Chapter 2

Why such patterns? A few motivation points

While getting familiar with results on patterns in later chapters, one not only has a chance to appreciate the beauty of combinatorics of patterns on words and permutations (which is interesting in its own right), but also to learn about connections of the field to other branches of combinatorics, mathematics, and theoretical computer science. The main application of patterns so far is that in many situations they provide a convenient language for describing various (combinatorial) objects. Such descriptions can be used in establishing properties (e.g., equidistribution results) of objects related to words or permutations restricted somehow by patterns (e.g., avoiding certain patterns). However, in most of the cases considered in the literature, the prime interest in linking pattern-restricted permutations/words to other objects is finding a bijection between the structures involved rather than also trying to find immediate applications.

In either case, the current chapter contains several of the most striking connections between patterns and other objects. Many more such connections will be given in later chapters.

2.1 Sorting permutations with stacks and other devices

There is a long line of papers in the computer science literature dedicated to problems of *sorting permutations* (arranging permutations in increasing order) with different devices, e.g., *stacks*, *queues*, and *deques* (see, for example, [18, 21, 33, 54, 56, 44, 45, 373, 52, 49, 63, 35, 325, 351, 148, 256, 59, 136, 134, 133, 135, 157, 156, 206, 205, 339,

S. Kitaev, *Patterns in Permutations and Words*, Monographs in Theoretical Computer Science. An EATCS Series, DOI 10.1007/978-3-642-17333-2_2,

423, 435, 441, 539, 658, 703, 705, 729, 716, 727, 762, 799, 801, 800, 813, 815, 774, 775]). We refer to [136] and [137, Chapter 8] by Bóna for introductions to the area of sorting we are interested in, and to [539] by Knuth for general sorting algorithms. The original appearance of the theory of patterns was due to its connections to sorting with stacks as discovered by Knuth [540, pp. 242–243] in 1968. Even these days, one of the main applications of our patterns is providing a language for describing sets of permutations sortable by given devices.

In this section we will illustrate how patterns can be used in the sorting industry. Notice that even though the results presented in this section on relations between patterns and sorting are rather comprehensive, we still leave aside some of them. For example, we do not discuss in any details the fact that the matrices corresponding to $\mathrm{Av}(3142, 2413)$ are exactly those which do not fill up under "*bootstrap percolation*" [716] (as we will see in Subsection 2.2.5, $\mathrm{Av}(3142, 2413)$ is enumerated by the *large Schröder numbers*; see Subsection A.2.1 for definitions).

2.1.1 Sorting with k stacks in series

A *stack* is a last-in first-out linear sorting device with *push* and *pop operations* (also known as *insert* and *remove operations*). In other words, a stack is a container for a linear sequence (in our case, for a permutation) that one is allowed to change by inserting new items (one at a time) at its tail and by removing tail items (again, one at a time). Initially the stack is empty and then a sequence of insertions interleaved with removals is made. Thus an input permutation is transformed thereby into an output permutation.

The *greedy* algorithm we are interested in for stack sorting a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ works as follows. We start with pushing π_1 onto the stack. Next, if $\pi_2 < \pi_1$ then we push π_2 onto the stack to be on top of π_1; on the other hand, if $\pi_2 > \pi_1$, we pop π_1 off and let π_2 enter the stack. More generally, suppose, at some point, the letters $\pi_1, \pi_2, \ldots, \pi_i$ have all been added to the stack (some of them could be still in the stack, others have been popped off), so we are reading π_{i+1}. We push π_{i+1} onto the stack if and only if π_{i+1} is less than the top element of the stack (which is easily seen to be π_i). Otherwise, we pop elements off the stack, one by one, until π_{i+1} is less than the top remaining stack element and then we push π_{i+1} onto the stack. When no more elements remain to be pushed onto the stack, we pop off all elements of the stack until it is empty. This produces a permutation $S(\pi)$ as output.

Definition 2.1.1. If $S(\pi)$ is the identity permutation $12\cdots n$, we say that π is *stack-sortable*. More generally, if $S^k(\pi)$ is the identity permutation (S^k is the result of application of S k times), we say that π is *k-stack sortable* (π is sorted with k

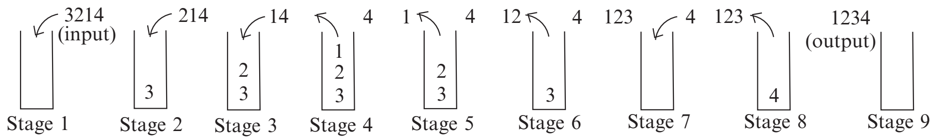


Figure 2.1: Stack sorting the permutation 3214.

stacks in series). We let $W_{n,k}$ denote the set of all k-stack sortable n-permutations.

Remark 2.1.2. There are other notions of a “k-stack sortable permutation” in the literature, which are different from that introduced in Definition 2.1.1 (see [54, 634]). To distinguish the permutations in Definition 2.1.1, they are sometimes called *West-k-stack sortable permutations* (West considered them in [799]). However, since we do not discuss the other notions of k-stack sortable permutations in much detail in this book, we omit “West-” in Definition 2.1.1, which should not cause any confusion.

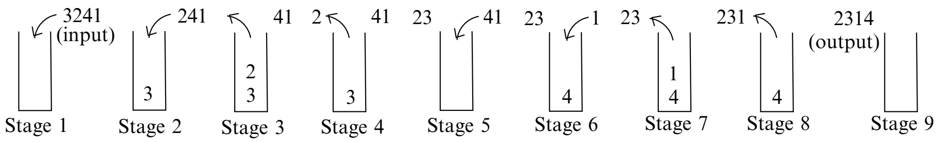


Figure 2.2: Stack sorting the permutation 3241.

Figure 2.1 shows an example of a stack-sortable permutation (3214), while Figure 2.2 shows an example of a non-stack-sortable permutation (3241). As a matter of fact, the permutation 3241 is 3-stack sortable but not 2-stack sortable which is illustrated in Figure 2.3 (a single adjacent transposition makes the permutation 3214 much harder to sort).

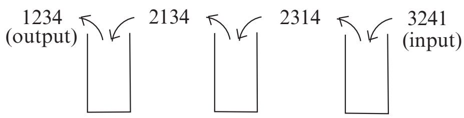


Figure 2.3: Sorting the permutation 3241 with 3 stacks in series.

It is clear that the set of stack-sortable permutations is closed under pattern containment (meaning that any subsequence of a stack-sortable permutation in reduced form is stack-sortable), since removing a letter from the input and ignoring the insertion and removal operations on it gives a proper computation that applies to the shorter input. However, before stating results on patterns related to stack sorting, we would like to define the stack sorting procedure in an alternative (equivalent) way (introducing relevant notions/notations), that is not only normally more convenient to deal with, but also allows several modifications of the procedure, not to be discussed here.

We define the *stack sorting operator* S recursively on an n-permutation as follows. For the empty permutation ε, $S(\varepsilon) = \varepsilon$. If $\pi \neq \varepsilon$ is an n-permutation, decompose π as $\pi = LnR$, where L and R are, possibly empty, factors to the left and to the right of n, respectively. Then

$$S(\pi) = S(L)S(R)n.$$

Going again through the examples above we see that using the new definition, $S(3241) = 2314$ and $S(3214) = S^3(3241) = 1234$ which matches our previous computations.

It is easy to see that $W_{n,n-1} = \mathcal{S}_n = \mathcal{S}_n(\emptyset)$, that is, all n-permutations can be sorted by $n-1$ applications of the S operator, and this can be seen as avoidance of the empty set of patterns. The first part of the following proposition, that Knuth [540, pp. 242-243] left as an exercise to the reader, began the theory of patterns in permutations, and it provides the first explicit application of patterns in computer science. To give an idea of approaches to use, we will provide a proof of Proposition 2.1.3. However, almost all of the upcoming propositions/theorems in this section are stated without proofs (explicit references to the results are given though).

Proposition 2.1.3. *$W_{n,1} = \mathcal{S}_n(231)$. Moreover, $|W_{n,1}| = s_n(231) = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number.*

Proof. Let us prove that $W_{n,1} = \mathcal{S}_n(231)$. Suppose $\pi_i\pi_j\pi_k$ is an occurrence of the pattern 231 in an n-permutation π, that is, $\pi_k < \pi_i < \pi_j$. At some point π_i will enter the stack, and before π_j can do so, π_i must leave the stack so π_k cannot be to the left of it in $S(\pi)$, so π is not stack-sortable.

Conversely, suppose π is not stack-sortable. Then $S(\pi)$ contains a 2-letter subsequence $\pi_i\pi_k$ such that $\pi_i > \pi_k$ ($\pi_i\pi_k$ is an *inversion* in $S(\pi)$). Thus, π_i left the stack before π_k arrived there. This can only happen if there is a letter π_j such that $\pi_i\pi_j\pi_k$ is a subsequence in π and $\pi_j > \pi_i$. But then, $\pi_i\pi_j\pi_k$ is an occurrence of the pattern 231 in π.

To prove the second part of the proposition, decompose a 231-avoiding n-permutation π as $\pi = LnR$, where L and R are the possibly empty factors of π to the left and to the right of the largest letter n, respectively. To avoid 231 each letter of L must be smaller than any letter of R, and L and R in reduced form must be 231-avoiding permutations. This brings us to the following recursion in which i can be viewed as the length of L:

$$s_n(231) = \sum_{i=0}^{n-1} s_i(231)s_{n-i-1}(231).$$

This recursion is a well-known recursion for the Catalan numbers and we are done. □

In his PhD thesis West [799] proved the following theorem (notice the appearance of a barred pattern).

Theorem 2.1.4. *([799])* $W_{n,2} = \mathcal{S}_n(2341, 3\bar{5}241)$.

West [799] conjectured the following formula for the number of 2-stack sortable permutations

$$s_n(2341, 3\bar{5}241) = \frac{2(3n)!}{(2n+1)!(n+1)!} \tag{2.1}$$

which was first proved by Zeilberger [815], who found the functional equation

$$x^2G^3(x) + x(2+3x)G^2(x) + (1-14x+3x^2)G(x) + x^2 + 11x - 1 = 0$$

for the generating function $G(x) = \sum_{n\geq 0} s_n(2341, 3\bar{5}241)x^n$ and then used Lagrange's inversion formula to solve it. Two bijective proofs [323, 423] of the conjecture by West appeared later and they connected together different combinatorial objects involving several classes of pattern-restricted permutations. Both of the bijective proofs rely on the known result on the number of *rooted non-separable planar maps* [190, 191]. Some further details on the proofs and related things are to be discussed in Section 2.11.

It should be mentioned that refinements for the number of stack-sortable and 2-stack sortable permutations are know when *descents* (occurrences of the pattern $\underline{21}$) are taken into account. In the first case, one gets the *Narayana numbers* (see [745]). More precisely, the number of stack-sortable n-permutations with m descents is shown to be equal to

$$\frac{1}{n}\binom{n}{m}\binom{n}{m+1}. \tag{2.2}$$

For the number of 2-stack sortable n-permutations with m descents one gets the following formula (see [135] by Bóna and references therein):

$$\frac{(n+m)!(2n-m-1)!}{(m+1)!(n-m)!(2m+1)!(2n-2m-1)!}. \tag{2.3}$$

Another refinement for counting 2-stack sortable permutations is the following theorem by Dulucq et al [323].

Theorem 2.1.5. *([323]) The number of 2-stack sortable n-permutations having k right-to-left maxima (see Definition A.1.1) is*

$$\frac{k+1}{(2n-k+1)!}\sum_{j=k+1}^{\min\{n+1,2k+2\}}\frac{(3k-2j+2)(2j-k-1)(j-2)!(3n-j-k+1)!}{(n-j+1)!(j-k-1)!(j-k)!(2k-j+2)!}.$$

To complete the relevant enumeration story, one should mention a result of Bousquet-Mélou [156] related to study of 2-stack sortable permutations subject to five statistics (including permutation length and the number of descents). It was shown that the five-variable generating function in question is *algebraic* of degree 20.

Coming back to using our patterns in describing sortable sets, West [799] showed that $W_{n,n-2}$ are precisely those permutations that do not have suffix $n1$. We state this as the following proposition that involves a bivincular pattern.

Proposition 2.1.6. *([799]) $W_{n,n-2} = \mathcal{S}_n(\begin{smallmatrix}\ulcorner 1\,2 \urcorner\\ \underline{2\,1}\lrcorner\end{smallmatrix})$.*

It is straightforward to see from Proposition 2.1.6, that the number of n-permutations sortable by applying the operator S $n-2$ times is $s_n(\begin{smallmatrix}\ulcorner 1\,2 \urcorner\\ \underline{2\,1}\lrcorner\end{smallmatrix}) = n!-(n-2)!$.

To our best knowledge, there is no known "nice" pattern description of the set $W_{n,n-3}$, although the cardinality of this set is found by Claesson et al. [256]:

$$|W_{n,n-3}| = \frac{(n-3)!}{2}(2n^3-6n^2-5n+16) \tag{2.4}$$

which holds for $n \geq 4$. Permutations from $W_{n,n-4}$ that are sortable by $n-4$ passes through a stack are also studied in [256].

Regarding other ways to define the notion of a k-stack sortable permutation (different from West-k-stack sortable permutations), Atkinson et al. [54] considered permutations that can be sorted by two stacks in series with each stack *remaining sorted from top to bottom.* This set of permutations, M, cannot be characterized by a finite number of classical patterns, but it is given by avoiding the following infinite set of patterns:

$$\{2(2m-1)416385\cdots(2m)(2m-3) \mid m = 2,3,\ldots\}.$$

Further, Atkinson et al. [54] showed that M is equinumerous with Av(1342), which was counted by Bóna (see Table 6.2).

Finally, Murphy [634] considered sorting with k stacks in series in more generality (many more operations are allowed) and proved that for $k \geq 2$ stacks, the set of sortable permutations cannot be characterized by a finite set of forbidden classical patterns.

2.1.2 Sorting with k stacks in parallel

Figure 2.4 shows an example of sorting the permutation 2341 with 2 stacks in parallel.

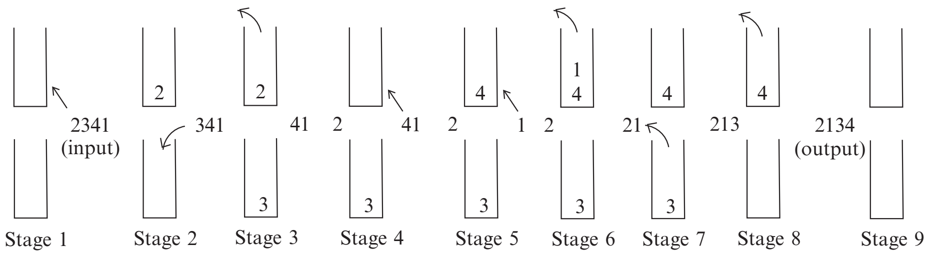


Figure 2.4: Sorting the permutation 2341 with 2 stacks in parallel.

As one sees, the permutation 2341 is not sortable this way, even though, of course, the class of such sortable permutations is larger than the class of 1-stack sortable permutations. We consider parallel sorting to give yet another example of a negative result: it is shown in [373, 762] that for $k \geq 2$, *no* finite set of forbidden *classical* patterns can characterize the set of permutations sortable with k stacks in parallel. It is no surprise that the enumeration problem related to the sortable permutations is unsolved for $k \geq 2$. What we do know [774, 775] is that for $k \leq 3$, the permutations sortable with k stacks in parallel can be recognized in time $O(n \log n)$ while for larger k, that recognition is NP-complete.

