# Chapter 2

# Maps on Linear Spaces

In this chapter various important classes of maps are considered for which one obtains interesting results in vector optimization. We especially consider convex maps and their generalizations and also several types of differentials. It is the aim of this chapter to present a brief survey on these maps.

## 2.1 Convex Maps

The importance of convex maps is based on the fact that the image set of such a map has useful properties. One of these properties is also valid for so-called convex-like maps which are investigated in this section as well.

First, recall the definition of a linear map.

**Definition 2.1.** Let $X$ and $Y$ be real linear spaces. A map $T : X \to Y$ is called *linear*, if for all $x, y \in X$ and all $\lambda, \mu \in \mathbb{R}$

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y).$$

The set of continuous (bounded) linear maps between two real normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is a linear space as well and

J. Jahn, *Vector Optimization: Theory, Applications, and Extensions*,
DOI 10.1007/978-3-642-17005-8_2, 

it is denoted $B(X,Y)$. With the norm $\|\cdot\| : B(X,Y) \to \mathbb{R}$ given by

$$\|T\| = \sup_{x \neq 0_X} \frac{\|T(x)\|_Y}{\|x\|_X} \text{ for all } T \in B(X,Y)$$

$(B(X,Y), \|\cdot\|)$ is even a normed space.

A linear map defines also a corresponding map as it may be seen in

**Definition 2.2.** Let $X$ and $Y$ be real separated locally convex linear spaces, and let $T : X \to Y$ be a linear map. A map $T^* : Y^* \to X^*$ given by

$$T^*(y^*)(x) = y^*(T(x)) \text{ for all } x \in X \text{ and all } y^* \in Y^*$$

is called the *adjoint* (or *conjugate* and *dual*, respectively) of $T$.

It is obvious that the adjoint $T^*$ is also a linear map. One can show that it is uniquely determined. Adjoints are useful for the solution of linear functional equations.

**Theorem 2.3.** *Let $X$ and $Y$ be real separated locally convex linear spaces, and let the elements $x \in X$, $x^* \in X^*$, $y \in Y$ and $y^* \in Y^*$ be given.*

*(a) If there is a linear map $T : X \to Y$ with $y = T(x)$ and $x^* = T^*(y^*)$, then $y^*(y) = x^*(x)$.*

*(b) If $x \neq 0_X$, $y^* \neq 0_{Y^*}$ and $y^*(y) = x^*(x)$, then there is a continuous linear map $T : X \to Y$ with $y = T(x)$ and $x^* = T^*(y^*)$.*

**Proof.**

(a) Let a linear map $T : X \to Y$ with $y = T(x)$ and $x^* = T^*(y^*)$ be given. Then we get

$$y^*(y) = y^*(T(x)) = T^*(y^*)(x) = x^*(x)$$

which completes the proof.

(b) Assume that for $x \neq 0_X$ and $y^* \neq 0_{Y^*}$ the functional equation

$$y^*(y) = x^*(x) \tag{2.1}$$

is satisfied. In the following we consider the two cases $x^*(x) \neq 0$ and $x^*(x) = 0$.

(i) First assume that $x^*(x) \neq 0$. Then we define a map $T : X \to Y$ by

$$T(z) = \frac{x^*(z)}{x^*(x)}\, y \text{ for all } z \in X. \tag{2.2}$$

Evidently, $T$ is linear and continuous. From (2.1) and (2.2) we conclude $T(x) = y$ and

$$y^*(T(z)) = \frac{x^*(z)}{x^*(x)}\, y^*(y) = x^*(z) \text{ for all } z \in X$$

which means $x^* = T^*(y^*)$.

(ii) Now assume that $x^*(x) = 0$. Because of $y^* \neq 0_{Y^*}$ there is a $\tilde{y} \neq 0_Y$ with $y^*(\tilde{y}) = 1$. Since in a separated locally convex space $X^*$ separates elements of $X$, $x \neq 0_X$ implies the existence of some $\tilde{x}^* \in X^*$ with $\tilde{x}^*(x) = 1$. Then we define the map $T : X \to Y$ as follows

$$T(z) = x^*(z)\tilde{y} + \tilde{x}^*(z)y \text{ for all } z \in X. \tag{2.3}$$

It is obvious that $T$ is a continuous linear map. With (2.3) we conclude

$$T(x) = x^*(x)\tilde{y} + \tilde{x}^*(x)y = y.$$

Furthermore, we obtain with (2.3) and (2.1)

$$y^*(T(z)) = x^*(z)y^*(\tilde{y}) + \tilde{x}^*(z)y^*(y) = x^*(z) \text{ for all } z \in X$$

which implies $x^* = T^*(y^*)$. □

The class of linear maps is contained in the class of convex maps.

**Definition 2.4.** Let $X$ and $Y$ be real linear spaces, $C_Y$ be a convex cone in $Y$, and let $S$ be a nonempty convex subset of $X$. A map $f : S \to Y$ is called *convex* (or $C_Y$-*convex*), if for all $x, y \in S$ and all $\lambda \in [0, 1]$

$$\lambda f(x) + (1 - \lambda)f(y) - f(\lambda x + (1 - \lambda)y) \in C_Y \tag{2.4}$$

(see Fig. 2.1 and 2.2). A map $f : S \to Y$ is called *concave* (or

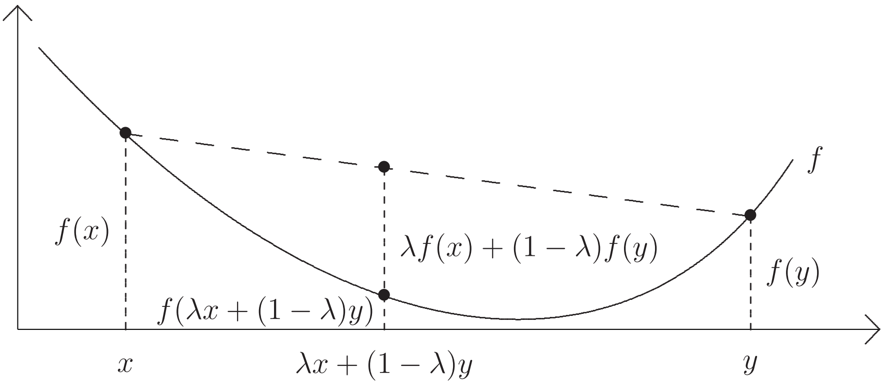


Figure 2.1: Convex functional.

