1

Basic Analysis

This chapter is devoted to the presentation of a few basic tools which will be used throughout this book. In the first section we state the Hölder and Minkowski inequalities. Next, we prove convolution inequalities in the general context of locally compact groups equipped with left-invariant Haar measures. The adoption of this rather general framework is motivated by the fact that these inequalities may be used not only in the $\mathbb{R}^d$ and $\mathbb{Z}^d$ cases, but also in other groups such as the Heisenberg group $\mathbb{H}^d$. Both Lebesgue and weak Lebesgue spaces are used. In the latter case, we introduce an atomic decomposition which will help us to establish a bilinear interpolation-type inequality. Finally, we give a few properties of the Hardy–Littlewood maximal operator.

The second section is devoted to a short presentation on the Fourier transform in $\mathbb{R}^d$. The third section is dedicated to homogeneous Sobolev spaces in $\mathbb{R}^d$. There, we state basic topological properties, consider embedding in Lebesgue, bounded mean oscillation, and Hölder spaces, and prove refined Sobolev inequalities. The classical Sobolev inequalities are of course invariant by translation and dilation. The refined versions of the Sobolev inequalities which we prove are, in addition, invariant by translation in the Fourier space. We also present some classes of examples to show that these inequalities are in some sense optimal. In the last section of this chapter, we focus on nonhomogeneous Sobolev spaces, with a special emphasis on trace theorems, compact embedding, and Moser–Trudinger and Hardy inequalities.

1.1 Basic Real Analysis

1.1.1 Hölder and Convolution Inequalities

We begin by recalling the classical Hölder inequality.

Proposition 1.1. *Let (X, μ) be a measure space and (p, q, r) in $[1, \infty]^3$ be such that*

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$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \cdot$$

If (f, g) belongs to $L^p(X, \mu) \times L^q(X, \mu)$, then fg belongs to $L^r(X, \mu)$ and

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Proof. The cases where $p = 1$ or $p = \infty$ being trivial, we assume from now on that p is a real number greater than 1. The concavity of the logarithm function entails that for any positive real numbers a and b and any θ in $[0, 1]$,

$$\theta \log a + (1 - \theta) \log b \leq \log(\theta a + (1 - \theta) b),$$

which obviously implies that

$$a^\theta b^{1-\theta} \leq \theta a + (1 - \theta) b.$$

Hence, assuming that $\|f\|_{L^p} = \|g\|_{L^q} = 1$, we can write

$$\begin{aligned}
\int_X |fg|^r \, d\mu &= \int_X (|f|^p)^{\frac{r}{p}} (|g|^q)^{\frac{r}{q}} \, d\mu \\
&\leq \frac{r}{p} \int_X |f|^p \, d\mu + \frac{r}{q} \int_X |g|^q \, d\mu \\
&\leq \frac{r}{p} + \frac{r}{q} = 1.
\end{aligned}$$

The proposition is thus proved. □

The following lemma states that Hölder's inequality is in some sense optimal.

Lemma 1.2. *Let (X, μ) be a measure space and $p \in [1, \infty]$. Let f be a measurable function. If*

$$\sup_{\|g\|_{L^{p'}} \leq 1} \int_X |f(x) g(x)| \, d\mu(x) < \infty,$$

then f belongs to L^p and[1]

$$\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}} \leq 1} \int_X f(x) g(x) \, d\mu(x).$$

Proof. Note that if f is in L^p, then Hölder's inequality ensures that

$$\sup_{\|g\|_{L^{p'}} \leq 1} \int_X f(x) g(x) \, d\mu(x) \leq \|f\|_{L^p}$$

so that only the reverse inequality has to be proven.

[1] Here, and throughout the book, p' denotes the *conjugate exponent* of p, defined by

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad \text{with the rule that} \quad \frac{1}{\infty} = 0.$$

We start with the case $p = \infty$. Let λ be a positive real number such that $\mu(|f| \geq \lambda) > 0$. Writing $E_\lambda \stackrel{\text{def}}{=} (|f| \geq \lambda)$, we consider a nonnegative function g_0 in L^1, supported in E_λ with integral 1. If we define

$$g(x) = \frac{f(x)}{|f(x)|} g_0,$$

then g is in L^1 so that fg is integrable by assumption, and we have

$$\int_X fg\, d\mu(x) = \int_X |f| g_0\, d\mu(x) \geq \lambda \int_X g_0\, d\mu(x) = \lambda.$$

The lemma is proved in this case. We now assume that $p \in\ ]1, \infty[$ and consider a nondecreasing sequence $(E_n)_{n\in\mathbb{N}}$ of subsets of finite measure of X, the union of which is X. Let[2]

$$f_n(x) = \mathbf{1}_{E_n \cap (|f| \leq n)} f \quad \text{and} \quad g_n(x) = \frac{f_n(x)|f_n(x)|^{p-1}}{|f_n(x)|\, \|f_n\|_{L^p}^{\frac{p}{p'}}}\cdot$$

It is obvious that f_n belongs to $L^1 \cap L^\infty$ and thus to L^p for any p. Moreover, we have

$$\|g_n\|_{L^{p'}}^{p'} = \frac{1}{\|f_n\|_{L^p}^p} \int_X |f_n(x)|^{(p-1)\frac{p}{p-1}}\, d\mu(x) = 1.$$

The definitions of the functions f_n and g_n ensure that

$$\begin{aligned}
\int_X f(x) \mathbf{1}_{E_n \cap (|f| \leq n)} g_n(x)\, d\mu(x) &= \int_X f_n(x) g_n(x)\, d\mu(x) \\
&= \left(\int_X |f_n(x)|^p\, d\mu(x) \right) \|f_n\|_{L^p}^{-\frac{p}{p'}} \\
&= \|f_n\|_{L^p}.
\end{aligned}$$

Thus, we have

$$\|f_n\|_{L^p} \leq \sup_{\|g\|_{L^{p'}} \leq 1} \int_X f(x) g(x)\, d\mu(x).$$

The monotone convergence theorem immediately implies that

$$\|f\|_{L^p} \leq \sup_{\|g\|_{L^{p'}} \leq 1} \int_X f(x) g(x)\, d\mu(x).$$

Finally, in order to treat the case where $p = 1$, we may consider the sequence $(g_n)_{n\in\mathbb{N}}$ defined by

$$g_n(x) = \mathbf{1}_{(f_n \neq 0)}(x) \frac{f_n(x)}{|f_n(x)|}\cdot$$

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[2] Throughout this book, the notation 1_A, where A stands for any subset of X, denotes the *characteristic function* of A.

We obviously have $\|g_n\|_{L^\infty} = 1$ and

$$\int_X f(x)g_n(x)\,d\mu(x) = \int_X |f_n(x)|\,d\mu(x).$$

Using the monotone convergence theorem, we get that

$$\int_X |f(x)|\,d\mu(x) < \infty \quad \text{and} \quad \int_X |f(x)|\,d\mu(x) = \lim_{n\to\infty} \int_X |f_n(x)|\,d\mu(x),$$

which completes the proof of the proposition. □

We now state *Minkowski's inequality.*

Proposition 1.3. *Let* (X_1,μ_1) *and* (X_2,μ_2) *be two measure spaces and* f *a nonnegative measurable function over* $X_1 \times X_2$. *For all* $1 \le p \le q \le \infty$, *we have*

$$\Big\| \|f(\cdot,x_2)\|_{L^p(X_1,\mu_1)} \Big\|_{L^q(X_2,\mu_2)} \le \Big\| \|f(x_1,\cdot)\|_{L^q(X_2,\mu_2)} \Big\|_{L^p(X_1,\mu_1)}.$$

Proof. The result is obvious if $q = \infty$. If q is finite, then, using Fubini's theorem and $r \stackrel{\text{def}}{=} (q/p)'$, we have

$$\begin{aligned}
\Big\| \|f(\cdot,x_2)\|_{L^p(X_1,\mu_1)} \Big\|_{L^q(X_2,\mu_2)} &= \left(\int_{X_2} \left(\int_{X_1} f^p(x_1,x_2)\,d\mu_1(x_1) \right)^{\frac{q}{p}} d\mu_2(x_2) \right)^{\frac{1}{q}} \\
&= \left(\sup_{\substack{\|g\|_{L^r(X_2,\mu_2)}=1 \\ g\ge 0}} \int_{X_1\times X_2} f^p(x_1,x_2)g(x_2)\,d\mu_1(x_1)\,d\mu_2(x_2) \right)^{\frac{1}{p}} \\
&\le \left(\int_{X_1} \left(\sup_{\substack{\|g\|_{L^r(X_2,\mu_2)}=1 \\ g\ge 0}} \int_{X_2} f^p(x_1,x_2)g(x_2)\,d\mu_2(x_2) \right) d\mu_1(x_1) \right)^{\frac{1}{p}}.
\end{aligned}$$

Using Hölder's inequality we may then infer that

$$\Big\| \|f(\cdot,x_2)\|_{L^p(X_1,\mu_1)} \Big\|_{L^q(X_2,\mu_2)} \le \left(\int_{X_1} \left(\int_{X_2} f^q(x_1,x_2)\,d\mu_2(x_2) \right)^{\frac{p}{q}} d\mu_1(x_1) \right)^{\frac{1}{p}},$$

and the desired inequality follows. □

The *convolution* between two functions will be used in various contexts in this book. The reader is reminded that convolution makes sense for real- or complex-valued measurable functions defined on some locally compact topological group G equipped with a left-invariant Haar measure[3] μ. The (formal) definition of convolution between two such functions f and g is as follows:

[3] This means that μ is a Borel measure on G such that for any Borel set A and element a of G, we have $\mu(a \cdot A) = \mu(A)$.

$$f \star g(x) = \int_G f(y)\, g(y^{-1} \cdot x)\, d\mu(y).$$

We can now state *Young's inequality* for the convolution of two functions.

Lemma 1.4. *Let G be a locally compact topological group endowed with a left-invariant Haar measure μ. If μ satisfies*

$$\mu(A^{-1}) = \mu(A) \text{ for any Borel set } A, \tag{1.1}$$

then for all (p, q, r) in $[1, \infty]^3$ such that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \tag{1.2}$$

and any (f, g) in $L^p(G, \mu) \times L^q(G, \mu)$, we have

$$f \star g \in L^r(G, \mu) \quad \text{and} \quad \|f \star g\|_{L^r(G,\mu)} \leq \|f\|_{L^p(G,\mu)} \|g\|_{L^q(G,\mu)}.$$

Proof. We first note that, owing to the left invariance and (1.1), for all $x \in G$ and any measurable function h on G, we have

$$\int_G h(y)\, d\mu(y) = \int_G h(y^{-1} \cdot x)\, d\mu(y).$$

Therefore, the case $r = \infty$ reduces to the Hölder inequality which was proven above.

We now consider the case $r < \infty$. Obviously, one can assume without loss of generality that f and g are nonnegative and nonzero. We write

$$(f \star g)(x) = \int_G f^{\frac{r}{r+1}}(y)\, g^{\frac{1}{r+1}}(y^{-1} \cdot x)\, f^{\frac{1}{r+1}}(y)\, g^{\frac{r}{r+1}}(y^{-1} \cdot x)\, d\mu(y).$$

Observing that (1.2) can be written $\displaystyle\frac{r}{r+1}\Big(\frac{1}{p} + \frac{1}{q}\Big) = 1$, Hölder's inequality implies that

$$(f \star g)(x) \leq \left(\int_G f^p(y) g^{\frac{p}{r}}(y^{-1} \cdot x)\, d\mu(y)\right)^{\frac{r}{(r+1)p}} \left(\int_G f^{\frac{q}{r}}(y) g^q(y^{-1} \cdot x)\, d\mu(y)\right)^{\frac{r}{(r+1)q}}.$$

Applying Hölder's inequality with $\alpha = rq/p$ (resp., $\beta = rp/q$) and the measure $f^p(y)\, d\mu(y)$ [resp., $g^q(y^{-1} \cdot x)\, d\mu(y)$], and using the invariance of the measure μ by the transform $y \mapsto y^{-1} \cdot x$, we get

$$(f \star g)(x) \leq \left(\int_G f^p(y) g^q(y^{-1} \cdot x)\, d\mu(y)\right)^{\frac{1}{r+1}\left(\frac{1}{p}+\frac{1}{q}\right)} \|f\|_{L^p(G,\mu)}^{\frac{r}{r+1}\left(1-\frac{p}{qr}\right)} \|g\|_{L^q(G,\mu)}^{\frac{r}{r+1}\left(1-\frac{q}{pr}\right)}.$$

Hence, raising the above inequality to the power r yields

$$\left|\left(\frac{f}{\|f\|_{L^p}}\star\frac{g}{\|g\|_{L^q}}\right)(x)\right|^r\leq\left(\frac{|f|^p}{\|f\|_{L^p}^p}\star\frac{|g|^q}{\|g\|_{L^q}^q}\right)(x).$$

Since the left invariance of the measure μ combined with Fubini's theorem obviously implies that the convolution maps $L^1(G,\mu)\times L^1(G,\mu)$ into $L^1(G,\mu)$ with norm 1, this yields the desired result in the case $r<\infty$. □

We now state a refined version of Young's inequality.

Theorem 1.5. *Let (G,μ) satisfy the same assumptions as in Lemma 1.4. Let (p,q,r) be in $]1,\infty[^3$ and satisfy (1.2). A constant C exists such that, for any $f\in L^p(G,\mu)$ and any measurable function g on G where*

$$\|g\|_{L^q_w(G,\mu)}^q\stackrel{def}{=}\sup_{\lambda>0}\lambda^q\mu(|g|>\lambda)<\infty,$$

the function $f\star g$ belongs to $L^r(G,\mu)$, and

$$\|f\star g\|_{L^r(G,\mu)}\leq C\|f\|_{L^p(G,\mu)}\|g\|_{L^q_w(G,\mu)}.$$

Remark 1.6. One can define the *weak L^q space* as the space of measurable functions g on G such that $\|g\|_{L^q_w(G,\mu)}$ is finite. We note that since

$$\lambda^q\mu(|g|>\lambda)\leq\int_{(|g|>\lambda)}|g(x)|^q\,d\mu(x)\leq\|g\|_{L^q(G,\mu)}^q, \tag{1.3}$$

the above theorem leads back to the standard Young inequality (up to a multiplicative constant).

We also that the weak L^q space belongs to the family of *Lorentz spaces* $L^{q,r}(G,\mu)$, which may be defined by means of *real interpolation*:

$$L^{q,r}(G,\mu)=[L^\infty(G,\mu),L^1(G,\mu)]_{1/q,r}\quad\text{for all}\quad 1<q<\infty\quad\text{and}\quad 1\leq r\leq\infty.$$

It turns out that the weak L^q space coincides with $L^{q,\infty}(G,\mu)$. From general real interpolation theory, we can therefore deduce a plethora of Hölder and convolution inequalities for Lorentz spaces (including, of course, the one which was proven above).

We also stress that the above theorem implies the well-known *Hardy–Littlewood–Sobolev inequality* on $\mathbb{R}^d$, given as follows.

Theorem 1.7. *Let α in $]0,d[$ and (p,r) in $]1,\infty[^2$ satisfy*

$$\frac{1}{p}+\frac{\alpha}{d}=1+\frac{1}{r}\cdot \tag{1.4}$$

A constant C then exists such that

$$\|\,|\cdot|^{-\alpha}\star f\|_{L^r(\mathbb{R}^d)}\leq C\|f\|_{L^p(\mathbb{R}^d)}.$$

Our proof of Theorem 1.5 relies on the atomic decomposition that we introduce in the next subsection.

1.1.2 The Atomic Decomposition

The *atomic decomposition* of an L^p function is described by the following proposition, which is valid for any measure space.

Proposition 1.8. *Let (X, μ) be a measure space and p be in $[1, \infty[$. Let f be a nonnegative function in L^p. A sequence of positive real numbers $(c_k)_{k\in\mathbb{Z}}$ and a sequence of nonnegative functions $(f_k)_{k\in\mathbb{Z}}$ (the* atoms*) then exist such that*

$$f = \sum_{k\in\mathbb{Z}} c_k f_k,$$

where the supports of the functions f_k are pairwise disjoint and

$$\mu(\mathrm{Supp}\ f_k) \leq 2^{k+1}, \tag{1.5}$$

$$\|f_k\|_{L^\infty} \leq 2^{-\frac{k}{p}}, \tag{1.6}$$

$$\frac{1}{2}\|f\|^p_{L^p} \leq \sum_{k\in\mathbb{Z}} c_k^p \leq 2\|f\|^p_{L^p}. \tag{1.7}$$

Remark 1.9. As implied by the definition given below, the sequence $(c_k f_k)_{k\in\mathbb{Z}}$ is independent of p and depends only on f.

Proof of Proposition 1.8. Define

$$\lambda_k \stackrel{\text{def}}{=} \inf\left\{\lambda\ /\mu(f > \lambda) < 2^k\right\},\ c_k \stackrel{\text{def}}{=} 2^{\frac{k}{p}}\lambda_k, \quad \text{and} \quad f_k \stackrel{\text{def}}{=} c_k^{-1}\mathbf{1}_{(\lambda_{k+1}<f\leq\lambda_k)}f.$$

It is obvious that $\|f_k\|_{L^\infty} \leq 2^{-\frac{k}{p}}$. Moreover, $(\lambda_k)_{k\in\mathbb{Z}}$ is a decreasing sequence which, owing to the fact that f is a nonnegative function in L^p, converges to 0 when k tends to infinity.