2.1.3 Input-restricted and output-restricted deques

An *input-restricted deque*, introduced by Knuth [540] is similar to a stack in that it has a push operation, but the pop operation can remove an element from either end of the deque. A successful sorting of a permutation requires the existence of a sequence involving the allowed operations that leads to the increasing permutation. Of course, we now have more possibilities to sort a permutation. For example, the reader may check that the permutation 2341 requires three stacks in series to be sorted while it can be sorted with a single input-restricted deque as shown in Figure 2.5.

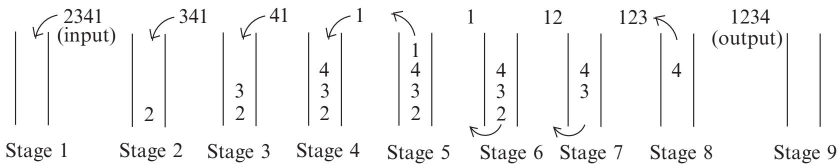


Figure 2.5: Sorting the permutation 2341 with an input-restricted deque.

On the other hand, not all permutations can be sorted with an input-restricted deque as shown in Figure 2.6.

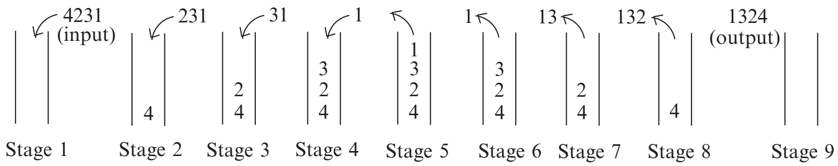


Figure 2.6: Sorting the permutation 4231 with an input-restricted deque.

Knuth [540] proved the following theorem where he used the so-called *kernel method* for proving the second part.

Theorem 2.1.7. *([540]) The set of n-permutations that can be sorted by an input-restricted deque is given by $\mathcal{S}_n(4231, 3241)$. The number $s_n(4231, 3241)$ of the sortable n-permutations is given by the $(n-1)$-th Schröder number S_{n-1}.*

We refer to [540] and to Subsection A.2.1 for more information on Schröder numbers. These numbers, no surprise, appear in the context of *output-restricted deques* as well when we are allowed to push letters at either end, but to pop them only from the top end. Knuth [540] show that the number of such permutations is given by the Schröder numbers, while West [801] shows the relation of these permutations to pattern-avoidance:

Theorem 2.1.8. *([801])* Av(2431, 4231) *is the set of all permutations that can be sorted using the output-restricted deque. Consequently, by the corresponding result of Knuth [540], $s_n(2431, 4231) = S_{n-1}$, the $(n-1)$-th Schröder number.*

We note the the second part of Theorem 2.1.8 can be obtained from Theorem 2.1.7 using the trivial bijections as discussed in the following remark.

Remark 2.1.9. Since $r.c.i(4231) = 4231$ and $r.c.i(3241) = 2431$, we have that a permutation π avoids the patterns 4231 and 3241 if and only if the permutation $r.c.i(\pi)$ avoids the patterns 4231 and 2431, that is,

$$\mathrm{Av}(2431, 4231) = \{r.c.i(\pi) | \pi \in \mathrm{Av}(4231, 3241)\}$$

and, in particular, $s_n(4231, 3241) = s_n(2431, 4231) = S_{n-1}$.

Definition 2.1.10. We call the sets $\mathrm{Av}(4231, 3241)$ and $\mathrm{Av}(2431, 4231)$ *input-restricted deque permutations* and *output-restricted deque permutations*, respectively.

Nothing is known on enumeration of (general) deque-sortable permutations (in such deques, we can push and pop letters at either end; Knuth posed the problem to study such permutations), although Pratt [658] proved that deque-sortable permutations are characterized by avoiding a certain *infinite* set of patterns.

2.1.4 Sorting with pop-stacks

Avis and Newborn [63] defined the following (less powerful) modification of the stack sorting procedure which they call "*sorting with pop-stacks*". A *pop-stack* is similar to a stack except that the pop operation unloads the entire stack (in the last-in, first-out manner). Figure 2.7 gives an example of a pop-sortable permutation (32154), while Figure 2.8 provides an example of a non-pop-sortable permutation (53412).

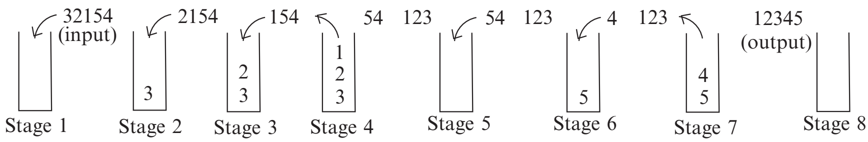


Figure 2.7: Pop-sorting the permutation 32154.

Definition 2.1.11. A permutation is called *layered* if it consists of a disjoint union of factors (the *layers*) so that the letters decrease within each layer, and increase between the layers. For example, 2136547 is a layered permutation with layers 21, 3, 654, and 7. It is an easy exercise to show that $\mathcal{S}_n(231, 312)$ is exactly the set of layered permutations of length n.

Proposition 2.1.12. *([63]) The set of pop-sortable permutations of length n is $\mathcal{S}_n(231, 312)$. Thus, the number of such permutations is $s_n(231, 312) = 2^{n-1}$.*

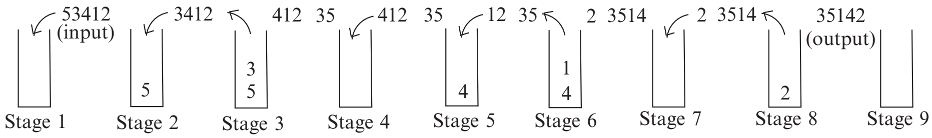


Figure 2.8: Pop-sorting the permutation 53412.

Proof. It is straightforward to see that any layered permutation is sortable with a pop-stack (the first layer of the form $i(i-1)\dots 1$, $i \geq 1$, will be turned into $12\dots i$ and the rest will be done by induction on length with the trivial base case — the permutation 1).

Conversely, assuming a permutation π is not pop-stack sortable, the output permutation must contain an inversion $\pi_i\pi_j$ ($\pi_i > \pi_j$). Thinking on what could make π_i precede π_j in the output permutation, we can see that either π contains a subsequence $\pi_i\pi_k\pi_j$ with $\pi_k < \pi_j$, or π contains a subsequence $\pi_i\pi_k\pi_j$ with $\pi_k > \pi_i$. Thus, π either contains the pattern 312, or the pattern 231, or both, and thus $\pi \notin \mathcal{S}_n(231, 312)$.

To enumerate $\mathcal{S}_n(231, 312)$, think of creating a layered permutation of length n by inserting the letters $1, 2, \dots, n$, one by one, starting with placing 1. Assuming $i-1$ letters have already been placed, $2 \leq i \leq n$, to avoid the patterns 231 and 312, we have two choices for placing i: either immediately to the left of $i-1$ or at the rightmost end of the $(i-1)$-permutation. Thus, $s_n(231, 312) = 2^{n-1}$. □

Proposition 2.1.12 justifies the fact that the permutation 32154 is sortable this way while 53412 is not.

Avis and Newborn [63] generalized Proposition 2.1.12 by enumerating those permutations that can be sorted with k *pop-stacks in series*. We note that by their interpretation, when the entire set of letters currently in the i-th pop-stack is popped, it is pushed onto the $(i+1)$-th pop-stack. To enumerate the objects, Avis and Newborn [63] proved that the set of permutations sortable by k pop-stacks in series can be characterized by a finite set of forbidden *classical* patterns. Even though the obtained formulas are rather complex, it is interesting that such enumeration can be done, taking into account the complexity of the case of (usual) stacks in series. See [58] for more results in this direction.

To complete the story, the situation for *pop-sorting in parallel* is as follows. Atkinson and Sack [56] proved that the set of permutations sortable with k *pop-stacks in parallel* is also (like in the series case) characterized by a finite set of forbidden *classical* patterns. For example, the following theorem holds.

Theorem 2.1.13. *([56])* $\mathcal{S}_n(3214, 2143, 24135, 41352, 14352, 13542, 13524)$ *is the set of permutations sortable with* 2 *pop-stacks in parallel. The number* s_n *of such permutations is defined by the conditions* $s_1 = 1$*,* $s_2 = 2$*,* $s_3 = 6$*, and the recurrence* $s_n = 6s_{n-1} - 10s_{n-2} + 6s_{n-3}$.

Moreover, Atkinson and Sack [56] conjectured that for all k, these permutations have a *rational* generating function. This conjecture was proved in [729] by Smith and Vatter.

Finally, the n-permutations sortable with k pop-stacks in parallel can be recognized in linear time [56].

2.1.5 A generalization of stack sorting permutations

In an (r, s)-stack defined by Atkinson [44], one is allowed to push into any of the first r positions and pop from any of the s positions at the top end of the stack. Notice that a usual stack corresponds to the case $r = s = 1$. Figure 2.9 gives an example of a (2,1)-stack sortable permutation (4231), which, by the way, is not stack sortable, and Figure 2.10 gives an example of a permutation (2341) that is not sortable with the (2,1)-stack. Notice that at Stage 3 in Figure 2.9 we intend to push 3 into the second position from the top instead of popping 2 out taking advantage of the new rules. Similarly, we used this trick (twice) at Stages 2 and 4 in Figure 2.10.

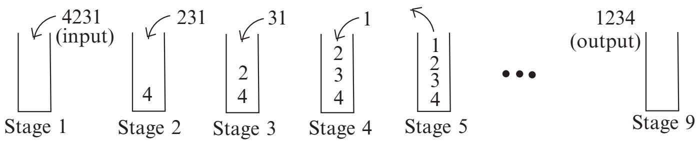


Figure 2.9: (2,1)-stack sorting the permutation 4231.

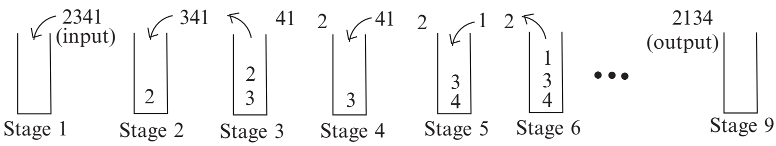


Figure 2.10: (2,1)-stack sorting the permutation 2341.

We conclude the section by stating several results (without proofs) on (r, s)-stack sortable permutations.

Proposition 2.1.14. *([44]) There is a one-to-one correspondence between* (r, s)*-stack sortable permutations and* (s, r)*-stack sortable permutations.*

Theorem 2.1.15. *([44]) A permutation is* $(r, 1)$*-stack sortable if and only if it avoids all* $r!$ *patterns of the form* $p_1p_2 \cdots p_r(r+2)1$. *Also, a permutation is* $(1, s)$*-stack sortable if and only if it avoids all* $s!$ *patterns of the form* $2p_1p_2 \cdots p_s1$.

Notice that Theorem 2.1.15 is the reason for the permutation 4231 being (2,1)-sortable in Figure 2.9, and for the permutation 2341 being not (2,1)-sortable in Figure 2.10.

Proposition 2.1.16. *([44]) The set of* $(r, 1)$*-stack sortable permutations, like the set of sortable permutations, has a* closure property*: any subsequence of an* $(r, 1)$*-stack sortable permutation is* $(r, 1)$*-stack sortable.*

Theorem 2.1.17. *([44]) If* $n \leq r$ *then there are* $n!$ $(r, 1)$*-stack sortable* n*-permutations, while otherwise, this number is the coefficient of* x^{n-r+2} *in*

$$-\frac{(r-1)!}{2}\sqrt{(r-1)^2x^2 - 2(r+1)x + 1}.$$

Theorem 2.1.18. *([44]) Asymptotically, the number of* $(r, 1)$*-stack sortable permutations is*

$$\frac{1}{2}(r-1)!\sqrt{r^{1/2}/(\pi n^3)}(1+\sqrt{r})^{2n-2r+3}.$$

Theorem 2.1.19. *([44]) A permutation is (2,2)-stack sortable if and only if it avoids all of the following* 8 *patterns:* 23451, 23541, 32451, 32541, 245163, 246153, 425163, *and* 426153.

2.2 Planar maps, trees, bipolar orientations

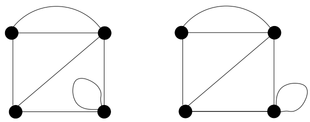
In Section 2.1 we have already mentioned the fact that proving formula (2.1) for the number of 2-stack sortable permutations combinatorially involved several objects. In this section, we will learn more about these objects and see that they build a layer in a hierarchy related to permutation patterns and studied in [260] in connection with embeddings of certain structures into $\beta(0, 1)$-*trees* (to be defined in Subsection 2.2.2). The variety of different classical combinatorial objects related to a single pattern class hierarchy is rather striking. Both the permutation patterns theory and the other structures involved benefit from the connection: for example, the number of 2-stack sortable permutations is obtained this way in a combinatorial fashion; an equidistribution result on non-separable permutations is obtained, to be discussed in Subsection 2.2.3; an alternative proof for the number of *planar bipolar* *orientations* defined in Subsection 2.2.1 is given via *Baxter permutations* discussed in Subsection 2.2.4.

In Subsections 2.2.1–2.2.6 we provide definitions of our objects of interest and their properties. Then we summarize all the connections in Subsection 2.2.7 (see Figure 2.32).

2.2.1 Planar maps and plane bipolar orientations

Definition 2.2.1. A *planar map* is a connected graph embedded in the plane with no edge-crossings. Such embeddings are considered up to continuous deformation. A map has *vertices* (points), *edges*, and faces (disjoint simply connected domains). The *outer face* is unbounded, the *inner faces* are bounded.

The two graphs below are the same as graphs, but they are different as planar maps since no continuous deformation transforms the first graph to the second one:



The maps we are dealing with are classical planar maps considered, for example, by Tutte [770] who founded the enumeration theory of planar maps in a series of papers in the 1960s (see [273] for references).

Definition 2.2.2. A *cut vertex* in a map is a vertex whose deletion disconnects the map. A *loop* is an edge whose endpoints coincide. A map is *non-separable* if it has no loops and no cut vertices.

The maps considered by us are *rooted*, meaning that a directed edge is distinguished as the root. Without loss of generality, we can assume that the root is incident to the outer face, and the outer face lies on its right side while following the root orientation. For such an orientation, the outer face will be the *root-face*. In general, the root face of a planar map is the face adjacent to the root that lies to the right of it while following the root direction. Also, the vertex from which the root comes out is called the *root-vertex*.

All rooted non-separable planar maps on 4 edges are given in Figure 2.11.

The number of rooted non-separable planar maps on $n+1$ edges was first determined by Tutte [769] and it is given by

$$\frac{4(3n)!}{n!(2n+2)!}, \tag{2.5}$$

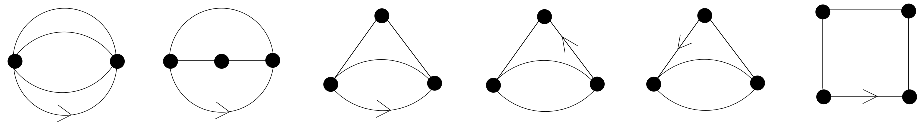


Figure 2.11: All rooted non-separable planar maps on 4 edges.

which was also proved differently by Brown [190].

Definition 2.2.3. A planar map is *cubic* if all its vertices are of degree 3. A cubic planar map is *bicubic* if it is bipartite, that is, if its vertices can be colored using two colors, say, black and white, so that each edge is incident to different colors.