$C_Y$-*concave*), if $-f$ is convex (see Fig. 2.3).

If $\leq_{C_Y}$ is the partial ordering in $Y$ induced by $C_Y$, then the condition (2.4) can also be written as

$$f(\lambda x + (1 - \lambda)y) \leq_{C_Y} \lambda f(x) + (1 - \lambda)f(y).$$

If $f$ is a linear map, then $f$ and $-f$ are convex maps.

**Definition 2.5.** Let $X$ and $Y$ be real linear spaces, let $C_Y$ be a convex cone in $Y$, let $S$ be a nonempty subset of $X$, and let $f : S \to Y$ be a given map. The set

$$\text{epi}(f) := \{(x, y) \mid x \in S,\ y \in \{(f(x)\} + C_Y\} \tag{2.5}$$

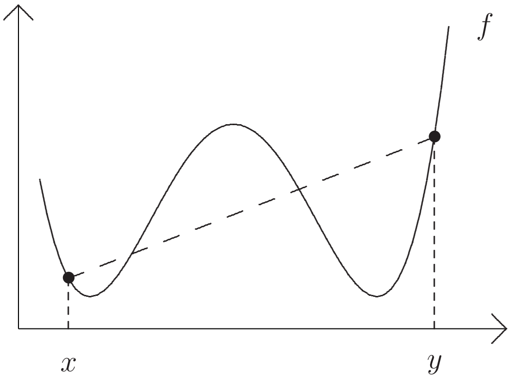


Figure 2.2: Non-convex functional.

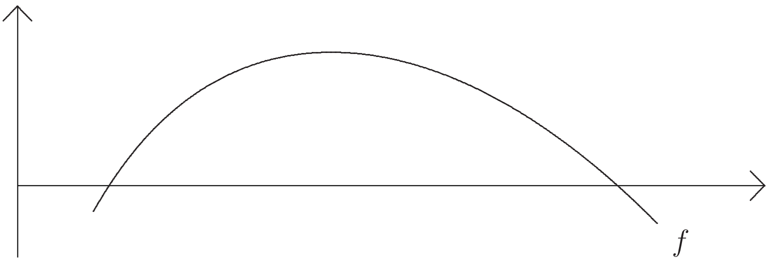


Figure 2.3: Concave functional.

is called the *epigraph* of $f$ (see Fig. 2.4).

Notice that the epigraph in (2.5) can also be written as

$$\text{epi}(f) = \{(x, y) \mid x \in S,\ f(x) \leq_{C_Y} y\}.$$

It turns out that a convex map can be characterized by its epigraph.

**Theorem 2.6.** *Let $X$ and $Y$ be real linear spaces, let $C_Y$ be a convex cone in $Y$, let $S$ be a nonempty subset of $X$ and let $f : S \to Y$ be a given map. Then $f$ is convex if and only if epi$(f)$ is a convex set.*

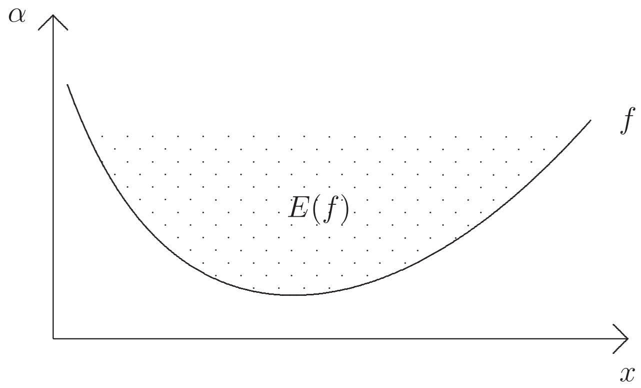


Figure 2.4: Epigraph of a functional.

**Proof.**

(a) Let $f$ be a convex map (then $S$ is a convex set). For arbitrary $z_1 = (x_1, y_1),\ z_2 = (x_2, y_2) \in \text{epi}(f)$ and $\lambda \in [0, 1]$ we obtain $\lambda x_1 + (1 - \lambda)x_2 \in S$ and

$$\begin{aligned}\lambda y_1 + (1-\lambda)y_2 &\in \lambda(\{f(x_1)\} + C_Y) + (1-\lambda)(\{f(x_2)\} + C_Y)\\ &= \{\lambda f(x_1) + (1-\lambda)f(x_2)\} + C_Y\\ &\subset \{f(\lambda x_1 + (1-\lambda)x_2)\} + C_Y.\end{aligned}$$

Consequently, we have $\lambda z_1 + (1-\lambda)z_2 \in \text{epi}(f)$. Thus, $\text{epi}(f)$ is a convex set.

(b) If $\text{epi}(f)$ is a convex set, then $S$ is convex as well. For arbitrary $x_1, x_2 \in S$ and $\lambda \in [0, 1]$ we obtain $\lambda(x_1, f(x_1)) + (1 - \lambda)(x_2, f(x_2)) \in \text{epi}(f)$ and

$$f(\lambda x_1 + (1-\lambda)x_2) \leq_{C_Y} \lambda f(x_1) + (1-\lambda)f(x_2).$$

Hence, $f$ is a convex map. □

Next, we list some other properties of convex maps.