By the definition of λ_k, we have $\mu(f > \lambda_k) \leq 2^k$ and thus $\mu(\mathrm{Supp}\ f_k) \leq 2^{k+1}$. This gives

$$\begin{aligned}\sum_{k\in\mathbb{Z}} c_k^p &= \sum_{k\in\mathbb{Z}} 2^k\lambda_k^p\\ &= p\sum_{k\in\mathbb{Z}}\int_0^\infty 2^k\mathbf{1}_{]0,\lambda_k[}(\lambda)\lambda^{p-1}\,d\lambda.\end{aligned}$$

Using Fubini's theorem, we get

$$\sum_{k\in\mathbb{Z}} c_k^p = p\int_0^\infty \lambda^{p-1}\Big(\sum_{k\,/\,\lambda_k>\lambda} 2^k\Big)\,d\lambda.$$

By the definition of the sequence $(\lambda_k)_{k\in\mathbb{Z}},\ \lambda < \lambda_k$ implies that $\mu(f > \lambda) \geq 2^k$. We thus infer that

$$\sum_{k\in\mathbb{Z}} c_k^p \le p\int_0^\infty \lambda^{p-1}\biggl(\sum_{k\,/\,2^k\le\mu(f>\lambda)} 2^k\biggr)\,d\lambda$$
$$\le 2p\int_0^\infty \lambda^{p-1}\mu(f>\lambda)\,d\lambda.$$

The right-hand inequality in (1.7) now follows from the fact that, by Fubini's theorem, we have

$$\|f\|_{L^p}^p = p\int_0^\infty \lambda^{p-1}\mu(|f|>\lambda)\,d\lambda. \tag{1.8}$$

In order to complete the proof of (1.7) it suffices to note that, because the supports of the functions $(f_k)_{k\in\mathbb{Z}}$ are pairwise disjoint, we may write

$$\|f\|_{L^p}^p = \sum_{k\in\mathbb{Z}} c_k^p\|f_k\|_{L^p}^p.$$

Taking advantage of inequalities (1.5) and (1.6), we find that

$$\|f_k\|_{L^p}^p \le 2 \quad \text{for all} \quad k\in\mathbb{Z}\,.$$

This yields the desired inequality. □

1.1.3 Proof of Refined Young Inequality

Let f and g be nonnegative measurable functions on (G,μ). Consider a nonnegative function h in $L^{r'}$ and define

$$I(f,g,h) \stackrel{\text{def}}{=} \int_{G^2} f(y)g(y^{-1}\cdot x)h(x)\,d\mu(x)\,d\mu(y).$$

Arguing by homogeneity, we can assume that $\|f\|_{L^p} = \|g\|_{L^q_w} = \|h\|_{L^{r'}} = 1$. Stating $C_j \stackrel{\text{def}}{=} \{y\in G\,,\ 2^j\le g(y)<2^{j+1}\}$, we can write

$$I(f,g,h)\ \le\ 2\sum_{j\in\mathbb{Z}} 2^j I_j(f,h) \quad \text{with}$$
$$I_j(f,h) \stackrel{\text{def}}{=} \int_{G^2} f(y)h(x)\mathbf{1}_{C_j}(y^{-1}\cdot x)\,d\mu(x)\,d\mu(y).$$

Because $\|g\|_{L^q_w}=1$, we have $\|\mathbf{1}_{C_j}\|_{L^s}\le 2^{-j\frac{q}{s}}$ for all $s\in[1,\infty]$. Thus, if we directly apply Young's inequality with p, q, and r, we find that $I_j(f,h)\le 2^{-j}$, so the series $\sum 2^{j+1}I_j(f,h)$ has no reason to converge. In order to bypass this difficulty, we may introduce the atomic decompositions of f and h, as given by Proposition 1.8. We then write

$$I_j(f,h) = \sum_{k,\ell} c_k d_\ell I_j(f_k,h_\ell).$$

Using Young's inequality, for any $(a,b) \in [1,\infty]^2$ such that $b \leq a'$ and for any $(\widetilde{f},\widetilde{h}) \in L^a \times L^b$, we get

$$I_j(\widetilde{f},\widetilde{h}) \leq \|\widetilde{f}\|_{L^a}\|\widetilde{h}\|_{L^b}\|\mathbf{1}_{C_j}\|_{L^{c'}} \quad \text{with} \quad \frac{1}{a}+\frac{1}{b} = 1+\frac{1}{c}\,\cdot$$

This gives

$$I_j(\widetilde{f},\widetilde{h}) \leq 2^{-jq\left(2-\frac{1}{a}-\frac{1}{b}\right)}\|\widetilde{f}\|_{L^a}\|\widetilde{h}\|_{L^b}.$$

Applying this for f_k and h_ℓ and using Proposition 1.8 now yields

$$2^j I_j(f_k,h_\ell) \leq 2^{jq\left(\frac{1}{q}-2+\frac{1}{a}+\frac{1}{b}\right)}2^{k\left(\frac{1}{a}-\frac{1}{p}\right)}2^{\ell\left(\frac{1}{b}-\frac{1}{r'}\right)}.$$

Using the condition (1.2) on (p,q,r) implies that

$$2^j I_j(f_k,h_\ell) \leq 2^{(jq+k)\left(\frac{1}{a}-\frac{1}{p}\right)}2^{(jq+\ell)\left(\frac{1}{b}-\frac{1}{r'}\right)}. \tag{1.9}$$

Take a and b such that

$$\frac{1}{a} \stackrel{\text{def}}{=} \frac{1}{p}-2\varepsilon\,\mathrm{sg}(jq+k) \quad \text{and} \quad \frac{1}{b} \stackrel{\text{def}}{=} \frac{1}{r'}-2\varepsilon\,\mathrm{sg}(jq+\ell) \quad \text{with} \quad \varepsilon \stackrel{\text{def}}{=} \frac{1}{4}\left(\frac{1}{p}-\frac{1}{r}\right),$$

where $\mathrm{sg}\,n = 1$ if $n \geq 0$, and $\mathrm{sg}\,n = -1$ if $n < 0$.

As $q > 1$, the condition (1.2) implies that $p < r$. Thus, by the definitions of ε, a, and b, we have $b \leq a'$. With this choice of a and b, (1.9) then becomes, using the triangle inequality,

$$\begin{aligned} 2^j I_j(f_k,h_\ell) &\leq 2^{-2\varepsilon|jq+k|-2\varepsilon|jq+\ell|} \\ &\leq 2^{-\varepsilon|jq+k|-\varepsilon|jq+\ell|-\varepsilon|k-\ell|}. \end{aligned}$$

Using Young's inequality for $\mathbb{Z}$ equipped with the counting measure, we may now deduce that

$$\begin{aligned} I(f,g,h) &\leq C\sum_{j,k,\ell} c_k d_\ell 2^{-\varepsilon|jq+k|-\varepsilon|jq+\ell|-\varepsilon|k-\ell|} \\ &\leq \frac{C}{\varepsilon}\sum_{k,\ell} c_k d_\ell 2^{-\varepsilon|k-\ell|} \\ &\leq \frac{C}{\varepsilon^2}\|(c_k)\|_{\ell^p}\|\|(d_\ell)\|_{\ell^{p'}}. \end{aligned}$$

The condition (1.2) implies that $r' \leq p'$ and thus

$$I(f,g,h) \leq \frac{C}{\varepsilon^2}\|(c_k)\|_{\ell^p}\|\|(d_\ell)\|_{\ell^{r'}}.$$

The theorem is thus proved. □

1.1.4 A Bilinear Interpolation Theorem

The following interpolation lemma, which will be useful in Chapter 8, provides another example of an application of atomic decomposition.

Proposition 1.10. *Let (X_1, μ_1) and (X_2, μ_2) be two measure spaces. Let $\mathcal{T}$ be a continuous bilinear functional on $L^2(X_1; L^{p_j}(X_2)) \times L^2(X_1; L^{q_j}(X_2))$ for j in $\{0,1\}$, where (p_j, q_j) is in $[1,2]^2$ and such that $p_0 \neq p_1$ and $q_0 \neq q_1$. For any $\theta \in [0,1]$, the bilinear functional $\mathcal{T}$ is then continuous on $L^2(X_1; L^{p_\theta}(X_2)) \times L^2(X_1; L^{q_\theta}(X_2))$ with*

$$\Big(\frac{1}{p_\theta}, \frac{1}{q_\theta}\Big) = (1-\theta)\Big(\frac{1}{p_0}, \frac{1}{q_0}\Big) + \theta\Big(\frac{1}{p_1}, \frac{1}{q_1}\Big) \cdot$$

Proof. Let $f \in L^2(X_1; L^{p_\theta}(X_2))$ and $g \in L^2(X_1; L^{q_\theta}(X_2))$. As in the proof of the refined Young's inequality, we will use the atomic decompositions of f and g. For any $(t,x) \in X_1 \times X_2$, we have

$$f(t,x) = \sum_{k\in\mathbb{Z}} c_k(t) f_k(t,x) \quad \text{and} \quad g(t,x) = \sum_{\ell\in\mathbb{Z}} d_\ell(t) g_\ell(t,x).$$

Let us write that

$$\mathcal{T}(f,g) = \sum_{k,\ell} \mathcal{T}(c_k f_k, d_\ell g_\ell).$$

Using the hypothesis on $\mathcal{T}$ and stating $\alpha \stackrel{\text{def}}{=} \Big(\frac{1}{p_0} - \frac{1}{p_1}\Big)^{-1}\Big(\frac{1}{q_0} - \frac{1}{q_1}\Big)$, we get

$$\begin{aligned} |\mathcal{T}(c_k f_k, d_\ell g_\ell)| &\leq C \min_{j\in\{0,1\}} \|c_k f_k\|_{L^2(X_1; L^{p_j}(X_2))} \|d_\ell g_\ell\|_{L^2(X_1; L^{q_j}(X_2))} \\ &\leq C \|c_k\|_{L^2(X_1)} \|d_\ell\|_{L^2(X_1)} \\ &\qquad \times \min\Big\{2^{-\theta\left(\frac{1}{p_0} - \frac{1}{p_1}\right)(k+\alpha\ell)}, 2^{(1-\theta)\left(\frac{1}{p_0} - \frac{1}{p_1}\right)(k+\alpha\ell)}\Big\}. \end{aligned}$$

Setting $\varepsilon \stackrel{\text{def}}{=} \Big|\frac{1}{p_0} - \frac{1}{p_1}\Big| \times \min\{\theta, (1-\theta)\}$, we deduce that

$$|\mathcal{T}(c_k f_k, d_\ell g_\ell)| \leq C \|c_k\|_{L^2(X_1)} \|d_\ell\|_{L^2(X_1)} 2^{-\varepsilon|k+\alpha\ell|}.$$

Using a weighted Cauchy–Schwarz inequality, we then get

$$\begin{aligned} |\mathcal{T}(f,g)| &\leq C_\varepsilon \Big(\sum_k \|c_k\|^2_{L^2(X_1)}\Big)^{\frac{1}{2}} \Big(\sum_\ell \|d_\ell\|^2_{L^2(X_1)}\Big)^{\frac{1}{2}} \\ &\leq C_\varepsilon \big\| \|(c_k)\|_{\ell^2(\mathbb{Z})} \big\|_{L^2(X_1)} \big\| \|(d_\ell)\|_{\ell^2(\mathbb{Z})} \big\|_{L^2(X_1)} \cdot \end{aligned}$$

Using the fact that p_θ and q_θ are less than 2, we infer that

$$|\mathcal{T}(f,g)| \leq C_\varepsilon \big\| \|(c_k)\|_{\ell^{p_\theta}(\mathbb{Z})} \big\|_{L^2(X_1)} \big\| \|(d_\ell)\|_{\ell^{q_\theta}(\mathbb{Z})} \big\|_{L^2(X_1)} \cdot$$

The inequality (1.7) from Proposition 1.8 then implies the proposition. □

1.1.5 A Linear Interpolation Result

We shall present here a basic result of linear *complex* interpolation theory which will be useful, particularly in Chapter 8.

Lemma 1.11. *Consider three measure spaces* $(X_k, \mu_k)_{1\leq k\leq 3}$ *and two elements* $(p_j, q_j, r_j)_{j\in\{0,1\}}$ *of* $[1,\infty]^3$*. Further, consider an operator* A *which continuously maps* $L^{p_j}(X_1; L^{q_j}(X_2))$ *into* $L^{r_j}(X_3)$ *for* j *in* $\{0,1\}$*. For any* θ *in* $[0,1]$*, if*

$$\left(\frac{1}{p_\theta}, \frac{1}{q_\theta}, \frac{1}{r_\theta}\right) \stackrel{def}{=} (1-\theta)\left(\frac{1}{p_0}, \frac{1}{q_0}, \frac{1}{r_0}\right) + \theta\left(\frac{1}{p_1}, \frac{1}{q_1}, \frac{1}{r_1}\right),$$

then A *continuously maps* $L^{p_\theta}(X_1; L^{q_\theta}(X_2))$ *into* $L^{r_\theta}(X_3)$ *and*

$$\|A\|_{\mathcal{L}(L^{p_\theta}(X_1;L^{q_\theta}(X_2));L^{r_\theta}(X_3))} \leq \mathcal{A}_\theta \quad \text{with}$$
$$\mathcal{A}_\theta \stackrel{def}{=} \|A\|^{1-\theta}_{\mathcal{L}(L^{p_0}(X_1;L^{q_0}(X_2));L^{r_0}(X_3))} \|A\|^{\theta}_{\mathcal{L}(L^{p_1}(X_1;L^{q_1}(X_2));L^{r_1}(X_3))}.$$

Proof. Consider f in $L^{p_\theta}(X_1; L^{q_\theta}(X_2))$ and φ in $L^{r_\theta}(X_3)$.[4] Using Lemma 1.2, it is enough to prove that

$$\int_{X_3} (Af)(x_3)\varphi(x_3)d\mu_3(x_3) \leq \mathcal{A}_\theta \|f\|_{L^{p_\theta}(L^{q_\theta})} \|\varphi\|_{L^{r'_\theta}}. \tag{1.10}$$

Let z be a complex number in the strip S of complex numbers whose real parts are between 0 and 1. Define

$$f_z(x_1,x_2) \stackrel{\text{def}}{=} \frac{f(x_1,x_2)}{|f(x_1,x_2)|}\left(\frac{|f(x_1,x_2)|}{\|f(x_1,\cdot)\|_{L^{q_\theta}}}\right)^{q_\theta\left(\frac{1-z}{q_0}+\frac{z}{q_1}\right)} \|f(x_1,\cdot)\|_{L^{q_\theta}}^{p_\theta\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)}$$

and

$$\varphi_z(x_3) = \frac{\varphi(x_3)}{|\varphi(x_3|}|\varphi(x_3)|^{r'_\theta\left(\frac{1-z}{r'_0}+\frac{z}{r'_1}\right)}.$$

Obviously, we have $f_\theta = f$ and $\varphi_\theta = \varphi$. It can be checked that the function defined by

$$F(z) \stackrel{\text{def}}{=} \int_{X_3} (Af_z)(x_3)\varphi_z(x_3)\, d\mu_3(x_3)$$

is holomorphic and bounded on S and continuous on the closure of S. From the Phragmen–Lindelhöf principle, we infer that

$$F(\theta) \leq M_0^{1-\theta} M_1^\theta \quad \text{with} \quad M_j \stackrel{\text{def}}{=} \sup_{t\in\mathbb{R}} |F(j+it)|. \tag{1.11}$$

[4] Throughout this proof, we write $L^{p_\theta}(X_1; L^{q_\theta}(X_2))$ simply as $L^{p_\theta}(L^{q_\theta})$ and $L^{r_\theta}(X_3)$ simply as L^{r_θ}.

We have

$$|f_{j+it}(x_1,x_2)| = \left(\frac{|f(x_1,x_2)|}{\|f(x_1,\cdot)\|_{L^{q_\theta}}}\right)^{\frac{q_\theta}{q_j}} \|f(x_1,\cdot)\|_{L^{q_\theta}}^{\frac{p_\theta}{p_j}}.$$

Thus, we have that f_{j+it} belongs to $L^{p_j}(L^{q_j})$ and

$$\|f_{j+it}\|_{L^{p_j}(L^{q_j})} = \|f\|_{L^{p_\theta}(L^{q_\theta})}^{\frac{p_\theta}{p_j}}.$$

In the same way, we get that $|\varphi_{j+it}(x_3)| = |\varphi(x_3)|^{\frac{r'_\theta}{r'_j}}$. Thus, thanks to Hölder's inequality, we get

$$\begin{aligned} M_j &\leq \sup_{t\in\mathbb{R}} \left| \int_{X_3} (Af_{j+it})(x_3)\varphi_{j+it}(x_3)\, d\mu_3(x_3) \right| \\ &\leq \|A\|^\theta_{\mathcal{L}(L^{p_j}(X_1;L^{q_j}(X_2));L^{r_j}(X_3))} \|f\|_{L^{p_\theta}(L^q_\theta)}^{\frac{p_\theta}{p_j}} |\|\varphi\|_{L^{r'_\theta}(L^{r'_\theta})}^{\frac{r'_\theta}{r'_j}}. \end{aligned}$$

Using (1.11), we then deduce (1.10) and the lemma is proved. □

From this lemma, taking $X_1 = \{a\}$ and then $X_3 = \{a\}$, we can infer the following two corollaries which will be used in Chapter 8.

Corollary 1.12. *Let $(X_k,\mu_k)_{1\leq k\leq 2}$ be two measure spaces and $(p_j,q_j)_{j\in\{0,1\}}$ be two elements of $[1,\infty]^2$. Consider a linear operator A which continuously maps $L^{p_j}(X_1)$ into $L^{q_j}(X_2)$ for $j\in\{0,1\}$. For any θ in $[0,1]$, if*

$$\left(\frac{1}{p_\theta},\frac{1}{q_\theta}\right) \stackrel{def}{=} (1-\theta)\left(\frac{1}{p_0},\frac{1}{q_0}\right) + \theta\left(\frac{1}{p_1},\frac{1}{q_1}\right),$$

then A continuously maps $L^{p_\theta}(X_1)$ into $L^{q_\theta}(X_2)$ and

$$\|A\|_{\mathcal{L}(L^{p_\theta}(X_1);L^{q_\theta}(X_2))} \leq \mathcal{A}_\theta \stackrel{def}{=} \|A\|^{1-\theta}_{\mathcal{L}(L^{p_0}(X_1);L^{q_0}(X_2))} \|A\|^\theta_{\mathcal{L}(L^{p_1}(X_1);L^{q_1}(X_2))}.$$

Corollary 1.13. *Let (X_1,μ_1), (X_2,μ_2) be two measure spaces and (p_0,q_0), (p_1,q_1) be two elements of $[1,\infty]^2$. Let A be a continuous linear functional on $L^{p_j}(X_1;L^{q_j}(X_2))$ for j in $\{0,1\}$. For any θ in $[0,1]$, if*

$$\left(\frac{1}{p_\theta},\frac{1}{q_\theta}\right) \stackrel{def}{=} (1-\theta)\left(\frac{1}{p_0},\frac{1}{q_0}\right) + \theta\left(\frac{1}{p_1},\frac{1}{q_1}\right),$$

then A is a continuous linear functional on $L^{p_\theta}(X_1;L^{q_\theta}(X_2))$ and

$$\|A\|_{\mathcal{L}(L^{p_\theta}(X_1;L^{q_\theta}(X_2));\mathbb{C})} \leq \mathcal{A}_\theta \quad \textit{with}$$
$$\mathcal{A}_\theta \stackrel{def}{=} \|A\|^{1-\theta}_{\mathcal{L}(L^{p_0}(X_1;L^{q_0}(X_2));\mathbb{C})} \|A\|^\theta_{\mathcal{L}(L^{p_1}(X_1;L^{q_1}(X_2));\mathbb{C})}.$$

1.1.6 The Hardy–Littlewood Maximal Function

In this subsection, we state a few elementary properties of the maximal function, which will be needed for proving Gagliardo–Nirenberg inequalities on the Euclidean space $\mathbb{R}^d$.

We first recall that the *maximal function* may be defined on any metric space (X,d) endowed with a Borel measure μ. More precisely, if $f : X \mapsto \mathbb{R}$ is in $L^1_{loc}(X,\mu)$, then we define

$$\forall x \in X,\ Mf(x) \stackrel{\text{def}}{=} \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, d\mu(y). \tag{1.12}$$

The following well-known continuity result for the maximal function is fundamental in harmonic analysis.

Theorem 1.14. *Assume that the measure metric space (X,d,μ) has the* doubling property.[5] *There then exists a constant C, depending only on the doubling constant D, such that for all $1 < p \leq \infty$ and $f \in L^p(X,\mu)$, we have $Mf \in L^p(X,\mu)$ and*

$$\|Mf\|_{L^p} \leq \frac{p}{p-1} C^{\frac{1}{p}} \|f\|_{L^p}. \tag{1.13}$$

Proof. First step: M maps L^∞ into L^∞. Indeed, we obviously have

$$\|Mf\|_{L^\infty} \leq \|f\|_{L^\infty} \quad \text{for all} \quad f \in L^\infty(X,\mu). \tag{1.14}$$

Second step: M maps L^1 into L^1_w. We claim that there exists some constant C_1, depending only on D, such that

$$\|Mf\|_{L^1_w} \leq C_1 \|f\|_{L^1} \quad \text{for all} \quad f \in L^1(X,\mu). \tag{1.15}$$

This is a mere consequence of the following *Vitali covering lemma* that we temporarily assume to hold.

Lemma 1.15. *Let (X,d) be a metric space endowed with a Borel measure μ with the doubling property. There then exists a constant c such that for any family $(B_i)_{1\leq i\leq n}$ of balls, there exists a subfamily $(B_{i_j})_{1\leq j\leq p}$ of pairwise disjoint balls such that*

$$\mu\Big(\bigcup_{j=1}^{p} B_{i_j}\Big) \geq c\,\mu\Big(\bigcup_{i=1}^{n} B_i\Big).$$

Fix some $f \in L^1(X,\mu)$ and some $\lambda > 0$. By definition of the function Mf, for any x in the set $E_\lambda \stackrel{\text{def}}{=} \{Mf > \lambda\}$, we can find some $r_x > 0$ such that

$$\int_{B(x,r_x)} |f|\, d\mu > \lambda \mu(B(x,r_x)). \tag{1.16}$$

[5] That is, there exists a positive constant D such that $\mu(B(x,2r)) \leq D\mu(B(x,r))$ for all $x \in X$ and $r > 0$.