The simplest cubic non-separable map is the map with two vertices and three edges joining them. It is a well-known fact that the faces of a bicubic map can be colored using three colors so that adjacent faces have distinct colors, say, colors 1, 2, and 3, in a counterclockwise order around white vertices. We can assume that the root vertex is black and the root face has color 3. All bicubic planar maps on 6 edges are given in Figure 2.12.

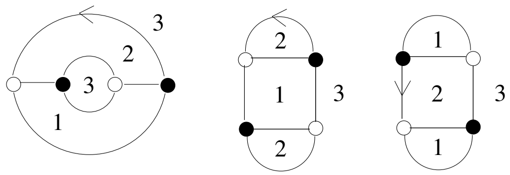


Figure 2.12: All bicubic planar maps on 6 edges.

The number of bicubic planar maps with $3n$ edges was given by Tutte [769]:

$$\frac{3 \cdot 2^{n-1}(2n)!}{n!(n+2)!}. \tag{2.6}$$

Definition 2.2.4. In a directed graph, a *source* is a vertex with no incoming edges and a *sink* is a vertex with no outgoing edges. A *plane bipolar orientation* O is an acyclic orientation of a planar map with a unique source s and a unique sink t, both located on the outer face. One of the oriented paths going from s to t has the outer face on its right: this path is the *right border* of O, and its length (the number of edges) is the *right outer degree* of O. The *left outer degree* can be defined similarly.

See the rightmost picture in Figure 2.23 for an example of a plane bipolar orientation with right degree 2 and left degree 3. The vertices s and t are called the *poles* of O.

The coefficient of x^1y^0 in the *Tutte polynomial* $T_M(x,y)$ of a non-separable planar map M having a fixed size, is the number of bipolar orientations of M [415, 431]. This number, up to a sign, is also the derivative of the *chromatic polynomial* of M, evaluated at 1 [555].

2.2.2 Description trees and skew ternary trees

Definition 2.2.5. A $\beta(1,0)$-*tree* is a rooted plane tree labeled with positive integers such that

1. Leaves have label 1.
2. The root has label equal to the sum of its children's labels.
3. Any other node has a label no greater than the sum of its children's labels.

All $\beta(1,0)$-trees on 3 edges are presented in Figure 2.13.

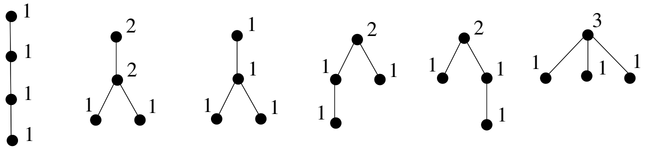


Figure 2.13: All $\beta(1,0)$-trees on 4 nodes.

To state some of the upcoming results, we first need to define several statistics on $\beta(1,0)$-trees. These are given in Table 2.1.

For the $\beta(1,0)$-tree T in Figure 2.14, the values of the statistics are as follows: leaves$(T) = 6$, int$(T) = 5$, root$(T) = 4$, sub$(T) = 3$, lpath$(T) =$ rpath$(T) = 2$, stem$(T) = 1$, lsub$(T) = 2$, and rsub$(T) =$ beta$(T) = 1$. For another example, the tree second from the left in Figure 2.13 has leaves$(T) =$ int$(T) =$ root$(T) = 2$, sub$(T) = 1$, lpath$(T) =$ rpath$(T) =$ stem$(T) = 2$, lsub$(T) =$ rsub$(T) = 1$, and beta$(T) = 2$.

A $\beta(1,0)$-tree T on at least two nodes is *indecomposable* if sub$(T) = 1$, that is, if the root of T has exactly one child; otherwise, T is *decomposable*. A $\beta(1,0)$-tree

Statistic	Description in a $\beta(1,0)$-tree T
leaves(T)	# leaves;
int(T)	# internal nodes (or nonleaves);
root(T)	root's label;
sub(T)	# children of (subtrees coming out from) the root;
lpath(T)	# edges from the root to the leftmost leaf = length of the leftmost path (left-path);
rpath(T)	# edges from the root to the rightmost leaf = length of the rightmost path (right-path);
stem(T)	# internal nodes common to the left- and the right-paths;
lsub(T)	# 1's below the root on the left-path;
rsub(T)	# 1's below the root on the right-path;
beta(T)	Let $\ell_1, \ldots, \ell_m$ be the leaves from left to right. If no node on the path from ℓ_1 to the root, except for ℓ_1, has label 1, reduce the labels on all nodes on that path by 1. Now look at ℓ_2 and repeat the process, until we come to a leaf ℓ_i whose path to the root has a node (other than ℓ_i) that now has label 1. Then beta$(T) = i$.

Table 2.1: Statistics on $\beta(1,0)$-trees as described in [262].

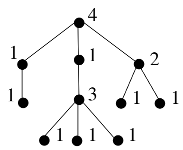


Figure 2.14: A $\beta(1,0)$-tree.

T on at least two nodes is *right-indecomposable* if rsub$(T) = 1$, that is, if the right-path has exactly one 1 below the root; otherwise, T is *right-decomposable*. The idea of the involution called h on $\beta(1,0)$-trees, defined in [262], is to turn $\beta(1,0)$-tree decompositions into right-decompositions, and vice versa. A recursive description of h is shown schematically in Figure 2.15: as the base case, we map the 1 node tree to itself. In the case of an indecomposable tree, we remove the top edge to get A (the root may need to be adjusted), apply h recursively to get $h(A)$, and adjoin the removed edge to the proper place on the right path of $h(A)$ (to make the *rpath* statistic in the obtained tree equal to x, the value of the *root* statistic in the original tree) increasing all the labels above the rightmost leaf by 1. One the other hand, if our tree is decomposable, we locate its rightmost subtree B (which includes the root of the original tree), apply h recursively on it to get $h(B)$, and glue its rightmost leaf with the root of $h(A)$ obtained recursively (the glue node will receive label 1). See Figure 2.16 for an example of applying the involution h together with some of the steps involved in the recursive procedure.

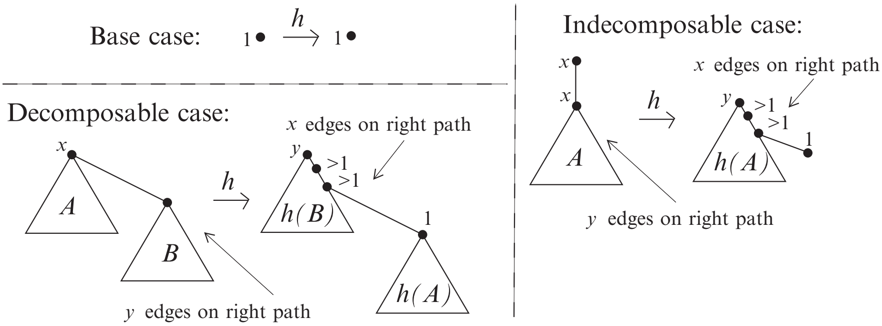


Figure 2.15: A schematic description of the involution h.

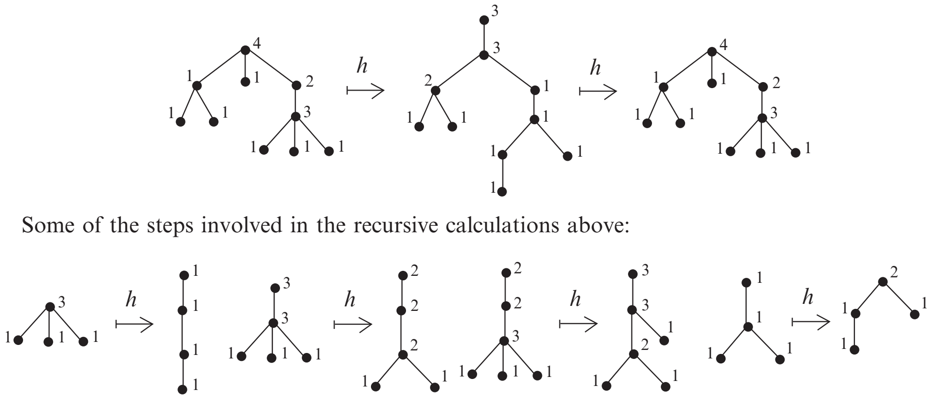


Figure 2.16: An example of applying the involution h together with some of the steps involved in the recursive procedure.

Theorem 2.2.6. *([262]) The map h is an involution (h^2 is the identity map) that sends the first tuple of statistics below to the second one (we refer to [262] for the definition of the gamma statistic; also, we can recall the notion of equidistribution of statistics in Definition 1.0.34):*

$$\begin{array}{lllllllll}
(& \text{leaves}, & \text{int}, & \text{root}, & \text{rpath}, & \text{sub}, & \text{rsub}, & \text{stem}, & \text{gamma} \;) \\
(& \text{int}, & \text{leaves}, & \text{rpath}, & \text{root}, & \text{rsub}, & \text{sub}, & \text{gamma}, & \text{stem} \;).
\end{array}$$

Another interesting property of h is that when restricted to $\beta(1,0)$-trees with all nodes labeled 1 (except for the root), which can be checked to be closed under h, the involution induces an involution on *unlabeled rooted plane trees*, very classical objects counted by the Catalan numbers (we can erase all labels from such $\beta(1,0)$-trees and reconstruct them, if needed). This involution is new and it is the subject of current studies in [261] by Claesson et al. One of the results that is a direct corollary to Theorem 2.2.6 is the following equidistribution fact; see Section 8.8 for more information on the subject.

Theorem 2.2.7. *([261]) On (unlabeled) rooted plane trees there is an automorphism (a bijection from the set of such trees to itself) that sends the first tuple of statistics below to the second one (in this case, the statistic* rpath *is identical to* rsub*, and* root *is identical to* sub*):*

$$\begin{array}{llllll} (& \text{leaves}, & \text{int}, & \text{rpath}, & \text{sub} &) \\ (& \text{int}, & \text{leaves}, & \text{sub}, & \text{rpath} &). \end{array}$$

Definition 2.2.8. A $\beta(0,1)$*-tree* is defined on non-negative integers in a similar way to $\beta(1,0)$-trees:

1. Leaves have label 0.
2. The root has label equal to 1 + the sum of its children's labels.
3. Any other node has label no greater than 1 + the sum of its children's labels.

All $\beta(0,1)$-trees on 3 edges are presented in Figure 2.17.

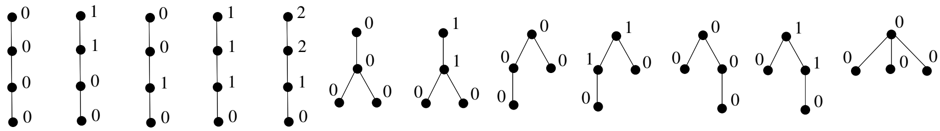


Figure 2.17: All $\beta(0,1)$-trees on 4 nodes.

Definition 2.2.9. A *ternary tree* is a rooted tree whose nodes have at most one son of each of the following three types: *left*, *middle*, and *right*. Vertices of ternary trees are labelled as follows: the root is labelled 0 and the non-root vertices take the label of their father, to which is added $+1$, $+0$, -1 when they are left, middle, or right sons, respectively. A ternary tree is *skew* if its labels are nonnegative.

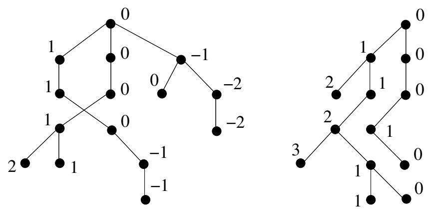


Figure 2.18: An example of a ternary tree and a skew ternary tree.

See Figure 2.18 for an example of a ternary tree and a skew ternary tree.

All but one statistics on skew ternary trees appearing below are straightforward to define: "*number of even labels*", "*number of odd labels*", and "*number of zeros*". The non-straightforward statistic is called "*number of first zeros*" in [476] and it is defined as the maximum number of vertices in the sequence of middle sons with label 0 starting from the root. For example, for the tree to the right in Figure 2.18, the number of even labels is 7, the number of odd labels is 6, the number of zeros is 5, and the number of first zeros is 3.

2.2.3 Relevant pattern-avoidance

A combinatorial proof of West's former conjecture (that the number of 2-stack sortable permutations is given by (2.1)) presented in [324] connects rooted non-separable planar maps with 2-stack sortable permutations through eight different sets of permutations (see Figure 2.19). These sets are in bijection, either because they have isomorphic *generating trees* (out of four generating trees involved only two are identical) or because they can be obtained from each other by applying one of the trivial bijections (r, c, or i). Each of these eight permutation classes could enter Table 2.3 below, and the more general picture in Figure 2.32. However, we are including there only the set of permutations $\mathrm{Av}(2413, 41\bar{3}52) = \mathrm{Av}(2413, 3\underline{14}2)$ that is connected directly to rooted non-separable planar maps and is called the set of *non-separable permutations*. A special interest of this particular set is that the reverse of these permutations was studied in [262] by Claesson et al. in connection with rooted non-separable planar maps, where another bijection, preserving more statistics, was found.

Theorem 2.2.10. *([262]) There is a bijection showing that the first tuple below has the same distribution on* $\beta(1,0)$*-trees as the second tuple does on* $\mathrm{Av}(3142, 2\underline{41}3)$

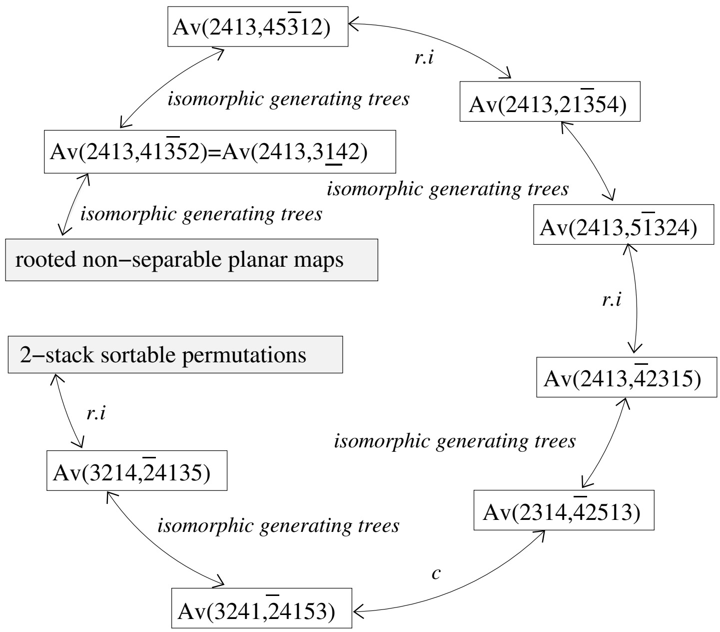


Figure 2.19: A connection between rooted non-separable planar maps and 2-stack sortable permutations through eight classes of pattern-restricted permutations.

(see Tables 2.1 and A.1 for definitions):

$$\begin{array}{lllllllll} (& \text{sub}, & \text{leaves}, & \text{root}, & \text{lpath}, & \text{rpath}, & \text{lsub}, & \text{beta} &) \\ (& \text{comp}, & 1+\text{asc}, & \text{lmax}, & \text{lmin}, & \text{rmax}, & \text{ldr}, & \text{lir} &). \end{array}$$

The idea of the bijection proving Theorem 2.2.10 is close to the idea of generating trees: one wants to show that the objects in question can be generated in similar ways. More precisely, irreducible $\beta(1,0)$-trees (having the statistic sub equal to 1) are mapped into irreducible permutations avoiding the patterns (that have the statistic comp equal to 1). To achieve this a non-trivial procedure was found on permutations based on inserting the new maximum letter in proper places of smaller permutations and rearranging the parts to the left and to the right of this letter keeping the same relative orders (see [262] for the actual construction). As a matter of fact, two procedures to do the task may be found in [262], but one of them preserves more statistics of interest than the second one.

