I

The Great Prize, the framework

> Bertinck has a chest wound. After a while a fragment smashes away his chin, and the same fragment has sufficient force to tear open Leer's hip. Leer groans as he supports himself on his arm, he bleeds quickly, no one can help him. Like an emptying tube, after a couple of minutes, he collapses.
> What use is it to him now that he was such a good mathematician at school?
>
> Erich Maria Remarque, *All Quiet on the Western Front* [1928]

In this chapter, we describe the setting for the beginning of our story: the topic of the Great Prize of mathematical sciences, the historical context in mathematics and the historical context in general, namely the First World War. After a quick introduction of the various characters and of their roles, we establish some notation and give the first pertinent examples of the mathematical question under consideration, that is, the iteration of rational functions, setting it as it was when the story began.

I.1 The iteration problem in 1915

In 1915, The Paris Academy of Sciences announced that it would award in 1918 a "Great Prize of mathematical sciences". This was a prize, financed by the French State, of 3 ,000 Francs. The topic was published on December 27$^{\text{th}}$, 1915 in [Académie 1915, p. 921], and was presented as follows:

> The *iteration* of a substitution with one or several variables, namely, the construction of a system of successive points $P_1, P_2, \ldots, P_n, \ldots$, each of them being deduced from the previous one by the same operation:
>
> $$P_n = \varphi(P_{n-1}) \qquad (n = 1, 2, \ldots, \infty)$$
>
> (where φ depends rationally, say, on the point P_{n-1}) and such that the first point P_0 is also given, appears in several classical theories and in some of the most famous papers of Poincaré.

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> Up to now, the well known works devoted to this investigation are mainly about the "local" point of view.
> The Academy considers that it would be interesting to proceed from here to the examination of the whole domain of the values taken by the variables. In this spirit, it opens a competition, for the year 1918, on the following question:
> *To improve in an important point the investigation of the successive powers of a same substitution, the exponent in the power increasing indefinitely.*
> *One will consider the effect of the choice of the initial element P_0, the substitution being given, and it will be possible to limit the investigation to the simplest cases, such as that of rational substitutions of one variable.*[1]

It would be interesting to know exactly how and why this topic was chosen. We know that some of the Academicians thought initially of another

[1] L'*itération* d'une substitution à une ou plusieurs variables, c'est-à-dire la construction d'un système de points successifs $P_1, P_2, \dots, P_n, \dots$, dont chacun se déduit du précédent par une même opération donnée:

$$P_n = \varphi(P_{n-1}) \qquad (n = 1, 2, \dots, \infty)$$

(φ dépendant rationnellement, par exemple, du point P_{n-1}) et dont le premier P_0 est également donné, intervient dans plusieurs théories classiques et dans quelques-uns des plus célèbres Mémoires de Poincaré.

Jusqu'ici les travaux bien connus consacrés à cette étude concernent surtout le point de vue "local".

L'Académie estime qu'il y aurait intérêt à passer de là à l'examen du domaine entier des valeurs que peuvent prendre les variables. Dans cet esprit, elle met au concours, pour l'année 1918, la question suivante:

Perfectionner en un point important l'étude des puissances successives d'une même substitution, l'exposant de la puissance augmentant indéfiniment.

On considérera l'influence du choix de l'élément initial P_0, la substitution étant donnée, et l'on pourra se borner aux cas les plus simples, tels que les substitutions rationnelles à une variable.

subject, nothing less than Fermat's Theorem[2], before they chose iteration. It had happened before, and was even a common practice, that a prize topic was announced because it was known that such and such a mathematician had made great progress on this topic[3]. This was apparently not the case here, where it seems, on the contrary, that the Prize itself would stimulate new research in the field. It is however possible that one of the Academicians, for instance Hadamard, thought of Fatou's note [Fatou 1906d] and of the fact that the latter had done further work on the topic (without writing it up) , when declaring the subject (see the excerpt of the letter quoted on page 18). The allusion to Poincaré might come from the same source: "as everybody", Hadamard admired Poincaré a lot, but *he himself* organised a seminar on his work in 1913 and should have known this work quite well[4]. The fact (note 2) that Darboux, Jordan and Picard thought of proposing Fermat's theorem is a hint that none of the three was at the source of the actual subject.

Note that, in the statement of the subject, the question is on iteration *per se* (in modern and anachronistic terms, on dynamics): mathematicians had iterates for a very long time, for instance in order to find approximate solutions of algebraic equations, and more than one of them had carried out

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[2] A little note, signed Darboux, Jordan and Picard, Prizes file, archives of the Academy of Sciences, reads:

> The Academy of Sciences opens a competition on the proof of the celebrated theorem of Fermat about the impossibility of the equation
>
> $$x^n + y^n = z^n.$$
>
> We are mostly looking forward to progress in number theory that could lead to this proof.

We cannot resist saying here that, a few years later, on September 19$^{\text{th}}$ 1923, in a letter to Pierre Gauja, the "secretary-archivist" of the Academy of Sciences, Picard asked Gauja to write to a correspondent

> that the Academy never declared a competition on this question [Fermat's Theorem], if this is indeed the case [que l'Académie n'a jamais mis la question [le théorème de Fermat] au concours, si toutefois il en est bien ainsi]

(Picard file, archives of the Academy of Sciences).

[3] Among the most famous examples, are that of the Bordin Prize of 1888, for the progress made by Sophie Kowalevski on the question of the rigid body (see [Audin 2008]), and that of the Great Prize of mathematical sciences, which was declared at the end of 1890 because Stieltjes thought he had a proof of the Riemann hypothesis (and which was eventually awarded to Hadamard). For a concise but efficient history of the mathematical prizes, see [Gray 2006].

[4] Note besides that Hadamard was the author of the paper [Hadamard 1921] on Poincaré's mathematical work that would be published by *Acta Mathematica* in its special issue of 1921.

such an activity[5]. One of the most interesting aspects of the subject was the global, "general", as they said at that time, nature of the expected research. Let us quote now, to whet the appetite, the start of the report [Académie 1918, p. 811] written by Émile Picard and Georges Humbert[6] for the award ceremony of the Prize in 1918. They mentioned the history of the subject and clarified, for instance, the allusion to Poincaré's works[7]:

> The Academy declared a competition on the investigation of the *iteration* of a substitution, recalling that only the *local* point of view had been considered until then and it invited the competitors to take a *general* point of view.
> Previous work, especially the fundamental works of M. Kœnigs, for a substitution S, $z_1 = \varphi(z)$ of one variable, led to the notion of *points of attraction:* if ζ is a point that is fixed under S or one of its powers (an *invariant point*), and if a corresponding quantity, called the *multiplier*[8], has absolute value *less* than unity, all the successive transforms (*consequents*[9]) of a point z, taken in a neighbourhood of ζ, tend to ζ or periodically tend to p points, one of which is ζ and the others its first $(p-1)$ consequents.
> These initial results raised many problems: are the attracting points of limited number; what exactly is the domain of attraction of one of them; what division of the plane is associated with a given function $\varphi(z)$?
> On these fundamental questions, we had only a Note of M. Fatou (October 1906), where the author showed that, in some examples, it could happen

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[5] To give a flavour of this: the "Newton method" is, according to Cajori [1893, p. 363–366], due to Raphson, so that he calls it the "Newton-Raphson method"; as for Cayley, whom we shall have the opportunity to meet again, he calls this same method the "Newton-Fourier method". Newton, Raphson, Fourier, one could even add two pages by Galois [1962, p. 379], following an Appendix in Legendre's book on number theory [1955, Appendice, Section I] (see also [Galuzzi 2001])... but let's remain serious. See [Alexander 1994] for a prehistory of the subject, from Newton's method to what we are discussing here.

[6] Georges Humbert (1859–1921), a member of the Academy of Sciences from 1901, plays an important supporting role in our story and in Gaston Julia's life. We shall see him settle the argument in a priority quarrel (below, in December 1917). Let us point out also that he, together with Painlevé, would back Julia in his election as a member of the French Mathematical Society (SMF) on March 13th 1919. Julia would participate in the publication of Humbert's Works, with Pierre Humbert (1891–1953), the son of Georges Humbert and a mathematical contemporary of Julia. See also Note 23 in Chapter II.

[7] It seems clear that the Academy of Sciences makes no connection with Poincaré's work on Kleinian groups, which we shall have the opportunity to mention again (see Note 14 and § IV.5.b), and this despite the fact that Fatou had noticed, as early as 1906, the analogy with automorphic forms [Fatou 1906d].

[8] The definition of the word multiplier, together with other useful definitions, can be found in § I.4.

[9] The beautiful word consequent (conséquent in French), which our protagonists will use quite a lot, deserves to be defined: the consequents of a point z are its successive images $z_n = R^n(z)$ $(n \geq 1)$, its iterates.

> that the regions of the division are bounded by non-analytic curves, thus highlighting the difficulty and the complexity of the question.
> Finally, from another point of view, Poincaré had established that, in certain cases, it is possible to associate with S a function $\theta(u)$, meromorphic in the whole plane, such that, if one puts $z = \theta(u)$, one has $z_1 = \theta(su)$, s being a constant of absolute value *greater* than 1, thus reducing the study of the iteration to that of $\theta(u)$; but no application was made of this *parametric iteration* method[10].