**Lemma 2.7.** *Let $X$, $Y$ and $Z$ be real linear spaces, let $C_Y$ and $C_Z$ be convex cones in $Y$ and $Z$, respectively, and let $S$ be a nonempty convex subset of $X$.*

(a) *If $g : S \to Y$ is an affine linear map (i.e. there is a $b \in Y$ and a linear map $L : S \to Y$ with $g(x) = b + L(x)$ for all $x \in S$) and $f : Y \to Z$ is a convex map, then the composition $f \circ g$ is a convex map.*

(b) *If $g : S \to Y$ is a convex map and $f : Y \to Z$ is a convex and monotonically increasing map (that is: $y_1 \leq_{C_Y} y_2 \Rightarrow f(y_1) \leq_{C_Z} f(y_2)$), then the composition $f \circ g$ is a convex map.*

**Proof.** Take arbitrary $x_1, x_2 \in S$ and $\lambda \in [0,1]$. Then we get for part (a)

$$\begin{aligned}
&\lambda(f \circ g)(x_1) + (1-\lambda)(f \circ g)(x_2) - (f \circ g)(\lambda x_1 + (1-\lambda)x_2) \\
&\quad = \lambda f(g(x_1)) + (1-\lambda) f(g(x_2)) - f(g(\lambda x_1 + (1-\lambda)x_2)) \\
&\quad = \lambda f(g(x_1)) + (1-\lambda) f(g(x_2)) - f(\lambda g(x_1) + (1-\lambda) g(x_2)) \\
&\quad \in C_Z.
\end{aligned}$$

For the proof of part (b) we obtain with the convexity of $g$

$$\lambda g(x_1) + (1-\lambda) g(x_2) - g(\lambda x_1 + (1-\lambda)x_2) \in C_Y$$

and with the monotonicity of $f$

$$f(\lambda g(x_1) + (1-\lambda) g(x_2)) - f(g(\lambda x_1 + (1-\lambda)x_2)) \in C_Z.$$

Since $f$ is also convex, we get

$$\lambda f(g(x_1)) + (1-\lambda) f(g(x_2)) - f(\lambda g(x_1) + (1-\lambda) g(x_2)) \in C_Z.$$

Consequently, we conclude

$$\lambda f(g(x_1)) + (1-\lambda) f(g(x_2)) - f(g(\lambda x_1 + (1-\lambda)x_2)) \in C_Z$$

and

$$\lambda(f \circ g)(x_1) + (1-\lambda)(f \circ g)(x_2) - (f \circ g)(\lambda x_1 + (1-\lambda)x_2) \in C_Z.$$

$\square$

In vector optimization one is often merely concerned with the convexity of the set $f(S)+C_Y$ instead of $\mathrm{epi}(f)$. In this case the notion of convexity of $f$ can be relaxed because the convexity of $f(S)+C_Y$ depends only on a property of the convex hull of $f(S)$.

**Lemma 2.8.** *Let $X$ and $Y$ be real linear spaces, let $C_Y$ be a convex cone in $Y$, let $S$ be a nonempty subset of $X$ and let $f : S \to Y$ be a given map. Then the set $f(S)+C_Y$ is convex if and only if*

$$co(f(S)) \subset f(S) + C_Y. \tag{2.6}$$

**Proof.**

(a) If the set $f(S)+C_Y$ is convex, then with Remark 1.7

$$\mathrm{co}(f(S)) \subset \mathrm{co}(f(S)) + C_Y = \mathrm{co}(f(S)+C_Y) = f(S) + C_Y.$$

(b) If the inclusion (2.6) is true, then

$$\mathrm{co}(f(S)+C_Y) = \mathrm{co}(f(S)) + C_Y \subset f(S) + C_Y$$

which implies that the set $f(S)+C_Y$ is convex. □

The inclusion (2.6) is used for the definition of convex-like maps.

**Definition 2.9.** Let $X$ and $Y$ be real linear spaces, let $C_Y$ be a convex cone, let $S$ be a nonempty subset of $X$ and let $f : S \to Y$ be a given map. Then $f$ is called *convex-like*, if for every $x, y \in S$ and every $\lambda \in [0,1]$ there is an $s \in S$ with

$$\lambda f(x) + (1-\lambda) f(y) - f(s) \in C_Y$$

(or: $f(s) \leq_{C_Y} \lambda f(x) + (1-\lambda) f(y)$).

**Example 2.10.**

(a) Obviously, every convex map is convex-like.

(b) Let the map $f : [\pi, \infty) \to \mathbb{R}^2$ be given by

$$f(x) = (x, \sin x) \text{ for all } x \in [\pi, \infty)$$

where $\mathbb{R}^2$ is partially ordered in the componentwise sense. The map $f$ is convex-like but it is not convex.

Example 2.10, (b) shows that the class of convex-like maps is even much larger than the class of convex maps. With Lemma 2.8 we get immediately the following

**Theorem 2.11.** *Let $X$ and $Y$ be real linear spaces, let $C_Y$ be a convex cone in $Y$, let $S$ be a nonempty set and let $f : S \to Y$ be a given map. Then the map $f$ is convex-like if and only if the set $f(S) + C_Y$ is convex (see Fig. 2.5).*

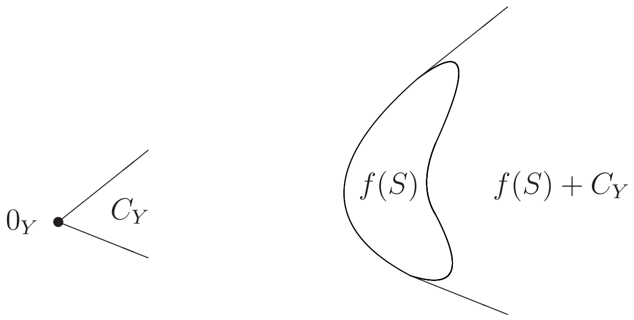


Figure 2.5: Convex-like map $f$.

## 2.2 Differentiable Maps

In the context with optimality conditions we have to work with generalized derivatives of maps. Therefore, we discuss various differentiability notions and we investigate the relationships among them.

**Definition 2.12.** Let $X$ be a real linear space, let $Y$ be a real topological linear space, let $S$ be a nonempty subset of $X$, and let $f : S \to Y$ be a given map.

(a) If for two elements $\bar{x} \in S$ and $h \in X$ the limit

$$f'(\bar{x})(h) := \lim_{\lambda \to 0_+} \frac{1}{\lambda}(f(\bar{x} + \lambda h) - f(\bar{x}))$$

exists, then $f'(\bar{x})(h)$ is called the *directional derivative* of $f$ at $\bar{x}$ in the direction $h$. If this limit exists for all $h \in X$, then $f$ is called *directionally differentiable* at $\bar{x}$ (see Fig. 2.6).

