\lambda\beta} = b_{\alpha\mu} b_{\beta}^{\mu} = c_{\beta\alpha}. \quad (2.112)$$

■

Remark 2.2.2. $\vec{a}_3 \cdot \vec{a}_3 = 1$ implies $\vec{a}_{3,\alpha} \cdot \vec{a}_3 = 0$, i.e. $\vec{a}_{3,\alpha}$ lies in the tangent plane. Hence we have

$$\vec{a}_{3,\alpha} = (\vec{a}_{3,\alpha} \cdot \vec{a}_{\lambda}) \vec{a}^{\lambda}, \quad (2.113)$$

and thus, recalling (2.104),

$$\vec{a}_{3,\alpha} = -b_{\alpha\lambda} \vec{a}^{\lambda} = -b_{\alpha}^{\lambda} \vec{a}_{\lambda}. \quad (2.114)$$

■

We now come to covariant differentiation of surface tensors. We start with a vector \underline{u} . In order to differentiate this vector, which, of course, can only be done along the surface (since the vector field is not defined outside the surface), we can see it as a vector of \mathcal{E} . We consider the quantity

$$\frac{d}{dx} [\underline{u}(\xi^1(x), \xi^2(x))] = (\xi^{\alpha})'(x) \frac{\partial \underline{u}}{\partial \xi^{\alpha}}, \quad (2.115)$$

i.e. the derivative of \underline{u} along a curve lying on the surface and described by $(\xi^1(x), \xi^2(x))$. We have

$$\begin{aligned}
\frac{\partial \underline{u}}{\partial \xi^\alpha} &= u_{\beta,\alpha} \vec{a}^\beta + u_\beta \vec{a}^\beta_{,\alpha} \\
&= u_{\beta,\alpha} \vec{a}^\beta + u_\lambda \vec{a}^\lambda_{,\alpha} \\
&= (u_{\beta,\alpha} + u_\lambda \vec{a}^\lambda_{,\alpha} \cdot \vec{a}_\beta) \vec{a}^\beta + (u_\lambda \vec{a}^\lambda_{,\alpha} \cdot \vec{a}_3) \vec{a}_3,
\end{aligned} \tag{2.116}$$

where we used the identity

$$\vec{w} = (\vec{w} \cdot \vec{a}_\beta) \vec{a}^\beta + (\vec{w} \cdot \vec{a}_3) \vec{a}_3, \tag{2.117}$$

since \vec{a}_3 is a unit-vector normal to the tangent plane, and we applied this identity with $\vec{w} = \vec{a}^\lambda_{,\alpha}$. Hence, defining the *surface Christoffel symbols*

$$\boxed{\Gamma^\lambda_{\beta\alpha} = \vec{a}_{\beta,\alpha} \cdot \vec{a}^\lambda} \tag{2.118}$$

and noting that $\vec{a}_\beta \cdot \vec{a}^\lambda = \delta^\lambda_\beta$ implies

$$\Gamma^\lambda_{\beta\alpha} = -\vec{a}_\beta \cdot \vec{a}^\lambda_{,\alpha}, \tag{2.119}$$

we obtain from (2.116)

$$\frac{\partial \underline{u}}{\partial \xi^\alpha} = (u_{\beta,\alpha} - \Gamma^\lambda_{\beta\alpha} u_\lambda) \vec{a}^\beta + b^\lambda_\alpha u_\lambda \vec{a}_3. \tag{2.120}$$

Finally, calling \underline{v} the tangent vector of the curve $(\xi^1(x), \xi^2(x))$, namely $v^\alpha = (\xi^\alpha)'$, we define the surface gradient of \underline{u} , denoted by $\underline{\nabla} \underline{u}$, as the second-order surface tensor which, acting on \underline{v} through the dot product, yields the tangential part of (2.115). Thus, denoting by $u_{\beta|\alpha}$ the covariant-covariant components of $\underline{\nabla} \underline{u}$, we have

$$\boxed{u_{\beta|\alpha} = u_{\beta,\alpha} - \Gamma^\lambda_{\beta\alpha} u_\lambda}, \tag{2.121}$$

and $u_{\beta|\alpha}$ is called a *surface covariant derivative* of u_β . Note that the expressions of surface Christoffel symbols and of surface covariant derivatives are very similar to their 3D counterparts, with Greek indices instead of Latin indices. Likewise, we can obtain the formulae for contravariant components and for higher-order tensors by adapting the corresponding 3D expressions.