Using Theorem 2.2.6 together with the bijection proving Theorem 2.2.10, the following two equidistribution results were obtained (in proving the second equidistribution result the mirror image on $\beta(1,0)$-trees is involved as well).

Theorem 2.2.11. *([262]) The following pairs of tuples of statistics are equidistributed on the set* $\mathrm{Av}(3142, 2\underline{41}3)$*, that is, there is a bijection (automorphism) from* $\mathrm{Av}(3142, 2\underline{41}3)$ *to itself sending the first tuple of statistics to the second one in each pair:*

$$\begin{array}{llll} (& \text{asc}, & \text{lmax}, & \text{rmax} &) \\ (& \text{des}, & \text{rmax}, & \text{lmax} &) \end{array}$$

and

$$\begin{array}{lllllll} (& \text{asc}, & \text{lmax}, & \text{lmin}, & \text{comp}, & \text{ldr} &) \\ (& \text{des}, & \text{lmin}, & \text{lmax}, & \text{ldr}, & \text{comp} &). \end{array}$$

Note that the first equidistribution result in Theorem 2.2.11, unlike the second one, is trivial on the set of *all* permutations: all one needs to do is to apply the reverse operation to the set of permutations. However, proving the same result on $\mathrm{Av}(3142, 2\underline{41}3)$ was unsuccessful for a long time before the involution h was invented. A direct (combinatorial) proof of results in Theorem 2.2.11 would be desirable. The diagrams in Figure 2.20 created by Anders Claesson may be of some help in solving the problem, though it is definitely not sufficient just to use them. The diagrams show translations of relevant statistics under compositions of two trivial bijections. We can use the fact that $i.r = c.i$, which is rather easy to prove, to obtain translations under compositions that are not present in the diagrams. The idea to approach finding a combinatorial proof of Theorem 2.2.11 is to use the fact that the set $\mathrm{Av}(3142, 2\underline{41}3)$ is closed under compositions of two trivial bijections, although it is not closed under any single such bijection.

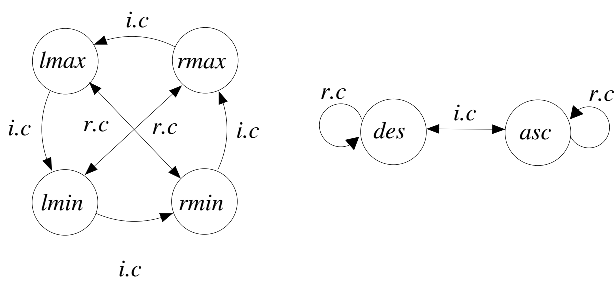


Figure 2.20: Translation of the statistics lmax, rmax, lmin, rmin, asc, and des under compositions involving two trivial bijections. To complete the picture, we can use $i.r = c.i$.

If one wishes to code rooted non-separable planar maps (equivalently, $\beta(1,0)$-trees) by permutations, it seems that the set $\mathrm{Av}(3142, 2\underline{41}3)$ (equivalently for non-separable permutations, $\mathrm{Av}(2413, 3\underline{14}2)$) is a better choice than 2-stack sortable permutations. Indeed, $\mathrm{Av}(3142, 2\underline{41}3)$ is more symmetric (it is closed under compositions of two trivial bijections while the set of 2-stack sortable permutations is not closed under any combination of trivial bijections), and $\mathrm{Av}(3142, 2\underline{41}3)$, under known bijections, captures better the structure of rooted non-separable planar maps by keeping track of more statistics than in the case of 2-stack sortable permutations. In either case, the following conjecture appears in [262] by Claesson et al. on relations between $\mathrm{Av}(3142, 2\underline{41}3)$ and 2-stack sortable permutations.

Conjecture 2.2.12. ([262]) The quadruple $(comp, asc, ldr, rmax)$ has the same distribution on $\mathcal{S}_n(3142, 2\underline{41}3)$ as it has on 2-stack sortable permutations of length n.

Remark 2.2.13. It is known [323] that the pair of statistics $(asc, lmax)$ on the class $\mathrm{Av}(3142, 2\underline{41}3)$ is equidistributed with the pair $(\mathrm{des}, \mathrm{rmax})$ on 2-stack sortable permutations; this fact also follows from Table 2.3 below. If Conjecture 2.2.12 is true, then it would strengthen the result in [323].

In discussing the coding of planar maps/description trees by permutations, we would like to mention [127] by Bóna who enumerated Av(1342) (see Table 6.2). This was the first case of enumeration of non-monotonic patterns of length more than 3 and this result is relevant to the hierarchy we will discuss in Subsection 2.2.7 (see Figure 2.32).

Theorem 2.2.14. *([127]) The following three sets of objects are in one-to-one correspondence:*

- Av(1342)*;*
- *Plane forests of* $\beta(0,1)$*-trees;*
- *Ordered collections of (rooted) bicubic planar maps.*

Using Theorem 2.2.14 and a known enumerative result from [769], the following theorem was proved.

Theorem 2.2.15. *([127]) One has the following enumeration results for* Av(1342)*:*

- *The generating function for* $s_n(1342)$ *is*

$$\sum_{n\geq 0} s_n(1342)x^n = \frac{32x}{-8x^2+12x+1-(1-8x)^{3/2}};$$

- *The exact formula for* $s_n(1342)$ *is*

$$(-1)^{n-1}\frac{7n^2-3n-2}{2}+3\sum_{i=2}^{n}(-1)^{n-i}2^{i+1}\frac{(2i-4)!}{i!(i-2)!}\binom{n-i+2}{2};$$

- $\sqrt[n]{s_n(1342)}$ *converges to 8 when* $n \to \infty$.

We conclude this subsection with another result relevant to the hierarchy to be discussed in Subsection 2.2.7.

Theorem 2.2.16. *([260]) For* $n \geq 0$, $s_n(2\underline{41}3) = s_n(3\underline{41}2)$. *Moreover, there is a bijection between the sets that sends a permutation in* $S_n(2\underline{41}3)$ *with* k *occurrences of* $3\underline{41}2$ *to a permutation in* $S_n(3\underline{41}2)$ *with* k *occurrences of* $2\underline{41}3$.

Proof. For this proof, we let $P_1 = 2\underline{41}3$ and $P_2 = 3\underline{41}2$.

If a permutation avoids P_1 *and* P_2, we map it to itself.

Now suppose that an n-permutation π avoids P_1 and it contains at least one occurrence of P_2. We consider the leftmost pair, say xy, of consecutive letters, depicted in Figure 2.21 by the solid circles, that play the role of 4 and 1 in an occurrence of the pattern P_2; notice that this pair is well-defined.

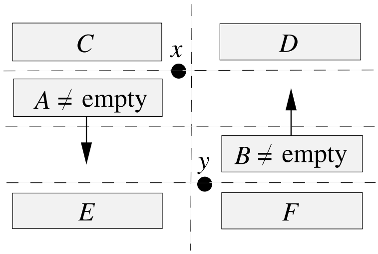


Figure 2.21: Sending a permutation from $S_n(P_1)$ to a permutation in $S_n(P_2)$.

One immediately realizes that if we restrict ourselves to the values of π between x and y, then in order to avoid P_1, everything to the left of xy must be larger than everything to the right of xy, which is shown schematically by the non-empty rectangles A and B in Figure 2.21. Notice that xy contributes $|A| \cdot |B|$ occurrences of P_2, where $|X|$ denotes the number of letters in X. The letters in A, C and E can be shuffled somehow, but they (together) do not contain an occurrence of P_1 or a pair of consecutive letters having the properties that xy above has (xy is the leftmost one with such properties). Also, D, B and F can be shuffled somehow in such a way that they (together) do not contain P_1.

We now decrease each letter of A by $|B|$ and increase each letter of B by $|A|$ thus turning each occurrence of P_2 involving xy into an occurrence of P_1 involving xy. We denote the resulting permutation π'.

Our claim is that no new occurrences of P_2 are introduced and no (new) occurrences of P_1, beyond those involving xy, are introduced after the described procedure is done. This follows easily from the fact that the elements of A (resp., B) do not change their relative position with respect to the elements of C and E (resp., D and F).

We can now proceed with π' and find $x'y'$, if there is one (otherwise we do not need to do anything), having the properties of xy. One then changes all the occurrences of P_2 involving $x'y'$ into occurrences of P_1 involving $x'y'$, in the way we did it above. There is only one difference between considering π and π': π contains no P_1, whereas π' does. However, the occurrences of P_1 in π' will not be affected by the procedure, again, because of the properties of A, C, and E. Indeed, either such an occurrence of P_1 is entirely in A, or in C, or in E, in which case it cannot disappear, or the occurrence has the letters corresponding to 2, 4, and 1 in P_1 either in C or in E, which, again, cannot cause the occurrence to disappear.

Thus we can go through all pairs xy from left to right and change all occurrences of P_2 to occurrences of P_1. The process terminates because of the fact that $x'y'$ is strictly to the right of xy. The map is easily seen to be injective and reversible, and it is easy to see that it sends a permutation in $S_n(P_1)$ with k occurrences of P_2 to a permutation in $S_n(P_2)$ with k occurrences of P_1. □

Remark 2.2.17. It is straightforward to see from the proof of Theorem 2.2.16 that the bijection there preserves (sends to themselves) an enormous number of permutation statistics, which includes (but is not limited to!) the following statistics (see Table A.1 for definitions): maj, lmax, lmin, rmax, rmin, des, peak, last $.i$, head $.i$, ldr, lir, rdr, rir, comp, ddes, and dasc.

Remark 2.2.18. It follows from Proposition 1.3.7 and Theorem 2.2.16 that

$$s_n(2\underline{41}3) = s_n(3\underline{41}2) = s_n(21\bar{3}54).$$

2.2.4 Baxter permutations

In 1964, Glen Baxter [98] introduced the following class of permutations that now bears his name.

Definition 2.2.19. A permutation $\pi = \pi_1\pi_2 \dots \pi_n$ is a *Baxter permutation* if there are no four indices $i < j < k < \ell$ such that

- $k = j + 1$;
- $\pi_i \pi_j \pi_k \pi_\ell$ is an occurrence of the pattern 2413 or 3142.

In the language of vincular patterns, Av$(2\underline{41}3, 3\underline{14}2)$ is the set of Baxter permutations.

Example 2.2.20. 25314 is a Baxter permutation, whereas 5327146 is not a Baxter permutation as it contains an occurrence of the pattern $3\underline{14}2$ (the subsequence 5274).

Gire [325, 419] showed that Av$(25\bar{3}14, 41\bar{3}52)$ is exactly the set of Baxter permutations. The motivation for introducing the permutations defined in Definition 2.2.19 was the following problem in analysis on commuting continuous functions.

Suppose f and g are continuous functions from $[0,1]$ to $[0,1]$ that commute, that it, $g(f(x)) = f(g(x))$. We let $h(x)$ denote $g(f(x)) = f(g(x))$. Suppose h has finitely many fixed points $x_1 < x_2 < \cdots < x_n$. A fixed point x_i of $h(x)$ may have one of the following three types:

1. x_i is *up-crossing* if the sign of $h(x) - x$ changes from negative to positive in the neighborhood of x_i;
2. x_i is *down-crossing* if the sign of $h(x) - x$ changes from positive to negative in the neighborhood of x_i;
3. x_i is *touching* if $h(x) - x$ does not change sign in the neighborhood of x_i.

Example 2.2.21. In Figure 2.22, d is an up-crossing fixed point, b and f are down-crossing fixed points, and a, c, and e are touching fixed points.

Theorem 2.2.22. *([98, 167]) For the objects defined above, we have the following facts ($f(x)$ can be substituted by $g(x)$ throughout):*

- *The function f maps the fixed points $x_1, x_2, \ldots, x_n$ of h bijectively onto themselves;*
- *The fixed point $f(x_i)$ has the same type as x_i has;*
- *The permutation of the up-crossing fixed points is determined by the permutation of the down-crossing fixed points;*
- *The permutation of the down-crossing fixed points is a Baxter permutation.*

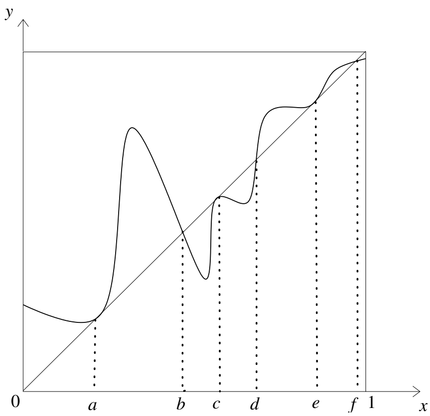


Figure 2.22: Fixed points of different types.

Baxter permutations are a widely studied class of permutations (see, for example, [98, 151, 167, 158, 272, 221, 257, 250, 379, 326, 419, 436, 620, 581, 787]). It is known that Baxter permutations of length n are equinumerous with several combinatorial objects, for example, with certain *rectangulations* with n points on the diagonal ([4, 379]) and with *plane bipolar orientations with* n *edges* (discussed in Subsection 2.2.1). Since the last set of objects is connected to maps, which is of special interest to us, we would like to sketch the idea of the bijection in [151] between Baxter permutations and plane bipolar orientations. We will explain the idea on the example in Figure 2.23 (we refer to [151] for a detailed description of the bijection).

Given a Baxter permutation, we start by drawing its permutation matrix using black circles. Next we add two white rectangles representing the poles, and we add white circles in certain places right after each ascent position (whenever we are coming from a smaller letter to a larger one while going from left to right) as shown in Figure 2.23. Then, starting from the source rectangle, we connect rectangles/circles (referred to as *nodes* in what follows) in the following way. Given a node x, draw an arrow from it to *each* node that is *visible directly* from x in the North-East direction. For example, from the leftmost white circle in Figure 2.23 we can see directly three black circles corresponding to the letters 7, 5, and 4 in the permutation 37568412. Thus, each directed path from the source to the sink is an alternation of black and white circles starting and ending with black circles. As the final step, we remove the black circles making the arrows going through them continuous as shown in Figure 2.23. The resulting object is a plane bipolar orientation.

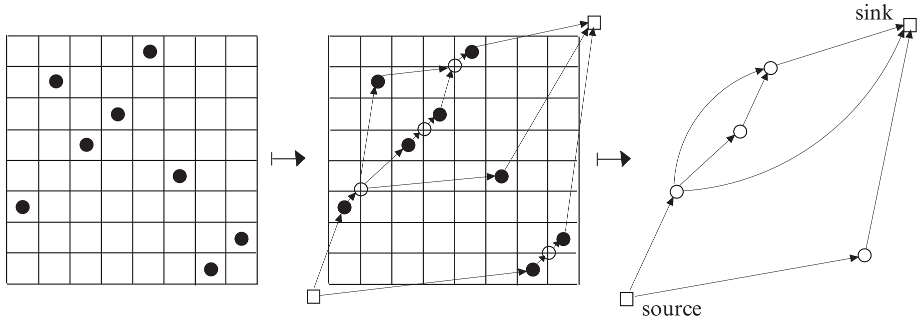


Figure 2.23: The Baxter permutation 37568412 and its transformation into a plane bipolar orientation under the bijection given in [151].