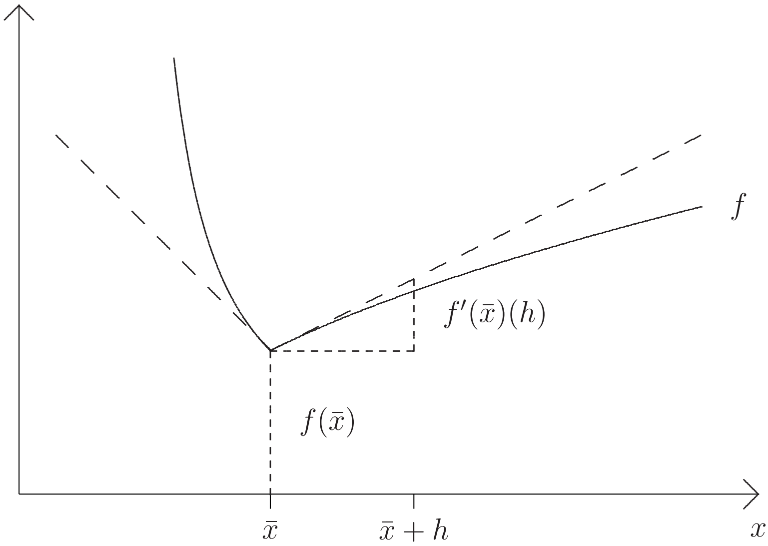


Figure 2.6: Directionally differentiable function.

(b) If for some $\bar{x} \in S$ and all $h \in X$ the limit

$$f'(\bar{x})(h) := \lim_{\lambda \to 0} \frac{1}{\lambda}(f(\bar{x} + \lambda h) - f(\bar{x}))$$

exists and if $f'(\bar{x})$ is a continuous linear map from $X$ to $Y$, then $f'(\bar{x})$ is called the *Gâteaux derivative* of $f$ at $\bar{x}$ and $f$ is called *Gâteaux differentiable* at $\bar{x}$.

Notice that for the limit defining the directional and Gâteaux derivative one considers arbitrary nets $(\lambda_i)_{i\in\mathbb{N}}$ converging to 0, $\lambda_i > 0$ for all $i \in \mathbb{N}$ in part (a), with the additional property that $\bar{x} + \lambda_i h$ belongs to the domain $S$ for all $i \in \mathbb{N}$. This restriction of the nets converging to 0 can be dropped, for instance, if $S$ equals the whole space $X$.

**Example 2.13.** For the function $f : \mathbb{R}^2 \to \mathbb{R}$ with

$$f(x_1, x_2) = \left\{ \begin{array}{ll} x_1^2(1 + \frac{1}{x_2}) & \text{if } x_2 \neq 0 \\ 0 & \text{if } x_2 = 0 \end{array} \right\} \text{ for all } (x_1, x_2) \in \mathbb{R}^2$$

which is not continuous at $0_{\mathbb{R}^2}$, we obtain the directional derivative

$$f'(0_{\mathbb{R}^2})(h_1, h_2) = \lim_{\lambda \to 0_+} \frac{1}{\lambda} f(\lambda(h_1, h_2)) = \left\{ \begin{array}{ll} \frac{h_1^2}{h_2} & \text{if } h_2 \neq 0 \\ 0 & \text{if } h_2 = 0 \end{array} \right\}$$

in the direction $(h_1, h_2) \in \mathbb{R}^2$. Notice that $f'(0_{\mathbb{R}^2})$ is neither continuous nor linear.

Sometimes it is very useful to have a derivative notion which does not require any topology in $Y$. A possible generalization of a directional derivative which will be used in the second part of this book is given by

**Definition 2.14.** Let $X$ and $Y$ be real linear spaces, let $S$ be a nonempty subset of $X$ and let $T$ be a nonempty subset of $Y$. Moreover, let a map $f : S \to Y$ and an element $\bar{x} \in S$ be given. A map $f'(\bar{x}) : S - \{\bar{x}\} \to Y$ is called a *directional variation* of $f$ at $\bar{x}$ with respect to $T$, if the following holds: Whenever there is an element $x \in S$ with $x \neq \bar{x}$ and $f'(\bar{x})(x - \bar{x}) \in T$, then there is a $\bar{\lambda} > 0$ with

$$\bar{x} + \lambda(x - \bar{x}) \in S \text{ for all } \lambda \in (0, \bar{\lambda}]$$

and

$$\frac{1}{\lambda}(f(\bar{x} + \lambda(x - \bar{x})) - f(\bar{x})) \in T \text{ for all } \lambda \in (0, \bar{\lambda}].$$

**Example 2.15.** Let $X$ be a real linear space, let $Y$ be a real topological linear space, and let $S$ be a nonempty subset of $X$. Further, let $f : S \to Y$ be a given map, and let $x, \bar{x} \in S$ with $x \neq \bar{x}$ be fixed. Assume that there is a $\bar{\lambda} > 0$ with

$$\bar{x} + \lambda(x - \bar{x}) \in S \text{ for all } \lambda \in (0, \bar{\lambda}].$$

(a) If $f'(\bar{x})$ is the directional derivative of $f$ at $\bar{x}$ in the direction $x - \bar{x}$, then $f'(\bar{x})$ is a directional variation of $f$ at $\bar{x}$ with respect to all nonempty open subsets of $Y$.

(b) Let $f$ be an affine linear map, i.e. there is a $b \in Y$ and a linear map $L : S \to Y$ with

$$f(x) = b + L(x) \text{ for all } x \in S.$$

If for some nonempty set $T \subset Y$ $\;L(x - \bar{x}) \in T$, then

$$\frac{1}{\lambda}(f(\bar{x} + \lambda(x - \bar{x})) - f(\bar{x})) = L(x - \bar{x}) \in T \text{ for all } \lambda \in (0, \bar{\lambda}].$$

Consequently, $L$ is the directional variation of $f$ at $\bar{x}$ with respect to all nonempty sets $T \subset Y$.

A less general but more satisfying derivative notion may be obtained in normed spaces.

**Definition 2.16.** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real normed spaces, let $S$ be a nonempty open subset of $X$, and let $f : S \to Y$ be a given map. Furthermore let an element $\bar{x} \in S$ be given. If there is a continuous linear map $f'(\bar{x}) : X \to Y$ with the property

$$\lim_{\|h\|_X \to 0} \frac{\|f(\bar{x} + h) - f(\bar{x}) - f'(\bar{x})(h)\|_Y}{\|h\|_X} = 0,$$

then $f'(\bar{x})$ is called the *Fréchet derivative* of $f$ at $\bar{x}$ and $f$ is called *Fréchet differentiable* at $\bar{x}$.