Since we are primarily concerned with shells (hence with surfaces), we will from now on omit the term “surface” when referring to quantities which pertain to surface differential geometry, and instead specify “3D” when referring to 3D differential geometry.

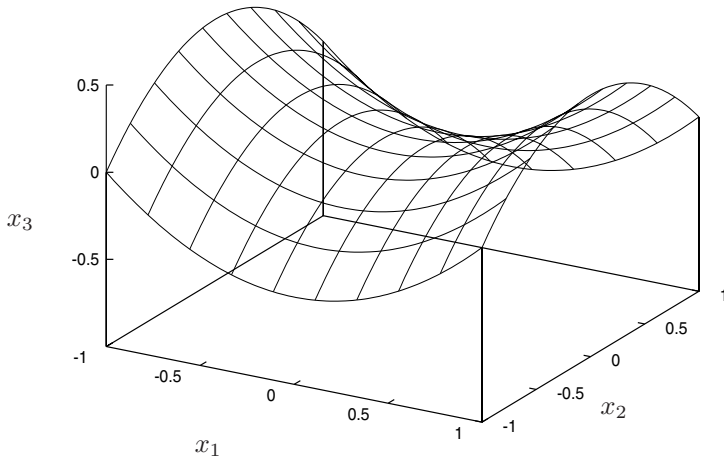


Fig. 2.4. Hyperbolic paraboloid

Example 2.2.1

Consider the surface described by the chart defined by its coordinates in the reference coordinate system as follows

$$\vec{\phi}(\xi^1, \xi^2) = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \frac{(\xi^1)^2 - (\xi^2)^2}{2} \end{pmatrix}, \quad \omega =]-1, 1[^2. \quad (2.122)$$

This surface, called a hyperbolic paraboloid, is shown in Figure 2.4. Note that – for dimensional (and physical) correctness – since a position vector has the dimension of a length we need to specify the unit in which (2.122) is to be understood. In other words, we can say that the chart is, in fact,

$$\vec{\phi} = \begin{pmatrix} L\xi^1 \\ L\xi^2 \\ L\frac{(\xi^1)^2 - (\xi^2)^2}{2} \end{pmatrix}, \quad \omega =]-1, 1[^2, \quad (2.123)$$

where L is a reference length that is taken equal to one in the unit considered for this example. For further considerations on dimensions, see also Remark 4.1.3. We then have

$$\vec{a}_1 = \begin{pmatrix} 1 \\ 0 \\ \xi^1 \end{pmatrix}, \quad \vec{a}_2 = \begin{pmatrix} 0 \\ 1 \\ -\xi^2 \end{pmatrix}, \quad (2.124)$$

$$(a_{\alpha\beta})_{\alpha,\beta=1,2} = \begin{pmatrix} 1 + (\xi^1)^2 & -\xi^1\xi^2 \\ -\xi^1\xi^2 & 1 + (\xi^2)^2 \end{pmatrix}, \quad (2.125)$$

$$a = \det(a_{\alpha\beta}) = 1 + (\xi^1)^2 + (\xi^2)^2, \quad (2.126)$$

$$\vec{a}_1 \wedge \vec{a}_2 = \begin{pmatrix} -\xi^1 \\ \xi^2 \\ 1 \end{pmatrix} \Rightarrow \vec{a}_3 = \frac{1}{\sqrt{a}} \begin{pmatrix} -\xi^1 \\ \xi^2 \\ 1 \end{pmatrix}. \quad (2.127)$$

Noting that

$$a_{\alpha\beta}a^{\beta\gamma} = \delta_{\alpha}^{\gamma} \quad (2.128)$$

implies that the matrices $(a_{\alpha\beta})$ and $(a^{\alpha\beta})$ are the inverses of each other, we infer

$$(a^{\alpha\beta})_{\alpha,\beta=1,2} = \frac{1}{a} \begin{pmatrix} 1 + (\xi^2)^2 & \xi^1\xi^2 \\ \xi^1\xi^2 & 1 + (\xi^1)^2 \end{pmatrix}. \quad (2.129)$$