It is a funny coincidence, pointed out by Bonichon et al. in [151], that another Baxter, the physicist *Rodney* Baxter, in his studies [99] came across objects equinumerous with Baxter permutations without realizing the fact that the numbers he was dealing with are known and moreover, they bear his name! Baxter, the physicist, studied the sum of the Tutte polynomials $T_M(x, y)$ of non-separable planar maps M having a fixed size (see [99]). He found the coefficient of x^1y^0 in $T_M(x, y)$, summed over all rooted non-separable planar maps M having $n+1$ edges, $m+2$ vertices, root-face of degree $i+1$ and a root-vertex of degree $j+1$. As was already mentioned in Subsection 2.2.1, this coefficient counts plane bipolar orientations of M (equivalently, Baxter permutations).

We will close our discussion of Baxter permutations for now by stating a few enumerative results on them. However, we will see Baxter permutations later in the book, in connection with pattern-avoidance in so-called *partial permutations*.

Theorem 2.2.23. *([250]) The number of Baxter permutations of length n is given by*

$$\sum_{i=0}^{n-1} \frac{\binom{n+1}{i}\binom{n+1}{i+1}\binom{n+1}{i+2}}{\binom{n+1}{1}\binom{n+1}{2}}. \tag{2.7}$$

While the proof of Theorem 2.2.23 given by Chung et al. [250] is analytical, Viennot [787] provided a bijective proof of formula (2.8). The following theorem is a refinement of Theorem 2.2.23.

Theorem 2.2.24. *([581]) The number of Baxter permutations of length n having m ascents, i left-to-right maxima and j right-to-left maxima (see Definition A.1.1)* *is given by*

(2.8)
$$\frac{ij}{n(n+1)}\binom{n+1}{m+1}\left[\binom{n-i-1}{n-m-2}\binom{n-j-1}{m-1}-\binom{n-i-1}{n-m-1}\binom{n-j-1}{m}\right].$$

Review Definition 1.0.18 for the notion of alternating permutations.

Definition 2.2.25. A permutation is *doubly alternating* if it is alternating *and* its inverse is alternating.

Example 2.2.26. 13254 and 354612 are examples of doubly alternating (in fact, Baxter) permutations, whereas the permutation 24153 is not doubly alternating, as its inverse, $i(24153) = 31524$, is not alternating (it starts with a descent, not an ascent).

Theorem 2.2.27. *([272]) The number of alternating Baxter permutations of length* $2n$ *and* $2n+1$ *is given by* C_n^2 *and* C_nC_{n+1}*, respectively, where* $C_n = \frac{1}{n+1}\binom{2n}{n}$ *is the* n*-th Catalan number.*

Theorem 2.2.28. *([436]) The number of doubly alternating Baxter permutations of length* $2n$ *or* $2n+1$ *is given by* $C_n = \frac{1}{n+1}\binom{2n}{n}$*, the* n*-th Catalan number.*

The research done in [620] by Mansour and Vajnovszki is of the type discussed in Problem 1.7.13, namely, one restricts the set of objects (Baxter permutations in our case) by some conditions (in this case, the permutations must avoid the pattern 123) and then additional avoidance or containment constraints are considered. We provide here just two theorems proved in [620]. For more results of this type see Subsections 6.1.5 and 7.1.4.

Theorem 2.2.29. *([620]) The generating function for the number of* 123*-avoiding Baxter permutations is given by*

$$\frac{(1-x)^2}{1-3x+2x^2-x^3}.$$

In other words, the number of 123*-avoiding* n*-permutations is given by the* $(3n+3)$*-th Padovan number.*

Theorem 2.2.30. *([620]) The number of* 123*-avoiding Baxter permutations containing exactly* r *occurrences of the vincular pattern* $\underline{132}$ *(or* $\underline{213}$*) is given by*

$$\sum_{i=0}^{n-3r} 2^{n-3r-i}\binom{i+r-2}{r-2}\binom{n-3r-i+r}{r}.$$

Finally, another result related to Problem 1.7.13 is the following theorem dealing with Baxter involutions, that is, Baxter permutations whose (usual group-theoretical) square is the identity permutation.

Theorem 2.2.31. *([151]) The number of fixed-point-free Baxter involutions of length $2n$ is*

$$\frac{3 \cdot 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n}.$$

2.2.5 Separable permutations

Definition 2.2.32. Suppose $\pi = \pi_1\pi_2\dots\pi_m \in \mathcal{S}_m$ and $\sigma = \sigma_1\sigma_2\dots\sigma_n \in \mathcal{S}_n$. We define the *direct sum* (or simply, *sum*) $\oplus$, and the *skew sum* $\ominus$ by building the permutations $\pi \oplus \sigma$ and $\pi \ominus \sigma$ as follows:

$$\begin{aligned}
(\pi \oplus \sigma)_i &= \begin{cases} \pi_i & \text{if } 1 \le i \le m, \\ \sigma_{i-m} + m & \text{if } m+1 \le i \le m+n, \end{cases} \\
(\pi \ominus \sigma)_i &= \begin{cases} \pi_i + n & \text{if } 1 \le i \le m, \\ \sigma_{i-m} & \text{if } m+1 \le i \le m+n. \end{cases}
\end{aligned}$$

Example 2.2.33. For example, $14325 \oplus 4231 = 143259786$ and $14325 \ominus 4231 = 587694231$. This example is best understood by looking at the permutation matrices in Figure 2.24 of the permutations involved.

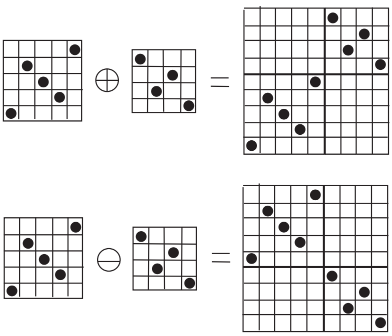


Figure 2.24: Permutation matrices illustration of the fact that $14325 \oplus 4231 = 143259786$ and $14325 \oplus 4231 = 587694231$.

Definition 2.2.34. The *separable permutations* are those which can be built from the permutation 1 by repeatedly applying the $\oplus$ and $\ominus$ operations.

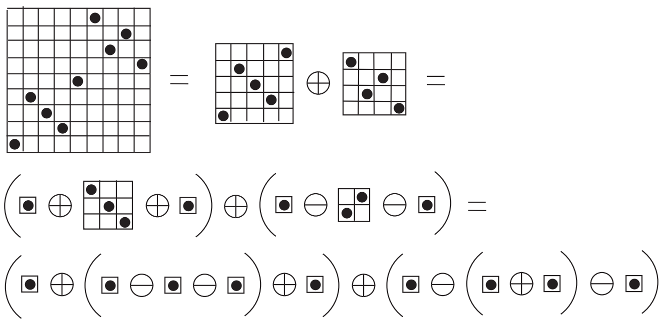


Figure 2.25: Decomposition of 143259786 using $\oplus$ and $\ominus$ operations.

Example 2.2.35. The permutations 143259786 and 587694231 appearing in Figure 2.24 are both separable. Figure 2.25 illustrates a step-by-step procedure to see that 143259786 is separable. All permutations of length 3 are separable, and only two permutations of length 4, 2413 and 3142, are not separable.

Bose et al. [155] introduced the notion of separable permutation in 1998, but the following well-known result is folkloric.

Theorem 2.2.36. *(folklore)* Av(2413, 3142) *is the set of all separable permutations.*

Remark 2.2.37. So far we introduced, in Definition 2.2.34, the class of separable permutations, Av(2413, 3142), and the class of non-separable permutations, Av(3142, $2\underline{41}3$), in Subsection 2.2.3. We note that Av(2413, 3142) $\subseteq$ Av(3142, $2\underline{41}3$) and thus each separable permutation is a non-separable one in our definitions (which sounds contradictory, but this is just a matter of names) but not vice versa. The word "separable" in the case of separable permutations came from the "process of separation," or decomposing permutations, whereas the word "non-separable" in the case of non-separable permutations came from a plain connection of the permutations to rooted non-separable planar maps, which has nothing to do with any separability on permutations themselves.

Throughout this book, by a "permutation class" we simply mean a set of permutations. In several places, however, to be mentioned explicitly, this expression has a stronger sense.

Definition 2.2.38. For permutations σ and π, we write $\sigma \leq \pi$ if σ occurs in π as a pattern (there exits a subsequence in π of the same length as σ that is order-isomorphic to σ). Thus we can define the *containment order* on the set of all permutations. Sets of permutations which are closed downward under this order are called *permutation classes* (or just *classes*). In other words, $\mathcal{C}$ is a class if for any $\pi \in \mathcal{C}$ and any $\sigma \leq \pi$, we have $\sigma \in \mathcal{C}$.

Example 2.2.39. It is easy to see that for any set P of classical patterns, $\mathrm{Av}(P)$ is a (permutation) class, whereas if other patterns are involved, that does not have to be the case. Indeed, consider the permutation $23154 \in \mathrm{Av}(1\underline{24}3)$. $\mathrm{red}(2354) = 1243 \leq 23154$ and $1243 \notin \mathrm{Av}(1\underline{24}3)$. Thus, $\mathrm{Av}(1\underline{24}3)$ is not a permutation class.

Definition 2.2.40. For two sets (classes) of permutations $\mathcal{C}$ and $\mathcal{D}$ we let

$$\begin{aligned} \mathcal{C} \oplus \mathcal{D} &= \{\pi \oplus \sigma | \pi \in \mathcal{C}, \sigma \in \mathcal{D}\}, \\ \mathcal{C} \ominus \mathcal{D} &= \{\pi \ominus \sigma | \pi \in \mathcal{C}, \sigma \in \mathcal{D}\}. \end{aligned}$$

We would like to discuss just a couple of results on separable permutations. For more results on them, consult [27, 338, 546, 734].

Proposition 2.2.41. *([27]) The class of separable permutations* $\mathrm{Av}(2413, 3142)$ *is the smallest nonempty class* $\mathcal{C}$ *which satisfies both* $\mathcal{C} \oplus \mathcal{C} \subseteq \mathcal{C}$ *and* $\mathcal{C} \ominus \mathcal{C} \subseteq \mathcal{C}$.

Since each of the patterns 2413 and 3142 contains every length 3 non-monotone pattern, all four of the classes Av(132), Av(213), Av(132) and Av(312) are contained in $\mathrm{Av}(2413, 3142)$, and each of these has a characterization similar to one given by the following proposition.

Proposition 2.2.42. *([27]) The class* $\mathrm{Av}(231)$ *is the smallest nonempty class* $\mathcal{C}$ *which satisfies both* $\mathcal{C} \oplus \mathcal{C} \subseteq \mathcal{C}$ *and* $1 \ominus \mathcal{C} \subseteq \mathcal{C}$.

One more similar result deals with so-called *skew-merged permutations* defined below in Definition 6.1.7.

Proposition 2.2.43. *([27]) The class* $\mathrm{Av}(2143, 2413, 3142, 3412)$ *of separable skew-merged permutations is the smallest nonempty class* $\mathcal{C}$ *which contains* $\mathcal{C} \oplus 1$, $1 \oplus \mathcal{C}$, $\mathcal{C} \ominus 1$ *and* $1 \ominus \mathcal{C}$.

The following theorem is the main result in [27] by Albert et al.

Theorem 2.2.44. *([27]) If* $\mathcal{C}$ *is a subclass of the separable permutations that does not contain any of* Av(132), Av(213), Av(231) *or* Av(312) *then* $\mathcal{C}$ *has a rational generating function.*

It was conjectured by Shapiro and Getu and, for the first time, proved by West [801] that the set of separable permutations of length n, $\mathcal{S}_n(3142, 2413)$, is counted by the $(n-1)$-th Schröder number. The proof involves studying the generating tree for the restricted permutations and it uses a well-known relation between the Schröder numbers S_n and the Catalan numbers C_n:

$$S_n = \sum_{i=0}^{n} \binom{2n-i}{i} C_{n-i}. \tag{2.9}$$

Theorem 2.2.45. *([801, 734, 27, 260]) The set of separable permutations of length n, $\mathcal{S}_n(3142, 2413)$, is counted by the $(n-1)$-th Schröder number.*

West asked for a more natural proof of the enumerative result, which was provided by Stankova [734]. In a recent paper, Albert et al. [27] provided a rather simple proof of the same result using decompositions involving the $\oplus$ and $\ominus$ operations. Using the same approach, the authors also gave an alternative proof of the fact that Av(231) is counted by the Catalan numbers. Two other proofs of the same result are presented in [260] by Claesson et al. In those bijective proofs, the Schröder paths (counted by the Schröder numbers) are mapped bijectively to plane rooted trees where some of the leaves may be marked, and then two bijections are found between the marked trees and Av(3142, 2413), simply by showing how to generate the permutations using relation (A.4) in two different ways.

The presentation in the rest of the subsection is based on [260].

Formula (2.9) is a standard one for calculating the Schröder numbers, but we can use another formula, which appears in [679]:

$$S_n = \sum_{k=0}^{n} 2^k C_{n,k}, \tag{2.10}$$

where $C_{n,k}$ is the number of Dyck paths of length $2n$ with k peaks (see Subsection A.2.2 for definitions). Indeed, if one takes a Dyck path of length $2n$ with k peaks then there are 2^k ways to decide which of the peaks will be turned into a double horizontal step, hh, thus ending up with a Schröder path of length $2n$. This procedure is obviously reversible.

There is an easy and standard correspondence between plane rooted trees with k leaves and Dyck paths with k peaks: one traverses a tree from the root (located, say, on top) using the *leftmost depth first algorithm*, and each step down in the tree corresponds to an up step in the Dyck path, whereas each step up in the tree corresponds to a down step in the Dyck path. See Figure 2.26 for an example of this correspondence.

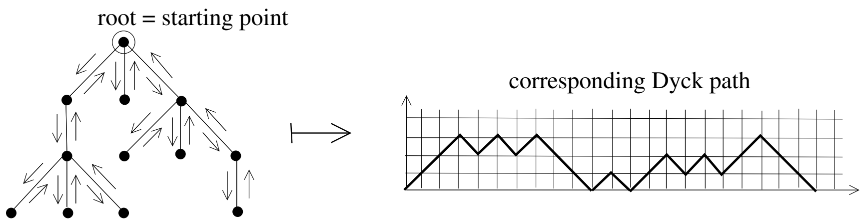


Figure 2.26: An example of a correspondence between plane rooted trees and Dyck paths.

To adapt the correspondence above for Schröder paths, we mark some of the leaves (maybe none, or all) in a tree with a star, which, once a marked leaf is reached, will instruct us to make a double horizontal step instead of creating a peak in the corresponding Schröder path. We call such trees *marked trees*. See Figure 2.27 for an example of the correspondence between Schröder paths and marked trees.

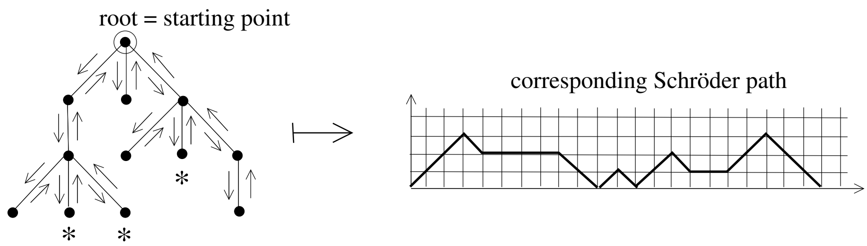


Figure 2.27: An example of a correspondence between plane rooted trees with marked leaves and Schröder paths.

We now interpret formula (A.4) ($S = 1 + hhS + uSdS$) generating the Schröder paths, as a generating relation for marked trees. Indeed, either a tree has one node (which cannot be marked by definition), or its root r has as its leftmost child a marked leaf (giving term hhS), or the leftmost child of r is the root of a tree (possibly a single node tree) and removing this tree leaves a tree with root r (this corresponds to the term $uSdS$ in (A.4)).