According to this definition we obtain for Fréchet derivatives with the notations used above

$$f(\bar{x} + h) = f(\bar{x}) + f'(\bar{x})(h) + o(\|h\|_X)$$

where the expression $o(\|h\|_X)$ of this Taylor series has the property

$$\lim_{\|h\|_X \to 0} \frac{o(\|h\|_X)}{\|h\|_X} = \lim_{\|h\|_X \to 0} \frac{f(\bar{x}+h) - f(\bar{x}) - f'(\bar{x})(h)}{\|h\|_X} = 0_Y.$$

With the next three assertions we present some known results on Fréchet differentiability.

**Lemma 2.17.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real normed spaces, let $S$ be a nonempty open subset of $X$, and let $f : S \to Y$ be a given map. If the Fréchet derivative of $f$ at some $\bar{x} \in S$ exists, then the Gâteaux derivative of $f$ at $\bar{x}$ exists as well and both are equal.*

**Proof.** Let $f'(\bar{x})$ denote the Fréchet derivative of $f$ at $\bar{x}$. Then we have

$$\lim_{\lambda \to 0} \frac{\|f(\bar{x}+\lambda h) - f(\bar{x}) - f'(\bar{x})(\lambda h)\|_Y}{\|\lambda h\|_X} = 0 \quad \text{for all } h \in X \backslash \{0_X\}$$

implying

$$\lim_{\lambda \to 0} \frac{1}{|\lambda|} \|f(\bar{x}+\lambda h) - f(\bar{x}) - f'(\bar{x})(\lambda h)\|_Y = 0 \quad \text{for all } h \in X \backslash \{0_X\}.$$

Because of the linearity of $f'(\bar{x})$ we obtain

$$\lim_{\lambda \to 0} \frac{1}{\lambda} [f(\bar{x}+\lambda h) - f(\bar{x})] = f'(\bar{x})(h) \quad \text{for all } h \in X.$$

□

**Corollary 2.18.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real normed spaces, let $S$ be a nonempty open subset of $X$, and let $f : S \to Y$ be a given map. If $f$ is Fréchet differentiable at some $\bar{x} \in S$, then the Fréchet derivative is uniquely determined.*

**Proof.** With Lemma 2.17 the Fréchet derivative coincides with the Gâteaux derivative. Since the Gâteaux derivative is as a limit uniquely determined, the Fréchet derivative is also uniquely determined. □

The following lemma says that Fréchet differentiability implies continuity as well.

**Lemma 2.19.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real normed spaces, let $S$ be a nonempty open subset of $X$, and let $f : S \to Y$ be a given map. If $f$ is Fréchet differentiable at some $\bar{x} \in S$, then $f$ is continuous at $\bar{x}$.*

**Proof.** To a sufficiently small $\varepsilon > 0$ there is a ball around $\bar{x}$ so that for all $\bar{x} + h$ of this ball

$$\|f(\bar{x} + h) - f(\bar{x}) - f'(\bar{x})(h)\|_Y \leq \varepsilon\|h\|_X.$$

Then we conclude for some $\alpha > 0$

$$\begin{aligned}
&\|f(\bar{x} + h) - f(\bar{x})\|_Y \\
&\quad = \|f(\bar{x} + h) - f(\bar{x}) - f'(\bar{x})(h) + f'(\bar{x})(h)\|_Y \\
&\quad \leq \|f(\bar{x} + h) - f(\bar{x}) - f'(\bar{x})(h)\|_Y + \|f'(\bar{x})(h)\|_Y \\
&\quad \leq \varepsilon\|h\|_X + \alpha\|h\|_X \\
&\quad = (\varepsilon + \alpha)\|h\|_X.
\end{aligned}$$

Consequently $f$ is continuous at $\bar{x}$. □

The following theorem gives a characterization of a convex Fréchet differentiable map.

**Theorem 2.20.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real normed spaces, let $S$ be a nonempty open convex subset of $X$, let $C_Y$ be a closed convex cone in $Y$, and let a map $f : S \to Y$ be given which is Fréchet differentiable at every $x \in S$. Then the map $f$ is convex if and only if*

$$f(y) + f'(y)(x - y) \leq_{C_Y} f(x) \text{ for all } x, y \in S$$

*(see Fig. 2.7).*

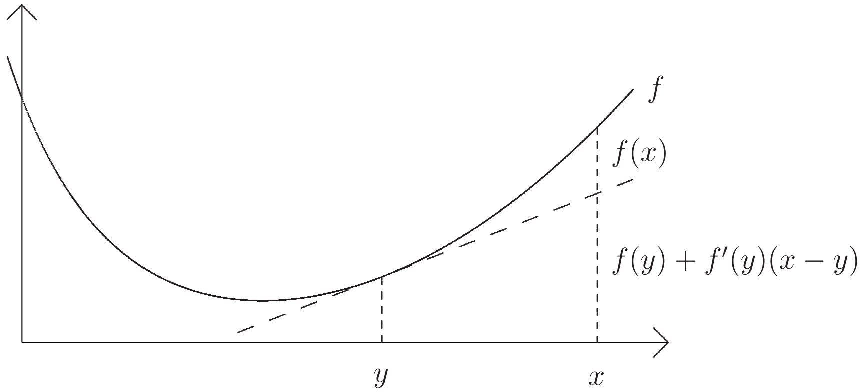


Figure 2.7: Illustration of the result of Thm. 2.20.