We can now easily compute the contravariant base vectors

$$\vec{a}^1 = a^{11}\vec{a}_1 + a^{12}\vec{a}_2 = \frac{1}{a} \begin{pmatrix} 1 + (\xi^2)^2 \\ \xi^1\xi^2 \\ \xi^1 \end{pmatrix}, \quad (2.130)$$

$$\vec{a}^2 = a^{21}\vec{a}_1 + a^{22}\vec{a}_2 = \frac{1}{a} \begin{pmatrix} \xi^1\xi^2 \\ 1 + (\xi^1)^2 \\ -\xi^2 \end{pmatrix}. \quad (2.131)$$

Furthermore

$$\vec{a}_{1,1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{a}_{1,2} = \vec{a}_{2,1} = \vec{0}, \quad \vec{a}_{2,2} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad (2.132)$$

hence

$$b_{11} = \frac{1}{\sqrt{a}}, \quad b_{12} = b_{21} = 0, \quad b_{22} = -\frac{1}{\sqrt{a}}, \quad (2.133)$$

$$\Gamma_{11}^1 = -\Gamma_{22}^1 = \frac{\xi^1}{a}, \quad \Gamma_{11}^2 = -\Gamma_{22}^2 = -\frac{\xi^2}{a}, \quad (2.134)$$

and all the other Christoffel symbols are equal to zero. Next we can use again (2.129) to compute the mixed components of the curvature tensor. We obtain

$$b_1^1 = \frac{1 + (\xi^2)^2}{a^{3/2}}, \quad b_1^2 = -b_2^1 = \frac{\xi^1 \xi^2}{a^{3/2}}, \quad b_2^2 = -\frac{1 + (\xi^1)^2}{a^{3/2}}, \quad (2.135)$$

Note that $b_2^1 \neq b_1^2$. The mean and Gaussian curvatures are

$$H = \frac{(\xi^2)^2 - (\xi^1)^2}{2a^{3/2}}, \quad K = -\frac{1}{a^2}. \quad (2.136)$$

Finally, the components of the third fundamental form are

$$c_{11} = \frac{1 + (\xi^2)^2}{a^2}, \quad c_{12} = c_{21} = -\frac{\xi^1 \xi^2}{a^2}, \quad c_{22} = \frac{1 + (\xi^1)^2}{a^2}. \quad (2.137)$$

■

The curvature tensor enjoys an additional symmetry property, which involves its covariant derivatives, namely

$$b_{\alpha\beta|\lambda} = b_{\alpha\lambda|\beta}, \quad \forall \alpha, \beta, \lambda = 1, 2.$$

(2.138)

This is called the *Codazzi Equation*.

Proof of (2.138).

$$\frac{\partial \vec{a}_{\alpha,\beta}}{\partial \xi^\lambda} = \frac{\partial \vec{a}_{\alpha,\lambda}}{\partial \xi^\beta} \quad (2.139)$$

implies

$$\begin{aligned} 0 &= \left(\frac{\partial \vec{a}_{\alpha,\beta}}{\partial \xi^\lambda} - \frac{\partial \vec{a}_{\alpha,\lambda}}{\partial \xi^\beta} \right) \cdot \vec{a}_3 \\ &= \frac{\partial}{\partial \xi^\lambda} (\vec{a}_{\alpha,\beta} \cdot \vec{a}_3) - \vec{a}_{\alpha,\beta} \cdot \vec{a}_{3,\lambda} - \left[\frac{\partial}{\partial \xi^\beta} (\vec{a}_{\alpha,\lambda} \cdot \vec{a}_3) - \vec{a}_{\alpha,\lambda} \cdot \vec{a}_{3,\beta} \right] \\ &= b_{\alpha\beta,\lambda} + b_{\lambda\mu} \vec{a}^\mu \cdot \vec{a}_{\alpha,\beta} - b_{\alpha\lambda,\beta} - b_{\beta\mu} \vec{a}^\mu \cdot \vec{a}_{\alpha,\lambda} \end{aligned} \quad (2.140)$$

using (2.101) and (2.114). Next, recalling the expression of Christoffel symbols (2.118), we have