Therefore, if K is a compact subset of E_λ, then we can find a finite covering $(B_i)_{1\leq i\leq n}$ of K by such balls. Denoting by $(B_{i_j})_{1\leq j\leq p}$ the subfamily supplied by the Vitali lemma and using (1.16), we can thus write

$$\lambda\,|K| \leq \frac{\lambda}{c}\mu\Big(\bigcup_{j=1}^{p} B_{i_j}\Big) \leq \frac{1}{c}\sum_{j=1}^{p}\lambda\mu(B_{i_j}) \leq \frac{1}{c}\sum_{j=1}^{p}\int_{B_{i_j}}|f|\,d\mu \leq \frac{1}{c}\int_X |f|\,d\mu,$$

which obviously leads to (1.15).

Third step: M maps L^p into L^p for all $p\in\,]1,\infty[$. The proof relies on arguments borrowed from real interpolation. Fix some function f in L^p and $\alpha\in\,]0,1[$. Since $M|f| = Mf$, we can assume that $f\geq 0$. Now, for all $\lambda>0$, we may write

$$f = f_\lambda + f^\lambda \quad \text{with} \quad f^\lambda \stackrel{\text{def}}{=} (f-\lambda\alpha)\mathbf{1}_{(f\geq\lambda\alpha)}.$$

Note that, thanks to (1.14), we have

$$(Mf>\lambda)\subset(Mf^\lambda>(1-\alpha)\lambda).$$

Hence the equality (1.8) implies that

$$\|Mf\|_{L^p}^p \leq p\int_0^{+\infty}\lambda^{p-1}\mu\big(Mf^\lambda>(1-\alpha)\lambda\big)\,d\lambda.$$

According to the inequality (1.15), we have

$$\mu\big(Mf^\lambda>(1-\alpha)\lambda\big) \leq \frac{C_1}{(1-\alpha)\lambda}\|f^\lambda\|_{L^1}.$$

So, finally, using the definition of f^λ and Fubini's theorem, we get

$$\begin{aligned}
\|Mf\|_{L^p}^p &\leq \frac{C_1 p}{1-\alpha}\int_0^{+\infty}\lambda^{p-2}\int_{(f\geq\lambda\alpha)}\big(f(x)-\lambda\alpha\big)\,d\mu(x)\\
&\leq \frac{C_1 p}{1-\alpha}\bigg(\int_X f(x)\int_0^{\frac{f(x)}{\alpha}}\lambda^{p-2}\,d\lambda\,d\mu(x) - \alpha\int_X\int_0^{\frac{f(x)}{\alpha}}\lambda^{p-1}\,d\lambda\,d\mu(x)\bigg)\\
&\leq \frac{C_1}{(p-1)(1-\alpha)\alpha^{p-1}}\|f\|_{L^p}^p.
\end{aligned}$$

Choosing $\alpha=(p-1)/p$ completes the proof of the inequality (1.13). □

Proof of Lemma 1.15. Without loss of generality, we can assume that $B_i = B(x_i,r_i)$ with $r_1\geq\cdots\geq r_n$. We can now construct the desired subfamily by induction. Indeed, for B_{i_1}, take the largest ball (i.e., B_1). Then, assuming that $B_{i_1},\ldots,B_{i_k}$ have been chosen, pick up the largest remaining ball which does not intersect the balls which have been taken so far.

Clearly, this process stops within a finite number of steps. In addition, if $i \notin \{i_1, \dots, i_p\}$, then there exists some index i_j such that $i_j < i$ and $B_i \cap B_{i_j}$ is not empty. Therefore, by virtue of the triangle inequality, B_i is included in $B(x_{i_j}, 3r_{i_j})$. This ensures that

$$\bigcup_{i=1}^{n} B_i \subset \bigcup_{j=1}^{p} B(x_{i_j}, 3r_{i_j}).$$

As the measure μ has the doubling property, this yields the desired result. $\square$

The following result is of importance for proving Gagliardo–Nirenberg inequalities.

Proposition 1.16. *Let G be a locally compact group with neutral element e, endowed with a distance d such that $d(e, y^{-1} \cdot x) = d(x, y)$ for all $(x, y) \in G^2$ and a left-invariant Haar measure μ satisfying (1.1).*

We assume, in addition, that for all $r > 0$ there exists a positive measure σ_r on the sphere $\Sigma_r \stackrel{def}{=} \{x \in G \,/\, d(e, x) = r\}$ such that for any L^1 function g on G, we have

$$\int_G g(z)\, d\mu(z) = \int_0^{+\infty} \biggl(\int_{\Sigma_r} g(z)\, d\sigma_r(z) \biggr)\, dr.$$

For all measurable functions f and any L^1 function K on G such that

$$\forall x \in G,\ K(x) = k(d(e, x))$$

for some nonincreasing function $k : \mathbb{R}^+ \mapsto \mathbb{R}^+$, we then have

$$\forall x \in G,\ \bigl|K \star f(x)\bigr| \le \|K\|_{L^1(G,\mu)}\, Mf(x).$$

Proof. Obviously we can restrict the proof to nonnegative functions f. Arguing by density we can also assume that k is C^1 and compactly supported. Owing to our assumptions on d and K, we have

$$\begin{aligned} K \star f(x) &= \int_G K(y) f(y^{-1} \cdot x)\, d\mu(y) \\ &= \int_0^{+\infty} k(r) \biggl(\int_{\Sigma_r} f(y^{-1} \cdot x)\, d\sigma_r(y) \biggr)\, dr. \end{aligned}$$

Therefore, integrating by parts with respect to r, we discover that

$$\begin{aligned} K \star f(x) &= \int_0^{+\infty} (-k'(r)) \biggl(\int_0^r \int_{\Sigma_s} f(y^{-1} \cdot x)\, d\sigma_s(y)\, ds \biggr) dr \\ &= \int_0^{+\infty} (-k'(r)) \biggl(\int_{B(x,r)} f(y)\, d\mu(y) \biggr) dr \\ &\le Mf(x) \int_0^{+\infty} (-k'(r)) \mu(B(x, r))\, dr. \end{aligned}$$

Finally, since

$$\mu(B(x,r)) = \mu(B(e,r)) = \int_0^r \int_{\Sigma_r} 1\, d\sigma_r(y)\, dr,$$

performing another integration by parts, we can write that

$$\int_0^{+\infty} (-k'(r))\mu(B(x,r))\, dr = \int_0^{+\infty} k(r)\Big(\int_{\Sigma_r} 1\, d\sigma_r(y)\Big) dr = \|K\|_{L^1(G,\mu)},$$

and the desired inequality follows. □

Remark 1.17. All the assumptions of the above proposition are satisfied if we take for G the group $(\mathbb{R}^d, +)$ endowed with the usual metric and the Lebesgue measure, or the Heisenberg group $(\mathbb{H}^d, \cdot)$ endowed with the Heisenberg distance and the Lebesgue measure of $\mathbb{R}^{2d+1}$.

We also note the following obvious generalization of the inequality stated in the above proposition:

$$\forall x \in G,\ \big|K \star f(x)\big| \leq \Big(\int_G \Big(\sup_{d(e,y') \geq d(e,y)} |K(y')|\Big)\, dy\Big) Mf(x),$$

which holds for any measurable function K on G. In fact, in Chapter 2 we shall use the above inequality rather than the above proposition.

1.2 The Fourier Transform

This section is devoted to a short presentation on the Fourier transform, a key tool in this monograph. In the first subsection we define the Fourier transform of a smooth function with fast decay at infinity. In the second subsection we then extend the definition (by duality) to tempered distributions. We conclude this section with the calculation of the Fourier transforms of some functions which play important roles in the following chapters.

1.2.1 Fourier Transforms of Functions and the Schwartz Space

The *Fourier transform* is defined on $L^1(\mathbb{R}^d)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i(x|\xi)} f(x)\, dx, \tag{1.17}$$

where $(x|\xi)$ denotes the inner product on $\mathbb{R}^d$. It is a continuous linear map from $L^1(\mathbb{R}^d)$ into $L^\infty(\mathbb{R}^d)$ because, obviously, $|\widehat{f}(\xi)| \leq \|f\|_{L^1}$. It is also clear that for any function $\phi \in L^1$ and automorphism L on $\mathbb{R}^d$, we have

$$\mathcal{F}(\phi \circ L) = \frac{1}{|\det L|} \widehat{\phi} \circ L^{-1}. \tag{1.18}$$

We now introduce the *Schwartz space* $\mathcal{S}(\mathbb{R}^d)$ (also denoted by $\mathcal{S}$ when no confusion is possible), which will be the basic tool for extending the Fourier transform to a very large class of distributions over $\mathbb{R}^d$. Let us first introduce the following notation. If α is a *multi-index* (i.e., an element of $\mathbb{N}^d$), x an element of $\mathbb{R}^d$, and f a smooth function of $\mathbb{R}^d$, then the *length* $|\alpha|$ of α is defined by $|\alpha| \stackrel{\text{def}}{=} \alpha_1 + \cdots + \alpha_d$. We also define $\partial^\alpha f \stackrel{\text{def}}{=} \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} f$ and $x^\alpha \stackrel{\text{def}}{=} x^{\alpha_1} \cdots x^{\alpha_d}$.

Definition 1.18. *The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is the set of smooth functions u on $\mathbb{R}^d$ such that for any $k \in \mathbb{N}$ we have*

$$\|u\|_{k,\mathcal{S}} \stackrel{def}{=} \sup_{\substack{|\alpha| \leq k \\ x \in \mathbb{R}^d}} (1+|x|)^k |\partial^\alpha u(x)| < \infty.$$

It is an easy exercise (left to the reader) to prove that, equipped with the family of seminorms $(\|\cdot\|_{k,\mathcal{S}})_{k\in\mathbb{N}}$, the set $\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space and that the space $\mathcal{D}(\mathbb{R}^d)$ of smooth compactly supported functions on $\mathbb{R}^d$ is dense in $\mathcal{S}(\mathbb{R}^d)$.

The way the Fourier transform $\mathcal{F}$ acts on the space $\mathcal{S}$ is described by the following theorem.

Theorem 1.19. *The Fourier transform continuously maps $\mathcal{S}$ into $\mathcal{S}$: For any integer k, there exist a constant C and an integer N such that*

$$\forall \phi \in \mathcal{S}\,,\ \|\widehat{\phi}\|_{k,\mathcal{S}} \leq C\|\phi\|_{N,\mathcal{S}}.$$

Moreover, the Fourier transform $\mathcal{F}$ is an automorphism of $\mathcal{S}$, the inverse of which is $(2\pi)^{-d}\check{\mathcal{F}}$, where $\check{\mathcal{F}}$ denotes the application $f \longmapsto \big\{\xi \mapsto (\mathcal{F}f)(-\xi)\big\}$.

Proof. Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$ with length k. Using Lebesgue's theorem and integration by parts, we get that, for any ϕ in $\mathcal{S}$,

$$(i\partial)^\alpha \widehat{f}(\xi) = \mathcal{F}(x^\alpha \phi)(\xi) \quad \text{and} \quad (i\xi)^\alpha \widehat{\phi}(\xi) = \mathcal{F}(\partial^\alpha \phi)(\xi). \tag{1.19}$$

From this, we deduce that

$$\begin{aligned} \Big|\xi^\beta \partial^\alpha \widehat{\phi}(\xi)\Big| &\leq \big|\mathcal{F}(\partial^\beta(x^\alpha\phi))(\xi)\big| \\ &\leq \|\partial^\beta(x^\alpha\phi)\|_{L^1} \\ &\leq c_d\|(1+|x|)^{d+1}\partial^\beta(x^\alpha\phi)\|_{L^\infty}. \end{aligned}$$

Hence, by the definition of the seminorms, we have $\|\widehat{\phi}\|_{k,\mathcal{S}} \leq C\|\phi\|_{k+d+1,\mathcal{S}}$.

We now prove the inverse formula, namely, $\mathcal{F}^{-1} = (2\pi)^{-d}\check{\mathcal{F}}$. The proof is based on the computation of Fourier transforms of Gaussian functions. If $d = 1$, we have, thanks to (1.19),

$$\frac{d}{d\xi}\left(\mathcal{F}(e^{-x^2})\right)(\xi) = \mathcal{F}(-ixe^{-x^2})(\xi)$$
$$= \mathcal{F}\Big(\frac{i}{2}\frac{d}{dx}e^{-x^2}\Big)(\xi)$$
$$= -\frac{\xi}{2}\mathcal{F}(e^{-x^2})(\xi).$$

As $\mathcal{F}\left(e^{-x^2}\right)(0) = \displaystyle\int e^{-x^2}\,dx = \pi^{\frac{1}{2}}$, we get that $\mathcal{F}\big(e^{-x^2}\big)(\xi) = \pi^{\frac{1}{2}}e^{-\frac{\xi^2}{4}}$.

From this and Fubini's theorem, we can now deduce that if d is any positive integer, then $\mathcal{F}\left(e^{-|x|^2}\right)(\xi) = \pi^{\frac{d}{2}}e^{-\frac{|\xi|^2}{4}}$. Using (1.18) we then infer that for any positive real number a,

$$\int_{\mathbb{R}^d} e^{-i(x|\xi)}e^{-a|x|^2}\,dx = \left(\frac{\pi}{a}\right)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4a}}. \tag{1.20}$$

Let ϕ be a function in $\mathcal{S}(\mathbb{R}^d)$ and ε any positive real number. Fubini's theorem applied to the function $(2\pi)^{-d}e^{i(x-y|\xi)}e^{-\varepsilon|\xi|^2}\phi(y)$, together with (1.20), implies that

$$(2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x|\xi)}e^{-\varepsilon|\xi|^2}\widehat{\phi}(\xi)\,d\xi = \left(\frac{1}{4\pi\varepsilon}\right)^{\frac{d}{2}}(e^{-\frac{|\cdot|^2}{4\varepsilon}}\star\phi)(x).$$

On the one hand, owing to Lebesgue's dominated convergence theorem, the left-hand side tends to $(2\pi)^{-d}\check{\mathcal{F}}\widehat{\phi}$. On the other hand, the right-hand side is the convolution of ϕ with an approximation of the identity. Letting ε tend to 0 thus completes the proof of the theorem. □

1.2.2 Tempered Distributions and the Fourier Transform

Definition 1.20. *A* tempered distribution *on* $\mathbb{R}^d$ *is any continuous linear functional*[6] *on* $\mathcal{S}(\mathbb{R}^d)$. *The set of tempered distributions is denoted by* $\mathcal{S}'(\mathbb{R}^d)$.

A sequence $(u_n)_{n\in\mathbb{N}}$ *of tempered distributions is said to* converge *to* u *in* $\mathcal{S}'(\mathbb{R}^d)$ *if*

$$\forall\phi\in\mathcal{S}(\mathbb{R}^d)\,,\ \lim_{n\to\infty}\langle u_n,\phi\rangle = \langle u,\phi\rangle.$$

Remark 1.21. The link with distributions on $\mathbb{R}^d$ is as follows: If T is a distribution on $\mathbb{R}^d$ such that for some integer k and positive real C we have

$$\forall\varphi\in\mathcal{D}(\mathbb{R}^d)\,,\ |\langle T,\varphi\rangle| \le C\|\varphi\|_{k,\mathcal{S}}, \tag{1.21}$$

then, as $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$, the linear functional T may be uniquely extended to a continuous linear functional. Moreover, if T belongs to $\mathcal{S}'(\mathbb{R}^d)$,

[6] That is, u is a tempered distribution if there exist a constant C and an integer k such that $|\langle u,\phi\rangle| \le C\|\phi\|_{k,\mathcal{S}}$ for all $\phi\in\mathcal{S}(\mathbb{R}^d)$.

then the restriction of T to $\mathcal{D}(\mathbb{R}^d)$ defines a distribution on $\mathbb{R}^d$ because, for any positive R and any function φ in $\mathcal{D}(B(0,R))$,

$$|\langle T, \varphi\rangle| \leq C\|\varphi\|_{k,\mathcal{S}} \leq C(1+R)^k \sup_{|\alpha|\leq k} \|\partial^\alpha \varphi\|_{L^\infty}.$$

Thus, the set of distributions T on $\mathbb{R}^d$ which satisfy (1.21) may be identified with $\mathcal{S}'(\mathbb{R}^d)$.

Example 1.22. – Let us denote by $L^1_{\mathcal{M}}$ the space of locally integrable functions f on $\mathbb{R}^d$ such that for some integer N, the function $(1+|x|)^{-N}f(x)$ is integrable. For any $f \in L^1_{\mathcal{M}}$, we can then define the tempered distribution T_f by the formula

$$\langle T_f, \phi\rangle = \int_{\mathbb{R}^d} f(x)\phi(x)\,dx.$$

In other words, we identify the function f with T_f.
- Any finite Borel measure may be seen as a tempered distribution. Indeed, we may take $k=0$ in (1.21).
- Any compactly supported distribution may be identified with an element of $\mathcal{S}'$.

Let us use L. Schwartz's idea of duality to define operators on the space of tempered distributions. It is based on the following proposition.

Proposition 1.23. *Let A be a linear continuous map from $\mathcal{S}$ into $\mathcal{S}$.*[7] *The formula*

$$\langle {}^t\!Au, \phi\rangle \stackrel{def}{=} \langle u, A\phi\rangle$$

then defines a tempered distribution. Moreover, ${}^t\!A$ is linear and continuous, in the sense that if $(u_n)_{n\in\mathbb{N}}$ is a sequence of distributions which converges to u in $\mathcal{S}'(\mathbb{R}^d)$, then $({}^t\!Au_n)_{n\in\mathbb{N}}$ converges to ${}^t\!Au$.

Proof. By the definition of a tempered distribution, an integer k and a constant C exist such that

$$\forall \theta \in \mathcal{S}\,,\ |\langle u, \theta\rangle| \leq C\|\theta\|_{k,\mathcal{S}}. \tag{1.22}$$

The linear map A is assumed to be continuous, hence there exist a constant C' and an integer N such that

$$\forall \phi \in \mathcal{S}\,,\ \|A\phi\|_{k,\mathcal{S}} \leq C'\|\phi\|_{N,\mathcal{S}}.$$

Applying (1.22) with $\theta = A\phi$ and the above inequality, we then get that ${}^t\!Au$ is a tempered distribution. By the definition of the convergence of a sequence of tempered distributions, we then write

[7] That is, for any integer k, there exist a constant C and an integer N such that $\|A\phi\|_{k,\mathcal{S}} \leq C\|\phi\|_{N,\mathcal{S}}$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$.

$$\langle {}^t\!A u_n, \phi\rangle = \langle u_n, A\phi\rangle \longrightarrow \langle u, A\phi\rangle = \langle {}^t\!A u, \phi\rangle.$$

The proposition is thus proved. □

We now list a few important examples to which Proposition 1.23 applies:

- We may take for A any operator $(-\partial)^\alpha$ or $x^\alpha \mapsto x^\alpha u$ with $\alpha \in \mathbb{N}^d$. Indeed, we have, for all ϕ in $\mathcal{S}$,

$$\|(-\partial)^\alpha \phi\|_{k,\mathcal{S}} \leq \|\phi\|_{k+|\alpha|,\mathcal{S}} \quad \text{and} \quad \|x^\alpha \phi\|_{k,\mathcal{S}} \leq \|\phi\|_{k+|\alpha|,\mathcal{S}}.$$

- Let L be a linear automorphism of $\mathbb{R}^d$ and define

$$A_L\phi \stackrel{\text{def}}{=} \frac{1}{\det L}\, \phi \circ L^{-1}.$$

  It is clear that A_L satisfies the hypothesis of Proposition 1.23.
- If we denote by $\Theta_{\mathcal{M}}$ the space of smooth functions on $\mathbb{R}^d$ such that, for any integer k, an integer N exists such that

$$\sup_{x\in\mathbb{R}^d} (1+|x|^k)^{-N} \sup_{|\alpha|\leq k} |\partial^\alpha f(x)| < \infty,$$

  then the operator A_f of multiplication by f satisfies the hypothesis of the proposition.
- If θ is a function of $\mathcal{S}$, it is left as an exercise for the reader to check that, for any $\phi \in \mathcal{S}$,

$$\|A_\theta \phi\|_{k,\mathcal{S}} \leq C_k \|\theta\|_{k+d+1,\mathcal{S}} \|\phi\|_{k,\mathcal{S}} \quad \text{with} \quad A_\theta\phi \stackrel{\text{def}}{=} \check{\theta} \star \phi.$$

- Theorem 1.19 guarantees, in particular, that the Fourier transform $\mathcal{F}$ satisfies the hypothesis of Proposition 1.23.