The Note [Fatou 1906d]

It seems that the only global work that had been undertaken in this field before the publication of the Prize topic was indeed Pierre Fatou's note [1906d].

> Nobody had dared to tackle the question in the whole plane when, in 1906, in a short *Comptes rendus* note, M. Fatou, giving the example of the extraordinary results met, showed at once the interest and the high difficulty of doing so,

Hadamard would comment in 1921 in a report that we shall have the opportunity to quote several times[11]. In this note, Fatou investigated the rational

[10] L'Académie avait mis au concours l'étude de l'*itération* d'une substitution, en rappelant que le point de vue *local* avait seul été considéré jusqu'alors et en invitant les concurrents à se placer d'un point de vue *général*.

Les travaux antérieurs, notamment les travaux fondamentaux de M. Kœnigs, avaient, pour une substitution S, $z_1 = \varphi(z)$, à une variable, conduit à la notion des *points d'attraction:* si ζ est un point laissé fixe par S ou par une de ses puissances (*point invariant*), et si une quantité correspondante, dite *multiplicateur*, est de module *inférieur* à l'unité, les transformés successifs (*conséquents*) d'un point z, pris au voisinage de ζ, tendent tous vers ζ, ou tendent périodiquement vers p points, dont l'un est ζ, et dont les autres sont ses $(p-1)$ premiers conséquents.

Ces résultats initiaux soulevaient bien des problèmes: les points attractifs sont-ils en nombre limité; quel est le domaine exact d'attraction de l'un d'eux; quelle division du plan est ainsi associée à une fonction $\varphi(z)$ donnée?

Sur ces questions fondamentales, on ne possédait qu'une Note de M. Fatou (octobre 1906), où l'auteur montrait, sur des exemples, que les régions de la division pouvaient être limitées par des courbes non analytiques, mettant ainsi en évidence les difficultés et la complexité de la question.

Enfin, à un autre point de vue, Poincaré avait établi que, dans certains cas, on peut associer à S une fonction méromorphe dans tout le plan, $\theta(u)$ telle que, si l'on pose $z = \theta(u)$, on ait $z_1 = \theta(su)$, s étant une constante de module *supérieur* à 1, ce qui ramène l'étude de l'itération à celle de $\theta(u)$; mais aucune application n'avait été faite de cette méthode d'*itération paramétrique*.

[11] Hand-written report, July 4th 1921, Fatou file, archives of the Academy of Sciences. The unabridged text (together with the French original) can be found in the Appendix at the end of this book.

maps the unique attracting[12] orbit of which is a fixed point. Despite the attention directed to functional equations, the questions Fatou raised are stated in dynamical terms: an attracting fixed point attracts a whole neighbourhood; what do the boundaries of these various convergence domains look like? He proves (with some additional assumptions) that the iterates of a rational function with a unique fixed point converge to this point... except on a set, which he denotes by E and of which he proves that it is totally discontinuous (*i.e.* its connected components are points) and perfect (*i.e.* it is closed and without isolated points). See, more precisely, Example I.4.2 below. In addition to being the first global result, it seems that this is also one of the first times that what we now call general topology was used in the iteration problem. Fatou also considered, in the second part of this note, the case where the rational function has several limit points, and proved that the lines between their convergence domains are not, in general, analytic, proving that this is the case for $R(z) = (z^2 + z)/2$ (see Example I.4.3 below). Fatou did continue to work on the subject, as he wrote in a letter to Fréchet on February 10$^{\text{th}}$ 1907:

> [...] I have undertaken more extensive research on iteration; but I lack the energy to write it all up [...]

(see the complete letter and the French original on page 256)—given the very close relationship between Fréchet and Hadamard, it is very probable that the latter was aware of this.

Digression (on general topology). What we today call "general topology" arose from a part of "set theory"[13] or "point set theory"—*Mengenlehre*, since

[12] The word "attracting" did not exist in 1906, neither did it exist when the Prize was declared. But it was used in the report on the Prize: we shall see an efficient terminology being set up as the work on the subject progresses (see page 78). We remind ourselves of the definition in § I.4.

[13] Regarding set theory in France at that time, see the very interesting paper [Gispert 1995].

this theory was invented in Germany and in German[14]: the paternity of Cantor is generally acknowledged.

Even if the term did not yet exist, general topology was used in analysis, even in France, at least since the work of Poincaré, then Borel[15].

> The impression produced on us by the articles of M. Cantor is appalling; to read them seems to all of us a genuine torture, and while we pay tribute to his merit, while we recognise that he has opened a new field of research, none of us is tempted to follow him[16],

Hermite had written [Dugac 1985, p. 209], but this was already very old (1883).

Let us mention for instance, the Borel-Lebesgue theorem, which was called "a lemma in set theory" (in particular in [Julia 1918f]). The work of Borel, Baire and Lebesgue uses set theory quite a lot. Since we have not much space here for that, we direct the reader, for a more detailed history, to Taylor's papers [1982 ; 1985] on Fréchet, to the introduction [Purkert 2002] to the edition of the book [Hausdorff 1914] contained in the complete Works [Hausdorff 2002], and to [James 1999]. It was Bourbaki who, much later, would make the separation between set theory (cardinals, and so on) and general topology[17].

Among the books devoted to this theory, let us quote here those of Borel [1898] and Baire [1905], then that of Grace Chisholm Young and her husband [1906], written with the blessing of Cantor himself (see the letter Cantor wrote to Grace Chisholm that can be found in the 1972 Chelsea edition of that book), and that of Schoenflies [1913]. There was also a book by Sierpinski, in Polish, a part of which was translated into French, but only in 1928.

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[14] It should be noticed that, the same year, Mittag-Leffler published in "his" journal French translations of some of Cantor's papers (among which [1884]) and the article of Poincaré on Kleinian groups, in which the latter writes [1883, p. 78]:

> The vertices of various polygons R form *eine undendliche Punktmenge* P and, to get the line L, we must add to this *Punktmenge* its *erste Ableitung* P'. One sees that the line L is *eine perfekte und zusammenhängende Punktmenge.* It is in this sense that it is a line. [Les sommets de divers polygones R forment *eine undendliche Punktmenge* P et pour obtenir la ligne L, il faut ajouter à cette *Punktmenge* son *erste Ableitung* P'. On voit que la ligne L est *eine perfekte und zusammenhängende Punktmenge.* C'est en ce sens que c'est une ligne.]

A passage which shows that the terminology (derived, perfect, connected) existed only in German... and the subject of which is not unrelated to that of this book, so that it deserves to be quoted here (see also § IV.5.b).

[15] See his book [Borel 1898].

[16] L'impression que nous produisent les mémoires de M. Cantor est désolante; leur lecture nous semble à tous un véritable supplice, et en rendant hommage à son mérite, en reconnaissant qu'il a ouvert comme un nouveau champ de recherches, personne de nous n'est tenté de le suivre.

[17] In 1965, Denjoy [1980] would complain that the students who learned under Bourbaki hardly knew the notions of power, order, and transfinite.

Neither the notion of metric space, nor, *a fortiori*, that of topological space, appear in these books that are, indeed, books on set theory—in which powers and cardinals are important. The notions of derived set (due to Cantor under the name of *Ableitung*), of perfect set and of boundary are set out.

It seems that in France, the analysis course of Jordan at École polytechnique (at least its second edition [1893]) played an important role in the popularisation of this subject among young mathematicians (this book was used by the generations of Lebesgue, Baire, Fatou... and at least until the thirties). It was essential to understanding measure theory and the Lebesgue integral. This is what one of our protagonists, Pierre Fatou, said, at the very beginning of his thesis [1906c, p. 335]:

> The problem of the measure of sets was first tackled by M. G. Cantor; his definitions were clarified by M. Jordan in his course on analysis; but it is M. E. Borel [...][18]

In this book by Jordan, one finds the word "écart" (gap), that Fréchet would still use and that would become, with Hausdorff, our "distance"[19].

The courses given by Borel at ENS from the spring of 1897 also played a not insignificant role. About Borel again: his series *Collection de monographies sur la théorie des fonctions* published by Gauthier-Villars had a notable effect on the spreading of what would become general topology.

Let us mention also the article of Zoretti [1912], under the influence of Borel (whose Peccot course of 1901–1902 on meromorphic functions he transcribed), in the *Encyclopédie des sciences mathématiques*[20]—and in which he introduces the measure theory of Borel and Lebesgue. It is worth noticing that the original German edition has no article on this subject:

– there is an article (by Schoenflies) on set theory in Volume I (arithmetic), the French version of which was published in 1909 and adapted by Baire, but this is devoted more to cardinals and ordinals than to point sets,

– there is an article by Dehn and Heegaard on the *Analysis situs* in Volume III (geometry), which is more on "geometry of situation" than on general

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[18] Le problème de la mesure des ensembles a été abordé pour la première fois par M. G. Cantor; ses définitions ont été précisées par M. Jordan dans son cours d'analyse; mais c'est M. E. Borel [...]

[19] Regarding Jordan's analysis course, see also [Gispert 1983].