**Proof.**

(a) First, we assume that the map $f$ is convex. Then it follows for all $x, y \in S$ and all $\lambda \in (0, 1]$

$$\lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda) y) \in C_Y$$

and

$$f(x) - f(y) - \frac{1}{\lambda}(f(y + \lambda(x-y)) - f(y)) \in C_Y.$$

Since $f$ is assumed to be Fréchet differentiable at $y$ and $C_Y$ is closed, we conclude

$$f(x) - f(y) - f'(y)(x-y) \in C_Y$$

or alternatively

$$f(y) + f'(y)(x-y) \leq_{C_Y} f(x).$$

(b) Next, we assume that

$$f(y) + f'(y)(x-y) \leq_{C_Y} f(x) \text{ for all } x, y \in S.$$

$S$ is convex and, therefore, we obtain for all $x, y \in S$ and all $\lambda \in [0, 1]$

$$f(x) - f(\lambda x + (1-\lambda) y) - f'(\lambda x + (1-\lambda) y)((1-\lambda)(x-y)) \in C_Y$$

and

$$f(y) - f(\lambda x + (1-\lambda)y) - f'(\lambda x + (1-\lambda)y)(-\lambda(x-y)) \in C_Y.$$

Since $C_Y$ is a convex cone and Fréchet derivatives are linear maps, we get

$$\begin{array}{l} \lambda f(x) - \lambda f(\lambda x + (1-\lambda)y) \\ -\lambda(1-\lambda)f'(\lambda x + (1-\lambda)y)(x-y) \\ +(1-\lambda)f(y) - (1-\lambda)f(\lambda x + (1-\lambda)y) \\ +(1-\lambda)\lambda f'(\lambda x + (1-\lambda)y)(x-y) \\ \in C_Y \end{array}$$

which implies

$$\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y) \in C_Y.$$

Hence, $f$ is a convex map.

□

The characterization of convex Fréchet differentiable maps presented in Theorem 2.20 is very helpful for the investigation of optimality conditions in vector optimization. This result leads to a generalization of the (Fréchet) derivative for convex maps which are not (Fréchet) differentiable.

**Definition 2.21.** Let $X$ and $Y$ be real topological linear spaces, let $C_Y$ be a convex cone in $Y$, and let $f : X \to Y$ be a given map. For an arbitrary $\bar{x} \in X$ the set

$$\partial f(\bar{x}) := \{T \in B(X,Y) \mid f(\bar{x}+h) - f(\bar{x}) - T(h) \in C_Y \text{ for all } h \in X\}$$

(where $B(X,Y)$ denotes the linear space of the continuous linear maps from $X$ to $Y$) is called the *subdifferential* of $f$ at $\bar{x}$. Every $T \in \partial f(\bar{x})$ is called a *subgradient* of $f$ at $\bar{x}$ (see Fig. 2.8).

**Example 2.22.** Let $X$ and $Y$ be real topological linear spaces, let $C_Y$ be a pointed convex cone in $Y$, and let $\|\!| \cdot |\!\| : X \to Y$ be a vectorial norm. Then we have for every $\bar{x} \in X$

$$\partial |\!\|\bar{x}|\!\| = \{T \in B(X,Y) \mid T(\bar{x}) = |\!\|\bar{x}|\!\| \text{ and } T(x) \leq_{C_Y} |\!\|x|\!\| \text{ for all } x \in X\}.$$

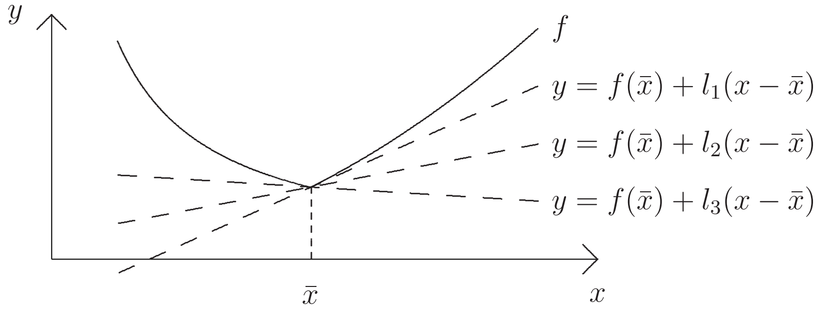


Figure 2.8: Subgradients of a convex functional.

**Proof.**

(a) First, choose an arbitrary $T \in B(X,Y)$ with $T(\bar{x}) = \|\bar{x}\|$ and

$$\|x\| - T(x) \in C_Y \text{ for all } x \in X.$$

Then we obtain for all $h \in X$

$$\begin{aligned}\|\bar{x}+h\| - \|\bar{x}\| - T(h) &= \|\bar{x}+h\| - T(\bar{x}+h) - \|\bar{x}\| + T(\bar{x}) \\ &\in C_Y\end{aligned}$$

which implies $T \in \partial\|\bar{x}\|$.

(b) Next, assume that any $T \in \partial\|\bar{x}\|$ is given. Then we get

$$\|\bar{x}\| - T(\bar{x}) = \|\bar{x}+\bar{x}\| - \|\bar{x}\| - T(\bar{x}) \in C_Y$$

and

$$-\|\bar{x}\| + T(\bar{x}) = \|\bar{x}-\bar{x}\| - \|\bar{x}\| - T(-\bar{x}) \in C_Y.$$

Since $C_Y$ is pointed, we conclude

$$\|\bar{x}\| - T(\bar{x}) \in (-C_Y) \cap C_Y = \{0_Y\}$$

which means $T(\bar{x}) = \|\bar{x}\|$. Finally, we obtain

$$\begin{aligned}\|x\| - T(x) &\in \{\|x+\bar{x}\| - \|\bar{x}\| - T(x)\} + C_Y \\ &\subset C_Y + C_Y = C_Y \text{ for all } x \in X.\end{aligned}$$

This completes the proof. $\square$

The next example is a special case of Example 2.22.

**Example 2.23.** Let $(X, \|\cdot\|_X)$ be a real normed space. Then we have for every $\bar{x} \in X$

$$\partial\|\bar{x}\|_X = \left\{ \begin{array}{l} \{x^* \in X^* \mid x^*(\bar{x}) = \|\bar{x}\|_X \text{ and } \|x^*\|_{X^*} = 1\} \text{ if } \bar{x} \neq 0_X \\ \{x^* \in X^* \mid \|x^*\|_{X^*} \leq 1\} \text{ if } \bar{x} = 0_X \end{array} \right\}.$$

**Proof.** The assertion follows directly from the preceding example for $Y = \mathbb{R}$ and $C_Y = \mathbb{R}_+$, if we notice that

$$\|x^*\|_{X^*} \leq 1 \iff x^*(x) \leq \|x\|_X \text{ for all } x \in X.$$

□

As a result of Example 2.23 the subdifferential of the norm at $0_X$ in a real normed space $X$ coincides with the closed unit ball of the dual space.