For all the above operators, we can apply Proposition 1.23. We now check briefly that this is a generalization of classical operations on functions. If u is an $L^1_{\mathcal{M}}$ function which is also C^1, then we have

$$\forall \phi \in \mathcal{S}\,,\ \langle {}^t(-\partial_j)u, \phi\rangle = \langle u, -\partial_j\phi\rangle = \int_{\mathbb{R}^d} u(x)(-\partial_j\phi)(x)\,dx.$$

An integration by parts ensures that ${}^t(-\partial_j)u = \partial_j u$, in the classical sense.

Next, we claim that ${}^t\!A_L f(y) = f(Ly)$ for all $f \in L^1_{\mathcal{M}}$. Indeed, a straightforward change of variables ensures that for all $\phi \in \mathcal{S}$ we have

$$\langle {}^t\!A_L f, \phi\rangle = \frac{1}{|\det L|} \int_{\mathbb{R}^d} f(x)\phi(L^{-1}x)\,dx = \int_{\mathbb{R}^d} f(Ly)\phi(y)\,dy.$$

In the particular case where $Lx = \lambda x$, we denote ${}^t\!A_L f$ by f_λ, and when $\lambda = -1$, the distribution ${}^t\!A_L f$ is denoted by $\check{f}$. In passing, let us recall that a tempered distribution f is said to be *homogeneous of degree m* if

$$f_\lambda = \lambda^m f \quad \text{for all} \quad \lambda > 0.$$

It is obvious that the operator A_f generalizes the classical multiplication of functions by f.

Finally, for any L^1 function f, we have, according to Fubini's theorem,

$$\begin{aligned}\langle {}^t A_\theta f, \phi\rangle &= \langle f, \check{\theta} \star \phi\rangle \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x)\theta(y-x)\phi(y)\, dy\, dx \\ &= \langle f \star \theta, \phi\rangle.\end{aligned}$$

Thus, the notion of convolution between a tempered distribution and a function of $\mathcal{S}$ coincides with the classical definition when the tempered distribution is an L^1 function.

In order to extend the definition of the Fourier transform to tempered distributions, we consider an L^1 function f. By Fubini's theorem and by definition of the Fourier transform on L^1, we have, for all $\phi \in \mathcal{S}$,

$$\begin{aligned}\langle {}^t\mathcal{F} f, \phi\rangle &= \int_{\mathbb{R}^d} f(x)\widehat{\phi}(x)\, dx \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) e^{-i(x|\xi)}\phi(\xi)\, dx\, d\xi \\ &= \langle \widehat{f}, \phi\rangle.\end{aligned}$$

In other words, the operator ${}^t\mathcal{F}$ restricted to L^1 functions coincides with the Fourier transform of functions. Thus, it will also be denoted by $\mathcal{F}$ in all that follows.

Proposition 1.24. *For any (u, θ) in $\mathcal{S}' \times \mathcal{S}$, $\lambda \in \mathbb{R} \setminus \{0\}$ and $(a, \omega) \in \mathbb{R}^d \times \mathbb{R}^d$, we have*[8]

$$(i\partial)^\alpha \widehat{u} = \mathcal{F}(x^\alpha u)\,,\ (i\xi)^\alpha \widehat{u} = \mathcal{F}(\partial^\alpha u)\,,\ e^{-i(a|\xi)}\widehat{u} = \mathcal{F}(\tau_a f)\,,$$
$$\tau_\omega \widehat{f} = \mathcal{F}(e^{i(x|\omega)} f)\,, \quad \lambda^{-d}\widehat{f}(\lambda^{-1}\xi) = \mathcal{F}(f(\lambda x)), \quad \textit{and} \quad \mathcal{F}(u \star \theta) = \widehat{\theta}\,\widehat{u}.$$

Proof. The first five equalities readily follow from (1.19) or direct computation once we observe that ${}^t(AB) = {}^tB{}^tA$. In order to prove the last identity, it suffices to use the fact that, by definition of the Fourier transform and convolution, we have

$$\langle \mathcal{F}(u \star \theta), \phi\rangle = \langle u \star \theta, \widehat{\phi}\rangle = \langle u, \check{\theta} \star \widehat{\phi}\rangle.$$

Fubini's theorem implies that

[8] Below, the notation τ_a stands for the *translation operator* $\tau_a : f \mapsto f(\cdot - a)$.

$$
\begin{aligned}
(\check{\theta} \star \widehat{\phi})(\xi) &= \int \check{\theta}(\xi - \eta)\Big(\int e^{-i(x|\eta)}\phi(x)\,dx\Big)d\eta \\
&= \int e^{-i(x|\xi)}\Big(\int e^{-i(x|\eta-\xi)}\theta(\eta-\xi)d\eta\Big)\phi(x)\,dx \\
&= \mathcal{F}(\widehat{\theta}\phi).
\end{aligned}
$$

We infer that $\langle \mathcal{F}(u \star \theta), \phi\rangle = \langle u, \mathcal{F}(\widehat{\theta}\phi)\rangle = \langle \widehat{u}, \widehat{\theta}\phi\rangle = \langle \widehat{\theta}\widehat{u}, \phi\rangle$. The proposition is thus proved. □

Theorem 1.25 (Fourier–Plancherel formula). *The Fourier transform is an automorphism of $\mathcal{S}'$ with inverse $(2\pi)^{-d}\check{\mathcal{F}}$. Moreover, $\mathcal{F}$ is also an automorphism of $L^2(\mathbb{R}^d)$ which satisfies, for any function f in L^2, $\|\widehat{f}\|_{L^2} = (2\pi)^{\frac{d}{2}}\|f\|_{L^2}$.*

Proof. On the space $\mathcal{S}$, we have $\mathcal{F}\check{\mathcal{F}} = \check{\mathcal{F}}\mathcal{F} = (2\pi)^d\,\mathrm{Id}$. Arguing by transposition, we discover that these two identities remain valid on $\mathcal{S}'$. Next, using the fact that for any function ϕ in $\mathcal{S}$ we have $\overline{\mathcal{F}\phi} = \check{\mathcal{F}}(\overline{\phi})$ and taking advantage of the inverse Fourier formula (see Theorem 1.19), we get, for any function ϕ in $\mathcal{S}$,

$$
\|\mathcal{F}\phi\|_{L^2}^2 = \langle \mathcal{F}\phi, \overline{\mathcal{F}\phi}\rangle = \langle \phi, \mathcal{F}\check{\mathcal{F}}\overline{\phi}\rangle = (2\pi)^d\|\phi\|_{L^2}^2.
$$

Combining the Riesz representation theorem with the density of $\mathcal{S}$ in L^2 enables us to complete the proof. □

Finally, let us define a subspace of $\mathcal{S}'(\mathbb{R}^d)$ which will play an important role in the following chapters.

Definition 1.26. *We denote by $\mathcal{S}'_h(\mathbb{R}^d)$ the space of tempered distributions u such that*[9]

$$
\lim_{\lambda\to\infty}\|\theta(\lambda D)u\|_{L^\infty} = 0 \quad \textit{for any} \;\; \theta \;\; \textit{in} \;\; \mathcal{D}(\mathbb{R}^d).
$$

Remark 1.27. It is clear that whether or not a tempered distribution u belongs to $\mathcal{S}'_h$ depends only on low frequencies. As a matter of fact, it is not hard to check that u belongs to $\mathcal{S}'_h(\mathbb{R}^d)$ if and only if one can find some smooth compactly supported function θ satisfying the above equality and such that $\theta(0) \neq 0$.

Examples

- If a tempered distribution u is such that its Fourier transform $\widehat{u}$ is locally integrable near 0, then u belongs to $\mathcal{S}'_h$. In particular, the space $\mathcal{E}'$ of compactly supported distributions is included in $\mathcal{S}'_h$.
- If u is a tempered distribution such that $\theta(D)u \in L^p$ for some $p \in [1, \infty[$ and some function θ in $\mathcal{D}(\mathbb{R}^d)$ with $\theta(0) \neq 0$, then u belongs to $\mathcal{S}'_h$.

[9] We agree that if f is a measurable function on $\mathbb{R}^d$ with at most polynomial growth at infinity, then the operator $f(D)$ is defined by $f(D)a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(f\mathcal{F}a)$.

– A nonzero polynomial P does not belong to $\mathcal{S}'_h$ because for any $\theta \in \mathcal{D}(\mathbb{R}^d)$ with value 1 at 0 and any $\lambda > 0$, we may write $\theta(\lambda D)P = P$. However, if η is in $\mathbb{R}^d \backslash \{0\}$, then $e^{i(\cdot|\eta)}P$ belongs to $\mathcal{S}'_h$ because the support of its Fourier transform is $\{\eta\}$. We note that this example implies that $\mathcal{S}'_h$ is not a closed subspace of $\mathcal{S}'$ for the topology of weak-$\star$ convergence, a fact which must be kept in mind in the applications.

1.2.3 A Few Calculations of Fourier Transforms

This subsection is devoted to the computation of the Fourier transforms of some functions which are definitely not in L^1.

Proposition 1.28. *Let z be a nonzero complex number with nonnegative real part. Then,*

$$\mathcal{F}\Big(e^{-z|\cdot|^2}\Big)(\xi) = \Big(\frac{\pi}{z}\Big)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4z}}$$

with $z^{-\frac{d}{2}} \stackrel{def}{=} |z|^{-\frac{d}{2}} e^{-i\frac{d}{2}\theta}$ *if* $z = |z|e^{i\theta}$ *with* $\theta \in [-\pi/2, \pi/2]$.

Proof. Let us remark that for any ξ in $\mathbb{R}^d$, the functions

$$z \longmapsto \int_{\mathbb{R}^d} e^{-i(x|\xi)} e^{-z|x|^2}\, dx \quad \text{and} \quad z \longmapsto \Big(\frac{\pi}{z}\Big)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4z}}$$

are holomorphic on the domain D of complex numbers with positive real part. Formula (1.20) states that these two functions coincide on the intersection of the real line with D. Thus, they also coincide on the whole domain D. Now, let $(z_n)_{n\in\mathbb{N}}$ be a sequence of elements of D which converges to it for $t \neq 0$. For any function ϕ in $\mathcal{S}$, we have, by virtue of Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} e^{-z_n|x|^2} \phi(x)\, dx = \int_{\mathbb{R}^d} e^{-it|x|^2} \phi(x)\, dx \quad \text{and}$$
$$\lim_{n\to\infty} \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4z_n}} \phi(\xi)\, d\xi = \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4it}} \phi(\xi)\, d\xi.$$

As we have

$$\mathcal{F}\left(e^{-z_n|\cdot|^2}\right) = \Big(\frac{\pi}{z_n}\Big)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4z_n}},$$

passing to the limit in $\mathcal{S}'(\mathbb{R}^d)$ when n tends to ∞ gives the result, thanks to Proposition 1.23. $\square$

Proposition 1.29. *If $\sigma \in\ ]0, d[$, then $\mathcal{F}(|\cdot|^{-\sigma}) = c_{d,\sigma}|\cdot|^{\sigma-d}$ for some constant $c_{d,\sigma}$ depending only on d and s.*

Proof. We only treat the case $d \geq 2$. The (easier) case $d = 1$ is left to the reader. Defining

$$R \stackrel{\text{def}}{=} \sum_{j=1}^{d} x_j \partial_j \quad \text{and} \quad Z_{j,k} \stackrel{\text{def}}{=} x_j \partial_k - x_k \partial_j \,,$$

we have $R(|\cdot|^{-\sigma}) = -\sigma|\cdot|^{-\sigma}$ and $Z_{j,k}(|\cdot|^{-\sigma}) = 0$. Then, using Proposition 1.24, we infer that $Z_{j,k}\mathcal{F}|\cdot|^{-\sigma} = 0$ and

$$R\mathcal{F}|\cdot|^{-\sigma} = \sum_{j=1}^{d} \partial_j \big(\xi_j \mathcal{F}|\cdot|^{-\sigma}\big) - d\mathcal{F}|\cdot|^{-\sigma} = (\sigma - d)\mathcal{F}|\cdot|^{-\sigma}.$$

By restricting to $\mathbb{R}^d \setminus \{0\}$, we then see that

$$R\Big(|\cdot|^{d-\sigma}\mathcal{F}|\cdot|^{-\sigma}\Big) = Z_{j,k}\Big(|\cdot|^{d-\sigma}\mathcal{F}|\cdot|^{-\sigma}\Big) = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d \setminus \{0\}).$$

We note that for any k,

$$|x|^2 \partial_k = \sum_{j=1}^{d} x_j^2 \partial_k = x_k R + \sum_{j=1}^{d} x_j Z_{j,k}.$$

Therefore, $\nabla\big(|\cdot|^{d-\sigma}\mathcal{F}|\cdot|^{-\sigma}\big)$ is supported in $\mathbb{R}^d \setminus \{0\}$. Because $d \geq 2$, we deduce that there exists some constant $c_{d,\sigma}$ such that $|\cdot|^{d-\sigma}\mathcal{F}|\cdot|^{-\sigma} - c_{d,\sigma}$ is also supported in $\mathbb{R}^d \setminus \{0\}$ and, owing to $\sigma > 0$, so is $\mathcal{F}|\cdot|^{-\sigma} - c_{d,\sigma}|\cdot|^{\sigma-d}$. The conclusion then follows easily from the following lemma. □

Lemma 1.30. *Let T be a distribution on $\mathbb{R}^d$ supported in $\{0\}$ and such that $RT = sT$ for some real number s.*

- *If s is not an integer less than or equal to $-d$, then $T = 0$.*
- *If s is an integer less than or equal to $-d$, then there exist some real numbers a_α such that*

$$T = \sum_{|\alpha| = -s-d} a_\alpha \partial^\alpha \delta_0.$$

Proof. We first observe that a distribution supported in $\{0\}$ is of the form $T = \displaystyle\sum_{|\alpha|\leq N} a_\alpha \partial^\alpha \delta_0$. We thus have

$$\begin{aligned} RT &= \sum_{j=1}^{d} \sum_{|\alpha|\leq N} a_\alpha x_j \partial_j \partial^\alpha \delta_0 \\ &= -\sum_{|\alpha|\leq N} (d + |\alpha|) a_\alpha \partial^\alpha \delta_0. \end{aligned}$$

As $(\partial^\alpha \delta_0)_{\alpha \in \mathbb{N}^d}$ is a family of linearly independent distributions, the fact that $RT = sT$ implies that $(d + |\alpha|)a_\alpha = -s a_\alpha$. The lemma is thus proved. □

1.3 Homogeneous Sobolev Spaces

This section is concerned with homogeneous Sobolev spaces. We first establish classical properties for these spaces, then we focus on embedding in Lebesgue, BMO and Hölder spaces.

1.3.1 Definition and Basic Properties

Definition 1.31. *Let s be in $\mathbb{R}$. The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ (also denoted by $\dot{H}^s$) is the space of tempered distributions u over $\mathbb{R}^d$, the Fourier transform of which belongs to $L^1_{loc}(\mathbb{R}^d)$ and satisfies*

$$\|u\|^2_{\dot{H}^s} \stackrel{def}{=} \int_{\mathbb{R}^d} |\xi|^{2s}|\widehat{u}(\xi)|^2\, d\xi < \infty.$$

We note that the spaces $\dot{H}^s$ and $\dot{H}^{s'}$ *cannot* be compared for the inclusion. Nevertheless, we have the following proposition.

Proposition 1.32. *Let $s_0 \leq s \leq s_1$. Then, $\dot{H}^{s_0} \cap \dot{H}^{s_1}$ is included in $\dot{H}^s$, and we have*

$$\|u\|_{\dot{H}^s} \leq \|u\|^{1-\theta}_{\dot{H}^{s_0}} \|u\|^{\theta}_{\dot{H}^{s_1}} \quad \textit{with} \quad s = (1-\theta)s_0 + \theta s_1.$$

Proof. It suffices to apply Hölder's inequality with $p = 1/(1-\theta)$ and $q = 1/\theta$ to the functions $\xi \mapsto |\xi|^{2(1-\theta)s_0}$, $\xi \mapsto |\xi|^{2\theta s_1}$ and the Borel measure $|\widehat{u}(\xi)|^2\, d\xi$. □

Using the Fourier–Plancherel formula, we observe that $L^2 = \dot{H}^0$ and that if s is a positive integer, then $\dot{H}^s$ is the subset of tempered distributions with locally integrable Fourier transforms and such that $\partial^\alpha u$ belongs to L^2 for all α in $\mathbb{N}^d$ of length s.

In the case where s is a negative integer, the Sobolev space $\dot{H}^s$ is described by the following proposition.

Proposition 1.33. *Let k be a positive integer. The space $\dot{H}^{-k}(\mathbb{R}^d)$ consists of distributions which are the sums of derivatives of order k of $L^2(\mathbb{R}^d)$ functions.*

Proof. Let u be in $\dot{H}^{-k}(\mathbb{R}^d)$. Using the fact that for some integer constants A_α, we have

$$|\xi|^{2k} = \sum_{1\leq j_1,\dots,j_k\leq d} \xi^2_{j_1}\cdots\xi^2_{j_k} = \sum_{|\alpha|=k} A_\alpha (i\xi)^\alpha(-i\xi)^\alpha, \tag{1.23}$$

we get that

$$\widehat{u}(\xi) = \sum_{|\alpha|=k} (i\xi)^\alpha v_\alpha(\xi) \quad \text{with} \quad v_\alpha(\xi) \stackrel{\text{def}}{=} A_\alpha \frac{(-i\xi)^\alpha}{|\xi|^{2k}}\widehat{u}(\xi).$$

As u is in $\dot{H}^{-k}$, the functions v_α belong to L^2. Defining $u_\alpha \stackrel{\text{def}}{=} \mathcal{F}^{-1} v_\alpha$, we then obtain

$$u = \sum_{|\alpha|=k} \partial^\alpha u_\alpha \quad \text{with} \quad u_\alpha \in L^2(\mathbb{R}^d).$$

This concludes the proof of the proposition. □

Proposition 1.34. *$\dot{H}^s(\mathbb{R}^d)$ is a Hilbert space if and only if $s < \dfrac{d}{2}\cdot$*

Proof. We first assume that $s < d/2$. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\dot{H}^s(\mathbb{R}^d)$. Then, $(\widehat{u}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $L^2(\mathbb{R}^d; |\xi|^{2s}\, d\xi)$. Because $|\xi|^{2s}\, d\xi$ is a measure on $\mathbb{R}^d$, there exists a function f in $L^2(\mathbb{R}^d; |\xi|^{2s}\, d\xi)$ such that $(\widehat{u}_n)_{n\in\mathbb{N}}$ converges to f in $L^2(\mathbb{R}^d; |\xi|^{2s}\, d\xi)$. Because $s < d/2$, we have

$$\int_{B(0,1)} |f(\xi)|\, d\xi \le \Big(\int_{\mathbb{R}^d} |\xi|^{2s} |f(\xi)|^2\, d\xi\Big)^{\frac12} \Big(\int_{B(0,1)} |\xi|^{-2s}\, d\xi\Big)^{\frac12} < \infty.$$

This ensures that $\mathcal{F}^{-1}(1_{B(0,1)} f)$ is a bounded function. Now, $1_{^c B(0,1)} f$ clearly belongs to $L^2(\mathbb{R}^d; (1+|\xi|^2)^s\, d\xi)$ and thus to $\mathcal{S}'(\mathbb{R}^d)$, so f is a tempered distribution. Define $u \stackrel{\text{def}}{=} \mathcal{F}^{-1} f$. It is then obvious that u belongs to $\dot{H}^s$ and that $\lim\limits_{n\to\infty} u_n = u$ in the space $\dot{H}^s$.