Using the interpretation above, we can easily see that all marked trees on n nodes can be generated from smaller marked trees using two operations: $\gamma_t^*(T)$ which adjoins to the tree T a marked leaf as the leftmost child of the root, and the $\oplus_t$ operation taking two trees as arguments and making the root of the left tree be the leftmost child of the root of the right tree (this adds an extra edge). In Figure 2.28, we show how to generate all marked trees on 3 nodes using the operations $\oplus_t$ and γ_t^*.

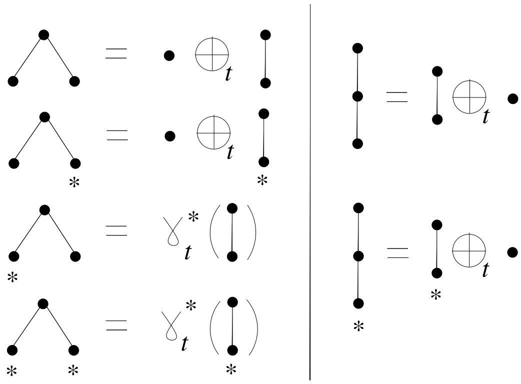


Figure 2.28: Generating all marked trees on 3 nodes.

The induced operation $\gamma_s^*(P)$ on Schröder paths corresponding to $\gamma_t^*(T)$ adjoins hh to the left of P. The operation $\oplus_s$ on paths corresponding to $\oplus_t$ on trees is defined as follows: for paths P_1 and P_2, $P_1 \oplus_s P_2$ is the Schröder path obtained by beginning with an up-step, then following P_1, then making a down-step and, finally, following P_2.

We distinguish two types of marked trees: trees of *type* 1 have the leftmost leaf marked, and all other trees are of *type* 2. Clearly, γ_t^* produces type 1 trees, while the type of $T_1 \oplus_t T_2$ is determined by T_1. Note that the induced definition for the Schröder paths is that if the leftmost increasing run of up-steps ends with a horizontal step (in particular, if a path begins with a horizontal step) then we have a *type* 1 *Schröder path*; otherwise we deal with a *type* 2 *Schröder path*. The number of type 1 trees/paths is easily seen to be the same as the number of type 2 trees/paths through a trivial bijection (involution) removing/adding a mark on the leftmost leaf for trees, and changing the leftmost peak to a double horizontal step and vice versa for paths. Thus the number of objects of each type is given by the *small Schröder numbers* (see Subsection A.2.1 for definition).

The statistic $\text{lpath}(T)$, the length of the leftmost path, for a plane rooted tree T is defined as for the $\beta(1,0)$-trees (see Table 2.1). We slightly change this definition for marked trees to define the statistic *lpath*$^*(T)$, the number of *non-marked* nodes on the leftmost path of a marked tree T below the root. Note that this statistic corresponds to the *length of the leftmost increasing run*, *lirun*, on the Schröder paths, that is, to the maximal number of the consecutive up-steps beginning a path. For the tree T in Figure 2.27, $lpath^*(T) = 3$ which matches the value $lirun(P) = 3$ for the corresponding Schröder path P, whereas in Figure 2.29, $lpath^*(T) = lirun(P) = 2$.

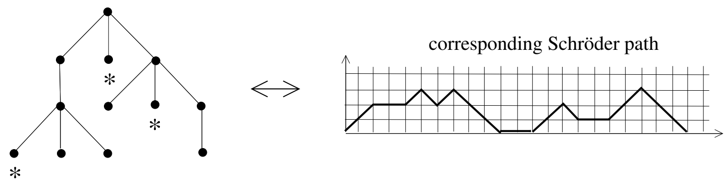


Figure 2.29: A marked tree and the Schröder path corresponding to it.

Another statistic of interest is 1 plus the number of marked leaves directly connected to the root in a marked tree, which we call $comp_t$, the *number of components*. This statistic obviously corresponds to 1 plus the number of hh steps on the ground level (x-axis) on the Schröder paths, which we call $comp_s$. For the tree T and the corresponding Schröder path P in Figure 2.27, we have $comp_t(T) = comp_s(P) = 1$, whereas in Figure 2.29, $comp_t(T) = comp_s(P) = 2$.

We can see that any $\pi \in \mathcal{S}_n(2413, 3142)$ has the following structure (see also Figure 2.30 for a schematic view of corresponding permutation matrices):

$$\pi = L_1 L_2 \cdots L_m n R_m R_{m-1} \cdots R_1 \tag{2.11}$$

where

- for $1 \leq i \leq m$, L_i and R_i are non-empty, with a possible exception for L_1 and R_m, separable permutations which are intervals in π;
- $L_1 < R_1 < L_2 < R_2 < \cdots < L_m < R_m$, where $A < B$, for two permutations A and B, means that each letter of A is less than every letter of B. In particular, L_1, if it is not empty, contains 1.

For example, if $\pi = 215643$ then $L_1 = 21$, $L_2 = 5$, $R_1 = 43$ and $R_2 = \emptyset$.

If $\pi \in \mathcal{S}_n(2413, 3142)$ and $\pi = L_1 L_2 \cdots L_m n R_m R_{m-1} \cdots R_1$, then the number of *left components* $\mathrm{lcomp}(\pi) = \mathrm{comp}(\mathrm{red}(L_1 L_2 \cdots L_m n))$, that is, lcomp gives the number of components to the left of the largest letter in a separable permutation. For example, $\mathrm{lcomp}(21376854) = 3$ which is the number of components in $\mathrm{red}(21376) = 21354 = 21 - 3 - 54$. Note that if the largest letter is the leftmost letter in a permutation, then lcomp is 0. In the bijection we will describe below, lcomp will correspond to lpath^* and to lirun.

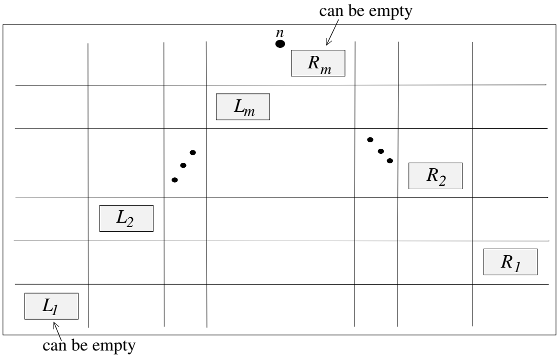


Figure 2.30: Schematic view of permutation matrices corresponding to separable permutations. Each L_i and R_j is a separable permutation.

We say that a separable permutation is of *type 1* if $L_1 = \emptyset$ in its decomposition (2.11), and it is of *type 2* otherwise. Clearly, by applying the reverse operation r, we see that the number of separable permutations of type 1 is the same as that of type 2 and is thus given by the small Schröder numbers.

Theorem 2.2.46. *([260]) There is a bijection between the separable permutations in $\mathcal{S}_{n+1}(2413, 3142)$ and the Schröder paths of length $2n$ (resp. plane rooted marked trees with n edges) such that the statistic* lcomp *on permutations corresponds to* lirun *on paths (resp.,* lpath* *on trees).*

Proof. We follow formula (A.4) to show a way to generate all separable permutations so that type 1 and type 2 permutations will correspond to type 1 and type 2 trees and paths, respectively. Also, from the generation it will be clear that lcomp on separable permutations will correspond to lpath* on marked trees and lirun on paths.

For separable permutations an analogue of the operations γ_t^*/γ_s^* on trees/paths, denoted γ_p^*, inserts the new largest letter in front of a given separable permutation. For example, $\gamma_p^*(21543) = 621543$. Clearly, this operation does not introduce any of the prohibited patterns. Also, it turns any permutation into a type 1 permutation. Finally, $\mathrm{lcomp}(\gamma_p^*(\pi)) = 0$ for any permutation π which agrees well with the behavior of lpath* and lirun under γ_t^* and γ_s^*, respectively.

We now introduce $\oplus_p$, an analogue of the $\oplus_t/\oplus_s$ operations, on separable permutations. Suppose $\pi = KnP$ and $\sigma = LR$ are two separable permutations where n is the largest letter in π and $L \neq \emptyset$ is the leftmost irreducible component of σ (it is possible that $R = \emptyset$). Then, by definition,

$$\pi \oplus_p \sigma = KL^+n^+R^+P,$$

where L^+ and R^+ are obtained from L and R, respectively, by increasing all of their letters by $|\pi|-1$ and $n^+ = |\pi|+|\sigma|$. For example, $23154 \oplus_p 21543 = 23165(10)9874$.

We now make a few remarks. First of all, the operation $\oplus_p$ on separable permutations does not introduce occurrences of the prohibited patterns since the resulting permutation has structure (2.11). Second, the operation is proper with respect to lengths. Indeed, we would like permutations of length $(n+1)$ to correspond to trees with n edges; if T_1 and T_2 are trees corresponding to permutations π and σ, respectively, then $T_1 \oplus_t T_2$ has one more edge than the number of edges in T_1 and T_2, which is consistent with the fact that $|\pi \oplus_p \sigma| = |\pi| + |\sigma|$ (π (resp., σ) has one more element than the number of edges in T_1 (resp., T_2)). Moreover, obviously $\mathrm{lcomp}(\pi \oplus_p \sigma) = \mathrm{lcomp}(\pi) + 1$ which agrees well with the $\oplus_t$ operation on trees and $\oplus_s$ on paths: for instance, for trees, this operation applied to trees T_1 and T_2 produces a tree with lpath* statistic one more than $\mathrm{lpath}^*(T_1)$. Finally, it is not hard to see that $\oplus_p$ is reversible like $\oplus_t$ and $\oplus_s$ are on trees and paths, respectively. □

Remark 2.2.47. Based on computer experiments, we get that the result in Theorem 2.2.46 is maximal with respect to statistics, in the sense that no additional statistics from Table A.1, and their variations under trivial bijections, can be preserved if we require the statistics lcomp, lpath*, and lirun to correspond to each other. However, we can modify the bijection in Theorem 2.2.46 by generating the separable permutations differently, to prove Theorem 2.2.48 dealing with other, even more natural statistics, and again, providing a maximal result with respect to statistics in Table A.1.

Theorem 2.2.48. *([260]) There is a bijection between the separable permutations in* $\mathcal{S}_{n+1}(2413, 3142)$ *and the Schröder paths of length* $2n$ *(resp. plane rooted marked trees with* n *edges) such that the statistic* comp *on permutations corresponds to* comp_s *on paths (resp.,* comp_t *on trees).*

Proof. Notice how the statistics $\mathrm{comp}_s/\mathrm{comp}_t$ on the Schröder paths/marked trees (counting hh steps on the ground level/marked leaves directly connected to the root) change while generating the objects: $\mathrm{comp}_s(\gamma_s^*(P)) = 1 + \mathrm{comp}_s(P)$, $\mathrm{comp}_s(P_1 \oplus_s P_2) = \mathrm{comp}_s(P_2)$, $\mathrm{comp}_t(\gamma_t^*(T)) = 1 + \mathrm{comp}_t(T)$, and $\mathrm{comp}_t(T_1 \oplus_t T_2) = \mathrm{comp}_t(T_2)$.

We now introduce the following modifications to γ_p^* and $\oplus_p$ defined in the proof of Theorem 2.2.46 which we will call γ_p^{**} and $\oplus_p'$, respectively. The operation γ_p^{**} inserts the new largest letter at the end of a given separable permutation. For example, $\gamma_p^{**}(21543) = 215436$. Notice that γ_p^{**} increases the number of components by 1 (the largest letter is a component by itself) as it is supposed to mimic the behavior of comp_s and comp_t.

Next, for two separable permutations $\pi \neq \emptyset$ and $\sigma = LnR \neq \emptyset$ (n is the maximum letter in σ) we define

$$\pi \oplus_p' \sigma = \begin{cases} L\pi^+n^+R & \textit{if } (n-1) \in R, \\ Ln^+\pi^+R & \textit{otherwise}, \end{cases}$$

where π^+ is obtained from π by increasing each of its letters by $|LR|$, while $n^+ = 1 + |L\pi R|$. In particular, if $\sigma = 1$, we use the second line in the definition. For example,

$$312 \oplus_p' 1423 = 1645723 \ \text{ and } \ 312 \oplus_p' 3412 = 3764512.$$

The outcome of the $\oplus_p'$ operation for two separable permutations is a separable permutation (which is easy to see since the structure is proper) with at least one letter to the right of the largest letter. Moreover, if $n-1$ was to the right of n in σ then the next largest letter is to the left of the largest letter in $\pi \oplus_p' \sigma$, and it is to the right of the largest letter in the sum otherwise. Thus, given $\tau \in \mathrm{Av}(2413, 3142)$, we can either conclude that it was obtained using γ_p^{**} if the largest element is the rightmost letter, or, depending on the position of the next largest letter we can easily find π and σ such that $\pi \oplus_p' \sigma = \tau$.

Finally, we can see that $\mathrm{comp}(\pi \oplus_p' \sigma) = \mathrm{comp}(\sigma)$ as desired. □

2.2.6 Schröder permutations

Kremer [546] showed that $s_n(1243, 2143)$ is given by the $(n-1)$-th Schröder number. For this reason, Egge and Mansour [338] called the set $\mathrm{Av}(1243, 2143)$ the *Schröder permutations*. However, Kremer [546] actually proved that ten inequivalent (modulo trivial bijections) classes of permutations are counted by the Schröder numbers. Representatives from these classes are given in Table 2.2.

Since we have

$$\mathrm{Av}(1324, 2314) = c(\mathrm{Av}(4231, 3241)) \text{ and } \mathrm{Av}(1324, 2314) = i.r(\mathrm{Av}(4231, 2431)),$$

we can see that class II contains the input- and output-restricted deque permutations introduced in Definition 2.1.10. Thus, one could call class II *deque-restricted permutations*. Class X represents the separable permutations. However, it is not so clear why class I should be called *the* Schröder permutations (class I is not any better than any other of the not yet mentioned classes counted by the Schröder numbers). In either case, since in the literature only classes I, II, and X seem to be studied, the following definition is justified.

I. Av(1234, 2134)	II. Av(1324, 2314)	III. Av(1342, 2341)
IV. Av(3124, 3214)	V. Av(3142, 3214)	VI. Av(3412, 3421)
VII. Av(1324, 2134)	VIII. Av(3124, 2314)	IX. Av(2134, 3124)
	X. Av(2413, 3142)	

Table 2.2: Classes of permutations counted by the Schröder numbers.

Definition 2.2.49. A permutation is a *Schröder permutation* if it avoids the patterns 1243 and 2143. Thus, Av(1243, 2143) is the class of all Schröder permutations.

Example 2.2.50. The permutation 264315 is a Schröder permutation, whereas 263514 is not as it contains an occurrence of the pattern 1243 (the subsequence 2354).

Remark 2.2.51. Once it comes to considering other, not yet studied classes of permutations counted by the Schröder numbers, one could invent something like the *Schröder permutation of the i-th kind*, where i is the class number in Table 2.2; following this scenario, for example, the *Schröder permutations of the first kind* would be simply the *Schröder permutations*.

Let us state a couple of results related to Schröder permutations coming from [338] by Egge and Mansour. Some of the open questions related to work in [338] are answered by Reifegerste [679] using so-called *essential sets* in permutation diagrams. We refer to [338, 679] for more results/details on that.