[20] This is the French edition, "written and published following the German edition under the direction of Jules Molk" of the *Encyclopädie der Mathematischen Wissenschaften mit Einschuss ihrer Anwendungen, Herausgegeben im Auftrage* (under the auspices) *des Akademien des Wissenschaften zu Göttingen, Leipzig, München und Wien, so wie unter Mitwirkung zahlreicher Fachgenossen* (with the collaboration of numerous scientists), started under the direction of Felix Klein, planned as an international collaboration... the French edition has been the only non-German edition to appear, before it stopped for good in 1916, because of the war. An example of international co-operation that was interrupted, brutally and for a long time.

topology—it seems to me that this paper had no analogue in the French version.

The additional chapter that contains Zoretti's paper is inserted between the adaptations of that of Pringsheim (on the fundamental principles of function theory) and that of Voss (on differential calculus). Slightly biased (!) information on the organisation of the writing of this encyclopaedia can be found in [Lebesgue 1991]. Baire also was asked by Molk to contribute. In a letter to Borel [1990], he complains he must read

> some spiels of Hausdorff to which the Germans attach such great importance[21].

If I am not mistaken, Lebesgue did not contribute to this work.

Besides, the book which is now considered as the first "true" topology book would be written by a German mathematician, namely Felix Hausdorff [1914] (and he would dedicate it to Georg Cantor, "creator of set theory"). The title is still *Grundzüge der Mengenlehre* (foundations of set theory). Of course, and even if he amply quotes Borel and even Baire, this book, which appeared in Germany in 1914, was not used in France at the time we are interested in: we shall see that the war would interrupt, for a long time, all communication between French and German mathematicians[22]. On the reception and the use of Hausdoff's work, see also § IV.2.

Hausdorff also mentioned the words "*Analysis situs*" (Latin root, as in the famous paper of Poincaré and in its no less famous supplements, and as in the chapter by Dehn and Heegaard cited above) and "Topologie" (Greek root[23]), neither of the two having already been adopted. The first one would become, more or less, algebraic topology, the second would have to take on the adjective "general" before it would replace "point sets"[24].

⋆

As this digression shows, the beginning of the 20$^{\text{th}}$ century is not a kind of appendix to a 19$^{\text{th}}$ century which would never finally terminate. The period

[21] certains topos de Hausdorff dont les Allemands font le plus grand cas.

[22] Also note that the Borel series *Collection de monographies* has no German contributor.

[23] It seems that the word "Topologie" was first used (at least first published) as early as in 1847 by the German mathematician Johann Benedict Listing [1847]. See a reproduction of its front page in [James 1999].

[24] The terminology "ensembles de points" has completely disappeared from French today, but the English *point set topology* persists in being synonymous with *general topology*.

just before the war was indeed, in mathematics as well[25], the beginning of a modern time, which war and its consequences would retard.

I.2 The protagonists around 1917–1918

The main people working on iteration between 1915 and 1918 were, in alphabetical order, Pierre Fatou (1878–1929), Gaston Julia (1893–1978) and Samuel Lattès (1873–1918), and also, as we shall see, the American mathematician Joseph Fels Ritt (1893–1951). The work of Fatou and Julia would make great use of the notion of *normal family*, due to Paul Montel (1876–1975). Here are a few words on these protagonists at that time, again in alphabetical order.

Pierre Fatou

Born in 1878 in Lorient (in Brittany), he entered the ENS in 1898, graduated in 1901, became assistant-astronomer ("astronome-adjoint") at the Paris Observatory, and passed his thesis in 1907. Of this thesis, he "often talked" with Henri Lebesgue [Lebesgue 1991, p. 112] (so that it is not surprising that he left his name to a lemma in integration theory). He was promoted to astronomer (permanent) in 1928 and died the following year. See Chapter V for portraits and for more information on his life and his work.

Gaston Julia

Born in 1893 in Sidi Bel Abbès, in Algeria, he entered (as the top-student) the ENS in 1911 (he was also ranked first at the École polytechnique) after only one year of preparation (while it usually takes two) at the lycée Janson de Sailly, and graduated in 1914. He was severely injured in the face (his nose was obliterated, his jaw was smashed) at "Chemin des Dames"[26] in 1915, had to undergo numerous operations (he was a "gueule cassée" (broken face), and would wear for the rest of his life a leather mask[27]). He passed his thesis in 1917 (this was on a different subject, the theory of forms, see page 64[28]), he was rewarded with the Bordin Prize of the Academy of Sciences and began to

[25] This is here an allusion to the modernity, for instance of a Picasso, a Schönberg or an Apollinaire. Thinking of the modernist mathematicians of the beginning of the 20$^{\text{th}}$ century, Borel, Baire, Lebesgue and Fatou, for instance, we quoted, as an opening to this book, another who was wounded in the head in the 1914–18 war, "trepanned under chloroform", and who was, indeed, a modernist.

[26] Several battles took place at Chemin des Dames (not far from Soissons, 150 km northeast of Paris), the most bloody of which was that of April 1917.

[27] There is a (later) photograph of Julia on page 210.

[28] The first paper published by Julia [1913] is called "On the singular lines of some analytical functions" [Sur les lignes singulières de certaines fonctions analytiques]. It appeared in Volume 41 of the *Bulletin* of the SMF... in the same volume two

work on the topic of the Prize we are discussing here. See Chapter VI to find out what happened next.

Samuel Lattès

Born in Nice in 1873, he entered the ENS in 1892 after only one year of preparation in Marseilles, graduated in 1895, defended his thesis in 1906, was professor at Toulouse University from 1911, died from typhoid fever during the summer of 1918. Regarding his work and his life, see Note 60 in this chapter, and Chapter II (where there is a photograph of Samuel Lattès), especially Note 65 and the references given there.

Paul Montel

Born in 1876, also in Nice, he entered the ENS in 1894, graduated in 1897, he liked to travel and to teach[29], he took his time before working on a thesis[30], passed it in 1907, taught in secondary schools then, from 1911, at the University of Paris. He was awarded the Gustave Roux Prize by the Academy of Sciences in 1913. He would die as an almost hundred-year-old, but we shall speak of him again (in Chapter VI). More precise biographic information can be found in the papers [Cassin 1966 ; Beer 1966].

As the readers will certainly have noticed, all the protagonists of this story, the son of a mechanic from Algeria (Julia), the sons of a photographer and a shopkeeper in Nice (Montel and Lattès), and the son of a Breton sailor as well (Fatou), all received the same scientific education, through preparatory classes and the École normale supérieure. We could think that they acquired the same knowledge, a common corpus—notice however that neither Lattès nor Montel benefitted from the courses of Borel at the École normale supérieure.

Digression (Reports on the theses). The reports on the theses of our protagonists (except for that of Julia, which was defended too late and of which we shall speak again in the next chapter) can be found in [Gispert 1991]: written by Painlevé on Fatou on p. 397 and Montel on p. 399, and by Hadamard on Lattès on p. 396 (Lattès' thesis was called "On the functional equations that define a curve or a surface that is invariant under a transformation" [Sur les équations fonctionnelles qui définissent une courbe ou une surface invariante par une transformation]).

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papers of Fatou [1913a ; 1913d] appeared a well, the title of one of them being "On the singular lines of analytic functions" [Sur les lignes singulières des fonctions analytiques]. The similarity between the titles is, taking the rest of the story into account, rather surprising, but it conceals deep differences: Julia was twenty and produced a classical work on complex analysis, as for Fatou, he proceeded in his using the Lebesgue integral.

29 One of his pupils in Poitiers in 1898 was Raoul Dautry, who became a politician and with whom he kept good relations for the rest of his life.

30 At the instigation of the historian Albert Mathiez, his fellow student at the ENS.

I.3 The war

The First World War has already made an appearance in this text, when we discussed the references on general topology. And indeed, this story takes place during the final two years of a war which was an absolute slaughter: it killed eight million people and produced six million disabled people, among whom were 1,400,000 French victims, that is, approximately one tenth of the male working population, and almost as many disabled, among whom were numerous "broken faces".

France sent its elite to the front line. The students of the French "grandes écoles" were most often in the infantry, in general with the rank of second lieutenant, a rank that put them at the head of their soldiers so that they were especially vulnerable. Both Gaston Julia and René Gateaux (of whom we shall speak more below) were infantry second lieutenants, the first in the 34$^{\text{th}}$ and the second in the 69$^{\text{th}}$ regiment. This was an effect of a 1905 law, called "the two year law"[31]. This "egalitarian" French policy had deeply unequal effects. If the proportion of mobilised soldiers who were killed was a dreadful 16,8%, this proportion was 30% for the infantry officers and 41% for the students of the ENS (figures given in [Audoin-Rouzeau 1992 ; Becker 1992]). It was also reckoned that 40% of the students who had registered at a French university in 1914 were killed or mutilated (figure from [Beaulieu 1990, p. 41]).