If $s \ge d/2$, observe that the function

$$N : u \longmapsto \|\widehat{u}\|_{L^1(B(0,1))} + \|u\|_{\dot{H}^s}$$

is a norm over $\dot{H}^s(\mathbb{R}^d)$ and that $(\dot{H}^s(\mathbb{R}^d), N)$ is a Banach space.

Now, if $\dot{H}^s(\mathbb{R}^d)$ endowed with $\|\cdot\|_{\dot{H}^s}$ were also complete, then, according to Banach's theorem, there would exist a constant C such that $N(u) \le C\|u\|_{\dot{H}^s}$. Of course, this would imply that

$$\|\widehat{u}\|_{L^1(B(0,1))} \le C\|u\|_{\dot{H}^s}. \tag{1.24}$$

This inequality is violated by the following example. Let $\mathcal{C}$ be an annulus included in the unit ball $B(0,1)$ and such that $\mathcal{C} \cap 2\mathcal{C} = \emptyset$. Define

$$\Sigma_n \stackrel{\text{def}}{=} \mathcal{F}^{-1} \sum_{q=1}^{n} \frac{2^{q(s+\frac{d}{2})}}{q} \mathbf{1}_{2^{-q}\mathcal{C}}.$$

We have

$$\|\widehat{\Sigma}_n\|_{L^1(B(0,1))} = C \sum_{q=1}^{n} \frac{2^{q(s-\frac{d}{2})}}{q} \quad \text{and} \quad \|\Sigma_n\|^2_{\dot{H}^s} \le C \sum_{q=1}^{n} \frac{1}{q^2} \le C_1.$$

As $s \ge d/2$, we deduce that $\|\widehat{\Sigma}_n\|_{L^1(B(0,1))}$ tends to infinity when n goes to infinity. Hence, the inequality (1.24) is false. □

Proposition 1.35. *If $s < d/2$, then the space $\mathcal{S}_0(\mathbb{R}^d)$ of functions of $\mathcal{S}(\mathbb{R}^d)$, the Fourier transform of which vanishes near the origin, is dense in $\dot{H}^s$.*

Proof. Consider u in $\dot{H}^s$ such that

$$\forall \phi \in \mathcal{S}_0(\mathbb{R}^d)\,,\ (u|\phi)_{H^s} = \int_{\mathbb{R}^d} |\xi|^{2s}\widehat{u}(\xi)\overline{\widehat{\phi}(\xi)}\,d\xi = 0.$$

This implies that the L^1_{loc} function $\widehat{u}$ vanishes on $\mathbb{R}^d\setminus\{0\}$. Thus, $\widehat{u} = 0$. Thanks to Theorem 1.25, we infer that $u = 0$. As we are considering the case where $\dot{H}^s$ is a Hilbert space, we deduce that $\mathcal{S}_0(\mathbb{R}^d)$ is dense in $\dot{H}^s$. □

The following proposition explains how the space $\dot{H}^{-s}$ can be considered as the dual space of $\dot{H}^s$.

Proposition 1.36. *If $|s| < d/2$, then the bilinear functional*

$$\mathcal{B} : \begin{cases} \mathcal{S}_0 \times \mathcal{S}_0 \to \mathbb{C} \\ \ (\phi,\varphi)\ \mapsto \displaystyle\int_{\mathbb{R}^d} \phi(x)\varphi(x)\,dx \end{cases}$$

can be extended to a continuous bilinear functional on $\dot{H}^{-s}\times\dot{H}^s$. Moreover, if L is a continuous linear functional on $\dot{H}^s$, then a unique tempered distribution u exists in $\dot{H}^{-s}$ such that

$$\forall \phi \in \dot{H}^s\,,\ \langle L,\phi\rangle = \mathcal{B}(u,\phi) \quad \textit{and} \quad \|L\|_{(\dot{H}^s)'} = \|u\|_{\dot{H}^{-s}}.$$

Proof. Let ϕ and φ be in $\mathcal{S}_0$. We can write

$$\begin{aligned} \left|\int_{\mathbb{R}^d}\phi(x)\varphi(x)\,dx\right| &= \left|\int_{\mathbb{R}^d}(\mathcal{F}^{-1}\phi)(\xi)(\mathcal{F}\varphi)(\xi)\,d\xi\right| \\ &= (2\pi)^{-d}\left|\int_{\mathbb{R}^d}|\xi|^{-s}\widehat{\phi}(-\xi)|\xi|^s\widehat{\varphi}(\xi)\,d\xi\right| \\ &\le (2\pi)^{-d}\|\phi\|_{\dot{H}^{-s}}\|\varphi\|_{\dot{H}^s}. \end{aligned}$$

As $\mathcal{S}_0$ is dense in $\dot{H}^\sigma$ when $|\sigma| < d/2$, we can extend $\mathcal{B}$ to $\dot{H}^{-s}\times\dot{H}^s$. Of course, if $(u,\phi)\in\dot{H}^{-s}\times\mathcal{S}$, then $\mathcal{B}(u,\phi) = \langle u,\phi\rangle$.

Let L be a linear functional on $\dot{H}^s$. Consider the linear functional L_s defined by

$$L_s : \begin{cases} L^2(\mathbb{R}^d) \longrightarrow \mathbb{C} \\ \quad f \quad\ \longmapsto \langle L, \mathcal{F}^{-1}(|\cdot|^{-s}f)\rangle. \end{cases}$$

It is obvious that

$$\begin{aligned} \sup_{\|f\|_{L^2}=1}|\langle L_s,f\rangle| &= \sup_{\|f\|_{L^2}=1}|\langle L,\mathcal{F}^{-1}(|\cdot|^{-s}f)\rangle| \\ &= \sup_{\|\phi\|_{\dot{H}^s}=1}|\langle L,\phi\rangle| \\ &= \|L\|_{(\dot{H}^s)'}. \end{aligned}$$

The Riesz representation theorem implies that a function g exists in L^2 such that

$$\forall h \in L^2\,,\ \langle L_s, h\rangle = \int_{\mathbb{R}^d} g(\xi)h(\xi)\,d\xi.$$

We obviously have $|\cdot|^s g \in L^2(\mathbb{R}^d; |\xi|^{-2s}\,d\xi)$. Now, as $|s| < d/2$, this implies that $|\cdot|^s g$ is in $\mathcal{S}'(\mathbb{R}^d)$ and thus we can define $u \stackrel{\text{def}}{=} \mathcal{F}(|\cdot|^s g)$. For any ϕ in $\mathcal{S}(\mathbb{R}^d)$, we then have

$$\langle u, \phi\rangle = \int_{\mathbb{R}^d} g(\xi)|\xi|^s \widehat{\phi}(\xi)\,d\xi = \langle L_s, |\cdot|^s\widehat{\phi}\rangle.$$

By the definition of L_s, we have $\langle u, \phi\rangle = \langle L, \phi\rangle$ and the proposition is thus proved. □

For s in the interval $]0,1[$, the space $\dot{H}^s$ can be described in terms of finite differences.

Proposition 1.37. *Let s be a real number in the interval $]0,1[$ and u be in $\dot{H}^s(\mathbb{R}^d)$. Then,*

$$u \in L^2_{loc}(\mathbb{R}^d) \quad \textit{and} \quad \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x+y)-u(x)|^2}{|y|^{d+2s}}\,dx\,dy < \infty.$$

Moreover, a constant C_s exists such that for any function u in $\dot{H}^s(\mathbb{R}^d)$, we have

$$\|u\|^2_{\dot{H}^s} = C_s \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x+y)-u(x)|^2}{|y|^{d+2s}}\,dx\,dy.$$

Proof. In order to see that u is in $L^2_{loc}(\mathbb{R}^d)$, it suffices to write

$$u = \mathcal{F}^{-1}\big(1_{B(0,1)}\widehat{u}\big) + \mathcal{F}^{-1}\big(1_{^cB(0,1)}\widehat{u}\big).$$

The rest of the proof relies on the Fourier–Plancherel formula (see Theorem 1.25), which implies that

$$\int_{\mathbb{R}^d} \frac{|u(x+y)-u(x)|^2}{|y|^{d+2s}}\,dx = (2\pi)^{-d}\int_{\mathbb{R}^d} \frac{|e^{i(y|\xi)}-1|^2}{|y|^{d+2s}}|\widehat{u}(\xi)|^2\,d\xi.$$

Therefore,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x+y)-u(x)|^2}{|y|^{d+2s}}\,dx\,dy = (2\pi)^{-d}\int_{\mathbb{R}^d} F(\xi)|\widehat{u}(\xi)|^2\,d\xi$$

with

$$F(\xi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \frac{|e^{i(y|\xi)}-1|^2}{|y|^{2s}}\,\frac{dy}{|y|^d}\cdot$$

It may be easily checked that F is a radial and homogeneous function of degree $2s$. This implies that the function $F(\xi)$ is proportional to $|\xi|^{2s}$ and thus completes the proof. □

1.3.2 Sobolev Embedding in Lebesgue Spaces

In this subsection, we investigate the embedding of $\dot{H}^s(\mathbb{R}^d)$ spaces in $L^p(\mathbb{R}^d)$ spaces. We begin with a classical result.

Theorem 1.38. *If s is in $[0, d/2[$, then the space $\dot{H}^s(\mathbb{R}^d)$ is continuously embedded in $L^{\frac{2d}{d-2s}}(\mathbb{R}^d)$.*

Proof. First, let us note that the critical index $p = 2d/(d-2s)$ may be found by using a scaling argument. Indeed, if v is a function on $\mathbb{R}^d$ and v_λ stands for the function $v_\lambda(x) \stackrel{\text{def}}{=} v(\lambda x)$, then we have

$$\|v_\lambda\|_{L^p} = \lambda^{-\frac{d}{p}} \|v\|_{L^p} \quad \text{and} \quad \|v_\lambda\|_{\dot{H}^s} = \lambda^{-\frac{d}{2}+s} \|v\|_{\dot{H}^s}.$$

If an inequality of the type $\|v\|_{L^p} \leq C\|v\|_{\dot{H}^s}$ is true for any smooth function v, then it is also true for v_λ for any λ. Hence, we must have $p = 2d/(d-2s)$.

Consider a function ϕ in $\mathcal{S}_0(\mathbb{R}^d)$. Defining $\widehat{\phi}_s(\xi) \stackrel{\text{def}}{=} |\xi|^s \widehat{\phi}(\xi)$ and using Propositions 1.24 and 1.29, we get that

$$\phi = \frac{(2\pi)^{-d} c_{d,s}}{|\cdot|^{d-s}} \star \phi_s \quad \text{with} \quad \|\phi_s\|_{L^2} = (2\pi)^{-\frac{d}{2}} \|\phi\|_{\dot{H}^s}.$$

Theorem 1.7 thus implies that $\|\phi\|_{L^p} \leq C\|\phi_s\|_{L^2}$. Now, according to Proposition 1.35, the space $\mathcal{S}_0(\mathbb{R}^d)$ is dense in $\dot{H}^s$. The proof is therefore complete. □

Corollary 1.39. *If p belongs to $]1, 2]$, then $L^p(\mathbb{R}^d)$ embeds continuously in $\dot{H}^s(\mathbb{R}^d)$ with $s = \dfrac{d}{2} - \dfrac{d}{p}$.*

Proof. We use the duality between $\dot{H}^s$ and $\dot{H}^{-s}$ described by Proposition 1.36. Write

$$\|a\|_{\dot{H}^s} = \sup_{\|\varphi\|_{\dot{H}^{-s}} \leq 1} \langle a, \varphi \rangle.$$

As $s = d\left(\dfrac{1}{2} - \dfrac{1}{p}\right)$, by Theorem 1.38 we have $\|\varphi\|_{L^{p'}} \leq C\|\varphi\|_{\dot{H}^{-s}}$ and thus

$$\|a\|_{\dot{H}^s} \leq C \sup_{\|\varphi\|_{L^{p'}} \leq 1} \langle a, \varphi \rangle \leq C\|a\|_{L^p}.$$

The corollary is thus proved. □

According to Proposition 1.24, the Fourier transform changes dilation into reciprocal dilation and translation into multiplication by a character $e^{i(x|\omega)}$ (and vice versa). Obviously, the inequality

$$\|u\|_{L^p(\mathbb{R}^d)} \leq C\|u\|_{\dot{H}^s(\mathbb{R}^d)} \quad \text{with} \quad p = 2d/(d-2s)$$

provided by Theorem 1.38 is invariant under translation and dilation.

We claim, however, that it is *not* invariant under multiplication by a character. Indeed, consider a function ϕ in $\mathcal{S}(\mathbb{R}^d)$ such that $\widehat{\phi}$ belongs to $\mathcal{D}(\mathbb{R}^d)$. For all positive ε, define the function

$$\phi_\varepsilon(x) = e^{i\frac{x_1}{\varepsilon}}\phi(x). \tag{1.25}$$

By the definition of $\|\cdot\|_{\dot{H}^s}$, we have

$$\begin{aligned}\|\phi_\varepsilon\|^2_{\dot{H}^s} &= \int_{\mathbb{R}^d} |\xi|^{2s}\Big|\widehat{\phi}\Big(\xi - \frac{e_1}{\varepsilon}\Big)\Big|^2\, d\xi\\ &= \int_{\mathbb{R}^d}\Big|\xi + \frac{e_1}{\varepsilon}\Big|^{2s}|\widehat{\phi}(\xi)|^2\, d\xi \quad \text{with} \;\; e_1 \stackrel{\text{def}}{=} (1,0,\dots,0).\end{aligned}$$

Hence, $\|\phi_\varepsilon\|_{\dot{H}^s}$ is equivalent to ε^{-s} when ε tends to 0, while $\|\phi_\varepsilon\|_{L^p}$ does not depend on ε.

In what follows, we want to improve the estimate of Theorem 1.38 so that it becomes also invariant if u is multiplied by any character $e^{i(x|\omega)}$. In fact, we shall construct a family of Banach spaces E_s, the norm of which is invariant under translation, satisfying

$$\|a(\lambda\cdot)\|_{E_s} \sim \lambda^{s-\frac{d}{2}}\|a\|_{E_s}\,, \quad f\|a(\lambda\cdot)\|_{E_s} \leq C_{s,d}\|a\|_{\dot{H}^s}\,,$$

and, for some real number $\beta \in\;]0,1[$,

$$\|a\|_{L^p} \leq C_{s,d}\|a\|^{1-\beta}_{\dot{H}^s}\|a\|^{\beta}_{E_s}.$$

In order to do this, we introduce the following definition.

Definition 1.40. *Let θ be a function in $\mathcal{S}(\mathbb{R}^d)$ such that $\widehat{\theta}$ is compactly supported, has value* 1 *near* 0, *and satisfies* $0 \leq \widehat{\theta} \leq 1$. *For u in $\mathcal{S}'(\mathbb{R}^d)$ and $\sigma > 0$, we set*

$$\|u\|_{\dot{B}^{-\sigma}} \stackrel{def}{=} \sup_{A>0} A^{d-\sigma}\|\theta(A\cdot)\star u\|_{L^\infty}.$$

It is left to the reader to check that the space $\dot{B}^{-\sigma}$ of tempered distributions u such that $\|u\|_{\dot{B}^{-\sigma}}$ is finite is a Banach space. It is also clear that changing the function θ gives the same space with the equivalent norm. These spaces come up in the next chapter in a more general context. We shall see that $\dot{B}^{-\sigma}$ coincides with the *homogeneous Besov space* $\dot{B}^{-\sigma}_{\infty,\infty}$.

For the time being, we will compare $\dot{B}^{-\sigma}$ with Sobolev spaces.

Proposition 1.41. *For any s less than $d/2$, the space $\dot{H}^s$ is continuously embedded in $\dot{B}^{s-\frac{d}{2}}$ and there exists a constant C, depending only on* Supp $\widehat{\theta}$ *and d, such that*

$$\|u\|_{\dot{B}^{s-\frac{d}{2}}} \leq \frac{C}{\big(\frac{d}{2}-s\big)^{\frac{1}{2}}}\|u\|_{\dot{H}^s} \quad \textit{for all} \quad u \in \dot{H}^s.$$

Proof. As $\widehat{u}$ is locally in L^1, the function $\widehat{\theta}(A^{-1}\cdot)\widehat{u}$ is in L^1. The inverse Fourier theorem implies that

$$\begin{aligned}\|A^d\theta(A\cdot)\star u\|_{L^\infty} &\leq (2\pi)^{-d}\|\widehat{\theta}(A^{-1}\cdot)\widehat{u}\|_{L^1}\\ &\leq (2\pi)^{-d}\int_{\mathbb{R}^d}\widehat{\theta}(A^{-1}\xi)|\xi|^{-s}|\xi|^s|\widehat{u}(\xi)|\,d\xi.\end{aligned}$$

Using the fact that $\widehat{\theta}$ is compactly supported, the Cauchy–Schwarz inequality implies that

$$\|A^d\theta(A\cdot)\star u\|_{L^\infty} \leq \frac{C}{\left(\frac{d}{2}-s\right)^{\frac{1}{2}}}A^{\frac{d}{2}-s}\|u\|_{\dot{H}^s}$$

and the proposition is thus proved. □

The difference between the $\dot{H}^s$ norm the $\dot{B}^{s-\frac{d}{2}}$ norm is emphasized by the following proposition.