With the following sequence of assertions it can be shown under appropriate assumptions that the subdifferential of a vectorial norm can be used in order to characterize the directional derivative of such a norm.

**Lemma 2.24.** *Let $X$ be a real linear space, let $Y$ be a real topological linear space, let $C_Y$ be a convex cone in $Y$ which is Daniell, and let $\|\!|\cdot|\!\| : X \to Y$ be a vectorial norm. Then the directional derivative of the vectorial norm exists at every $\bar{x} \in X$ and in every direction $h \in X$.*

**Proof.** Let $f : X \to Y$ be an arbitrary convex map with $f(0_X) = 0_Y$. Then we obtain for all $x \in X$ and all $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha \leq \beta$

$$\frac{\alpha}{\beta}f(\beta x) - f(\alpha x) = \frac{\alpha}{\beta}f(\beta x) + \frac{\beta-\alpha}{\beta}f(0_X) - f\Big(\frac{\alpha}{\beta}\beta x + \frac{\beta-\alpha}{\beta}0_X\Big) \in C_Y$$

resulting in

$$\frac{1}{\beta}f(\beta x) - \frac{1}{\alpha}f(\alpha x) \in C_Y.$$

If we take especially

$$f(x) = \|\bar{x} + x\| - \|\bar{x}\| \text{ for all } x \in X,$$

then $f$ is convex and $f(0_X) = 0_Y$. Hence, the above result applies to this special $f$, that is

$$\begin{aligned}&\frac{1}{\beta}(\|\bar{x} + \beta x\| - \|\bar{x}\|) - \frac{1}{\alpha}(\|\bar{x} + \alpha x\| - \|\bar{x}\|) \in C_Y \\ &\text{for all } x \in X \text{ and all real numbers } \alpha, \beta \text{ with } 0 < \alpha \leq \beta.\end{aligned} \tag{2.7}$$

Next, we show that the difference quotient which appears in the definition of the directional derivative is bounded. Since the vectorial norm is a convex map, we get for all $x \in X$ and all $\lambda > 0$

$$\begin{aligned}&\frac{1}{1+\lambda}\|\bar{x} + \lambda x\| + \frac{\lambda}{1+\lambda}\|\bar{x} - x\| - \|\bar{x}\| \\ = \; &\frac{1}{1+\lambda}\|\bar{x} + \lambda x\| + \frac{\lambda}{1+\lambda}\|\bar{x} - x\| \\ &- \left\| \frac{1}{1+\lambda}(\bar{x} + \lambda x) + \frac{\lambda}{1+\lambda}(\bar{x} - x) \right\| \\ \in \; &C_Y\end{aligned}$$

implying

$$\frac{1}{\lambda}(\|\bar{x} + \lambda x\| - \|\bar{x}\|) \in \{\|\bar{x}\| - \|\bar{x} - x\|\} + C_Y.$$

This condition means that $\|\bar{x}\| - \|\bar{x} - x\|$ is, for every $\lambda > 0$, a lower bound of the difference quotient $\frac{1}{\lambda}(\|\bar{x} + \lambda x\| - \|\bar{x}\|)$. Since $C_Y$ is assumed to be Daniell, we conclude with the condition (2.7) and the boundedness property that the directional derivative of the vectorial norm exists at every $\bar{x} \in X$ and in every direction $h \in X$. $\square$

**Lemma 2.25.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real reflexive Banach spaces, and let $C_Y$ be a closed convex cone in $Y$ which is Daniell and has a weakly compact base. If $\|\cdot\| : X \to Y$ is a vectorial norm which* *is continuous at an* $\bar{x} \in X$*, then we have for the directional derivative at* $\bar{x} \in X$ *in every direction* $h \in X$

$$T(h) \leq_{C_Y} |||\bar{x}|||'(h) \text{ for all } T \in \partial|||\bar{x}|||.$$

**Proof.** Notice that with Lemma 2.24 the directional derivative $|||\bar{x}|||'(h)$ exists for all $\bar{x}, h \in X$. By a result of Zowe [370] the subdifferential $\partial|||\bar{x}|||$ is nonempty. For every $\bar{x}, h \in X$ we get

$$\begin{aligned} |||\bar{x} + \lambda h||| - |||\bar{x}||| &\in \{T(\bar{x} + \lambda h) - T(\bar{x})\} + C_Y \\ &= \{\lambda T(h)\} + C_Y \text{ for all } \lambda > 0 \text{ and all } T \in \partial|||\bar{x}|||. \end{aligned}$$

Consequently, we have

$$\frac{1}{\lambda}(|||\bar{x} + \lambda h||| - |||\bar{x}|||) \in \{T(h)\} + C_Y \text{ for all } \lambda > 0 \text{ and all } T \in \partial|||\bar{x}|||.$$

Since $C_Y$ is closed, we conclude

$$|||\bar{x}|||'(h) \in \{T(h)\} + C_Y$$

which leads to the assertion. □

For the announced characterization result of the directional derivative of a vectorial norm we need a special lemma on subdifferentials.

**Lemma 2.26.** *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be real reflexive Banach spaces, and let* $C_Y$ *be a convex cone in* $Y$ *with a weakly compact base. If* $f : X \to Y$ *is a convex map which is continuous at some* $\bar{x} \in X$*, then*

$$t \circ \partial f(\bar{x}) = \partial(t \circ f)(\bar{x}) \text{ for all } t \in C_{Y^*}.$$

A proof of this lemma may be found in a paper of Zowe [370] even in a more general form (compare also Valadier [336] and Borwein [40, p. 437]).