Recall from Chapter 1 that, for a pattern p, $p(\pi)$ denotes the number of occurrences of p in π. In particular, $12\cdots k(\pi)$ denotes the number of increasing subsequences of length k in π.

Theorem 2.2.52. *([338])*

$$\sum_{\pi\in \mathrm{Av}(1243,2143)} \prod_{k\geq 1} x_k^{12\cdots k(\pi)} = 1 + \cfrac{x_1}{1 - x_1 - \cfrac{x_1x_2}{1 - x_1x_2 - \cfrac{x_1x_2^2x_3}{1 - x_1x_2^2x_3 - \cdots}}}.$$

One of the specializations of the variables x_i leads to the following result.

Theorem 2.2.53. *([338]) For $k \geq 1$,*

$$\sum_{n\geq 0} s_n(1243, 2143, 12\cdots k)x^n = 1 + \cfrac{x}{1 - x - \cfrac{x}{1 - x - \cfrac{x}{1 - x - \cdots}}},$$

where the continued fraction has $k-1$ denominators.

Example 2.2.54. If $k=2$ in Theorem 2.2.53 then we get

$$\sum_{n\geq 0} s_n(1243, 2143, 12)x^n = 1 + \frac{x}{1-x} = 1 + x + x^2 + x^3 + \dots$$

which makes sense as for each n there is only one permutation, the decreasing permutation $n(n-1)\dots 1$, that avoids the pattern 12 (the other two prohibited patterns will be avoided automatically).

The formal power series in Theorem 2.2.53 admits another expression, in terms of the Chebyshev polynomials of the second kind (see Definition B.2.1), as shown in the following theorem.

Theorem 2.2.55. *([338]) For $k \geq 1$,*

$$\sum_{n\geq 0} s_n(1243, 2143, 12\cdots k)x^n = 1 + \frac{\sqrt{x}U_{k-2}\left(\frac{1-x}{2\sqrt{x}}\right)}{\left(U_{k-1}\frac{1-x}{2\sqrt{x}}\right)}.$$

A similar result is recorded in the following theorem.

Theorem 2.2.56. *([338]) For $k \geq 1$,*

$$\sum_{n\geq 0} s_n(1243, 2143, 2134\cdots k)x^n = 1 + \frac{\sqrt{x}U_{k-2}\left(\frac{1-x}{2\sqrt{x}}\right)}{U_{k-1}\left(\frac{1-x}{2\sqrt{x}}\right)}.$$

From Theorems 2.2.55 and 2.2.56,

$$\{1243, 2143, 12\cdots k\} \sim \{1243, 2143, 2134\cdots k\}$$

(these sets are Wilf-equivalent).

As particular cases of much more general theorems, the following theorem on restricted Schröder permutations is obtained.

Theorem 2.2.57. *([338, 679]) For $n \geq 2$,*

$$s_n(1243, 2143, 231) = (n+2)2^{n-3},$$

and for $n \geq 1$,

$$s_n(1243, 2143, 321) = n + 2\binom{n}{3}.$$

For a final example of results in [338], we state the following theorem.

Theorem 2.2.58. *([338])*

$$\sum_{\pi} x^{|\pi|} = \frac{x(1+x)(1-x)^2}{\left(U_{k-1}\left(\frac{1-x}{2\sqrt{x}}\right)\right)^2}$$

where the sum on the left is over all permutations in $\mathrm{Av}(1243, 2143)$ *which contain exactly one occurrence of the pattern* $2134\cdots k$*.*

We close the subsection by mentioning a result related to Class II in Table 2.2. A reason to do this is that Bandlow et al. [73] define "A Schröder permutation is a permutation that is both 4132- and 4231-avoiding". As a matter of fact, $\mathrm{Av}(4132, 4231) = r.c(\mathrm{Av}(4231, 3241))$, and $\mathrm{Av}(4231, 3241)$ is defined by us as the set of the input-restricted deque permutations (a particular case of deque-restricted permutations); thus, in our terminology, $\mathrm{Av}(4132, 4231)$ belongs to Class II, not to Class I as the permutations' name suggests in [73]. Keeping this little inconsistency in mind, we now describe the result.

As it is defined in Table A.1, the *inversion* statistic for permutations, denoted inv, is the number of pairs $i < j$ such that $\pi_i > \pi_j$ in a permutation $\pi = \pi_1\pi_2\ldots\pi_n$. For example, if $\pi = 42513$ then $\mathrm{inv}(\pi) = 6$ since each of the 2-letter subsequences 42, 41, 43, 21, 51, and 53 contributes 1 to the total value of the statistic. Any permutation statisic that is equidistributed with inv is said to be *Mahonian*. The generating function for the inversion statistic on $\mathcal{S}_n(4231, 4132)$ is defined as

$$I_n(q) = \sum_{\pi\in\mathcal{S}_n(4231,4132)} q^{\mathrm{inv}(\pi)}.$$

Given a Schröder path P, the *area* statistic, $a(P)$, is the number of full squares and "upper" triangles (equivalently, triangles whose sides do not coincide with a double horizontal step in P) that lie below the path and above the x-axis. The definition is best understood by looking at the path P in Figure 2.31 and convincing yourself that $a(P) = 27$.

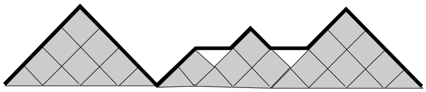


Figure 2.31: Illustration of the area statistic on a Schröder path.

Assuming S_n denotes the set of all Schröder paths on $2n$ steps, the generating function for the area statistic on Schröder paths is given by

$$\sum_{P\in S_n} q^{a(P)} = S_n(q)$$

and is known as the *Schröder polynomial* [152]. Specializing $q = 1$ in the Schröder polynomials gives usual Schröder numbers. Thus, the Schröder polynomials is a *q-analogue* to the Schröder numbers.

Using rather technical machinery, Barcucci et al. [81] show that

$$I_{n+1}(q) = S_n(q).$$

Bandlow et al. [73] give a constructive bijection from Schröder paths to $\mathrm{Av}(4231, 4132)$ that takes the area statistic on Schröder paths to the inversion number on permutations in $A(4231, 4132)$.

2.2.7 A hierarchy of permutation classes

Let us first summarize our knowledge of the following equinumerous objects: rooted non-separable planar maps, $\beta(1,0)$-trees, skew ternary trees, 2-stack sortable permutations, and non-separable permutations. Everything but the last row in Table 2.3 essentially came from the corresponding table in [476]: we refer to this paper and references therein for further details (all but one of the statistics for objects involved are defined above and in Table A.1). The last row came from [262] as a particular case of Theorem 2.2.10 stated in Subsection 2.2.3 when the reverse operation is applied to get non-separable permutations from $\mathrm{Av}(3142, 2\underline{41}3)$. Note that "nodes" has the same meaning as "vertices" in Table 2.3 as opposed to the corresponding table in [476] where "nodes" actually means "non-leaf vertices". Finally, in Table 2.3, there is dependence between n, i, and j: $n = i + j + 1$.

We now let the equinumerous objects in Table 2.3 form a layer in a hierarchy (by set inclusion) of sets of permutations avoiding vincular patterns based on the permutations 2413 and 3142 (see Figure 2.32). This hierarchy is considered in [260] by Claesson et al. and its basic idea is as follows. Consider the set of permutations $\mathrm{Av}(3\underline{14}2)$. We can make the restriction $3\underline{14}2$ stronger either by removing the underline thus arriving at the set $\mathrm{Av}(3142) \subseteq \mathrm{Av}(3\underline{14}2)$, or by adding an extra pattern to avoid, say, $2\underline{41}3$ thus arriving at the set $\mathrm{Av}(2\underline{41}3, 3\underline{14}2) \subseteq \mathrm{Av}(3\underline{14}2)$. Then we can build other sets of permutations shown in Figure 2.32 in the same way. We would have a slightly different picture if instead of $\mathrm{Av}(3\underline{14}2)$ we started with $\mathrm{Av}(2\underline{41}3)$: instead of the chain

$$\mathrm{Av}(2413, 3142) \subseteq \mathrm{Av}(3142, 2\underline{41}3) \subseteq \mathrm{Av}(3142) \subseteq \mathrm{Av}(3\underline{14}2)$$

presented in Figure 2.32, we would get the chain

$$\mathrm{Av}(2413, 3142) \subseteq \mathrm{Av}(2413) \subseteq \mathrm{Av}(2\underline{41}3).$$

rooted non-sep. planar maps	# edges $= n + 1$	# nodes $= i + 2$	# faces $= j + 2$	# cut-vert. after remov. root $= m$	# edges on outer face $= k + 1$
$\beta(1,0)$-trees	# nodes $= n + 1$	leaves $= i + 1$	int $= j + 1$	sub $= m + 1$	root $= k$
skew ternary trees	# nodes $= n$	even labels $= i + 1$	odd labels $= j$	first zeros $= m + 1$	zeros $= k$
2-stack sort. permutations	length $= n$	des $= i$	asc $= j$	see [423] for definition	rmax $= k$
non-separable permutations	length $= n$	des $= i$	asc $= j$	comp $.r$ $= m + 1$	rmax $= k$

Table 2.3: Statistics translated under bijections between rooted non-separable planar maps, $\beta(1,0)$-trees, skew ternary trees, 2-stack sortable permutations, and non-separable permutations.

Once the hierarchy on sets of permutations is built, we can add to each layer other combinatorial objects related to the pattern-restricted classes and discussed in this section to see a "big" picture of relations between objects and to enjoy the variety of structures involved. In Figure 2.32 we use "$\sim$" to show that one class of objects is equinumerous with another one, while "=" is used to show that the objects are actually the same. Finally, note that Figure 2.32 could accommodate more relevant objects, e.g. certain rectangulations equinumerous with the Baxter permutations and studied in [4, 379] (not discussed in this book), the deque-restricted permutations mentioned in Subsection 2.2.6, or any class of permutations, other than I, II, and X, in Table 2.2.

2.3 Schubert varieties and Kazhdan-Lusztig polynomials

We start by sketching several algebraic notions appearing in this section. However, if needed, one should consult other sources for precise definitions, for example, the book "Singular loci of Schubert varieties" by Billey and Lakshmibai [117].

A *Kazhdan-Lusztig polynomial* $P_{x,w}$ is a member of a family of integral polynomials introduced by Kazhdan and Lusztig [497] in 1979 (see [180] by Brenti for an introduction to the polynomials). These polynomials play an important role in Lie theory. Kazhdan and Lusztig originally defined the polynomials in terms of a complicated recurrence relation. While there are many uses for, and interpretations of,

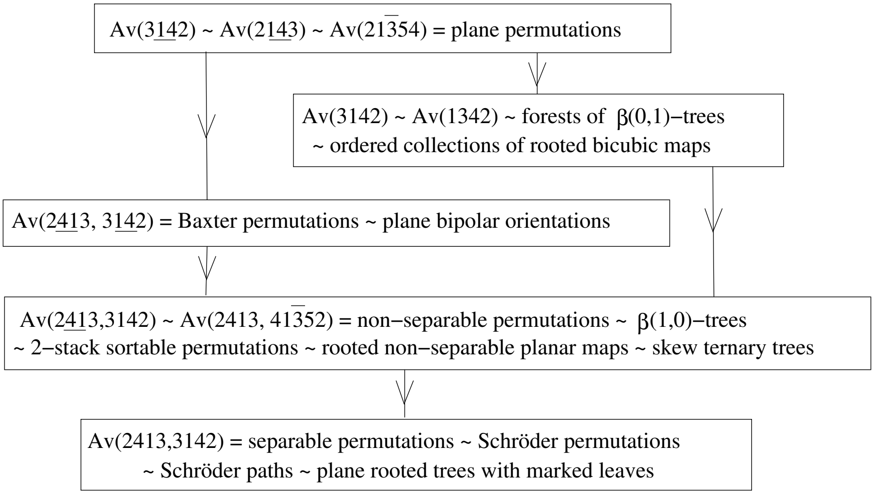


Figure 2.32: A hierarchy of permutation classes and related combinatorial objects.

Kazhdan-Lusztig polynomials, their combinatorial structure is not yet understood; in particular, there has been limited success in finding non-recursive formulas for them. However, for particular x and w, explicit formulas for Kazhdan-Lusztig polynomials can be obtained (see, e.g. [120] by Billey and Warrington for an overview of relevant results). One such particular case related to our pattern-avoidance is considered in Subsection 2.3.2.

An *algebraic variety* is the set of solutions of a system of polynomial equations. More precisely an algebraic variety is a space that is locally a set of solutions of a system of polynomial equations. Algebraic varieties are one of the central objects of study in algebraic geometry.

A *singularity* is a point at which a given object, e.g. a function, is not defined. *Local rings* are certain rings that are comparatively simple, and are used to describe what is called "local behaviour", in the sense of functions defined on varieties. The notion of local rings was introduced by Wolfgang Krull in 1938 under the name *Stellenringe.*

Below we will consider special subsets of the *flag variety* called *Schubert varieties*. A *flag* in $\mathbb{C}^n$ is an increasing sequence of subspaces in $\mathbb{C}^n$,

$$F_\bullet = \{0\} \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = \mathbb{C}^n,$$

such that $\dim F_i = i$. The flag variety $\mathcal{F}l_n(\mathbb{C})$ is the set of all such flags. There is also an alternative description of this set which goes as follows: Consider the general linear group Gl_n consisting of all invertible $n \times n$ matrices and let B be the subset of invertible upper triangular matrices. Then given a matrix M from Gl_n we can construct a flag by letting F_i be the span of the first i columns. Two matrices M_1 and M_2 will correspond to the same flag if and only if there is a matrix $N \in B$ such that $M_2 = N \cdot M_1$. Thus $\mathcal{F}l_n(\mathbb{C}) = \mathrm{Gl}_n/B$. For the next definition we assume we have fixed a basis $e_1, e_2, \ldots, e_n$ of $\mathbb{C}^n$ and fixed a *reference flag* $E_\bullet$ such that E_i is the span of the first i basis vectors.

Definition 2.3.1. For each permutation $\pi \in S_n$ we define a subset of flags

$$X_\pi = \{F_\bullet \,|\, \dim(F_p \cap E_q) \geq \#\{i \leq p \,|\, \pi(i) \leq q \text{ for } 1 \leq p \leq q \leq n\}, \forall\, p, q\},$$

called a *Schubert variety.*

Example 2.3.2. Consider now the flag variety $\mathcal{F}l_3(\mathbb{C})$ and the Schubert variety X_{231}. The only non-trivial dimension condition becomes

$$\dim(F_1 \cap E_2) \geq 1.$$

Geometrically, this implies that the line F_1 should lie in the plane E_2. This implies that $X_{231} \cong \mathbb{C}^2$ and is therefore two-dimensional. Note that the permutation 231 has two inversions and it is a general fact that the number of inversions equals the dimension of the corresponding variety.

Schubert varieties form one of the most important and best-studied classes of algebraic varieties, and they are often used to test conjectures about more general varieties. A certain measure of the singularity of Schubert varieties is provided by Kazhdan-Lusztig polynomials, which encode their local *Goresky-MacPherson intersection cohomology.* These varieties are indexed by permutations and many properties of the varieties are encoded in the patterns that the permutations either contain or avoid. We will discuss relevant results in the next subsection.

2.3.1 Schubert varieties

Theorem 2.3.3. *([706, 551, 807, 745]) For $\pi \in \mathcal{S}_n$, the variety X_π is* smooth *(i.e., has no singularities) if and only if* $\pi \in \mathrm{Av}(4231, 3412)$.

Following [160] by Bousquet-Mélou and Butler, which is influenced by Theorem 2.3.3, we give the following definition.