Proposition 1.42. *Let $\sigma \in\ ]0,d]$ and let $(\phi_\varepsilon)_{\varepsilon>0}$ be defined according to (1.25). There then exists a constant C such that $\|\phi_\varepsilon\|_{\dot{B}^{-\sigma}} \leq C\varepsilon^\sigma$ for all $\varepsilon > 0$.*

Proof. By Hölder's inequality, we have

$$A^d\|\theta(A\cdot)\star\phi_\varepsilon\|_{L^\infty} \leq \|\theta\|_{L^1}\|\phi\|_{L^\infty}.$$

From this we deduce that if $A\varepsilon \geq 1$, then we have

$$A^{d-\sigma}\|\theta(A\cdot)\star\phi_\varepsilon\|_{L^\infty} \leq \varepsilon^\sigma\|\theta\|_{L^1}\|\phi\|_{L^\infty}. \tag{1.26}$$

If $A\varepsilon \leq 1$, then we perform integration by parts. More precisely, using the fact that

$$(-i\varepsilon\partial_1)^d e^{i\frac{x_1}{\varepsilon}} = e^{i\frac{x_1}{\varepsilon}}$$

and the Leibniz formula, we get

$$\begin{aligned}A^d(\theta(A\cdot)\star\phi_\varepsilon)(x) &= (iA\varepsilon)^d\int_{\mathbb{R}^d}\partial_{y_1}^d\left(\theta(A(x-y))\phi(y)\right)e^{i\frac{y_1}{\varepsilon}}\,dy\\ &= (iA\varepsilon)^d\sum_{k\leq d}\binom{d}{k}A^k((-\partial_1)^k\theta)(A\cdot)\star(e^{i\frac{y_1}{\varepsilon}}\partial_1^{d-k}\phi)(x).\end{aligned}$$

Using Hölder's inequality, we get that

$$A^k\left\|((-\partial_1)^k\theta)(A\cdot)\star(e^{i\frac{y_1}{\varepsilon}}\partial_1^{d-k}\phi)\right\|_{L^\infty} \leq \|\partial_1^k\theta\|_{L^{\frac{d}{k}}}\|\partial_1^{d-k}\phi\|_{L^{(\frac{d}{k})'}}.$$

Thus, we get $A^d\|\theta(A\cdot)\star\phi_\varepsilon\|_{L^\infty} \leq C(A\varepsilon)^d$. As we are considering the case where $A\varepsilon \leq 1$, we get, for any $\sigma \leq d$,

$$A^d\|\theta(A\cdot)\star\phi_\varepsilon\|_{L^\infty} \leq C(A\varepsilon)^\sigma.$$

Together with (1.26), this concludes the proof of the proposition. □

We can now state the so-called *refined Sobolev inequalities.*

Theorem 1.43. *Let s be in $]0,d/2[$. There exists a constant C, depending only on d and $\widehat{\theta}$, such that*

$$\|u\|_{L^p} \leq \frac{C}{(p-2)^{\frac{1}{p}}}\|u\|^{1-\frac{2}{p}}_{\dot{B}^{s-\frac{d}{2}}}\|u\|^{\frac{2}{p}}_{\dot{H}^s} \quad \textit{with} \quad p=\frac{2d}{d-2s}\cdot$$

Proof. Without loss of generality, we can assume that $\|u\|_{\dot{B}^{s-\frac{d}{2}}}=1$. As will be done quite often in this book, we shall decompose the function into low and high frequencies. More precisely, we write

$$u=u_{\ell,A}+u_{h,A} \quad \text{with} \quad u_{\ell,A}=\mathcal{F}^{-1}(\widehat{\theta}(A^{-1}\cdot)\widehat{u}), \tag{1.27}$$

where θ is the function from Definition 1.40. The triangle inequality implies that

$$\big(|u|>\lambda\big)\subset\big(|u_{\ell,A}|>\lambda/2\big)\cup\big(|u_{h,A}|>\lambda/2\big)\cdot$$

By the definition of $\|\cdot\|_{\dot{B}^{s-\frac{d}{2}}}$ we have $\|u_{\ell,A}\|_{L^\infty}\leq A^{\frac{d}{2}-s}$. From this we deduce that

$$A=A_\lambda \stackrel{\text{def}}{=} \Big(\frac{\lambda}{2}\Big)^{\frac{p}{d}} \Longrightarrow \mu\big(|u_{\ell,A}|>\lambda/2\big)=0.$$

From the identity (1.8) we deduce that

$$\|u\|^p_{L^p}\leq p\int_0^\infty \lambda^{p-1}\mu\big(|u_{h,A_\lambda}|>\lambda/2\big)\,d\lambda.$$

Using the fact that

$$\mu\big(|u_{h,A_\lambda}|>\lambda/2\big)\leq 4\frac{\|u_{h,A_\lambda}\|^2_{L^2}}{\lambda^2},$$

we get

$$\|u\|^p_{L^p}\leq 4p\int_0^\infty \lambda^{p-3}\|u_{h,A_\lambda}\|^2_{L^2}\,d\lambda.$$

Because the Fourier transform is (up to a constant) an isometry on $L^2(\mathbb{R}^d)$ and the function $\widehat{\theta}$ has value 1 near 0, we thus get, for some $c>0$ depending only on $\widehat{\theta}$,

$$\|u\|^p_{L^p}\leq 4p\,(2\pi)^{-d}\int_0^\infty \lambda^{p-3}\int_{(|\xi|\geq cA_\lambda)}|\widehat{u}(\xi)|^2\,d\xi\,d\lambda. \tag{1.28}$$

Now, by definition of A_λ, we have

$$|\xi|\geq cA_\lambda \Longleftrightarrow \lambda\leq C_\xi \stackrel{\text{def}}{=} 2\Big(\frac{|\xi|}{c}\Big)^{\frac{d}{p}}.$$

Fubini's theorem thus implies that

$$\begin{aligned}\|u\|_{L^p}^p &\leq 4p\,(2\pi)^{-d}\int_{\mathbb{R}^d}\biggl(\int_0^{C_\xi}\lambda^{p-3}d\lambda\biggr)\,|\widehat{u}(\xi)|^2\,d\xi\\ &\leq (2\pi)^{-d}\frac{p2^p}{p-2}\int_{\mathbb{R}^d}\Bigl(\frac{|\xi|}{c}\Bigr)^{\frac{d(p-2)}{p}}\,|\widehat{u}(\xi)|^2\,d\xi.\end{aligned}$$

As $s = d\Bigl(\dfrac{1}{2}-\dfrac{1}{p}\Bigr)$, the theorem is proved. □

Remark 1.44. Combining Proposition 1.41 and Theorem 1.43, we see that if $0 < s < d/2$, then we have, for all $u \in \dot{H}^s$,

$$\|u\|_{L^p} \leq C_d \frac{p}{\sqrt{p-2}}\,\|u\|_{\dot{H}^s} \quad \text{with} \quad p = \frac{2d}{d-2s}\cdot \tag{1.29}$$

Of course, since we have $\|u\|_{L^2} = (2\pi)^{-\frac{d}{2}}\|u\|_{\dot{H}^0}$, we do not expect the constant to blow up when p goes to 2. In fact, combining this latter inequality with the inequality (1.29) (with, say, $p = 4$) and resorting to a complex interpolation argument, we get

$$\|u\|_{L^p} \leq C_d\sqrt{p}\,\|u\|_{\dot{H}^s} \quad \text{with} \quad p = \frac{2d}{d-2s}\cdot \tag{1.30}$$

By taking advantage of Proposition 1.42 and the computations that follow (1.25), it is not difficult to check that the inequality stated in Theorem 1.43 is indeed invariant (up to an irrelevant constant) under multiplication by a character. We now want to consider whether our refined inequalities are sharp. Obviously, according to Proposition 1.42, we have

$$\lim_{\varepsilon\to 0}\frac{\|\phi_\varepsilon\|_{L^p}}{\|\phi_\varepsilon\|^{\beta}_{\dot{B}^{s-\frac{d}{2}}}\,\|\phi_\varepsilon\|^{1-\beta}_{\dot{H}^s}} = +\infty \quad \text{for any} \quad \beta > 1-2/p.$$

Therefore, the exponent $1-2/p$ cannot be improved. We claim that even under a sign assumption, the above refined Sobolev inequalities are sharp. More precisely, we shall exhibit a sequence $(f_n)_{n\in\mathbb{N}}$ of *nonnegative* functions such that

$$\lim_{n\to\infty}\frac{\|f_n\|_{L^{\frac{2d}{d-2s}}}}{\|f_n\|^{\beta}_{\dot{B}^{s-\frac{d}{2}}}\,\|f_n\|^{1-\beta}_{\dot{H}^s}} = +\infty \quad \text{for any} \quad \beta > 1-2/p. \tag{1.31}$$

Constructing such a family may be done by means of an iterative process. At each step of the process, we use a linear transform T (defined below) which duplicates any function f into 2^d copies of the same function, at the scale $1/4$.

Definition 1.45. *Define* $Q \stackrel{def}{=} [-1/2, 1/2]^d$ *and let* $x_J = 3/8\,J$ *for any element* J *of* $\{-1,1\}^d$*. We then define the transform* T *by*

$$T : \begin{cases} \mathcal{D}(Q) \longrightarrow \mathcal{D}(Q) \\ \;f \quad \longmapsto Tf \overset{def}{=} 2^d \displaystyle\sum_{J\in\{-1,1\}^d} T_J f \qquad \textit{with} \quad T_J f(x) \overset{def}{=} f(4(x-x_J)). \end{cases}$$

For $B \subset Q$, *we define* $T_J(B) \overset{def}{=} x_J + \frac{1}{4}B$, $T(B) \overset{def}{=} \displaystyle\bigcup_{J\in\{-1,1\}^d} T_J(B)$ *and denote* $T_J(Q)$ *by* Q_J.

Using the fact that for any $f \in \mathcal{D}(Q)$, the support of $T_J f$ is included in Q_J and the fact that if $J \neq J'$, then $Q_J \cap Q_{J'} = \emptyset$, we immediately get

$$\|Tf\|_{L^p} = 2^{d\left(1-\frac{1}{p}\right)}\|f\|_{L^p}. \tag{1.32}$$

For the sake of simplicity we restrict our attention here to the case where s is an integer.[10] Then, observing that

$$\partial_j(Tf)(x) = 2^d \sum_{J\in\{-1,1\}^d} 4(\partial_j f)(4(x-x_J)) = 4T(\partial_j f)(x)$$

and using (1.32), we get

$$\|Tf\|_{\dot H^s} = 2^{\frac{d}{2}+2s}\|f\|_{\dot H^s}. \tag{1.33}$$

The estimate of Tf in terms of the $\dot B^{-\sigma}$ norm is described by the following proposition.

Proposition 1.46. *For* $\sigma \in \;]0,d]$, *a constant* C *exists such that*

$$\|Tf\|_{\dot B^{-\sigma}} \le 2^{d-2\sigma}\|f\|_{\dot B^{-\sigma}} + C\|f\|_{L^1}.$$

Proof. Since, thanks to (1.32), we have

$$\lambda^{d-\sigma}\|\theta(\lambda\cdot)\star(Tf)\|_{L^\infty} \le \lambda^{d-\sigma}\|\theta\|_{L^\infty}\|Tf\|_{L^1} \le \lambda^{d-\sigma}\|\theta\|_{L^\infty}\|f\|_{L^1},$$

we get

$$\sup_{\lambda\le 1}\lambda^{-\sigma}\|\lambda^d\theta(\lambda\cdot)\star(Tf)\|_{L^\infty} \le \|\theta\|_{L^\infty}\|f\|_{L^1}. \tag{1.34}$$

The case where λ is large (which corresponds to high frequencies) is more intricate. We first estimate $\lambda^d(\theta(\lambda\cdot)\star(Tf))(x)$ when x is not too close to $T(Q)$, namely, $x \in \widetilde{Q}^c \overset{\text{def}}{=} \{x \in Q \,/\, d(x,T(Q)) \ge 1/8\}$. As the function θ belongs to $\mathcal{S}(\mathbb{R}^d)$, we have, for any positive integer N,

$$\begin{aligned} \left|\lambda^d(\theta(\lambda\cdot)\star(Tf))(x)\right| &\le \lambda^d\|\theta\|_{N,\mathcal{S}}\int_{\mathbb{R}^d}\frac{1}{\lambda^N|x-y|^N}|Tf(y)|\,dy \\ &\le C\|\theta\|_{N,\mathcal{S}}\lambda^{d-N}\|f\|_{L^1}. \end{aligned}$$

[10] The general case follows by interpolation.

This gives, for sufficiently large N,

$$\sup_{\lambda\geq 1} \lambda^{-\sigma}\|\lambda^d\theta(\lambda\cdot)\star(Tf)\|_{L^\infty(\widetilde{Q}^c)} \leq C\|\theta\|_{N,\mathcal{S}}\|f\|_{L^1}. \tag{1.35}$$

We now investigate the case where $x \in \widetilde{Q}$. By definition, an element J_x of $\{-1,1\}^d$ and a point y of Q_{J_x} exist such that $d(x,y) \leq 1/8$. For any $J' \neq J_x$, we have

$$d(x,Q_{J'}) \geq d(y,Q_{J'}) - d(x,y) \geq \frac{1}{2} - \frac{1}{8} \geq \frac{3}{8}\cdot$$

We now write

$$\begin{aligned}\left|\lambda^d\theta(\lambda\cdot)\star(Tf)\right|(x) \leq 2^d\left|\lambda^d\theta(\lambda\cdot)\star(T_{J_x}f)\right|(x)&\\ +\sum_{J'\in\{-1,1\}^d\setminus\{J_x\}} 2^d\left|\lambda^d\theta(\lambda\cdot)\star(T_{J'}f)\right|(x).&\end{aligned}$$

Again using the fact that the function θ belongs to $\mathcal{S}(\mathbb{R}^d)$, we have, for any positive integer N and any $J' \neq J_x$,

$$\begin{aligned}\left|\lambda^d(\theta(\lambda\cdot)\star(T_{J'}f))(x)\right| &\leq \|\theta\|_{N,\mathcal{S}}\lambda^d\int_{\mathbb{R}^d}\frac{1}{\lambda^N|x-y|^N}|T_{J'}f(y)|\,dy\\ &\leq C\|\theta\|_{N,\mathcal{S}}\lambda^{d-N}\|T_{J'}f\|_{L^1}.\end{aligned}$$

Using (1.32), we infer that, for $\lambda \geq 1$ and N sufficiently large,

$$\begin{aligned}\sum_{J'\in\{-1,1\}^d\setminus\{J_x\}}\left|\lambda^d\theta(\lambda\cdot)\star(T_{J'}f)\right|(x) &\leq C\|\theta\|_{N,\mathcal{S}}\sum_{J'\in\{-1,1\}^d\setminus\{J_x\}}\|T_{J'}f\|_{L^1}\\ &\leq C\|\theta\|_{N,\mathcal{S}}\|f\|_{L^1}.\end{aligned} \tag{1.36}$$

For any J, we have, by definition of T_J,

$$\sup_{\lambda>0}\lambda^{-\sigma}\|\lambda^d\theta(\lambda\cdot)\star(T_Jf)\|_{L^\infty} \leq \sup_{\lambda>0}\lambda^{-\sigma}\Big\|\Big(\frac{\lambda}{4}\Big)^d\theta\Big(\frac{\lambda}{4}\cdot\Big)\star f\Big\|_{L^\infty} \leq 2^{-2\sigma}\|f\|_{\dot{B}^{-\sigma}}.$$

Together with (1.34), (1.35), and (1.36), this gives

$$\sup_{\lambda\geq 1}\lambda^{-\sigma}\|\lambda^d\theta(\lambda\cdot)\star(Tf)\|_{L^\infty} \leq 2^{d-2\sigma}\|f\|_{\dot{B}^{-\sigma}} + C\|f\|_{L^1}.$$

This completes the proof. □

We can now construct a sequence $(f_n)_{n\in\mathbb{N}}$ of functions satisfying (1.31). For that purpose, we consider a smooth nonnegative function f_0, supported in Q, and define $f_n = T^nf_0$. Iterating the inequality from Proposition 1.46 yields

$$\|f_n\|_{\dot{B}^{-\sigma}} \leq 2^{n(d-2\sigma)}\|f_0\|_{\dot{B}^{-\sigma}} + C\Big(\sum_{m=0}^{n-1}2^{m(d-2\sigma)}\Big)\|f_0\|_{L^1}.$$

Taking $\sigma = d/2 - s$ with $s \in]0,d/2[$, we deduce that

$$\|f_n\|_{\dot{B}^{s-\frac{d}{2}}} \leq C_{f_0}2^{2ns}.$$

Using (1.32) and (1.33), we can now conclude that (1.31) is satisfied.

1.3.3 The Limit Case $\dot{H}^{\frac{d}{2}}$

The space $\dot{H}^{\frac{d}{2}}(\mathbb{R}^d)$ is not included in $L^\infty(\mathbb{R}^d)$. We give an explicit counterexample in dimension two. Let the function u be defined by

$$u(x) = \varphi(x)\log(-\log|x|)$$

for some smooth function φ supported in $B(0,1)$ with value 1 near 0. On the one hand, u is not bounded. On the other hand, we have, near the origin,

$$|\partial_j u(x)| \leq \frac{C}{|x|\,|\log|x|\,|}$$

so that u belongs to $\dot{H}^1(\mathbb{R}^2)$.

This motivates the following definition.

Definition 1.47. *The space $BMO(\mathbb{R}^d)$ of bounded mean oscillations is the set of locally integrable functions f such that*

$$\|f\|_{BMO} \stackrel{def}{=} \sup_B \frac{1}{|B|}\int_B |f - f_B|\,dx < \infty \quad \textit{with} \quad f_B \stackrel{def}{=} \frac{1}{|B|}\int_B f\,dx.$$

The above supremum is taken over the set of Euclidean balls.

We point out that the seminorm $\|\cdot\|_{BMO}$ vanishes on constant functions. Therefore, this is not a norm. We now state the critical theorem for Sobolev embedding.

Theorem 1.48. *The space $L^1_{loc}(\mathbb{R}^d) \cap \dot{H}^{\frac{d}{2}}(\mathbb{R}^d)$ is included in $BMO(\mathbb{R}^d)$. Moreover, there exists a constant C such that*

$$\|u\|_{BMO} \leq C\|u\|_{\dot{H}^{\frac{d}{2}}}$$

for all functions $u \in L^1_{loc}(\mathbb{R}^d) \cap \dot{H}^{\frac{d}{2}}(\mathbb{R}^d)$.

Proof. We use the decomposition (1.27) into low and high frequencies. For any Euclidean ball B we have

$$\int_B |u - u_B|\frac{dx}{|B|} \leq \|u_{\ell,A} - (u_{\ell,A})_B\|_{L^2(B,\frac{dx}{|B|})} + \frac{2}{|B|^{\frac{1}{2}}}\|u_{h,A}\|_{L^2}.$$

Let R be the radius of the ball B. We have

$$\begin{aligned}
\|u_{\ell,A} - (u_{\ell,A})_B\|_{L^2(B,\frac{dx}{|B|})} &\leq R\|\nabla u_{\ell,A}\|_{L^\infty} \\
&\leq CR\int_{\mathbb{R}^d} |\xi|^{1-\frac{d}{2}}|\xi|^{\frac{d}{2}}|\widehat{u}_{\ell,A}(\xi)|\,d\xi \\
&\leq CRA\|u\|_{\dot{H}^{\frac{d}{2}}}.
\end{aligned}$$

We infer that

$$\int_B |u - u_B|\frac{dx}{|B|} \leq CRA\|u\|_{\dot{H}^{\frac{d}{2}}} + C(AR)^{-\frac{d}{2}}\left(\int_{|\xi|\geq A} |\xi|^d|\widehat{u}(\xi)|^2\,d\xi\right)^{\frac{1}{2}}.$$

Choosing $A = R^{-1}$ then completes the proof. □

1.3.4 The Embedding Theorem in Hölder Spaces

Definition 1.49. *Let (k,ρ) be in $\mathbb{N}\times]0,1]$. The* Hölder space $C^{k,\rho}(\mathbb{R}^d)$ *(or $C^{k,\rho}$, if no confusion is possible) is the space of C^k functions u on $\mathbb{R}^d$ such that*

$$\|u\|_{C^{k,\rho}} = \sup_{|\alpha|\leq k}\Big(\|\partial^\alpha u\|_{L^\infty} + \sup_{x\neq y}\frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|}{|x-y|^\rho}\Big) < \infty.$$

Proving that the sets $C^{k,\rho}$ are Banach spaces is left as an exercise. We point out that $C^{0,1}$ is the space of bounded Lipschitz functions.

Theorem 1.50. *If $s>\frac{d}{2}$ and $s-\frac{d}{2}$ is not an integer, then the space $\dot H^s(\mathbb{R}^d)$ is included in the Hölder space of index*

$$(k,\rho) = \left(\left[s-\frac{d}{2}\right], s-\frac{d}{2}-\left[s-\frac{d}{2}\right]\right),$$

and we have, for all $u\in\dot H^s(\mathbb{R}^d)$,

$$\sup_{|\alpha|=k}\sup_{x\neq y}\frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|}{|x-y|^\rho} \leq C_{d,s}\|u\|_{\dot H^s}.$$

Proof. We prove the theorem only in the case where the integer part of $s-d/2$ is 0. As s is greater than $d/2$, writing

$$\widehat u = \mathbf{1}_{B(0,1)}\widehat u + (\mathbf{1}-\mathbf{1}_{B(0,1)})\widehat u,$$

we get that $\widehat u$ belongs to $L^1(\mathbb{R}^d)$, and thus u is a bounded continuous function. We again use the decomposition (1.27) into low and high frequencies. The low-frequency part of u is of course smooth. By Taylor's inequality, we have

$$|u_{\ell,A}(x)-u_{\ell,A}(y)| \leq \|\nabla u_{\ell,A}\|_{L^\infty}|x-y|.$$

Using the Fourier inversion formula and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\|\nabla u_{\ell,A}\|_{L^\infty} &\leq C\int_{\mathbb{R}^d}|\xi|\,|\widehat u_{\ell,A}(\xi)|\,d\xi\\ &\leq C\left(\int_{|\xi|\leq CA}|\xi|^{2-2s}\,d\xi\right)^{\frac12}\|u\|_{\dot H^s}\\ &\leq \frac{C}{(1-\rho)^{\frac12}}A^{1-\rho}\|u\|_{\dot H^s}\quad\text{with}\quad \rho = s-d/2.\end{aligned}$$

Reasoning along exactly the same lines, we also have that

$$\begin{aligned}
\|u_{h,A}\|_{L^\infty} &\le \int_{\mathbb{R}^d} |\widehat{u}_{h,A}(\xi)|\,d\xi \\
&\le \left(\int_{|\xi|\ge A} |\xi|^{-2s}\,d\xi\right)^{\frac12} \|u\|_{\dot H^s} \\
&\le \frac{C}{\rho^{\frac12}} A^{-\rho}\|u\|_{\dot H^s}.
\end{aligned}$$

It is then obvious that

$$\begin{aligned}
|u(x)-u(y)| &\le \|\nabla u_{\ell,A}\|_{L^\infty}|x-y| + 2\|u_{h,A}\|_{L^\infty} \\
&\le C_s\left(|x-y|A^{1-\rho} + A^{-\rho}\right)\|u\|_{\dot H^s}.
\end{aligned}$$

Choosing $A = |x-y|^{-1}$ then completes the proof of the theorem. □

1.4 Nonhomogeneous Sobolev Spaces on $\mathbb{R}^d$

In this section, we focus on nonhomogeneous Sobolev spaces. As in the previous section, the emphasis is on embedding properties in Lebesgue and Hölder spaces. We also establish a trace theorem and provide an elementary proof for a Hardy inequality.