**Theorem 2.27.** *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be real reflexive Banach spaces, and let* $C_Y$ *be a closed convex cone in* $Y$ *which is*

*Daniell and has a weakly compact base. If* $\|\!|\cdot|\!\|: X \to Y$ *is a vectorial norm which is continuous at an* $\bar{x} \in X$*, then the directional derivative of* $f$ *at* $\bar{x}$ *in every direction* $h$ *is given by*

$$\begin{aligned} \|\!|\bar{x}|\!\|'(h) \;\; &= \;\; \max\,\{T(h) \mid T \in B(X,Y),\ T(\bar{x}) = \|\!|\bar{x}|\!\| \\ &\qquad\qquad\qquad \textit{and } \|\!|x|\!\| - T(x) \in C_Y \textit{ for all } x \in X\} \end{aligned}$$

*which means that there is a* $\bar{T} \in B(X,Y)$ *with* $\bar{T}(\bar{x}) = \|\!|\bar{x}|\!\|$ *and*

$$\|\!|x|\!\| - \bar{T}(x) \in C_Y \textit{ for all } x \in X$$

*so that*

$$\|\!|\bar{x}|\!\|'(h) = \bar{T}(h)$$

*and*

$$\begin{aligned} \|\!|\bar{x}|\!\|'(h) \in \{T(h)\} + C_Y \;\; & \textit{for all } T \in B(X,Y) \textit{ with } T(\bar{x}) = \|\!|\bar{x}|\!\| \\ & \textit{and } \|\!|x|\!\| - T(x) \in C_Y \textit{ for all } x \in X. \end{aligned}$$

**Proof.** Take any direction $h \in X$. From Example 2.22 and Lemma 2.25 we obtain immediately

$$\begin{aligned} \|\!|\bar{x}|\!\|'(h) \in \{T(h)\} + C_Y \;\; & \text{for all } T \in B(X,Y) \text{ with } T(\bar{x}) = \|\!|\bar{x}|\!\| \\ & \text{and } \|\!|x|\!\| - T(x) \in C_Y \text{ for all } x \in X. \end{aligned}$$

Therefore, we have only to show that there is a $\bar{T} \in \partial\|\!|\bar{x}|\!\|$ with $\|\!|\bar{x}|\!\|'(h) = \bar{T}(h)$.

With Corollary 3.19 (which will be stated later) there is a continuous linear functional $t \in C^{\#}_{Y^*}$. Then we consider the functional $f := t \circ \|\!|\cdot|\!\| : X \to \mathbb{R}$. $f$ is continuous at $\bar{x}$ and with Lemma 2.7, (b) it is even convex. With Lemma 2.25 we conclude

$$f'(\bar{x})(h) \geq \sup\,\{x^*(h) \mid x^* \in \partial f(\bar{x})\},$$

and since $\partial f(\bar{x})$ is weak*-compact in $X^*$, this supremum is actually attained, that is

$$f'(\bar{x})(h) \geq \max\,\{x^*(h) \mid x^* \in \partial f(\bar{x})\}.$$

In order to prove the equality we assume that there is an $\alpha \in \mathbb{R}$ with

$$f'(\bar{x})(h) > \alpha > \max\{x^*(h) \mid x^* \in \partial f(\bar{x})\}. \tag{2.8}$$

If $S$ denotes the linear hull of $\{h\}$, we define a linear functional $l : S \to \mathbb{R}$ by

$$l(\lambda h) = \lambda\alpha \text{ for all } \lambda \in \mathbb{R}.$$

Then we get

$$l(\lambda h) \leq \lambda f'(\bar{x})(h) = f'(\bar{x})(\lambda h) \text{ for all } \lambda \in \mathbb{R}.$$

Since $f'(\bar{x})$ is sublinear, there is a continuous extension $\bar{l}$ of $l$ on $X$ with

$$\bar{l}(x) \leq f'(\bar{x})(x) \text{ for all } x \in X$$

which implies $\bar{l} \in \partial f(\bar{x})$. But with $\bar{l}(h) = \alpha$ we arrive at a contradiction to (2.8).

Summarizing these results we obtain

$$f'(\bar{x})(h) = \max\{x^*(h) \mid x^* \in \partial f(\bar{x})\}.$$

Consequently, there is an $x^* \in \partial f(\bar{x})$ with

$$f'(\bar{x})(h) = x^*(h).$$

With Lemma 2.26 there is a $\bar{T} \in \partial\|\bar{x}\|$ with $x^* = t \circ \bar{T}$ and we get

$$t \circ \|\bar{x}\|'(h) = (t \circ \|\bar{x}\|)'(h) = t \circ \bar{T}(h). \tag{2.9}$$

Assume that $\|\bar{x}\|'(h) \neq \bar{T}(h)$. Then we get from Lemma 2.25

$$\|\bar{x}\|'(h) - \bar{T}(h) \in C_Y \backslash \{0_Y\}$$

and, therefore,

$$t \circ \|\bar{x}\|'(h) - t \circ \bar{T}(h) > 0$$

which contradicts (2.9). Hence, $\|\bar{x}\|'(h) = \bar{T}(h)$ and this completes the proof. $\square$

It should be noted that the assumptions of Theorem 2.27 are very restrictive (they are fulfilled, for instance, for $Y = \mathbb{R}^n$ and $C_Y = \mathbb{R}^n_+$). The assertion remains valid under even weaker conditions and for these investigations we refer to Borwein [40, p. 437].

# Notes

A lot of material on convex functions may be found in the books of Rockafellar [284] and Roberts-Varberg [282]. For investigations on convex relations in analysis we refer to a paper of Borwein [37]. Convex-like maps were first introduced by Vogel [341, p. 165] who also formulated Theorem 2.11. In connection with a minisup theorem Aubin [10, § 13.3] presented a similar statement like Theorem 2.11 for so-called $\gamma$-convex functionals.

A survey on differentials in nonlinear functional analysis may be found in the extensive paper of Nashed [255]. The so-called directional variation was introduced by Kirsch-Warth-Werner [188, p. 33] in a more general form; they called it “B-Variation”. The differentiability concept used in this book is based on a paper of Jahn-Sachs [172]. For a further generalized differentiability notion compare also the paper of Sachs [293]. The results on Fréchet differentiation can also be found in the books of Luenberger [238] and Jahn [164]. Subdifferentials were introduced by Moreaux and Rockafellar. We restrict ourselves to refer to the lecture notes of Rockafellar [286]. The books of Holmes [140], Ekeland-Temam [101] and Ioffe-Tihomirov [144] also present an interesting overview on subdifferentials and their use in optimization. Theorems on subdifferentials in partially ordered linear spaces may be found in the papers of Valadier [336], Zowe [370], Elster-Nehse [102], Penot [271] and Borwein [40].

Much of the work on vectorial norms described in the second section is based on various results of Holmes [140] and Borwein [40].

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