$$\begin{aligned}
 0 &= b_{\alpha\beta,\lambda} + b_{\lambda\mu}\Gamma_{\alpha\beta}^{\mu} - b_{\alpha\lambda,\beta} - b_{\beta\mu}\Gamma_{\alpha\lambda}^{\mu} \\
 &= b_{\alpha\beta,\lambda} - b_{\beta\mu}\Gamma_{\alpha\lambda}^{\mu} - b_{\alpha\mu}\Gamma_{\beta\lambda}^{\mu} - (b_{\alpha\lambda,\beta} - b_{\lambda\mu}\Gamma_{\alpha\beta}^{\mu} - b_{\alpha\mu}\Gamma_{\beta\lambda}^{\mu}) \\
 &= b_{\alpha\beta|\lambda} - b_{\alpha\lambda|\beta},
 \end{aligned} \tag{2.141}$$

from the covariant differentiation rule of second-order tensors (see (2.74)). ■

We recall that a surface is called *elliptic*, *parabolic* or *hyperbolic* according to whether its Gaussian curvature K is positive, zero, or negative, respectively. For example, an ellipsoid is an elliptic surface; a cylinder, a cone, and developable surfaces in general are all parabolic surfaces; a hyperbolic paraboloid (see Example 2.2.1) is a hyperbolic surface, as reflected in its name. Of course, general surfaces need not be of uniform nature in this respect, but this distinction can always be made pointwise.

For a parabolic or a hyperbolic surface, it is clear that the ratio (2.106) vanishes at least for one direction in the tangent plane. Of course, this occurs when

$$b_{\alpha\beta}v^{\alpha}v^{\beta} = 0, \tag{2.142}$$

for a vector \underline{v} corresponding to the direction considered. We call these specific directions the *asymptotic directions* of the surface. If we perform a change of coordinates such that the first coordinate corresponds to an asymptotic direction at the point in consideration, which is always possible locally for a smooth chart, we obtain

$$b_{11} = 0 \tag{2.143}$$

in the new coordinate system. Recalling our discussion on the second fundamental form, we can interpret (2.143) and conclude that the curve obtained by intersecting the surface with the plane defined by the asymptotic direction and the vector normal to the surface has zero curvature at this point.

Furthermore, since the Gaussian curvature is the product of the two principal curvatures, i.e. the product of the minimum and maximum values of (2.106), we can be more specific about the number of asymptotic directions. For a hyperbolic surface, there are exactly two asymptotic directions, which lie in between the directions of principal curvatures. For a parabolic surface, there is either one single asymptotic direction which is also a direction of principal curvature if only one of the extreme values is zero, or all directions are asymptotic directions if the two principal curvatures are zero, namely if the second fundamental form is the zero tensor (e.g. for a plane).

For a hyperbolic surface, we can give an interesting characterization of the asymptotic directions as follows.

Proposition 2.2.1 *For a hyperbolic surface, the tangential plane intersects the surface along two directions which are the asymptotic directions.*

Proof. Consider a point, given by the curvilinear coordinates $(\bar{\xi}^1, \bar{\xi}^2)$, at which the surface is hyperbolic. Recalling the definition of the covariant basis (2.81), the second order Taylor expansion of the chart around this point gives

$$\begin{aligned} \vec{\phi}(\xi^1, \xi^2) &= \vec{\phi}(\bar{\xi}^1, \bar{\xi}^2) \\ &\quad + (\xi^\alpha - \bar{\xi}^\alpha) \vec{a}_\alpha(\bar{\xi}^1, \bar{\xi}^2) \\ &\quad + \frac{1}{2} (\xi^\alpha - \bar{\xi}^\alpha) (\xi^\beta - \bar{\xi}^\beta) \frac{\partial^2 \vec{\phi}}{\partial \xi^\alpha \partial \xi^\beta}(\bar{\xi}^1, \bar{\xi}^2) \\ &\quad + O(\|\xi^1 - \bar{\xi}^1, \xi^2 - \bar{\xi}^2\|^3), \end{aligned} \tag{2.144}$$

