Chapter 2
Preferences Modeling

2.1 Introduction

Dealing with preferences means that one needs to make a choice from a collection of possible choices. The generic term *object* is used as a formal term for choice. An object may be a trip, a candidate, a course, a time interval, etc.

Choosing an object from a set of objects implicitly means needing to compare the objects, e.g., this candidate is better than the other one, this course is more interesting than the others, I don't have a preference between giving my talk today or tomorrow, these two candidates are incomparable, etc.

Preferences modeling is formally describing these comparisons. It is the first and basic step of all works dealing with preferences in various disciplines [1, 2, 3, 5, 7]. Formally speaking, the basic ingredient of preferences modeling is a binary relation over the set of objects called preference relation. The latter expresses that an object is at least as preferred as another object. Three specific relations can be induced by a preference relation: A strict preference relation which describes a strict preference between two objects, an indifference relation which describes a similarity between two objects and an incomparability relation which holds when two objects cannot be compared in terms of a strict preference or indifference. These three relations form a preference structure. We survey different kinds of preference structures depending on the properties of the underlying preference relation.

In the last section of this chapter, we make precise the notion of an object. In this book it is generally described by a set of attributes, each attribute taking its values in a finite domain. Exceptions are made in Chapters 6 and 8 due to the context.

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2.2 Crisp Preference Relations

Preferences modeling is based on a finite set of objects, denoted by $\mathscr{O}$, to be compared or evaluated [4]. The basic ingredient in this framework is a binary relation, denoted by $\succeq$, over $\mathscr{O} \times \mathscr{O}$. The notation $o \succeq o'$ stands for "o is at least as preferred as o'". Thus $\succeq$ is referred to as *preference relation*.

Given a preference relation $\succeq$ and two objects $o, o' \in \mathscr{O}$, we distinguish between three relations over o and o':

- o is strictly preferred to o', denoted by $o \succ o'$, when $o \succeq o'$ holds but $o' \succeq o$ does not. $\succ$ is called a *strict preference relation*.

- o is indifferent to o', denoted by $o \approx o'$, when both $o \succeq o'$ and $o' \succeq o$ hold. $\approx$ is called an *indifference relation*.

- o is incomparable to o', denoted by $o \sim o'$, when neither $o \succeq o'$ nor $o' \succeq o$ holds. $\sim$ is called an *incomparability relation*.

Let us now recall some basic properties concerning binary relations.

- $\succeq$ is reflexive if and only if $\forall o \in \mathscr{O}$, $o \succeq o$.

- $\succeq$ is irreflexive if and only if $\forall o \in \mathscr{O}$, $o \succeq o$ does not hold.

- $\succeq$ is complete if and only if $\forall o, o' \in \mathscr{O}$, we have $o \succeq o'$ or $o' \succeq o$.

- $\succeq$ is transitive if and only if $\forall o, o', o'' \in \mathscr{O}$, if $o \succeq o'$ and $o' \succeq o''$ then $o \succeq o''$.

- $\succeq$ is symmetric if and only if $\forall o, o' \in \mathscr{O}$, if $o \succeq o'$ then $o' \succeq o$.

- $\succeq$ is antisymmetric if and only if $\forall o \in \mathscr{O}$, $\forall o' \in \mathscr{O} \backslash \{o\}$, we have $not(o \succeq o'$ and $o' \succeq o)$.

- $\succeq$ is asymmetric if and only if $\forall o, o' \in \mathscr{O}$, we have $not(o \succeq o'$ and $o' \succeq o)$.

Therefore, $\succ$ is asymmetric and $\approx$, $\sim$ are symmetric. While the reflexivity of $\succeq$ may be considered as a natural property, it is not a necessary condition. If $\succeq$ is reflexive then $\approx$ is reflexive and $\sim$ is irreflexive.

Given $\succeq$, the triple $(\succ, \approx, \sim)$ is called a preference structure induced by $\succeq$. We also say that each relation in the triple (i.e., $\succeq$, $\approx$ or $\sim$) is associated with (or induced by) $\succeq$.

Given the properties of a preference relation $\succeq$, we distinguish between different types of preference structures. As far as the next few chapters are concerned, we recall two structures:

- *A total preorder:* this corresponds to a reflexive, complete and transitive preference relation $\succeq$. The associated strict preference relation and indifference relation

are transitive while the associated incomparability relation is empty.
When $\succeq$ is antisymmetric, $\approx$ is the set of pairs (o,o) and the preference structure is called a *total order*. Lastly, when $\succeq$ is asymmetric, $\approx$ is empty and the preference structure is called a *strict total order*.