The other warring countries had different policies. This does not mean that the young German intellectuals were not called up, nor does this mean that they did not enlist. For instance, Richard Courant was wounded in the trenches [Reid 1976][32], Max Dehn enlisted and was in the war from 1915 to 1918, Heinz Hopf enlisted as well, was in the war as a lieutenant on the western front and was wounded twice, Emil Artin was enlisted in the Austrian army[33] (see also the examples of Siegel and Hasse on page 126), the narrator of *All Quiet on the Western Front* [Remarque 1928] was also a student. Let us mention also, briefly, the British case. The army engaged by the United Kingdom in the war was at first a professional one. It then called for volunteers. The first wave of conscription began after the *Military Service Act* of January 1916: single men from 18 to 41 were enlisted. This was of course

[31] Despite strong opposition from the socialists, the two year law was replaced, in July 1913, by a "three year law", which announced the forthcoming war and which modified the duration of military service and the age of call-up (20 instead of 21), but which did not modify the status of the students of the grandes écoles.

[32] See the same book, pages 47–49, for information and comments on the mobilisation in Göttingen at the beginning of the war.

[33] Max Dehn was rewarded by the *Ehrenkreuz*, a military decoration, but this did not prevent him in 1935 from having to leave, first his position at Frankfort, then Germany. See [Burde et al. 2002 ; Siegel 1978]. As for Artin, who was a teenager when he was enlisted, it was Hamburg that he would have to leave in 1937. See [Brauer 1967]. Regarding Heinz Hopf, see [Frei & Stammbach 1999]. He was a professor in Zurich from 1931.

very different from the situation of the French army. On the other hand, there were, for instance in Cambridge, during the whole war, active pacifists (often people who remembered the Boer war), conscientious objectors (often for religious reasons), and even, after the Military Service Act, associations fighting against conscription (Hardy was the secretary of one of these associations, *The Union of Democratic Control*)—such phenomena had no analogue in France after the start of the war. If this war slaughtered 800,000 British soldiers (among them the eldest son of Grace Chisholm and William Young, who were mentioned above) and if two million people were wounded (among them the mathematician Ralph Fowler, wounded at Gallipoli), this was not a massacre of the intellectual elite comparable to that which affected French students. See [Barrow-Green 2008].

But let us come back to the French mathematicians. Paul Lévy and Émile Borel served in the artillery, Maurice Fréchet, born in 1878, was mobilised with the rank of sergeant and served as an interpreter with the British troops (but on the front), André Bloch[34] was injured, René Thiry was injured twice and was taken as a prisoner, Louis Antoine, second lieutenant[35] in the 151st, was wounded three times and became blind, Louis Sartre, another 1911 student of the ENS, was taken prisoner, Paul Flamant, second lieutenant in the 77th, was wounded in Charleroi and taken prisoner, André Marchaud, a 1909 graduate of the ENS, second lieutenant in the 344th, was taken prisoner as early as August 20th 1914, Henri Mineur, who entered the ENS at eighteen in 1917, enlisted in the army.

Many young scientists died. According to [Guiraldenq 1999], of 265 students who entered the ENS between 1910 and 1913, 109 were killed (this is the 41% mentioned above)[36]. Of most of them, the names have been forgotten—they are nevertheless still visible, in golden letters engraved in the marble of the Memorial at the École normale supérieure, as well as on the yellowing pages of the yearbook of former students. One thinks for instance of the brilliant young mathematician René Gateaux, killed as early as October 3rd 1914, at the age of twenty-five, when he had not quite finished his thesis[37], of Joseph

[34] André Bloch and Paul Lévy are the only former students of the École polytechnique in our list of young mathematician soldiers, all the others being from the École normale. As was Julia, André Bloch was born in 1893. He entered the École polytechnique in 1912. For more information about this uncommon mathematician, who, by the way, was also a specialist in the Picard theorems that we mention here and there in this text, see [Cartan & Ferrand 1988].

[35] The ranks and the numbers of the regiments given here come from the yearbook of the association of former students of the École normale supérieure.

[36] I don't know the proportion of students of the École polytechnique who were killed. In his book [1970], Paul Lévy writes that the fact that he chose the École polytechnique (rather than the ENS) may have saved his life.

[37] Regarding René Gateaux' life, death and destiny, see the paper of Laurent Mazliak [2007]. The Academy of Sciences posthumously awarded the Francœur Prize to Gateaux in 1916.

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The Memorial, unveiled on December 9th 1923,
at the École normale supérieure

M. Gateaux, formerly a student of the École normale, then disciple of our illustrious correspondent M. Senator Volterra, who recognised his high value, also died for France, thus disappointing the legitimate expectations created by his first work [M. Gateaux, naguère élève de l'École normale, puis disciple de notre Illustre correspondant M. le Sénateur Volterra, qui en avait reconnu la haute valeur, est aussi mort pour la France, trompant ainsi les légitimes espérances que suscitaient ses premiers travaux]

Jordan [1916] would say during the public session in which the prizes were announced. Two of his posthumous papers [1919a ; 1919b] would be published by Hadamard and Paul Lévy, in the same volume as an article [Fatou 1919b] we shall have the opportunity to mention again.

Marty, a specialist in Fredholm equations, who died for France in 1914[38], of Roger Vidil, a 1911 student of the ENS as was Julia, who had written up the notes he took during the Peccot course of Châtelet [1913], killed near Arras on November 27th 1914 and reported missing until his body was identified in 1917, of Paul Lambert, an algebraist, ranked second at seventeen in the same year 1911, the author of a paper on Gauss integers, corporal in the 60th, killed at the front in 1915, of Jean Piglowski, who was always in a good mood and had written a small paper on the motion of projectiles in 1911, killed in in the Vosges on February 18th 1915, of Louis Néollier, second lieutenant in the 258th killed in 1914, whose disastrous agrégation oral exam was recounted by Lebesgue [1991, p. 308] (in 1913)[39], and who was still reported missing[40] after the armistice in 1918, or of Roger Félix, ranked first at the ENS, at seventeen, in 1916, who enlisted before call-up because he wanted to be with his classmates and who "died for France" shortly before the armistice[41].

This is how the destiny of these young people is related, at the Academy of Sciences, at the end of 1915:

> They were pacifists, so to speak, by purpose, because scientific works are, more than anything else, works of peace and quiet; they were also pacifists by reason, because an intelligence which has been charmed by the enchantments of Science, delighted by its wonders, refuses to understand that men should use all the resources of their mind to collect the most effective ways of killing one another. They enroled willingly among the disciples of the naïve school which pretended "to declare peace to the world".
> But now sounds the call to arms. The Homeland is attacked, and all these pacifists stand: no one will miss the call. Farewell! the quiet work in the laboratory; they are now only soldiers; they do not even think—and it might be a pity that nobody thought of it for them—to take advantage of their knowledge to obtain special positions; had they not been called to handle the shovel and the gun like their friends who just had left the file or the plough, they would think themselves demeaned[42]. They are all brothers; all

[38] See also Note 81 in Chapter VI.

[39] The algebraists Lambert and Vidil are mentioned in [Dubreil 1982]. Julia mentioned Lambert in a speech [1970, p. 169] he gave in 1950. Lambert, Piglowski and Néollier appear, very much alive, in [Lebesgue 1991]. The article [Piglowski 1911] of Piglowski is referenced in the *Jahrbuch über die Fortschritte der Mathematik*. I found mention of the paper [Lambert 1912] in the notice [Julia 1919a] which Gaston Julia devoted to his friend Paul Lambert.

[40] This war also caused numerous disappearances. The record of Louis Néollier in the database "memory of men" [mémoire des hommes] of the French Ministry of Defence says that he "died for France between September 20th and 26th 1914"—his body was never found and it was a judgement dated May 21st 1920 that declared him dead.

[41] Regarding Roger Félix, see the memoires of his sister in [Félix 2005].

[42] Let us listen to another point of view, that of Camille Marbo [1968, p. 165 and 172], regarding her adopted son Fernand Lebeau, a student at the ENS, a socialist who was an opponent of the war:

> will walk hand in hand, under the horizon blue uniform, the uniform of equality that mistakes them for the sky! Only, when the assault time comes, they will remember that the more educated have to set the example; they will be the first to jump on the embankments, the first to run to the barbed wires, the first to die [Perrier 1915, p. 803][43].

Note that the rhetoric that would accompany the mention of the war wound of Julia all the way through speeches by himself or by others that are collected in [Julia 1970] is very close to that which we have just had the opportunity to

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> As a physicist with a great future ahead of him, he was posted in a sound reconnaissance section [...] Without mentioning it to anybody, he managed to get a position of lieutenant in the infantry, like almost all his colleagues. During his first leave, he told me
>
> "We socialists, who want to work for harmony between peoples and peace, we decided to be sent to the front line in order to prove that we are as brave as anybody. Those who survive will have the right to speak loudly in front of the shirkers." [...]
>
> "Never forget it", Fernand told me.
> I never forgot it.

[En tant qu'agrégé physicien promis à un brillant avenir, il avait été affecté à une section de repérage par le son [...] Sans en parler à personne, il avait fait des démarches pour obtenir un poste de lieutenant d'infanterie, comme la presque totalité de ses camarades. À sa dernière permission, il m'avait dit: " Nous, socialistes, désireux de travailler pour l'entente des peuples et la paix, nous avons décidé de nous faire envoyer en première ligne afin de prouver que nous sommes aussi courageux que n'importe qui. Ceux qui survivront auront le droit de parler haut devant les embusqués." [...] "Ne l'oublie pas, m'avait dit Fernand". Je ne l'ai pas oublié.]