with $\|\xi^1, \xi^2\| = \sqrt{(\xi^1)^2 + (\xi^2)^2}$. We then have

$$\begin{aligned} \frac{\partial^2 \vec{\phi}}{\partial \xi^\alpha \partial \xi^\beta} &= \vec{a}_{\alpha, \beta} \\ &= (\vec{a}_{\alpha, \beta} \cdot \vec{a}^\lambda) \vec{a}_\lambda + (\vec{a}_{\alpha, \beta} \cdot \vec{a}_3) \vec{a}_3 \\ &= \Gamma_{\alpha\beta}^\lambda \vec{a}_\lambda + b_{\alpha\beta} \vec{a}_3 \end{aligned} \tag{2.145}$$

recalling (2.101) and (2.118). Hence

$$\begin{aligned} \vec{\phi}(\xi^1, \xi^2) &= \vec{\phi}(\bar{\xi}^1, \bar{\xi}^2) + (\xi^\alpha - \bar{\xi}^\alpha) \vec{a}_\alpha \\ &\quad + \frac{1}{2} (\xi^\alpha - \bar{\xi}^\alpha) (\xi^\beta - \bar{\xi}^\beta) (\Gamma_{\alpha\beta}^\lambda \vec{a}_\lambda + b_{\alpha\beta} \vec{a}_3) \\ &\quad + O(\|\xi^1 - \bar{\xi}^1, \xi^2 - \bar{\xi}^2\|^3), \end{aligned} \tag{2.146}$$

where the covariant base vectors, Christoffel symbols and coefficients of the second fundamental form are all taken at the point of coordinates $(\bar{\xi}^1, \bar{\xi}^2)$.

The position of the surface with respect to the plane tangent to this surface at the point $(\bar{\xi}^1, \bar{\xi}^2)$ is then characterized by the quantity

$$\begin{aligned} [\vec{\phi}(\xi^1, \xi^2) - \vec{\phi}(\bar{\xi}^1, \bar{\xi}^2)] \cdot \vec{a}_3(\bar{\xi}^1, \bar{\xi}^2) &= \frac{1}{2} (\xi^\alpha - \bar{\xi}^\alpha) (\xi^\beta - \bar{\xi}^\beta) b_{\alpha\beta}(\bar{\xi}^1, \bar{\xi}^2) \\ &\quad + O(\|\xi^1 - \bar{\xi}^1, \xi^2 - \bar{\xi}^2\|^3), \end{aligned} \tag{2.147}$$

since $\vec{a}_3(\bar{\xi}^1, \bar{\xi}^2)$ is normal to the plane in consideration. Moreover, if we use a coordinate system aligned with the asymptotic directions at the point considered, we have $b_{11} = b_{22} = 0$ (and $b_{12} \neq 0$), hence

$$\begin{aligned}
[\vec{\phi}(\xi^1, \xi^2) - \vec{\phi}(\bar{\xi}^1, \bar{\xi}^2)] \cdot \vec{a}_3(\bar{\xi}^1, \bar{\xi}^2) &= (\xi^1 - \bar{\xi}^1)(\xi^2 - \bar{\xi}^2)b_{12}(\bar{\xi}^1, \bar{\xi}^2) \\
&+ O(\|\xi^1 - \bar{\xi}^1, \xi^2 - \bar{\xi}^2\|^3). \quad (2.148)
\end{aligned}$$

It is then clear that the surface locally “behaves like” $(\xi^1 - \bar{\xi}^1)(\xi^2 - \bar{\xi}^2)b_{12}(\bar{\xi}^1, \bar{\xi}^2)$, i.e. it intersects the tangential plane along two curves that are tangent to the coordinate curves, hence to the asymptotic directions. ■

Example 2.2.2

We consider again the surface used in Example 2.2.1, and we change the coordinate system by a rotation of angle $\pi/4$, namely we set

$$\begin{cases} y^1 = \frac{\sqrt{2}}{2}(\xi^1 + \xi^2) \\ y^2 = \frac{\sqrt{2}}{2}(-\xi^1 + \xi^2) \end{cases} \Leftrightarrow \begin{cases} \xi^1 = \frac{\sqrt{2}}{2}(y^1 - y^2) \\ \xi^2 = \frac{\sqrt{2}}{2}(y^1 + y^2) \end{cases} \quad (2.149)$$