- *A partial preorder:* this corresponds to a reflexive and transitive preference relation $\succeq$. The associated strict preference relation and indifference relation are transitive while the associated incomparability relation is not empty.
When $\succeq$ is antisymmetric, $\approx$ is composed of pairs (o,o) only and the preference structure is called a *partial order*. Lastly, when $\succeq$ is asymmetric, $\approx$ is empty and the preference structure is called a *strict partial order*.

A preference relation $\succeq$ is cyclic if and only if its induced strict preference relation $\succ$ is cyclic, i.e., there exists a chain of objects $o, \cdots, o'$ such that $o \succ \cdots \succ o' \succ o$. Otherwise $\succeq$ is acyclic.

$\succ$, $\approx$ and $\sim$ will always refer to the strict preference relation, the indifference relation and the incomparability relation, respectively.

Given a transitive indifference relation $\approx$ and a transitive strict preference relation $\succ$ we have the following combinations: $\forall o, o' \in \mathscr{O}$,

$$\text{if } o \succ o' \text{ and } o' \approx o'' \text{ then } o \succ o''$$

and

$$\text{if } o \approx o' \text{ and } o' \succ o'' \text{ then } o \succ o''.$$

From now on, we suppose that $\succeq$ is transitive. By abuse of language we sometimes say that $\succeq$ is a total or partial (pre)order or (strict) total or partial (pre)order. When no confusion is possible, an order is equivalently denoted by $\succeq$ or $\succ$.

When the preference relation $\succeq$ is a total preorder, the indifference relation $\approx$ induced by $\succeq$ is an equivalence relation (reflexive, symmetric and transitive). The set of equivalence classes of $\mathscr{O}$ given $\approx$ is totally ordered w.r.t. $\succ$. Let $E_1, \cdots, E_n$ be the set of equivalence classes induced by $\approx$. Then,

(i) $\forall i = 1, \cdots, n$, $E_i \neq \emptyset$,
(ii) $E_1 \cup \cdots \cup E_n = \mathscr{O}$,
(iii) $\forall i, j$, $E_i \cap E_j = \emptyset$ for $i \neq j$,
(iv) $\forall o, o' \in E_i$, $o \approx o'$.

$(E_1, \cdots, E_n)$ is an ordered partition of $\mathscr{O}$ given $\succeq$ iff ($\forall o, o' \in \mathscr{O}$, $o \in E_i$, $o' \in E_j$ with $i < j$ if and only if $o \succ o'$).

Example 2.1. Let $\mathscr{O} = \{o_0, o_1, o_2, o_3\}$ be the set of objects. Let $\succeq$ be a total preorder over $\mathscr{O} \times \mathscr{O}$ defined by $o_1 \succ o_3$, $o_3 \approx o_0$ and $o_3 \succ o_2$. Then, the ordered partition of $\mathscr{O}$ is (E_1, E_2, E_3) with $E_1 = \{o_1\}$, $E_2 = \{o_0, o_3\}$ and $E_3 = \{o_2\}$.

A total order can also be written as an ordered partition of $\mathscr{O}$ where each equivalence class is composed of a single object.

Note that an ordered partition of $\mathscr{O}$ associated with $\succeq$ is acyclic. For example, the preference relation $o_1 \succ o_2$, $o_2 \succ o_3$, $o_3 \succ o_2$ and $o_3 \succ o_4$ cannot be written in terms of an ordered partition.

Let $\succeq$ and $\succeq'$ be two preference relations. We say that $\succeq$ extends $\succeq'$ if and only if $\forall o, o' \in \mathscr{O}$, if $o \succeq' o'$ then $o \succeq o'$.

We compare total preorders on the basis of specificity principles [10]. The minimal specificity principle, used in system Z [8], gravitates towards the least-specific preorder, while the maximal specificity principle gravitates towards the most-specific preorder.