Numerous pacifists and socialists enlisted, like Henri Barbusse who described his experience in *Under Fire* (*le Feu*) [Barbusse 1916], Goncourt Prize 1916.

43 Ils étaient, pour ainsi dire, pacifistes par destination, parce que les œuvres scientifiques sont avant tout des œuvres de calme et de sérénité; ils l'étaient aussi, par raison, parce qu'une intelligence séduite par les enchantements de la Science, ravie par ses merveilles, se refuse à comprendre que des hommes emploient toutes les ressources de leur esprit à rassembler les plus sûrs moyens de s'entre-tuer. Volontiers, ils se rangeaient parmi les disciples de cette école naïve qui prétendait "déclarer la paix au monde".

Mais voilà que résonne l'appel aux armes. La Patrie est attaquée et tous ces pacifistes sont debout: pas un ne manquera à l'appel. Adieu! le tranquille travail du laboratoire; ils ne sont plus que des soldats désormais; ils ne songent même pas — et il fut peut-être dommage qu'on n'y ait pas songé pour eux — à se réclamer de leur savoir pour obtenir des postes spéciaux; ils croiraient déchoir s'ils n'étaient pas appelés à manier la pelle ou le fusil, tout comme les camarades qui viennent de déposer la lime ou de quitter la charrue. Tous sont frères; tous vont marcher la main dans la main, sous l'uniforme bleu horizon, l'uniforme d'égalité qui les confond avec le ciel! Seulement, quand arrivera l'heure de l'assaut, ils se souviendront que les plus instruits doivent l'exemple; ils seront les premiers à sauter sur les glacis, les premiers à courir aux fils barbelés, les premiers à mourir.

admire. More of a politician, Painlevé would say, in his victory speech [1918c, p. 799]:

> Ah! Gentlemen, the horrifying holocaust demanded by the monstrous Moloch erected by pangermanist ambition before he was annihilated! However stoical we decide to be, our heart becomes heavy when we think of our deserted laboratories, of our university chairs from which eloquent and deep voices will not be heard anymore, of so many young and powerful brains whose fertile thought was interrupted forever by a stupid iron fragment. Our grandes Écoles, the breeding grounds of our engineers and scientists,—École polytechnique, École centrale, École normale supérieure, to quote only these,—how empty will be their audience[44] when they meet for the first time![45]

From this evocation, we understand that a whole generation of intellectuals, of scientists and, for what interests us here, a generation of mathematicians, was sacrificed and decimated. According to [Leloup 2009], the last thesis in mathematics defended in Paris by a French mathematician in 1914 was that of Georges Valiron, on June 20$^{\text{th}}$, there were three of them in 1915, including that of Joseph Pérès, one in 1916, then the next one was that of Julia, of which we shall speak at length later (see page 64), in December 1917. There was then only one in 1918, that of Pierre Humbert (Georges Humbert's son) on June 18$^{\text{th}}$ 1918 (in Paris)[46]. One should note that he was wounded during the war and, like Julia, started his research again after his injury.

Fathers bury their sons. Julia and the mathematicians

This is one of the reasons why Gaston Julia, a brilliant young former student of the ENS, seriously and atrociously injured on January 25$^{\text{th}}$ 1915, was the darling child of the mathematicians of the previous generation.

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[44] Let us quote Camille Marbo again [1968, p. 171]:

> Back in his position of scientific director [at the ENS], Émile Borel found the school filled with ghosts. [Revenu prendre son poste de directeur scientifique [à l'ENS], Émile Borel trouva l'École peuplée d'ombres.]

[45] Ah! Messieurs, l'effroyable holocauste qu'a exigé, avant d'être anéanti, le Moloch monstrueux dressé par l'ambition pangermaniste! Si stoïques que nous voulions être, notre cœur se serre quand nous songeons à nos laboratoires déserts, à nos chaires où des voix éloquentes et graves ne se feront plus entendre, à tant de cerveaux jeunes et puissants dont un éclat de fer stupide a interrompu pour jamais la pensée féconde. Nos grandes Écoles, pépinières de nos ingénieurs et de nos savants, — École polytechnique, École centrale, École normale supérieure, pour ne citer que celles-là, — quels vides présenteront leurs auditoires quand ils se réuniront pour la première fois!

[46] I owe this information to Juliette Leloup.

He was seen as a substitute son for a lot of them, who had lost theirs. Hadamard, for instance, lost two of his sons[47], Jordan three sons and one grandson.

Closer to Julia, let us mention Émile Borel, who was the scientific director of the ENS, and who lost his nephew and adopted son, also a former student of the ENS, in the storm on September 29th 1915 (Fernand Lebeau has already been mentioned in Note 42, see the memories book [Marbo 1968]), Émile Picard, whose elder son was killed[48] in Crouy in January 1915 (just before Julia was wounded), Georges Humbert, whose son was wounded. All took it in turns at his bedside in the military hospital of Val-de-Grâce.

The atmosphere in which Julia's work took place is perfectly described in the report that Picard would deliver at the end of the viva for his thesis (see page 67): through Julia, the sacrificed generation[49] would be glorified[50].

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[47] While their mother, Louise Hadamard, was a nurse, Pierre Hadamard, a student of the École polytechnique, was killed in February 1916 in Verdun at the age of 22, and Étienne Hadamard, who passed the exam to enter the École centrale, was killed in June 1916, also in Verdun, at the age of 19. Let us add that the third son of Jacques and Louise Hadamard, Mathieu, a member of the FFL (the French resistance army), would be killed in 1944. See [Maz'ya & Shaposhnikova 1998].

[48] The elder son of Picard, Charles (1884–1915), was killed in Crouy, near Soissons, on January 8th 1915; his youngest daughter Madeleine (1892–1915) was killed during the war as well, she was a nurse. A few years later, Picard would lose his last son Henry (1886–1926) killed by tuberculosis. Two daughters would survive him, Jeanne and Suzanne (Picard's biographical file, archives of the Academy of Sciences). Jeanne and Charles Picard (born in 1882 and 1884) were childhood friends (and cousins) of Marguerite Borel (born in 1883), see her book [Marbo 1968].

[49] A more detailed discussion of the notion of "generation" (and of the generation in question here) is given in [Sirinelli 1992].

[50] Julia played very well the role of a representative of this generation. The way Paul Dubreil [1950, p. 149], a 1923 graduate of the ENS, remembered the inauguration of the Memorial shown on page 26 shows this:

> Shortly after the start of the academic year, the "École" was awarded the War Cross, during a ceremony in which the sacrifice of its Sons, dead and alive, was exalted. When Dupuy finished reading the long list of those who fell, we saw you [Dubreil is addressing Julia] move forward to receive the Cross of the École and to carry it to the Memorial. Among the memories I have that are related to the 1914–1918 war, this is one of the deepest; it still gives me a striking impression of seriousness and greatness. [Peu après la rentrée, l'École, cette année-là, reçut la Croix de Guerre, au cours d'une cérémonie où fut exalté le sacrifice de ses Fils, morts et vivants. Quand Dupuy eut fini de lire la longue liste de ceux qui étaient tombés, nous vous [Dubreil s'adresse à Julia] vîmes vous avancer pour recevoir la Croix de l'École et la porter au Monument aux Morts. Dans ceux de mes souvenirs qui se rapportent à la guerre 1914–1918, celui-ci est l'un des plus profonds; il me laisse encore une impression saisissante de gravité et de grandeur.]

Volume 6 of the Complete Works of Julia [1970], the one in which his speeches and non-mathematical texts are collected, is full of hints of the fact that Picard was, for Julia, a fatherly figure. The very catholic Gaston Julia loved to compare Picard with Saint Christopher (see for instance [Julia 1970, p. 50 and 262]). One can find in the archives of the Academy of Sciences a letter sent by Julia to Picard in February 1936 to give him an account of the state of the International Mathematical Union (IMU) (Julia was a member of the Commission in charge of the preparation of the refounding of the IMU at the Oslo conference[51]), a letter the main part of which is devoted to a description of the pain and health problems of Julia. The way Julia, who, at the time of this letter, was a man of 43, complains, shows the kind of relationship he had with his correspondent. In addition, Julia was the one who gave a speech "on behalf of the students of M. Émile Picard" during the ceremony of awarding the medal of the Mittag-Leffler Institute to Picard on July 6th 1937 (see [Julia 1970, p. 39]). It is probable that, conversely, Julia was a kind of son for Picard.

There was a similar relationship between Julia and Borel. All the students of the ENS, and in particular Julia, who was writing up Borel's lectures, were, even before the war, the "children" of the assistant director Borel[52]. The letters of Julia to Borel[53] show that Borel sent money to Julia, in 1914, to help him to equip himself before leaving for the front. Borel was also, for quite a long time, a kind of father for Julia[54].

War effort

Montel, who finished his military service in 1898 with the rank of corporal [Cassin 1966] was mobilised, despite some "eye problems":

> Montel writes to me that his eyes begin again to worry him[55]

Lebesgue [1991, p. 144] wrote on March 23th 1906, and again on October 7th 1910,

> The state of Montel's eyes has not improved, on the contrary[56] [Lebesgue 1991, p. 268].

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[51] Regarding the IMU, see also page 209.