In this new coordinate system, the chart becomes

$$\vec{\phi} = \begin{pmatrix} \frac{\sqrt{2}}{2}(y^1 - y^2) \\ \frac{\sqrt{2}}{2}(y^1 + y^2) \\ -y^1 y^2 \end{pmatrix} \quad (2.150)$$

and we can see that all the coordinate curves (obtained by freezing either y^1 or y^2) are in fact straight lines. We infer that, considering a unit vector \underline{n} directed along any coordinate curve, the intersection of the surface with the plane defined by \underline{n} and \vec{a}_3 is a straight line, hence $b_{\alpha\beta}n^\alpha n^\beta = 0$ since we recall that this quantity gives the curvature of the intersection. Therefore the coordinate curves defined by (y^1, y^2) are directed along the asymptotic directions at all points of the surface. We show the hyperbolic paraboloid with the (y^1, y^2) coordinate curves in Figure 2.5. ■

Remark 2.2.3. Note that the argument used in Example 2.2.2 can easily be extended to more generally show that, when a straight line is contained in a surface, this line provides an asymptotic direction at all of its points. A line which is tangent to an asymptotic direction at all points is called an *asymptotic line*. ■

2.2.3 3D differential geometry for shells

We now focus on the natural 3D curvilinear coordinate system based on a parametrization of the midsurface of the shell. Using the definition of the 3D

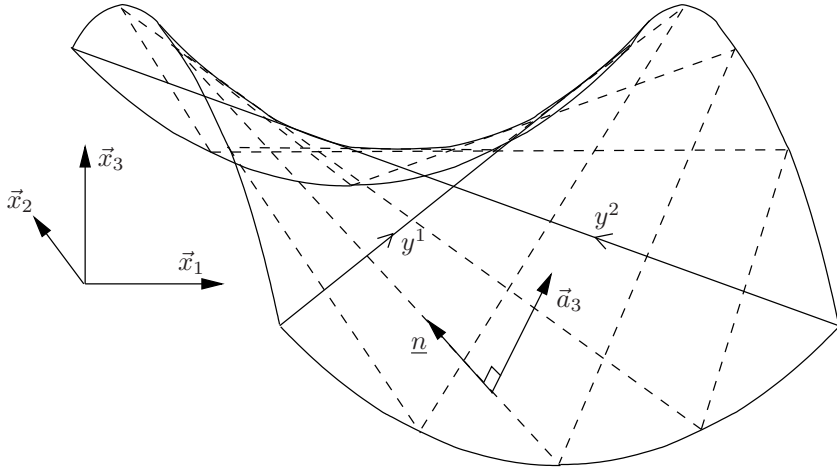


Fig. 2.5. Hyperbolic paraboloid with straight coordinate curves

chart given in (2.83), we can derive the 3D covariant base vectors. We have, recalling (2.114),

$$\vec{g}_\alpha = \frac{\partial \vec{\Phi}}{\partial \xi^\alpha} = \vec{a}_\alpha + \xi^3 \vec{a}_{3,\alpha} = \vec{a}_\alpha - \xi^3 b_\alpha^\lambda \vec{a}_\lambda, \quad (2.151)$$

hence

$$\boxed{\vec{g}_\alpha = (\delta_\alpha^\lambda - \xi^3 b_\alpha^\lambda) \vec{a}_\lambda.} \quad (2.152)$$

Moreover,

$$\boxed{\vec{g}_3 = \frac{\partial \vec{\Phi}}{\partial \xi^3} = \vec{a}_3.} \quad (2.153)$$

From (2.152) and (2.153) we can directly derive the components of the 3D metric tensor. We obtain

$$\boxed{\begin{cases} g_{\alpha\beta} = \vec{g}_\alpha \cdot \vec{g}_\beta = a_{\alpha\beta} - 2\xi^3 b_{\alpha\beta} + (\xi^3)^2 c_{\alpha\beta} \\ g_{\alpha 3} = \vec{g}_\alpha \cdot \vec{g}_3 = 0 \\ g_{33} = \vec{g}_3 \cdot \vec{g}_3 = 1 \end{cases}} \quad (2.154)$$

Note that the expression of $g_{\alpha\beta}$ suggests an interpretation of the “fundamental form” terminology, since the components of the three fundamental forms compose the coefficients of the polynomial expansion (along the transverse coordinate) of the tangential components of the 3D metric tensor.