Definition 2.1 (Minimal or Maximal specificity principle).
Let $\succeq$ and $\succeq'$ be two total preorders over a set of objects $\mathscr{O}$ represented by ordered partitions $(E_1, \cdots, E_n)$ and $(E'_1, \cdots, E'_{n'})$, respectively. We say that $\succeq$ is less specific than $\succeq'$, written as $\succeq \sqsubseteq \succeq'$, iff $\forall o \in \mathscr{O}$, if $o \in E_i$ and $o \in E'_j$ then $i \leq j$. $\succeq$ belongs to the set of the least- (or most-) specific preorders in a set of preorders $\mathscr{D}$ if there is no $\succeq'$ in $\mathscr{D}$ such that $\succeq' \sqsubset \succeq$ (or $\succeq \sqsubset \succeq'$), where $\succeq' \sqsubset \succeq$ iff $\succeq' \sqsubseteq \succeq$ holds but $\succeq \sqsubseteq \succeq'$ does not.

In other words, $\succeq$ is less specific than $\succeq'$ when the rank of each object in $\succeq$ is not greater than its rank in $\succeq'$. Also, $\succeq$ is more specific than $\succeq'$ when the rank of each object in $\succeq$ is not smaller than its rank in $\succeq'$. Recall that the smaller the rank i, the more preferred the objects in E_i.

The following example illustrates the minimal and maximal specificity principles.

Example 2.2. Let $\mathscr{O} = \{o_0, o_1, o_2, o_3\}$ be the set of objects. Let $\succeq$, $\succeq'$ and $\succeq''$ be three preorders such that $\succeq = (\{o_3, o_2, o_0\}, \{o_1\})$, $\succeq' = (\{o_3, o_2\}, \{o_1, o_0\})$ and $\succeq'' = (\{o_3, o_0\}, \{o_1, o_2\})$. $\succeq$ is less specific than both $\succeq'$ and $\succeq''$. $\succeq'$ and $\succeq''$ are incomparable w.r.t. the specificity principle. The least-specific preorder is $\succeq$ while the most-specific preorders are $\succeq'$ and $\succeq''$.

The aim of preferences modeling is to guide a choice from the set $\mathscr{O}$ or a subset of $\mathscr{O}$. When objects are totally rank-ordered (i.e., the preference relation $\succeq$ is a total (pre)order), rationality of choice suggests that the users make a choice from among the most preferred objects w.r.t. $\succeq$, i.e., from among objects which belong to E_1 given the ordered partition $(E_1, \cdots, E_n)$ associated with $\succeq$. However, when $\succeq$ is a partial (pre)order, speaking about "most" preferred objects does not make sense. Therefore, we use the notion of undominated objects, which is suitable for both total and partial (pre)orders. We say that o dominates o' when $o \succ o'$ holds. The set of undominated (or best) objects of $\mathscr{O}' \subseteq \mathscr{O}$ w.r.t. $\succeq$, denoted by $\max(\mathscr{O}', \succeq)$, is defined by

$$\max(\mathscr{O}', \succeq) = \{o \mid o \in \mathscr{O}', \nexists o' \in \mathscr{O}', o' \succ o\}.$$

The set of the worst objects of $\mathscr{O}' \subseteq \mathscr{O}$ w.r.t. $\succeq$, denoted by $\min(\mathscr{O}', \succeq)$, is defined by

$$\min(\mathscr{O}', \succeq) = \{o \mid o \in \mathscr{O}', \nexists o' \in \mathscr{O}', o \succ o'\}.$$

The notions of best or worst objects also allow us to deal with a cyclic preference relation. For example, although the preference relation $o_1 \succ o_2$, $o_2 \succ o_3$, $o_3 \succ o_2$ and $o_3 \succ o_4$ cannot be written in terms of an ordered partition, we can conclude that o_1 is the best object and o_4 is the worst object.

2.3 Fuzzy Preference Relations

In some situations the preference relation is provided with additional information expressing the degree of plausibility of the preferences. This is called a *fuzzy preference relation*. More specifically, a fuzzy preference relation, denoted by $\mathscr{R}$, is a function from $\mathscr{O} \times \mathscr{O}$ to the unit interval $[0,1]$ such that $\mathscr{R}(o,o')$ corresponds to the degree to which the assertion "o is at least as preferred as o'" is true. Therefore, $\succeq$ is a crisp preference relation. It is a special case of a fuzzy preference relation $\mathscr{R}$ when $o \succeq o'$ holds if and only if $\mathscr{R}(o,o') = 1$.

In this extended framework, a fuzzy preference structure is defined by including a fuzzy strict preference relation denoted by P, a fuzzy indifference relation denoted by I and a fuzzy incomparability relation denoted by J.