[52] Regarding the family atmosphere at the ENS, invitation of the students by the scientific director, and so on, see again [Marbo 1968].

[53] Borel Collection, archives of the Academy of Sciences.

[54] These mathematicians constituted a kind of "parental generation", even if it is not absolutely correct, in terms of their influence, to put Borel and Hadamard in the same generation as Picard and Jordan. Actually, Borel, as the scientific director of the ENS and with his *Collection de monographies sur la théorie des fonctions*, and Hadamard, setting up his legendary seminar at the Collège de France in 1913, would have a more direct scientific influence than Picard and Jordan.

[55] Montel m'écrit que ses yeux recommencent à l'inquiéter

[56] L'état des yeux de Montel ne s'est pas amélioré au contraire

After a few months, Painlevé called him to the Department of Inventions and he contributed to the war effort by Lebesgue's side (see below).

Other mathematicians were not called up, Baire for instance, due to failing psychological health[57], Denjoy, who was declared unfit for military life because of his very bad eyesight, was only called up for the auxiliary service [Cartan 1974] and worked on the mathematical problems of ballistics (see below)[58]. The two other protagonists of the story of iteration, Fatou and Lattès, did not fight in the war.

Pierre Fatou had a very weak constitution (see Chapter V and the references there), which explains why he was not called up.

Samuel Lattès also had some health problems. For him too, we have a testimony of Lebesgue. In 1910, he wrote about the question of academic positions:

> Lattès' health is such that he might not agree to go to Clermont, but it should be offered to him[59] [Lebesgue 1991, p. 251][60],

and, a few days later:

> Lattès is neurasthenic[61], true, but not in his work [Lebesgue 1991, p. 256].[62]

It seems that he stayed in Toulouse for the whole duration of the war. If he contributed to the war effort and how, we do not know.

⋆

This is not the place to investigate in great detail and systematically how the scientists contributed to the war effort. This question goes far beyond the scope of the present study and deserves to be left to "real" historians—who do indeed investigate it: see the very interesting study [Aubin & Bret 2003] and, more specifically on mathematics and mathematicians, the works of the "project on the history of mathematics" of the Institut mathématique de Jussieu (for instance [Mazliak 2007 ; Goldstein 2009]).

We nevertheless mention briefly (based solely on published sources which are easily accessible[63] and which concern, for the most part, mathematicians

[57] Declared unfit for duty after a few weeks of military service at the end of 1897, he was exempted permanently on December 1st 1914 (see [Dugac 1990]).

[58] According to [Choquet 1975], he was sent on a mission to Utrecht in 1917 and was torpedoed twice on the way there.

[59] La santé de Lattès est telle qu'il n'acceptera peut-être pas d'aller à Clermont, mais on devrait le lui offrir

[60] Lattès was a teacher in a secondary school in Algiers, then in Dijon, in preparatory classes in Nice, then in Aix after a sick leave. He was then, after his thesis in 1906, appointed in Montpellier in 1908, "chargé de cours" (assistant-professor) at the Faculty of sciences in Besançon in 1911 and eventually, a few months later, professor in Toulouse.

[61] In a letter to Borel, dated March 11th 1902, Baire described Lattès as a "fellow in neurasthenia" [confrère en neurasthénie] [Baire 1990, p. 51].

[62] Lattès est neurasthénique, soit, mais pas dans ses travaux.

[63] To browse the *Comptes rendus* is rather informative. In 1915, the following titles can be found:

who appear elsewhere in this text) that some scientists, and in particular some mathematicians, contributed to the war effort without actually fighting in the war. The mobilisation started in France on August 2nd 1914. As early as August 3rd, the President of the Academy of Sciences (who, in 1914, was Paul Appell) declared:

> My dear Fellows
>
> In the serious situation the country is facing, I am certain that I am the spokesman of all the Members of the Academy who have not been called up to a public service, when I declare, on their behalf, that they are at the disposal of the Government to help in the national defence, each within his own special field[64].

And our commentator Perrier, who was the president the following year, confirmed, in his inexhaustible speech [1915, p. 809]:

> Since August 3rd 1914, its members [of the Academy of Sciences] split up into four big Commissions corresponding to their specific competence, with respect to the various aspects of the war. A Mechanics Commission prepares for the study of the possible improvements of the Air Force, electric traction, destruction of barbed wires and even of artillery. Numerous and especially delicate are the problems the Physics Commission deals with. The Chemistry Commission prepares to know everything concerning explosives and gas, either tear gas, suffocating gas, or deadly gas, by the use of which the Germans have managed to deepen further the barbarity of their war; it thinks of organising, not without feeling nauseous, the way of paying back dishonourable enemies, but who should nevertheless be controlled by their own means, gas for gas[65], asphyxiation for asphyxiation, as our threatened soldiers and the neutrals themselves urge us to do [... the fourth Commission is that of hygiene, health and diet][66].

New treatments of the injuries of nerves by projectiles, On the ration of the soldier in wartime, On the wounds of the external genital organs, Feeding armies in campaign, Radioscopic methods to locate projectiles, On an induction device for searching for projectiles...

During 1916, just Ernest Esclangon published three notes under the titles:

On the air trajectories of projectiles, On cannon shots and silence zones, On the Doppler principle and the whistling of projectiles.

64 Mes chers Confrères,

Dans la situation grave où se trouve la Patrie, je suis assuré d'être l'interprète de tous les Membres de l'Académie non mobilisés dans un service public, en déclarant en leur nom qu'ils se tiennent à la disposition du Gouvernement, pour aider à la défense nationale, chacun selon sa spécialité.

65 There was, in the ENS, a chemistry laboratory that produced deadly gas [Lebesgue 1991, note 996].

66 Dès le 3 août 1914, ses membres [de l'Acadmie des sciences] se répartissent en quatre grandes Commissions correspondant à leurs compétences particulières, relativement aux divers aspects de la guerre. Une Commission de Mécanique s'apprête à étudier les perfectionnements qui peuvent être apportés à l'Aviation,

As we shall see (in § II.2), at the beginning of 1918, Painlevé would use a more sober style to describe this contribution of the Academy of Sciences. In what we have already called his "victory speech" [1918c, p. 808], he draws up a list and recalls the "mobilisation of science":

> All the problems raised by the war, on land, on sea or in the air, the war of mines, the submarine war, all the attack and defence methods in the war of trenches, and so on, have been studied, explored, by a multitude of researchers, scientists, engineers, workers. Applications and improvement of the T.S.F. [wireless transmission (radio)]; long distance land communications; sound reconnaissance of enemy batteries and saps; radiowave tracking or guiding of dirigibles and aeroplanes; reconnaissance of enemy positions by aerial photographs; new explosives; smoke projectiles; toxic gas (as attack or protection means); aircraft engines; trench mortar shells; infantry cannons; aeroplane cannons; and lastly tanks[67], all subjects (and how many subjects do I not forget!) that required the intervention of the most diverse intelligence and to which all the sciences contributed: chemistry, mechanics, thermodynamics, optics, acoustics, electricity, meteorology, up to the investigation of new problems the interest of which will appear in the future. The most abstract or the most subtle mathematics contributed to the solution of reconnaissance problems and to the computation of very new range tables which increased by 25 per cent the efficiency of the artillery.[68]

à la traction électrique ou à vapeur, à la destruction des fils barbelés ou même à l'Artillerie. Nombreux et particulièrement délicats sont les problèmes qui doivent occuper la Commission de Physique. Celle de Chimie se dispose à connaître de tout ce qui concerne les explosifs et ces gaz lacrymogènes, asphyxiants ou meurtriers par l'emploi desquels les Allemands ont trouvé moyen d'avilir encore la barbarie de leur guerre; elle songe à organiser, non sans un haut-le-cœur, les moyens de rendre à des ennemis déshonorés, mais qu'il fallait cependant contenir par leurs propres moyens, gaz pour gaz, asphyxie pour asphyxie, comme le réclament instamment nos soldats menacés et les neutres eux-mêmes

[67] The production of tanks was determined in June 1917 and played a decisive role in the last phase of the war. Jules Breton, who would be elected as a free Academician in 1920, contributed, at the Department of Inventions, to developing this vehicle. Regarding this topic, see the speech [Perrier 1940] of the President of the Academy of Sciences, another Perrier, during another war.

[68] Tous les problèmes que posent la guerre sur terre, sur mer ou dans les airs, la guerre de mines, la guerre sous-marine, tous les moyens d'attaque et de défense dans la guerre de tranchées, etc., ont été étudiés, fouillés, approfondis par une multitude de chercheurs, savants, ingénieurs, artisans, ouvriers. Applications et perfectionnements de la T.S.F.; communications à distance par le sol; repérage par le son des batteries ou des sapes ennemies; repérage ou guidage par les ondes hertziennes des dirigeables ou des avions; repérage des positions ennemies par photographies aériennes; explosifs nouveaux; projectiles fumigènes; gaz toxiques (moyens d'attaque ou de protection); moteurs d'avions; mortiers de tranchées; canons d'infanterie; canons d'avions; enfin tanks, autant de sujets (et combien j'en oublie!) qui ont sollicité les intelligences les plus diverses et mis à contribution toutes les sciences: chimie, mécanique, thermodynamique, optique, acoustique,

Let us thus come to the mathematicians[69]. Here is, for instance, what Montel [1941] said in his obituary of Lebesgue[70] in 1941:

> During the 1914–1918 war, he chaired the Mathematics Commission of the Service of Inventions, Investigations and Scientific Experiments the director of which is our fellow member M. Maurain[71] in this Department of Inventions which was created by Painlevé. With a tireless energy, he worked on the solution of the problems raised by the computation and correction of the trajectories of projectiles, sound reconnaissance, and so on. With the help of a large team of volunteers, he prepares a triple entry collection of trajectories[72], which would be used, by interpolation, for the fast computation of range tables[73].