1.4.1 Definition and Basic Properties

Definition 1.51. *Let s be a real number. The Sobolev space $H^s(\mathbb{R}^d)$ consists of tempered distributions u such that $\widehat{u} \in L^2_{loc}(\mathbb{R}^d)$ and*

$$\|u\|^2_{H^s} \stackrel{def}{=} \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2\,d\xi < \infty.$$

As the Fourier transform is an isometric linear operator from the space $H^s(\mathbb{R}^d)$ onto the space $L^2(\mathbb{R}^d;(1+|\xi|^2)^s\,d\xi)$, the space $H^s(\mathbb{R}^d)$ equipped with the scalar product

$$(u \mid v)_{H^s} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} (1+|\xi|^2)^s \widehat{u}(\xi)\overline{\widehat{v}}(\xi)\,d\xi \tag{1.37}$$

is a Hilbert space.

It is obvious that the family of H^s spaces is decreasing with respect to s. Moreover, we have the following proposition, the proof of which is strictly analogous to that of Proposition 1.32.

Proposition 1.52. *If $s_0 \le s \le s_1$, then we have*

$$\|u\|_{H^s} \le \|u\|^{1-\theta}_{H^{s_0}} \|u\|^{\theta}_{H^{s_1}} \quad \textit{with} \quad s = (1-\theta)s_0 + \theta s_1.$$

When s is a nonnegative integer, the Fourier–Plancherel formula ensures that the space H^s coincides with the set of L^2 functions u such that $\partial^\alpha u$ belongs to L^2 for any α in $\mathbb{N}^d$ with $|\alpha| \leq s$. In the case where s is a negative integer, the space H^s is described by the following proposition, the proof of which is analogous to that of Proposition 1.33.

Proposition 1.53. *Let k be a positive integer. The space $H^{-k}(\mathbb{R}^d)$ consists of distributions which are sums of an $L^2(\mathbb{R}^d)$ function and derivatives of order k of $L^2(\mathbb{R}^d)$ functions.*

Remark 1.54. The Dirac mass δ_0 belongs to $H^{-\frac{d}{2}-\varepsilon}$ for any positive ε but does not belong to $H^{-\frac{d}{2}}$. Moreover, δ_0 is not in $\dot{H}^s$ for any s.

It is obvious that when s is nonnegative, H^s is included in $\dot{H}^s$, and that the opposite happens when s is negative. Further, $\dot{H}^s \neq H^s$ for $s \neq 0$. In the following proposition, we state that the two spaces coincide for compactly supported distributions and nonnegative s.

Proposition 1.55. *Let s be a nonnegative real number and K a compact subset of $\mathbb{R}^d$. Let $H^s_K(\mathbb{R}^d)$ be the space of those distributions of $H^s(\mathbb{R}^d)$ which are supported in K. There then exists a positive constant C such that*

$$\forall u \in H^s_K(\mathbb{R}^d)\,,\ \frac{1}{C}\|u\|_{H^s} \leq \|u\|_{\dot{H}^s} \leq \|u\|_{H^s}.$$

Proof. We simply have to prove that $\|u\|_{L^2} \leq C_K \|u\|_{\dot{H}^s}$. Using the Fourier–Plancherel formula and the inverse formula, we have[11]

$$|\widehat{u}(\xi)| \leq \|u\|_{L^1} \leq \sqrt{|K|}\,\|u\|_{L^2} \leq (2\pi)^{-\frac{d}{2}}\sqrt{|K|}\,\|\widehat{u}\|_{L^2}.$$

For any positive ε we then get

$$\begin{aligned}\|\widehat{u}\|^2_{L^2} &\leq (2\pi)^{-d}|K|\|\widehat{u}\|^2_{L^2}\big|B(0,\varepsilon)\big| + \int_{\mathbb{R}^d\setminus B(0,\varepsilon)} |\xi|^{-2s}|\xi|^{2s}|\widehat{u}(\xi)|^2\,d\xi\\ &\leq (2\pi)^{-d}c_d\varepsilon^d\,|K|\,\|\widehat{u}\|^2_{L^2} + \frac{1}{\varepsilon^{2s}}\|u\|^2_{\dot{H}^s}.\end{aligned}$$

Taking ε such that $(2\pi)^{-d}c_d\varepsilon^d\,|K| = 1/2$, we see that

$$\|\widehat{u}\|_{L^2} \leq \frac{\sqrt{2}}{(2\pi)^s}\big(2c_d|K|\big)^{\frac{s}{d}}\|u\|_{\dot{H}^s}, \tag{1.38}$$

and the result follows. □

From the above proposition, we can infer the following Poincaré-type inequality, which is relevant for functions supported in small balls.

[11] From now on, we agree that $|K|$ denotes the Lebesgue measure of the set K.

Corollary 1.56. *Let $0 \leq t \leq s$. A constant C exists such that for any positive δ and any function $u \in H^s(\mathbb{R}^d)$ supported in a ball of radius δ, we have*

$$\|u\|_{\dot{H}^t} \leq C\delta^{s-t}\|u\|_{\dot{H}^s} \quad \textit{and} \quad \|u\|_{H^t} \leq C\delta^{s-t}\|u\|_{H^s}.$$

Proof. Using the fact that the $\|\cdot\|_{H^s}$ norm is invariant under translation, we can suppose that the ball is centered at the origin. If we set $v(x) = u(\delta x)$, then v is supported in the unit ball and obviously satisfies $\|v\|_{H^t} \leq C\|v\|_{H^s}$, hence also $\|v\|_{\dot{H}^t} \leq C\|v\|_{\dot{H}^s}$, due to the previous proposition.

Using the fact that $\widehat{v}(\xi) = \delta^{-d}\widehat{u}\Big(\dfrac{\xi}{\delta}\Big)$, we thus get $\|u\|_{\dot{H}^t} \leq C\delta^{s-t}\|u\|_{\dot{H}^s}$. Using (1.38) we then get the inequality pertaining to nonhomogeneous norms. □

We have the following density result, strictly analogous to Proposition 1.35.

Proposition 1.57. *The space $\mathcal{S}$ is dense in H^s.*

The duality between H^s and H^{-s} is described by the following proposition, the proof of which is analogous to that of Proposition 1.36.

Proposition 1.58. *For any real s, the bilinear functional*

$$\mathcal{B} : \begin{cases} \mathcal{S} \times \mathcal{S} \to \mathbb{C} \\ (\phi, \varphi) \mapsto \displaystyle\int_{\mathbb{R}^d} \phi(x)\varphi(x)\,dx \end{cases}$$

can be extended to a continuous bilinear functional on $H^{-s} \times H^s$. Moreover, if L is a continuous linear functional on H^s, a unique tempered distribution u exists in H^{-s} such that

$$\forall \phi \in \mathcal{S}\,,\ \langle L, \phi\rangle = \mathcal{B}(u, \phi).$$

In addition, we have $\|L\|_{(H^s)'} = \|u\|_{H^{-s}}$.

The following proposition can be very easily deduced from Proposition 1.37.

Proposition 1.59. *Let $s = m + \sigma$ with $m \in \mathbb{N}$ and $\sigma \in\ ]0, 1[$. We then have*

$$\begin{aligned} H^s(\mathbb{R}^d) = \Big\{ & u \in L^2(\mathbb{R}^d) \,/\, \forall \alpha \in \mathbb{N}^d \ /\ |\alpha| \leq m\,,\ \partial^\alpha u \in L^2(\mathbb{R}^d) \\ & \textit{and, for } \ \alpha \,/\, |\alpha| = m\,, \ \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\partial^\alpha u(x+y) - \partial^\alpha u(x)|^2}{|y|^{d+2\sigma}}\,dx\,dy < +\infty \Big\}, \end{aligned}$$

and there exists a constant C such that

$$\begin{aligned} C^{-1}\|u\|_{H^s}^2 \leq \sum_{|\alpha|=m} \int_{\mathbb{R}^d \times \mathbb{R}^d} & \frac{|\partial^\alpha u(x+y) - \partial^\alpha u(x)|^2}{|y|^{d+2\sigma}}\,dx\,dy \\ & + \sum_{|\alpha|\leq m} \|\partial^\alpha u\|_{L^2}^2 \leq C\|u\|_{H^s}^2. \end{aligned}$$

The above characterization of Sobolev spaces is suitable for establishing invariance under diffeomorphism. In what follows, it is understood that a *global k-diffeomorphism* on $\mathbb{R}^d$ is any C^k diffeomorphism φ from $\mathbb{R}^d$ onto $\mathbb{R}^d$ whose derivatives of order less than or equal to k are bounded and which satisfies, for some constant C,

$$\forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d\,,\ |\varphi(x) - \varphi(y)| \geq C|x-y|.$$

Corollary 1.60. *Let φ be a global k-diffeomorphism on $\mathbb{R}^d$, $0 \leq s < k$, and $u \in H^s(\mathbb{R}^d)$. Then, $u \circ \varphi \in H^s(\mathbb{R}^d)$.*

Proof. By virtue of the chain rule, it is enough to consider the case where s is in $[0,1[$. The result follows easily from the identity

$$\begin{aligned}
J(u) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(\varphi(x)) - u(\varphi(y))|^2}{|x-y|^{d+2s}}\, dx\, dy \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|\psi(x) - \psi(y)|^{d+2s}} |\det(D\psi(x))|^{-1} |\det(D\psi(y))|^{-1}\, dx\, dy \\
&\leq C \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}}\, dx\, dy,
\end{aligned}$$

where it is understood that $\psi = \varphi^{-1}$. This proves the corollary. $\square$

The following density theorem will be useful.

Theorem 1.61. *For any real s, the space $\mathcal{D}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$.*

Proof. In order to prove this theorem, we consider a distribution u in $H^s(\mathbb{R}^d)$ such that for any test function φ in $\mathcal{D}(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \widehat{\varphi}(\xi)(1+|\xi|^2)^s \overline{\widehat{u}(\xi)}\, d\xi = 0.$$

Knowing that $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$ and that the Fourier transform is an automorphism of $\mathcal{S}(\mathbb{R}^d)$, we have, for any function f in $\mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} f(\xi)(1+|\xi|^2)^s \overline{\widehat{u}(\xi)}\, d\xi = 0.$$

This implies that $(1+|\cdot|^2)^s \widehat{u} = 0$ as a tempered distribution. Thus, $\widehat{u} = 0$, and then $u = 0$. $\square$

The Sobolev spaces are not stable under multiplication by C^∞ functions; nevertheless, they are *local*. This is a consequence of the following result.

Theorem 1.62. *Multiplication by a function of $\mathcal{S}(\mathbb{R}^d)$ is a continuous map from $H^s(\mathbb{R}^d)$ into itself.*

Proof. As we know that $\widehat{\varphi u} = (2\pi)^{-d}\widehat{\varphi} \star \widehat{u}$, the proof of Theorem 1.62 is reduced to the estimate of the $L^2(\mathbb{R}^d)$ norm of the function U_s defined by

$$U_s(\xi) \stackrel{\text{def}}{=} (1+|\xi^2|)^{\frac{s}{2}} \int_{\mathbb{R}^d} |\widehat{\varphi}(\xi-\eta)| \times |\widehat{u}(\eta)|\, d\eta.$$

We will temporarily assume that

$$(1+|\xi|^2)^{\frac{s}{2}} \leq 2^{\frac{|s|}{2}} (1+|\xi-\eta|^2)^{\frac{|s|}{2}} (1+|\eta|^2)^{\frac{s}{2}}. \tag{1.39}$$

We then infer that

$$|U_s(\xi)| \leq 2^{\frac{|s|}{2}} \int_{\mathbb{R}^d} (1+|\xi-\eta|^2)^{\frac{|s|}{2}} |\widehat{\varphi}(\xi-\eta)| (1+|\eta|^2)^{\frac{s}{2}} |\widehat{u}(\eta)|\, d\eta.$$

Using Young's inequality, we get

$$\|\varphi u\|_{H^s} \leq 2^{\frac{|s|}{2}} \|(1+|\cdot|^2)^{\frac{|s|}{2}} \widehat{\varphi}\|_{L^1} \|u\|_{H^s},$$

and the desired result follows.

For the sake of completeness, we now prove the inequality (1.39). Interchanging ξ and η, we see that it suffices to consider the case $s \geq 0$. We have

$$\begin{aligned}(1+|\xi|^2)^{\frac{s}{2}} &\leq (1+2(|\xi-\eta|^2+|\eta|^2))^{\frac{s}{2}}\\ &\leq 2^{\frac{s}{2}}(1+|\xi-\eta|^2)^{\frac{s}{2}}(1+|\eta|^2)^{\frac{s}{2}}.\end{aligned}$$

This completes the proof of the theorem. □

We will now consider the problem of *trace* and *trace lifting* operators for the Sobolev spaces. Consider the hyperplane $x_1 = 0$ in $\mathbb{R}^d$. Because this has measure zero, we cannot give any reasonable sense to the trace operator γ formally defined by $\gamma u(x') = u(0, x')$ if u belongs to a Lebesgue space. For instance, there exist elements of $L^2(\mathbb{R}^d)$ which are continuous for $x_1 \neq 0$ and tend to infinity when x_1 goes to 0. This obviously precludes us from defining the trace of a general L^2 function.

The following theorem shows that defining γu makes sense for $u \in H^s(\mathbb{R}^d)$ with s greater than $1/2$. Extending the usual trace operator by continuity provides us with the relevant definition.

Theorem 1.63. *Let s be a real number strictly larger than $1/2$. The restriction map γ defined by*

$$\gamma : \begin{cases} \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^{d-1}) \\ \quad \phi \quad \longmapsto \gamma(\phi) : (x_2, \ldots, x_d) \mapsto \phi(0, x_2, \ldots, x_d) \end{cases}$$

can be continuously extended from $H^s(\mathbb{R}^d)$ onto $H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$.

Proof. We first prove the existence of γ. Arguing by density, it suffices to find a constant C such that

$$\forall \phi \in \mathcal{S}\,,\ \|\gamma(\phi)\|_{H^{s-\frac{1}{2}}} \leq C\|\phi\|_{H^s}. \tag{1.40}$$

To achieve the above inequality, we may rewrite the trace operator in terms of a Fourier transform:

$$\begin{aligned}\phi(0,x') &= (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x'|\xi')}\widehat{\phi}(\xi_1,\xi')\,d\xi_1\,d\xi'\\ &= (2\pi)^{1-d}\int_{\mathbb{R}^{d-1}} e^{i(x'|\xi')}\Big((2\pi)^{-1}\int_{\mathbb{R}}\widehat{\phi}(\xi_1,\xi')\,d\xi_1\Big)\,d\xi'.\end{aligned}$$

We thus have

$$\widehat{\gamma(\phi)}(\xi') = (2\pi)^{-1}\int_{\mathbb{R}}\widehat{\phi}(\xi_1,\xi')\,d\xi_1.$$

By multiplication and division by $(1+|\xi_1|^2+|\xi'|^2)^{\frac{s}{2}}$ and the Cauchy–Schwarz inequality, we have

$$|\widehat{\gamma(\phi)}(\xi')|^2 \leq \frac{1}{4\pi^2}\Big(\int_{\mathbb{R}}(1+\xi_1^2+|\xi'|^2)^{-s}\,d\xi_1\Big)\Big(\int_{\mathbb{R}}(|\widehat{\phi}(\xi)|^2(1+|\xi|^2)^s\,d\xi_1\Big).$$

Having $s > \frac{1}{2}$ ensures that the first integral is finite. In order to compute it, we make the change of variables $\xi_1 = (1+|\xi'|^2)^{\frac{1}{2}}\lambda$. We obtain

$$\int(1+\xi_1^2+|\xi'|^2)^{-s}\,d\xi_1 = C_s(1+|\xi'|^2)^{-s+\frac{1}{2}} \quad\text{with}\quad C_s = \int(1+\lambda^2)^{-s}d\lambda.$$

We deduce that $\|\gamma(\phi)\|^2_{H^{s-\frac{1}{2}}} \leq C_s\|\phi\|^2_{H^s}$, which completes the proof of the first part of the theorem.

We now define the trace lifting operator. Let χ be a function in $\mathcal{D}(\mathbb{R})$ such that $\chi(0)=1$. We define

$$Rv(x) \stackrel{\text{def}}{=} (2\pi)^{-d+1}\int_{\mathbb{R}^{d-1}} e^{i(x'|\xi')}\chi(x_1\langle\xi'\rangle)\widehat{v}(\xi')\,d\xi' \quad\text{with}\quad \langle\xi'\rangle = \sqrt{1+|\xi'|^2}.$$

It is clear that

$$\begin{aligned}\mathcal{F}Rv(\xi) &= \int_{\mathbb{R}} e^{-it\xi_1}\chi(t\langle\xi'\rangle)\widehat{v}(\xi')\,dt\\ &= \langle\xi'\rangle^{-1}\widehat{\chi}\Big(\frac{\xi_1}{\langle\xi'\rangle}\Big)\widehat{v}(\xi').\end{aligned}$$

Taking N sufficiently large, we deduce that

$$\begin{aligned}\|Rv\|^2_{H^s} &= \int_{\mathbb{R}^d}(1+|\xi_1|^2+|\xi'|^2)^s\langle\xi'\rangle^{-2}\big|\widehat{\chi}\big(\langle\xi'\rangle^{-1}\xi_1\big)\big|^2|\widehat{v}(\xi')|^2\,d\xi\\ &\leq C_N\int_{\mathbb{R}^{d-1}}\Big(\int_{\mathbb{R}}\Big(1+\frac{|\xi_1|^2}{\langle\xi'\rangle^2}\Big)^{s-N}\langle\xi'\rangle^{-1}\,d\xi_1\Big)(1+|\xi'|^2)^{s-\frac{1}{2}}|\widehat{v}(\xi')|^2\,d\xi'\\ &\leq C_N\|v\|^2_{H^{s-\frac{1}{2}}}.\end{aligned}$$

Of course, we have $\gamma Rv = v$. This completes the proof of the theorem. $\square$

We infer the following corollary.

Corollary 1.64. *Let $s > m + \frac{1}{2}$ with $m \in \mathbb{N}$. The map*

$$\Gamma : \begin{cases} H^s(\mathbb{R}^d) \longrightarrow \displaystyle\bigoplus_{j=0}^{m} H^{s-j-\frac{1}{2}}(\mathbb{R}^{d-1}) \\ \qquad u \longmapsto (\gamma_j(u))_{0\leq j\leq m} \end{cases}$$

with $\gamma_j(u) = \gamma(\partial_{x_1}^j u)$ is then continuous and onto.