In order to extend the properties of a crisp preference relation to the fuzzy case, the above fuzzy relations are defined from a De Morgan triple $\langle T,S,n\rangle$, where T is a t-norm, S is a t-conorm and n is a negation [6, 9] such that $S(a,b) = nT(na,nb)$.

Usually the functions T and n are respectively defined by $T(a,b) = \min(a,b)$ and $n(a) = 1-a$. Therefore, $S(a,b) = \max(a,b)$. The properties of a preference relation listed in the previous section can be extended to the fuzzy case as follows:

- $\mathscr{R}$ is reflexive if and only if $\forall o \in \mathscr{O}$, $\mathscr{R}(o,o) = 1$.
- $\mathscr{R}$ is complete if and only if $\forall o,o' \in \mathscr{O}$, we have $\max(\mathscr{R}(o,o'),\mathscr{R}(o',o)) = 1$.
- $\mathscr{R}$ is transitive if and only if $\forall o,o',o'' \in \mathscr{O}$, $\min(\mathscr{R}(o,o'),\mathscr{R}(o',o'')) \le \mathscr{R}(o,o'')$.
- $\mathscr{R}$ is symmetric if and only if $\forall o,o' \in \mathscr{O}$, $\mathscr{R}(o,o') = \mathscr{R}(o',o)$.
- $\mathscr{R}$ is antisymmetric if and only if $\forall o \in \mathscr{O}$, $\forall o' \in \mathscr{O}\backslash\{o\}$, we have $\min(\mathscr{R}(o,o'),\mathscr{R}(o',o)) = 0$.
- $\succeq$ is asymmetric if and only if we have $\forall o,o' \in \mathscr{O}$, $\min(\mathscr{R}(o,o'),\mathscr{R}(o',o)) = 0$.

Different proposals have been made to compute a fuzzy preference structure (P,I,J) from a fuzzy preference relation $\mathscr{R}$. We refer the reader to [9] for a detailed exposition.

2.4 The Language

Generally, users compare objects on the basis of their characteristics. For example, a car can be characterized by its color, cost and capacity. Its color may be red, white or blue. Its price may be 15,900 euros or 18,000 euros and its capacity may be five or nine persons. Therefore, the user has to compare 12 cars. However, sometimes not all possible objects are feasible due to integrity constraints. For example, a red car with capacity of nine persons and price of 15,900 euros does not exist. So the user has to make a choice from 11 feasible cars. For simplicity and without loss of generality, we do not explicitly refer to integrity constraints and suppose that all possible objects are feasible. We refer to feasibility only when it is important.

Characteristics considered to describe an object are called attributes or variables. We suppose that they take their values from a finite set.

We denote variables by uppercase letters (possibly subscripted), e.g., A, B, X_1, X_2. The domain of a variable X is denoted by $Dom(X)$.

Given a binary variable X, its values are denoted by x and $\neg x$, i.e., $Dom(X) = \{\neg x, x\}$. Sometimes, for clarity, the values $\neg x$ and x are explicitly given, e.g., *fish* and *meat* for dish, *white* and *red* for wine, etc. V denotes the set of all variables at hand.

An outcome (or object, choice, alternative), denoted by ω, is the result of assigning a value to each variable in V. Ω is the set of all possible outcomes, i.e., the Cartesian product of the domain of variables in V. $Asst(V')$, with $V' \subseteq V$, is the set of all possible assignments to variables in V'. Therefore, $\Omega = Asst(V)$.

Let $\mathscr{L}$ be a language based on V. Formulas are built on $\mathscr{L}$ using logical connectors $\wedge$, $\vee$ and $\neg$, which respectively stand for conjunction, disjunction and negation. $Mod(\varphi)$ denotes the set of outcomes that make the formula φ true. We write $\omega \models \varphi$ when $\omega \in Mod(\varphi)$. We say that ω satisfies φ. If ω does not satisfy φ, i.e., $\omega \notin Mod(\varphi)$, we write $\omega \not\models \varphi$ and say that ω is a countermodel of φ.

Given a set of formulas $\mathscr{F}$, ω satisfies $\mathscr{F}$ if and only if ω satisfies each formula in $\mathscr{F}$. ω is called $\mathscr{F}$-outcome. ω falsifies $\mathscr{F}$ if and only if ω falsifies at least one formula in $\mathscr{F}$. We say that ω is a countermodel of $\mathscr{F}$.

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