This Commission was created by Painlevé, then the minister of Public Education and the minister of Inventions regarding national defence, in November 1915. He called Émile Borel (who was a second lieutenant in the artillery, in a fighting company although he was forty-four) to take care of it, together with Maurain and Lebesgue. Montel does not say so, but he was there too, with Lebesgue[74]:

> Besides the examination of the inventions, they had to establish the range tables of the enemy cannons for the sound reconnaissance of their position. With the help of the information given by the intelligence service, they had

électricité, météorologie, jusqu'à l'étude de phénomènes nouveaux dont l'intérêt apparaîtra dans l'avenir. Les mathématiques les plus abstraites ou les plus subtiles ont participé à la solution des problèmes de repérage et au calcul des tables de tir toutes nouvelles qui ont accru de 25 pour 100 l'efficacité de l'artillerie.

69 See [Barrow-Green 2008] for the considerable contribution of British mathematicians to the war effort.

70 Other mathematicians contributed in a different way, for instance Sergeant Élie Cartan who was director of the "auxiliary hospital 103" set up in the premises of the École normale supérieure (see, for instance [Julia 1970, p. 59]). Montel probably died too late for his own obituary [Mandelbrojt 1975] to be concerned about telling us such old stories about him.

71 Charles Maurain, mobilised in 1914, was sent in 1915 to a sound reconnaissance station (like many others), then to the Department of Inventions (see [Coulomb 1968]).

72 According to [Félix 1974], Lebesgue corrected obvious (and dangerous) errors, one of which was foreseeing that the projectile could fall behind the gunman.

73 Pendant la guerre de 1914–1918, il préside la Commission de Mathématiques du Service des Inventions, Études et Expériences scientifiques que dirige notre confrère M. Maurain dans cette Direction des Inventions que Painlevé avait créée. Avec une énergie inlassable, il travaille à la résolution des problèmes soulevés par la détermination et la correction des trajectoires des projectiles; le repérage par le son etc. Aidé par une nombreuse équipe de travailleurs bénévoles, il prépare un recueil de trajectoires, à triple entrée, qui doit servir par interpolation à l'établissement rapide des tables de tir.

74 According to a letter quoted in [Taylor 1985, p. 289], René Garnier, then in Poitiers, used to come to Paris to make computations "at the Artillery Section".

> to reconstruct the trajectory of the projectile, piece by piece. They also had to modify some of their tables and to study the monstrous sketch which led them to the tank[75] [Beer 1966, p. 67].

Hadamard worked there too [Maz'ya & Shaposhnikova 1998, p. 100]. In [Guiraldenq 1999] is reproduced a journal article reporting Borel's nomination[76]. Also, in the last letters of [Lebesgue 1991], information is given on the work done under his direction. Jules Drach and Ernest Vessiot applied their research to problems in ballistics, the work of Drach was communicated to Denjoy, so that he could study it from a practical point of view, when he was at the "Grâves polygon[77]" (see [Drach 1920, note 1]). Henri Villat, who was then a private, computed, at the range centre of Bourg d'Oisans, range tables against aircraft [Leray 1973]. The report written by Hadamard on the work of the Commission of ballistics can be found in [Hadamard 1920].

The obituary of Gabriel Kœnigs [de Launay 1931], one of the main characters in the prehistory of the iteration problem, also mentions his mechanics laboratory (especially devoted to engine thermodynamics), opened in 1914 and which was very useful for national defence.

As for Paul Appell, he was the one who founded "National Relief" (Secours National) (see [Buhl 1931b]).

At the Observatory

Thinking of Pierre Fatou who worked there, let us come now to the Observatory. The Paris Observatory did not contribute as such to the war effort[78], but it is known that some astronomers of the Observatory did contribute. The staff gave their knowledge to help the army... but not during their working hours, as the annual report of the Observatory for 1915 points out. Charles Nordmann, for instance, who joined the army and who, as a second lieutenant in the Engineers, developed the first sound reconnaissance machines[79]

[75] En dehors de l'examen des inventions, ils eurent à établir les tables de tir des canons ennemis pour le service de repérage par le son de leur emplacement. À l'aide des éléments fournis par le service d'espionnage, il fallait reconstituer la trajectoire du projectile morceau par morceau. Ils durent aussi modifier certaines de leurs tables de tir et s'intéresser à la première et monstrueuse ébauche qui devait les conduire au tank.

[76] When Painlevé left the government in 1917, Borel was forty-six and went back to the army. Regarding Borel's activities, see also the obituary by Paul Montel [1956].

[77] Regarding the work on and investigation of ballistics done at the fort of Grâves, see [Aubin 2008].

[78] Regarding the astronomers in the war effort, see [Saint-Martin 2008, § 3.3.2].

[79] There was also a sound reconnaissance laboratory at the ENS, directed by Jacques Duclaux, the husband of Germaine Appell, herself a sister of Marguerite Borel, in short a brother-in-law of Borel. Jean Chazy worked there. See, in [Lebesgue 1991] the letter dated January 2nd 1915 and note 973 and following. Chazy located very precisely the position of the so-called "big Bertha" (a German cannon)—and for this he was awarded the War Cross (see [Denjoy 1956]).

in October 1914—and he actually located an enemy battery on December 8th 1914 [Lebesgue 1991, Note 978]. There is no precise information that Fatou participated in these experiments, but it is likely that he did so, since we know he was aware of them, he spoke of them with Lebesgue before January 15th 1915 [Lebesgue 1991, p. 317].

The annual reports of the Observatory[80] are extremely discreet on these questions. They do not even mention, for instance, this activity of Nordmann. The report for 1916 notes that the service of physical astronomy of Maurice Hamy was suspended and that Hamy was "given the responsibility of various missions related to national defence", but without further detail. That of 1919 would nevertheless list those people, working at the Observatory, who where not in the army but were rewarded for "astronomical work and the way they served, in the interest of the war" (among them, Pierre Fatou who was awarded the title "Officier de l'Instruction publique" (Public education officer)).

"Patriotic" atmosphere

This was a time of "patriotism" and above all of wild propaganda[81]. The popular novels that appeared at that time, for instance *The Shell Shard* [Leblanc 1916] or *Rouletabille at Krupp's* [Leroux 1917], to cite here only the best-selling authors[82], show the strength of anti-German feeling. The most reasonable among the French agreed with it: we need only think of *Noël des enfants qui n'ont plus de maison* (Christmas for children who have lost their home) by Claude Debussy in 1915:

> Christmas, little Christmas, do not visit them, never visit them again, punish them!
> Avenge children from France![83]

If Henri Barbusse was awarded, in 1916, the Goncourt Prize for *Under Fire* (*le Feu*), a dreadful account of life and death in the trenches, in the last pages of which one can read:

> "After all, why do we make war?" We don't know at all why, but we can say *who* we make it for. We shall be forced to see that if every nation everyday brings the fresh bodies of fifteen hundred young men to the God of

[80] Library of the Observatory.

[81] Concerning the effect, both of this patriotic and anti-German atmosphere and of the war experience on the "fire generation", driven to pacifism, communism, or collaborationism, see again the article [Sirinelli 1992].

[82] Maurice Leblanc was the creator of Arsène Lupin, a character who appears very briefly (and rather artificially) in a late version of [Leblanc 1916], and Gaston Leroux was the creator both of Rouletabille (*The Mystery of the yellow room*) and Chéri-Bibi (an enormous best-seller in France in 1913). These two writers were extremely popular in France at the beginning of the 20th century.

[83] Noël, petit Noël, n'allez pas chez eux, n'allez plus jamais chez eux, punissez-les! Vengez les enfants de France!

> War to be lacerated, it's for the pleasure of a few ringleaders that we could easily count; that if whole nations go to slaughter marshalled in armies in order that the gold-striped caste may write their princely names in history, so that other gilded people of the same rank can contrive more business, and expand in the way of employees and shops—and we shall see, as soon as we open our eyes, that the divisions between mankind are not what we thought, and those one did believe in are not divisions[84]. [Barbusse 1916]

it seems that this condemnation—of war rather than of Germany—was not much heard. Propaganda was stronger (notice that the French expression "bourrage de crâne" (brainwashing) was used in Barbusse's book for the first time). The realistic

> One believes he dies for his country. He dies for some manufacturers[85],

of Anatole France, would not be heard until 1922. As for the *Craonne Song*[86], it was forbidden after it was sung by the mutineers in 1917, and it would remain forbidden... until 1974.