Remark 1.65. More generally, the trace operator γ_Σ may be defined for any smooth hypersurface Σ of $\mathbb{R}^d$. Indeed, according to Theorem 1.62 and Corollary 1.60, the spaces $H^s(\mathbb{R}^d)$ are local and invariant under the action of diffeomorphism, so localizing and straightening Σ reduces the problem to the study of the trace operator defined in Theorem 1.63.

1.4.2 Embedding

In this subsection, we present a few properties concerning embedding in Lebesgue spaces. First, from Theorems 1.38 and 1.50 we can easily deduce the following result.

Theorem 1.66. *The space $H^s(\mathbb{R}^d)$ embeds continuously in:*

- *the Lebesgue space $L^p(\mathbb{R}^d)$, if $0 \leq s < d/2$ and $2 \leq p \leq 2d/(d-2s)$*
- *the Hölder space $C^{k,\rho}(\mathbb{R}^d)$, if $s \geq d/2+k+\rho$ for some $k \in \mathbb{N}$ and $\rho \in \;]0,1[$.*

As in the homogeneous case, the space $H^{\frac{d}{2}}$ fails to be embedded in L^∞. However, the following *Moser–Trudinger inequality* holds.

Theorem 1.67. *There exist two constants, c and C, depending only on the dimension d, such that for any function $u \in H^{\frac{d}{2}}(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d} \left(\exp\left(c\left(\frac{|f(x)|}{\|f\|_{H^{\frac{d}{2}}}} \right)^2 \right) - 1 \right) dx \leq C.$$

Proof. As usual, arguing by density and homogeneity, it suffices to consider the case where f is in $\mathcal{S}$ and satisfies $\|f\|_{H^{\frac{d}{2}}} = 1$.

Now, the proof is based on the fact that, according to the inequality (1.30) and the definition of nonhomogeneous Sobolev spaces, there exists some constant C_d (depending only on the dimension d) such that

$$\|f\|_{L^{2p}} \leq C_d \sqrt{p} \quad \text{for all} \quad p \geq 1. \tag{1.41}$$

For all $x \in \mathbb{R}^d$, we may write

$$\exp\bigl(c|f(x)|^2\bigr) - 1 = \sum_{p\geq 1} \frac{c^p}{p!} |f(x)|^{2p}.$$

Integrating over $\mathbb{R}^d$ and using the inequality (1.41) yields

$$\int_{\mathbb{R}^d} \Big(\exp(c|f(x)|^2) - 1\Big)\, dx = \sum_{p\geq 1} c^p C_d^{2p} \frac{p^p}{p!}\cdot$$

The theorem then follows from our choosing the constant c sufficiently small. $\square$

As stated before, the space $H^s(\mathbb{R}^d)$ is included in $H^t(\mathbb{R}^d)$ whenever $t \leq s$. If the inequality is strict, then the following statement ensures that the embedding is locally compact.

Theorem 1.68. *For $t < s$, multiplication by a function in $\mathcal{S}(\mathbb{R}^d)$ is a compact operator from $H^s(\mathbb{R}^d)$ in $H^t(\mathbb{R}^d)$.*

Proof. Let φ be a function in $\mathcal{S}$. We have to prove that for any sequence (u_n) in $H^s(\mathbb{R}^d)$ satisfying $\sup_n \|u_n\|_{H^s} \leq 1$, we can extract a subsequence (u_{n_k}) such that (φu_{n_k}) converges in $H^t(\mathbb{R}^d)$.

As $H^s(\mathbb{R}^d)$ is a Hilbert space, the weak compactness theorem ensures that the sequence $(u_n)_{n\in\mathbb{N}}$ converges weakly, up to extraction, to an element u of $H^s(\mathbb{R}^d)$ with $\|u\|_{H^s} \leq 1$. We continue to denote this subsequence by $(u_n)_{n\in\mathbb{N}}$ and set $v_n = u_n - u$. Thanks to Theorem 1.62, $\sup_n \|\varphi v_n\|_{H^s} \leq C$. Our task is thus reduced to proving that the sequence $(\varphi v_n)_{n\in\mathbb{N}}$ tends to 0 in $H^t(\mathbb{R}^d)$. We now have, for any positive real number R,

$$\begin{aligned}\int (1+|\xi|^2)^t |\mathcal{F}(\varphi v_n)(\xi)|^2\, d\xi &\leq \int_{|\xi|\leq R} (1+|\xi|^2)^t |\mathcal{F}(\varphi v_n)(\xi)|^2\, d\xi \\ &\qquad + \int_{|\xi|\geq R} (1+|\xi|^2)^{t-s} (1+|\xi|^2)^s |\mathcal{F}(\varphi v_n)(\xi)|^2\, d\xi \\ &\leq \int_{|\xi|\leq R} (1+|\xi|^2)^t |\mathcal{F}(\varphi v_n)(\xi)|^2\, d\xi + \frac{\|\varphi v_n\|_{H^s}^2}{(1+R^2)^{s-t}}\cdot\end{aligned}$$

As $(\varphi v_n)_{n\in\mathbb{N}}$ is bounded in $H^s(\mathbb{R}^d)$, for a given positive real number ε, we can choose R such that

$$\frac{1}{(1+R^2)^{s-t}} \|\varphi v_n\|_{H^s}^2 \leq \frac{\varepsilon}{2}\cdot$$

On the other hand, as the function ψ_ξ defined by

$$\psi_\xi(\eta) \stackrel{\text{def}}{=} (2\pi)^{-d} \mathcal{F}^{-1}\Big((1+|\eta|^2)^{-s} \widehat{\varphi}(\xi-\eta)\Big)$$

belongs to $\mathcal{S}(\mathbb{R}^d)$, we can write

$$\begin{aligned}\mathcal{F}(\varphi v_n)(\xi) &= (2\pi)^{-d} \int \widehat{\varphi}(\xi-\eta)\widehat{v_n}(\eta)\, d\eta \\ &= \int (1+|\eta|^2)^s \widehat{\psi}_\xi(\eta)\widehat{v_n}(\eta)\, d\eta \\ &= (\psi_\xi \mid v_n)_{H^s}.\end{aligned}$$

As $(v_n)_{n\in\mathbb{N}}$ converges weakly to 0 in $H^s(\mathbb{R}^d)$, we can thus conclude that

$$\forall \xi \in \mathbb{R}^d, \quad \lim_{n\to\infty} \mathcal{F}(\varphi v_n)(\xi) = 0.$$

Let us temporarily assume that

$$\sup_{\substack{|\xi|\leq R\\ n\in\mathbb{N}}} |\mathcal{F}(\varphi v_n)(\xi)| \leq M < \infty. \tag{1.42}$$

Lebesgue's theorem then implies that

$$\lim_{n\to\infty} \int_{|\xi|\leq R} (1+|\xi|^2)^t |\mathcal{F}(\varphi v_n)(\xi)|^2 \, d\xi = 0,$$

which leads to the convergence of the sequence $(\varphi v_n)_{n\in\mathbb{N}}$ to 0 in $H^t(\mathbb{R}^d)$.

To complete the proof of the theorem, let us prove (1.42). It is clear that

$$\begin{aligned}|\mathcal{F}(\varphi v_n)(\xi)| &\leq (2\pi)^{-d} \int_{\mathbb{R}^d} |\widehat{\varphi}(\xi-\eta)|\, |\widehat{v}_n(\eta)| \, d\eta\\ &\leq (2\pi)^{-d} \|v_n\|_{H^s} \biggl(\int (1+|\eta|^2)^{-s} |\widehat{\varphi}(\xi-\eta)|^2 \, d\eta \biggr)^{\frac{1}{2}}.\end{aligned}$$

Now, as $\widehat{\varphi}$ belongs to $\mathcal{S}(\mathbb{R}^d)$, a constant C exists such that

$$|\widehat{\varphi}(\xi-\eta)| \leq \frac{C_{N_0}}{(1+|\xi-\eta|^2)^{N_0}} \quad \text{with} \quad N_0 = \frac{d}{2} + |s| + 1.$$

We thus obtain

$$\begin{aligned}\int (1+|\eta|^2)^{-s} |\widehat{\varphi}(\xi-\eta)|^2 \, d\eta &\leq \int_{|\eta|\leq 2R} (1+|\eta|^2)^{-s} |\widehat{\varphi}(\xi-\eta)|^2 \, d\eta\\ &\qquad + \int_{|\eta|\geq 2R} (1+|\eta|^2)^{-s} |\widehat{\varphi}(\xi-\eta)|^2 \, d\eta\\ &\leq C \int_{|\eta|\leq 2R} (1+|\eta|^2)^{|s|} \, d\eta\\ &\qquad + C_{N_0} \int_{|\eta|\geq 2R} (1+|\eta|^2)^{|s|} (1+|\xi-\eta|^2)^{-N_0} \, d\eta.\end{aligned}$$

Finally, since $|\xi| \leq R$, we always have $|\xi-\eta| \geq \dfrac{|\eta|}{2}$ in the last integral, so we eventually get

$$\int (1+|\eta|^2)^{-s} |\widehat{\varphi}(\xi-\eta)|^2 \, d\eta \leq C(1+R^2)^{|s|+\frac{d}{2}} + C \int \frac{d\eta}{(1+|\eta|^2)^{\frac{d}{2}+1}}.$$

This yields (1.42) and completes the proof of the theorem. □

From the above theorem, we can deduce the following compactness result.

Theorem 1.69. *For any compact subset K of $\mathbb{R}^d$ and $s' < s$, the embedding of $H^s_K(\mathbb{R}^d)$ into $H^{s'}_K(\mathbb{R}^d)$ is a compact linear operator.*

Proof. It suffices to consider a function φ in $\mathcal{S}(\mathbb{R}^d)$ which is identically equal to 1 in a neighborhood of the compact K and then to apply Theorem 1.68. □

1.4.3 A Density Theorem

In this subsection we investigate the density of the space $\mathcal{D}(\mathbb{R}^d\backslash\{0\})$ in Sobolev spaces. This result is useful for proving Hardy inequalities and is related to the problem of the pointwise value of a function in $H^s(\mathbb{R}^d)$. Indeed, having $\mathcal{D}(\mathbb{R}^d\backslash\{0\})$ dense in $H^s(\mathbb{R}^d)$ precludes any reasonable definition of the "value at 0" of an element of $H^s(\mathbb{R}^d)$. We now state the result.

Theorem 1.70. *If $s \leq d/2$ (resp., $< d/2$), then the space $\mathcal{D}(\mathbb{R}^d\backslash\{0\})$ is dense in $H^s(\mathbb{R}^d)$ [resp., in $\dot{H}^s(\mathbb{R}^d)$]. If $s > d/2$, then the closure of the space $\mathcal{D}(\mathbb{R}^d\backslash\{0\})$ in $H^s(\mathbb{R}^d)$ is the set of functions u in $H^s(\mathbb{R}^d)$ such that $\partial^\alpha u(0) = 0$ for any $\alpha \in \mathbb{N}^d$ such that $|\alpha| < s - d/2$.*

Proof. As $H^s(\mathbb{R}^d)$ is a Hilbert space it is enough to study the orthogonal complement of $\mathcal{D}(\mathbb{R}^d\backslash\{0\})$ in $H^s(\mathbb{R}^d)$. For u in H^s we define

$$u_s \stackrel{\text{def}}{=} \mathcal{F}^{-1}((1+|\xi|^2)^s\widehat{u}).$$

If u belongs to the orthogonal complement of $\mathcal{D}(\mathbb{R}^d\backslash\{0\})$, then we have

$$\int_{\mathbb{R}^d} \widehat{u}_s(\xi)\overline{\widehat{\varphi}}(\xi)\,d\xi = \langle u_s, \varphi\rangle = 0 \quad \text{for any} \quad \varphi \quad \text{in} \quad \mathcal{D}(\mathbb{R}^d\backslash\{0\}).$$

This implies that the support of u_s is included in $\{0\}$. We infer that a sequence $(a_\alpha)_{|\alpha|\leq N}$ exists such that

$$u_s = \sum_{|\alpha|\leq N} a_\alpha \partial^\alpha \delta_0. \tag{1.43}$$

As u_s belongs to H^{-s}, Remark 1.54 implies that $a_\alpha = 0$ for $|\alpha| \geq s - d/2$. Thus, if $s \leq d/2$, then $u_s = u = 0$ and the density is proved in that case. The proof of the density in the homogeneous case follows the same lines and is left to the reader as an exercise.

When s is greater than $d/2$, the orthogonal complement of the space $\mathcal{D}(\mathbb{R}^d\backslash\{0\})$ is exactly the finite-dimensional vector space $\mathcal{V}_s$ spanned by the functions $(u_\alpha)_{|\alpha|\leq[s-d/2]}$ defined by

$$u_\alpha(x) \stackrel{\text{def}}{=} (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x|\xi)}\frac{(i\xi)^\alpha}{(1+|\xi|^2)^s}\,d\xi.$$

However, thanks to the relation (1.43), if the partial derivatives of order less than or equal to $s - d/2$ of a function v in H^s vanish at 0, then we have

$$(v|u_\alpha)_{H^s} = \langle v, \partial^\alpha \delta_0 \rangle = 0.$$

Thus, the function v belongs to the orthogonal complement of $\mathcal{V}_s$, which is the closure of $\mathcal{D}(\mathbb{R}^d \setminus \{0\})$. □

Remark 1.71. If $d = 1$, then the above result means that the map $u \mapsto u(0)$ cannot be extended to $H^{\frac{1}{2}}(\mathbb{R})$ functions. More generally, arguing as above, we can prove that the restriction map γ on the hyperplane $x_1 = 0$ cannot be extended to $H^{\frac{1}{2}}(\mathbb{R}^d)$ functions.[12]

1.4.4 Hardy Inequality

This brief subsection is devoted to proving a fundamental inequality with singular weight in Sobolev spaces: the so-called *Hardy inequality.* More general Hardy inequalities will be established in the next chapter (see Theorem 2.57).

Theorem 1.72. *If $d \geq 3$, then*

$$\left(\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} \, dx \right)^{\frac{1}{2}} \leq \frac{2}{d-2} \|\nabla f\|_{L^2} \quad \textit{for any} \quad f \quad \textit{in} \quad \dot{H}^1(\mathbb{R}^d). \tag{1.44}$$

Proof. Arguing by density, it suffices to prove the inequality for $f \in \mathcal{D}(\mathbb{R}^d \setminus \{0\})$. Let $\mathcal{R}$ be the radial vector field $\mathcal{R} = \sum_{i=1}^d x_i \partial_{x_i}$. Because $\mathcal{R}|x|^{-2} = -2|x|^{-2}$, integrating by parts yields

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} \, dx = \frac{1}{2} \int_{\mathbb{R}^d} \frac{2f(x)\mathcal{R}f(x)}{|x|^2} \, dx + \frac{d}{2} \int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} \, dx.$$

Thus, we have, by the Cauchy–Schwarz inequality,

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} \, dx &= \frac{2}{2-d} \int_{\mathbb{R}^d} \frac{f(x)\mathcal{R}f(x)}{|x|^2} \, dx \\ &\leq \frac{2}{d-2} \left(\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \frac{|\mathcal{R}f(x)|^2}{|x|^2} \, dx \right)^{\frac{1}{2}}, \end{aligned}$$

which implies that

---

[12] In fact, γu makes sense whenever u belongs to the *smaller* space

$$H^{\frac{1}{2}}_{0,0}(\mathbb{R}^d) \stackrel{\text{def}}{=} \left\{ u \in H^{\frac{1}{2}}(\mathbb{R}^d) \Big/ \frac{u}{|x_1|^{\frac{1}{2}}} \in L^2(\mathbb{R}^d) \right\}.$$

$$\left(\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2}\,dx\right)^{\frac{1}{2}} \leq \frac{2}{d-2}\left(\int_{\mathbb{R}^d} |\nabla f(x)|^2\,dx\right)^{\frac{1}{2}}. \qquad \square$$

Remark 1.73. Let us note that using Lorentz spaces provides an elementary proof of more general Hardy inequalities, namely,

$$\left\|\frac{f}{|x|^s}\right\|_{L^2} \leq C\|f\|_{\dot{H}^s} \quad \text{for} \quad 0 \leq s < \frac{d}{2}\cdot$$

Indeed, using real interpolation we can show that $\dot{H}^s$ not only embeds in the space L^p with $1/p = 1/2 - s/d$, but also in the Lorentz space $L^{p,2}$. Now, it is clear that the function $x \mapsto |\cdot|^{-s}$ belongs to the space $L_w^{d/s}$, so applying generalized Hölder inequalities in Lorentz spaces, we get

$$\left\|\frac{f}{|x|^s}\right\|_{L^2} \leq C\left\|\frac{1}{|\cdot|^s}\right\|_{L_w^{d/s}} \|f\|_{L^{p,2}} \leq C'\|f\|_{\dot{H}^s}.$$

1.5 References and Remarks

The Hölder and Young inequalities belong to mathematical folklore. Refined Young inequalities are special cases of convolution inequalities in Lorentz spaces. An exhaustive list of such inequalities can be found in [171] or the book by P.-G. Lemarié-Rieusset [205]. More about atomic decomposition and bilinear interpolation can be found in the book by L. Grafakos [150].

In the present chapter, we restricted ourselves to the very basic properties of the Fourier transform. For a more complete study of the Fourier transform of harmonic analysis methods for partial differential equations, the reader may refer to the textbooks [40] by J.-M. Bony, [122] by L.C. Evans, [275] by E.M. Stein, [167, vol. 1] by L. Hörmander and [282, 283] by M.E. Taylor.

The Sobolev embedding in Lebesgue spaces was first stated by S. Sobolev himself in [270, 271]. There is now a plethora of generalizations ($W^{s,p}$ spaces, metric spaces, etc.) Basic references for Sobolev spaces may be found in the books [3] by R. Adams and [146] by D. Gilbarg and N. Trudinger. Refined Sobolev inequalities were discovered by P. Gérard, Y. Meyer, and F. Oru in [140]. The proof which has been proposed here is borrowed from [77]. The fractal counterexample comes from [22]. The study of embedding of Sobolev spaces in Hölder spaces goes back to C. Morrey's work in [235]. The BMO space was first introduced by F. John and L. Nirenberg in [174].

Most of the results concerning nonhomogeneous Sobolev spaces are classical. Hardy inequalities go back to the pioneering work by G.H. Hardy in [153, 154]. In the next chapter, we shall state more general Hardy inequalities in Sobolev spaces with *fractional* indices of regularity.

For more details on the Moser–Trudinger inequality, see the pioneering works by J. Moser in [236] and N.S. Trudinger in [290]. For recent developments, see [2].

Note that combining the Sobolev embedding theorem with Theorem 1.68 ensures that the embedding of $\dot{H}^s(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$ is locally compact whenever $2 \leq p \leq \infty$ and $s > d/2 - d/p$. In contrast, due to the scaling invariance of the critical Sobolev embedding,[13] the fact that $\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^{p_s}(\mathbb{R}^d)$ when $0 \leq s < d/2$, and that fact that $p_s = 2d/(d-2s)$, no compactness properties may be expected in this case. Indeed, if $u \in \dot{H}^s \setminus \{0\}$, then for any sequence (y_n) of points in $\mathbb{R}^d$ tending to infinity and for any sequence (h_n) of positive real numbers tending to 0 or to infinity, the sequences $(\tau_{y_n} u)$ and $(\delta_{h_n} u)$ converge weakly to 0 in $\dot{H}^s$ but are not relatively compact in L^p since $\|\tau_{y_n} u\|_{L^p} = \|u\|_{L^p}$ and $\|\delta_{h_n} u\|_{L^p} = \|u\|_{L^p}$. The study of this *defect of compactness* was initiated by P.-L. Lions in [212] (see also the paper by P. Gérard [139]). In short, it has been shown that translational and scaling invariance are the only features responsible for the defect of compactness of the embedding of $\dot{H}^s$ into L^p.

[13] Throughout this book, we agree that whenever X and Y are Banach spaces, the notation $X \hookrightarrow Y$ means that $X \subset Y$ and that the canonical injection from X to Y is continuous.

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