Definition 2.3.4. Any permutation in $\mathrm{Av}(4231, 3412)$ is called a *smooth permutation.*

A recurrence relation for counting smooth permutations is obtained by Stankova in [734] and the corresponding generating function is given by the following theorem.

Theorem 2.3.5. *([130, 160, 444]) The g.f. for $s_n(4231, 3412)$, the number of smooth n-permutations, is given by*

$$\frac{1 - 5x + 3x^2 + x^2\sqrt{1-4x}}{1 - 6x + 8x^2 - 4x^3}.$$

The following result was obtained by Bóna [130].

Theorem 2.3.6. *([130]) One has the following equinumeration result for five inequivalent (modulo trivial bijections) classes for $n \geq 0$:*

$$s_n(4231, 3412) = s_n(2431, 1342) = s_n(2431, 1423) = s_n(2431, 4132) = s_n(4231, 3142).$$

The generating function for these classes is given by Theorem 2.3.5 and there are no other inequivalent pairs of patterns that are Wilf-equivalent to the five above.

A weakening of smoothness is the notion of a *factorial* variety, which means that the local rings are unique factorization domains. Bousquet-Mélou and Butler [160] proved a conjecture by Yong and Woo on a characterization of factorial varieties. We state this result in the following theorem, where we use Proposition 1.3.7 to turn the barred pattern into the vincular one.

Theorem 2.3.7. *([160]) For $\pi \in \mathcal{S}_n$, the Schubert variety X_π is factorial if and only if*

$$\pi \in \mathrm{Av}(4231, 45\bar{3}12) = \mathrm{Av}(4231, 3\underline{41}2).$$

Remark 2.3.8. As it was remarked in [160] by Bousquet-Mélou and Butler, results of Cortez [284], and independently of Manivel [583], show that avoidance of 4231 and $45\bar{3}12$ characterizes *generically locally factorial Schubert varieties*, where *generic* has the following meaning: The variety is smooth at almost all points but has a closed subset Y_π where it is not smooth, and in that closed subset it is factorial at *almost* all points.

Remark 2.3.9. $\mathrm{Av}(4231, 45\bar{3}12) = \mathrm{Av}(4231, 3\underline{41}2)$ coincides with the class of *forest-like permutations* studied in [160]. We will not provide the original definition of forest-like permutations here, just saying that the definition is based on permutation matrices and certain drawings on them. Looking at the prohibited patterns, one sees that the class of smooth permutations is a subclass of the class of forest-like permutations (which is reminiscent of relations between some of the objects considered in Figure 2.32). However, there are three other subclasses of forest-like permutations, namely, *path-like permutations*, *tree-like permutations*, and *rooted tree-like permutations* (the last one is also a subclass of smooth permutations) all of which were enumerated in [160].

Theorem 2.3.10. *([160]) The g.f. for* $\mathrm{Av}(4231, 3\underline{41}2)$, *the forest-like permutations, is given by*

$$F(x) = \frac{(1-x)(1-4x-2x^2) - (1-5x)\sqrt{1-4x}}{2(1-5x+2x^2-x^3)}.$$

Gasharov and Reiner [414] defined a subclass of the factorial varieties that they name *defined by inclusions*. They described these varieties with a geometric condition and also with pattern-avoidance of four classical patterns (4231, 35142, 42513 and 351624). Úlfarsson and Woo [773] have shown that a relaxation of these conditions gives the Schubert varieties that are *local complete intersections*.

A further weakening is to only require that the local rings of X_π be *Gorenstein local rings*, in which case we say that X_π is a *Gorenstein variety*. Woo and Yong [809] gave a characterization of such varieties in terms of certain *Bruhat restrictions* with additional constraints. They also gave a characterization in terms of the avoidance of interval patterns [810]. However, Úlfarsson [772] provided a characterization of Gorenstein varieties in terms of bivincular patterns. To state the respective result (in Theorem 2.3.11), we need to define two infinite families, $\mathcal{G}_1$ and $\mathcal{G}_2$, of bivincular patterns.

- The family $\mathcal{G}_1$ is defined as

  $$\mathcal{G}_1 = \left(\genfrac{}{}{0pt}{}{1\overline{2345}}{53241}, \genfrac{}{}{0pt}{}{1\overline{234567}}{7432651}, \genfrac{}{}{0pt}{}{1\overline{23456789}}{954328761}, \cdots \right)$$

  The general member of this family is of the form

  $$\genfrac{}{}{0pt}{}{1\overline{2\cdots\cdots\cdots k}}{k\ell\cdots 2\cdots \ell{+}1 1},$$

  where $\ell = (k-3)/2$.

- The family $\mathcal{G}_2$ is defined as

  $$\mathcal{G}_2 = \left(\genfrac{}{}{0pt}{}{\overline{1234}5}{52431}, \genfrac{}{}{0pt}{}{\overline{123456}7}{7326541}, \genfrac{}{}{0pt}{}{\overline{12345678}9}{943287651}, \cdots \right).$$

  The general member of this family is of the form

  $$\genfrac{}{}{0pt}{}{\overline{12\cdots\cdots\cdots}k}{k\ell\cdots 2\cdots \ell{+}1 1},$$

  where $\ell = (k-1)/2$.

Note that the two families are the reverse complement of each other.

Theorem 2.3.11. *([772]) For $\pi \in \mathcal{S}_n$, the Schubert variety X_π is* Gorenstein *if and only if*

- *each Grassmannian permutation associated with π (see [772] for definitions) avoids every bivincular pattern in the families $\mathcal{G}_1$ and $\mathcal{G}_2$ defined above;*

- *π avoids the bivincular patterns $\begin{smallmatrix} 12\overline{34}5 \\ 3\underline{51}42 \end{smallmatrix}$ and $\begin{smallmatrix} 1\overline{23}45 \\ 42\underline{51}3 \end{smallmatrix}$.*

2.3.2 Kazhdan-Lusztig polynomials

Definition 2.3.12. Permutations in

$$\text{Av}(321, 46718235, 46781235, 56718234, 56781234)$$

are called 321-*hexagon-avoiding permutations*. The reason for this name is that if the *heap* of such a permutation is calculated, it does not contain a hexagon [120].

As we have already mentioned in the introduction to the section, the Kazhdan-Lusztig polynomials $P_{x,w}$ are defined in a complicated way, and finding explicit formulas for these polynomials for various x and w is a challenging task. Deodhar [295] proposes a combinatorial framework for determining the Kazhdan-Lusztig polynomials for an arbitrary *Coxeter group*. However, the algorithm is impractical for routine computations. On the other hand, the algorithm can be utilized efficiently to calculate $P_{x,w}$ in some cases, in particular, in the case of 321-hexagon-avoiding n-permutations w, as is shown in [120] by Billey and Warrington – an explicit description of the polynomials is obtained in these cases (we skip here most of the definitions and related results instead referring to the original source, [120]):

$$P_{x,w} = \sum q^{d(\sigma)},$$

where w is 321-hexagon-avoiding, $x \leq w$, $d(\sigma)$ is the *defect statistic*, the sum is over all *masks* σ on $\mathbf{a}$ whose product is x, and $\mathbf{a} = s_{i_1} s_{i_2} \cdots s_{i_r}$ is a reduced expression for $w \in \mathcal{S}_n$.

Definition 2.3.13. For the pattern 3412, the *height* of its occurrence in a permutation is the difference between the first and the last letters.

Example 2.3.14. There are four occurrences of the pattern 3412 in the permutation 461523: 4612, 4613, 4623 and 4523 of heights 2, 1, 1 and 1, respectively.

As is shown by Deodhar [294], $P_{id,w} = 1$ (for $\mathcal{S}_n$; *id* denotes the identity permutation) if and only if the Schubert variety X_w is smooth, and, more generally, $P_{u,w}(q) = 1$ if and only if X_w is smooth over the *Schubert cell* X_u°. The following theorem involving patterns was proved in [808] by Woo.

Theorem 2.3.15. *([808]) The Kazhdan-Lusztig polynomial for w satisfies $P_{id,w}(1) = 2$ if and only if the following two conditions are both satisfied:*

- *The* singular locus *of X_w has exactly one* irreducible component*;*
- *The permutation w avoids the patterns* 653421*,* 632541*,* 463152*,* 526413*,* 546213 *and* 465132*.*

More precisely, when these conditions are satisfied, $P_{id,w}(q) = 1 + q^h$ where h is the minimum height of a 3412 *occurrence, with $h = 1$ if no such occurrence exists.*

Finally we note that Billey and Postnikov [118] showed that pattern-avoidance can be extended to all Coxeter groups. This is done in terms of root subsystems and flattening maps. See also [115] by Billey and Braden.

For other materials relevant to this subsection, see [115, 121, 284, 496, 584].

2.4 A link to computational biology

In the last few decades, much has been done in the study of *genome evolution*, a research direction in *computational biology*. We refer to [164, 239] for some references on the biological aspects related to this section; we provide here almost no details on these. One of the many models for genome evolution, which take into account various biological phenomena, is the *tandem duplication-random loss model*, or simply the *duplication-loss model*. In this model, genomes are represented by permutations, that can evolve through *duplication-loss steps* representing the biological phenomenon that duplicates fragments of genomes, and then loses one copy of every duplicated gene. This model is well-studied in the biology literature, where it has been shown to be perhaps the most important rearrangement process in the case of *animal mitochondrial genomes*. For more on the biological motivation to study the duplication-loss model see [239] by Chaudhuri et al.

In this model, permutations can be modified by duplication-loss steps. Each of these steps is composed of two elementary operations, which are, for a given permutation π, as follows:

1. A factor (a fragment of consecutive letters) of π is duplicated, and the newly created factor is inserted immediately after the original copy; this is *tandem duplication*.
2. *Random loss* then takes place, which removes (exactly) one copy of *every* duplicated letter, resulting in a permutation.

For any duplication-loss step, the number of letters that are duplicated (the length of the duplicated factor) is called the *width* of the step.

Example 2.4.1. One step of a tandem duplication-random loss of width 3 applied to the permutation 123456 is as follows:

$$\begin{array}{ccccc} 12\underbrace{345}\underbrace{345}6 & \mapsto & 123\underline{4}5\underline{3}4\underline{5}6 & \mapsto & 123546 \\ \text{(tandem duplication)} & & \text{(random loss)} & & \end{array}$$

As is mentioned in [166, 164], the duplication-loss model can be viewed as a particular case of *permuting machines* that sort and generate permutations, and they are defined in [19] by Albert et al. We will consider permutations that are obtained from the permutation $12\cdots n$ after a given number r of duplication-loss steps.

Various duplication-loss models can be defined depending on a given so-called *cost function* c. Although it is intuitively clear that the cost of a duplication should be some non-decreasing function of the length of the duplication, it is not clear what exactly this function should be. We assume that the cost $c(k)$ of a duplication-loss step is dependent only on the width k of the step. In the original model of Chaudhuri et al. [239], $c(k) = \alpha^k$, for a parameter $\alpha \geq 1$. In [166] by Bouvel and Rossin, the cost function is defined by $c(k) = 1$ if $k \leq K$, $c(k) = \infty$ otherwise, for a parameter $K \in \mathbb{N}\backslash\{0,1\}$. In the model of [164] by Bouvel and Pergola, for all k, one has $c(k) = 1$, which is a special case of both the model of [239] (the case $\alpha = 1$) and the model of [166] (the case $K = \infty$). The model with $c(k) = 1$ is called the *whole genome duplication-random loss model*, which is motivated by the following argument: Since any step has cost 1, regardless of its width, we can, without loss of generality, assume that the whole permutation is duplicated at any step.

Pattern-avoidance is used in [166] to describe the set of permutations obtainable from an identity permutation after a number of duplication-loss steps of bounded width and we discuss it briefly in the following subsection. On the other hand, in the description of obtainable permutations in the whole genome duplication-random loss model, descents in permutations are involved, which are occurrences of the vincular (consecutive) pattern $\underline{21}$. Besides, consecutive 4-patterns are involved in a characterization of *minimal permutations with width d* (see Definition 2.4.7), objects involved in describing obtainable permutations in the whole genome duplication-random loss model which we discuss in Subsection 2.4.2. We refer to [239, 166] for algorithmic aspects related to the duplication-loss model. Finally, we refer to Definition 2.2.38 for the notion of a permutation class and to Section 8.1 for the notion of the *basis* of a permutation class.

2.4.1 Duplication-loss steps of bounded width

The main object of interest in [166] is given by the following definition.

Definition 2.4.2. Let $\mathcal{C}(K,r)$ denote the class of all permutations obtainable from $12\cdots n$ (for any n) after r duplication-loss steps of width at most K, for some constant parameters r and K.

Remark 2.4.3. Following remarks in [239, 166], the duplication-loss steps are not reversible and thus $\mathcal{C}(K,r)$ is *not* the class of permutations that can be sorted to $12\cdots n$ in r duplication-loss steps of width at most K.

Theorem 2.4.4. *([166]) $\mathcal{C}(K,1)$ is a class of pattern-avoiding permutations $\mathrm{Av}(B)$ whose basis B is finite of size $2^{K-1}+3$. More precisely, $B=\{321,3142,2143\}\cup D$, where D is the set of all permutations in $\mathcal{S}_{K+1}$ that do not start with 1 nor end with $K+1$, and contain exactly one descent.*

In the general case, Bouvel and Rossin [166] obtained the following result for $\mathcal{C}(K,r)$.

Theorem 2.4.5. *([166]) $\mathcal{C}(K,r)$ is a class of pattern-avoiding permutations whose basis is finite and contains patterns of size at most $(Kr+2)^2-2$.*

2.4.2 The whole genome duplication-random loss model

Bouvel and Pergola [164] proved the following characterization theorem.

Theorem 2.4.6. *([164]) The permutations that can be obtained in at most r steps in the whole genome duplication-random loss model are exactly those whose number of descents is at most 2^r-1.*

Definition 2.4.7. A permutation is *minimal with d descents* if removing any of its letters and taking the reduced form gives a permutation with fewer descents.

Example 2.4.8. The permutation $\pi=31254$ has 2 descents but it is not minimal with 2 descents as we can remove the letter 2 obtaining the permutation 2143 still having two descents. On the other hand, the permutation 642197385 is minimal with 6 descents.

Theorem 2.4.9. *([166, 164]) The class of permutations obtainable in at most r steps in the whole genome duplication-random loss model is a class of pattern-avoiding permutations whose basis is finite and is composed of the minimal permutations with 2^r descents.*

Taking into account the importance of minimal permutations with a specified number of descents in Theorem 2.4.9, we provide selected known facts on these permutations (consult [164] for more facts). A characterization of such permutations involving consecutive patterns is given in [164] by Bouvel and Pergola.

Theorem 2.4.10. *([164]) A permutation π is minimal with d descents if and only if it has exactly d descents and its ascents $\pi_i\pi_{i+1}$ are such that $2 \leq i \leq n-2$ and $\pi_{i-1}\pi_i\pi_{i+1}\pi_{i+2}$ forms an occurrence of either the pattern $\underline{2143}$ or the pattern $\underline{3142}$.*

Theorem 2.4.11. *([164]) The minimal permutations with d descents and of size*

- *$2d$ are enumerated by the d-th Catalan number $C_d = \frac{1}{d+1}\binom{2d}{d}$;*
- *$d+2$ are enumerated by $2^{d+2} - (d+1)(d+2) - 2$.*

Further studies of minimal permutations with d descents are carried out in [163] by Bouvel and Ferrari.

Patterns in Permutations and Words
Kitaev, S.
2011, XXII, 494 p., Hardcover
ISBN: 978-3-642-17332-5