Next, we can show that the g quantity appearing in volume measures is given by

$$\boxed{g = a(1 - 2H\xi^3 + K(\xi^3)^2)^2.} \quad (2.155)$$

Proof of (2.98) and (2.155). We have from (2.82)

$$\|\vec{a}_1 \wedge \vec{a}_2\| = (\vec{a}_1 \wedge \vec{a}_2) \cdot \vec{a}_3 = [\vec{a}_1, \vec{a}_2, \vec{a}_3], \quad (2.156)$$

which is thus a positive number. Therefore, we infer from (2.152) and (2.153) that

$$\|\vec{a}_1 \wedge \vec{a}_2\| = (|[\vec{g}_1, \vec{g}_2, \vec{g}_3]|)_{\xi^3=0}. \quad (2.157)$$

Using (2.53) and (2.154), we get

$$\|\vec{a}_1 \wedge \vec{a}_2\| = (\sqrt{\det(g_{mn})})_{\xi^3=0} = \sqrt{\det(a_{\alpha\beta})} \quad (2.158)$$

and (2.98) is proved. Then, from (2.53) and (2.52) we have

$$g = [\vec{g}_1 \cdot (\vec{g}_2 \wedge \vec{g}_3)]^2 = [\vec{g}_3 \cdot (\vec{g}_1 \wedge \vec{g}_2)]^2. \quad (2.159)$$

Substituting (2.152)-(2.153) into (2.159) and using

$$\vec{a}_1 \wedge \vec{a}_2 = \sqrt{a} \vec{a}_3, \quad (2.160)$$

(directly inferred from (2.82) and (2.98)), recalling (2.107) and (2.108) a direct calculation gives (2.155). \blacksquare

We note from (2.155) that the mapping $\vec{\Phi}$ is well defined (hence so is the system of curvilinear coordinates) provided that the expression $1 - 2H\xi^3 + K(\xi^3)^2$ is always strictly positive. This is clearly equivalent to requiring that, $\forall(\xi^1, \xi^2) \in \bar{\omega}$,

$$\boxed{t(\xi^1, \xi^2) < 2|R_{\min}(\xi^1, \xi^2)|} \quad (2.161)$$

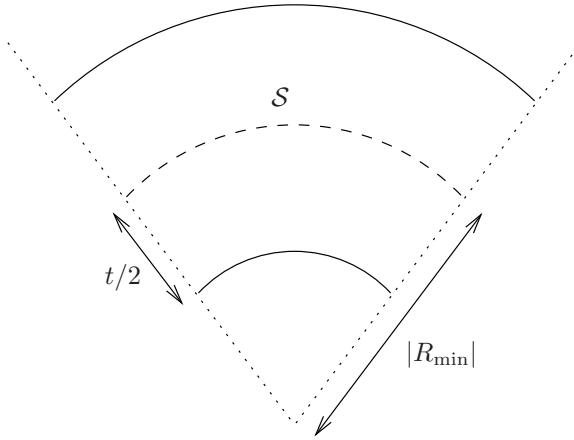


Fig. 2.6. Why “ $t/2 < |R_{\min}|$ ” must hold

where $R_{\min}(\xi^1, \xi^2)$ is the radius of curvature of smallest modulus of the surface at point $\vec{\phi}(\xi^1, \xi^2)$. We therefore henceforth suppose that Condition (2.161) is satisfied everywhere. See Figure 2.6 for a geometric interpretation of this condition.

Remark 2.2.4. The quantities relative to surface differential geometry are in fact restrictions to ω (i.e. calculated for $\xi^3 = 0$) of 3D quantities. In particular, we have

$$a_{\alpha\beta} = (g_{\alpha\beta})_{\xi^3=0}, \quad (2.162)$$

$$a = (g)_{\xi^3=0}, \quad (2.163)$$

$$\Gamma_{\alpha\beta}^{\gamma} = (\bar{\Gamma}_{\alpha\beta}^{\gamma})_{\xi^3=0}, \quad (2.164)$$

$$b_{\alpha\beta} = (\bar{\Gamma}_{\alpha\beta}^3)_{\xi^3=0}. \quad (2.165)$$

■

The Finite Element Analysis of Shells - Fundamentals

Chapelle, D.; Bathe, K.-J.

2011, XV, 410 p., Hardcover

ISBN: 978-3-642-16407-1