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[84] "Après tout, pourquoi fait-on la guerre?" Pourquoi, on n'en sait rien; mais pour qui, on peut le dire. On sera bien forcé de voir que si chaque nation apporte à l'Idole de la guerre la chair fraîche de quinze cent jeunes gens à égorger chaque jour, c'est pour le plaisir de quelques meneurs qu'on pourrait compter; que les peuples entiers vont à la boucherie, rangés en troupeaux d'armées, pour qu'une caste galonnée d'or écrive ses noms de princes dans l'Histoire; pour que des gens dorés aussi, qui font partie de la même gradaille, brassent plus d'affaires — pour des questions de personnes et des questions de boutiques. — Et on verra, dès qu'on ouvrira les yeux que les séparations qui se trouvent entre les hommes ne sont pas celles qu'on croit, et que celles qu'on croit ne sont pas.

[85] On croit mourir pour la patrie. On meurt pour des industriels

[86] *La Chanson de Craonne* is a song which described the life of the soldiers at the front, the chorus of which says:

> Farewell life, farewell love,
> Farewell all women
> It is over, it is forever
> With this infamous war
> It is at Craonne on the plateau
> That we shall leave our skin
> We are all sentenced
> We are the sacrificed.
> [Adieu la vie, adieu l'amour,
> Adieu toutes les femmes
> C'est bien fini, c'est pour toujours
> De cette guerre infâme
> C'est à Craonne sur le plateau
> Qu'on doit laisser sa peau
> Car nous sommes tous condamnés
> C'est nous les sacrifiés.]

This anti-German atmosphere was not at all lightened among our gentlemen of the Academy of Sciences. Moreover, it would last long after the war and we shall never stop noticing its consequences.

⋆

But, for now, we are in 1915. On March 15th 1915, the Academy of Sciences expelled its correspondent members who signed the "Manifesto of the Ninety-Three", or "Appeal of the German intellectuals to the civilised nations" dated October 4th 1914, a text which, to tell the truth, was rather calm, and which protested against the accusation of barbarity made against Germany after, in particular, the invasion of Belgium (six thousand civilians killed in August-September 1914). This exclusion was a very exceptional measure. In all the history of the Academy of Sciences, from its creation to the present day, there have been only three waves of "exclusions": Carnot and Monge were expelled from the Academy of Sciences at the time of the Restoration of the Monarchy, by a royal order in 1816 (by the political authorities); our German intellectuals are expelled by the Academicians themselves; there would also be the invalidation of the election of Georges Claude, a rather too visible collaborationist (who had been a big contributor to the war effort in 1914–18), at the time of the Liberation in 1944.

But let us go back to 1915 and let President Perrier [1915, p. 805] speak (once again); he "explains" to us the matter:

> Barbarity was spoken of, but conscious barbarity changes its name: it is called crime, and crime does not stop being crime when it is committed by crowned heads, when it becomes collective, when it is moreover disciplined. This is why the Academy of Sciences crossed off its lists, on March 15th 1915, the signatories of the sorry manifesto in which the German intellectuals tried to defend the cruelty and the treachery committed by their compatriots and inspired by those whom they serve: the chemist von Bayer [*sic*], from Munich, foreign associate member, and three correspondents: the mathematician Felix Klein[87], from Göttingen, the chemist Emil Fischer, from Berlin; the anatomist Waleyer, also from Berlin[88]

[87] Felix Klein was the only mathematician among the 93. According to tradition, he was asked by phone to sign the text and had no opportunity to read it (see for instance [James 2002, p. 228]—I don't know a more direct source).

[88] The chemist Adolf von Baeyer (1835–1917), Nobel Prize in 1905, was the one who, among other work, discovered tear gas. Emil Fischer (1852–1919), who was his assistant at Strasbourg, was also awarded the Nobel Prize in chemistry and in 1902.

> [...] With this gesture, the Academy wanted to stigmatise those who despise the moral values that were passed on to us by the generations who, during long centuries, lived, suffered, loved and thought on our soil[89]. [90]

Soon after this exclusion, the daily paper *le Figaro* published (on April 21st and 25th and May 9th, 18th and 26th 1915) a series of papers on the "Bluff of German science", written by distinguished scientists (among whom we shall not be surprised to find our Perrier). Still in 1915, on October 4th, Picard presented to the Academy his pamphlet *The history of science and the claims of German science* [*L'histoire des sciences et les prétentions de la science allemande*][91] with these words:

[89] It is not true, fortunately, that the Academy of Sciences excluded most of its German members, as can be read in [Lehto 1998, p. 16], and it is even less true that this happened *after* the election of Picard as Permanent Secretary, as the writing of the text under consideration implies. After the exclusion of the four signatories of the manifesto of the Ninety-Three, the "state of the Academy", published in the *Comptes rendus* in January 1916, attests the attendance of two German associates (including Dedekind), of fourteen German correspondents (including Schwarz, Max Noether and Hilbert) and of two Austrian correspondents.

[90] On a parlé de barbarie, mais la barbarie consciente change de nom: elle s'appelle le crime, et le crime ne cesse pas d'être le crime quand il est commis par des têtes couronnées, quand il devient collectif, surtout quand il est discipliné. C'est pourquoi l'Académie des sciences a rayé de ses listes, le 15 mars 1915, les signataires du triste manifeste où les intellectuels allemands ont essayé de défendre les cruautés et les félonies commises par leurs compatriotes et inspirées par eux à ceux qui les servent: le chimiste von Bayer [*sic*], de Munich, associé étranger, et trois correspondants: le mathematician Felix Klein, de Göttingue; le chimiste Emil Fischer, de Berlin; l'anatomiste Waldeyer, également de Berlin. [...] Par son geste l'Académie a voulu stigmatiser les contempteurs des conceptions morales que nous ont léguées les générations qui ont, durant de longs siècles, vécu, souffert, aimé et pensé sur notre sol.

[91] The anti-German feelings of Picard were deep and long-lasting. They would show themselves once again in the boycott of Einstein by the Academy of Sciences when Langevin, a convinced internationalist, would invite him to lecture at the Collège de France in March-April 1922. Borel, Appell, Cartan among any others, would warmly welcome the physicist, as Camille Marbo recounts [1968, p. 193].

The anti-German feelings of Permanent Secretary Picard against Einstein played an important role in this boycott, they have to be added in this case to a professed anti-Semitism and to a hatred of the Human Rights League and of left-wing people, including Langevin, Perrin and Hadamard, as his correspondence with Lacroix shows (archives of the Academy of Sciences, letters dated August 6th 1921, August 8th 1923, August 9th 1925, November 13th 1926...).

Coming back to Germans in general, let us quote another letter to Lacroix, dated April 17th 1922, in which Picard mentions a German colleague under the name of "kraut Hecker" [Boche Hecker] (Picard file, archives of the Academy of Sciences).

See also Note 36 in Chapter VI.

> Except for a specifically historical part, I emphasise in this study the often very formal nature of scientific German writings. This nature, in which a singular notion of reality and truth, and a sort of contempt for common sense sometimes appear, can, I believe, be linked to Kant subjectivism and formalism, and to the philosophical systems that more or less directly derive from them. The tendency to systematise everything is common in the German spirit. It can even be found in the most practical views, up to the concept of organisation, the new requirements that Germany would like, for its greatest profit, to impose on the world[92].[93]

And it was at the end of the same year 1915, on December 27$^{\text{th}}$, that President Perrier delivered the customary speech during the annual public session of the Academy, an interminable speech of nineteen printed pages, several excerpts of which have already been quoted here, a speech permeated with a patriotic hatred that readers have probably noticed... and from which we extract again a few well-chosen expressions: German felony, Germany, form now on separated from the whole civilised world, Germany's crimes, Germany who deserved all the curses, what Germany calls its Kultur, monstrous Germany [la félonie germanique, la Germanie, séparée désormais de tout le monde civilisé, les crimes de l'Allemagne, qui a mérité toutes les malédictions, ce qu'elle appelle sa Kultur, la monstrueuse Allemagne][94]...

From December 1915 to March 1916, the Belgian mathematician Charles de la Vallée Poussin[95], from Louvain, was invited to the Collège de France,

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[92] Would it not be worthwhile to do a comparative study of this text by Picard with the "types" in the Nazi journal *Deutsche Mathematik* of Bieberbach?

We are very far from the repeated assertions of Picard's father-in-law, Charles Hermite, in his correspondence, thirty years earlier, of his admiration for German mathematicians, with whom neither the war nor the political disagreements could prevent him from collaborating (see [Dugac 1984 ; 1985 ; 1989 ; Lampe 1916], and Note 64 of Chapter II).

[93] En dehors d'une partie plus particulièrement historique, j'insiste dans cette étude sur le caractère souvent si formel des écrits scientifiques allemands. Ce caractère, où apparaissent parfois une notion singulière du réel et du vrai, et une sorte de mépris pour le sens commun, peut, je crois, être rattaché au subjectivisme et au formalisme kantiens, et aux systèmes philosophiques qui en dérivent plus ou moins directement. La tendance à tout systématiser est habituelle à l'esprit germanique. On la retrouve même dans les vues les plus pratiques, jusque dans le concept d'organisation, nouvel Impératif que l'Allemagne voudrait, pour son plus grand profit, imposer au monde

[94] The final goal of Perrier's speech was to denounce the drop in the birth rate, alcoholism and the class struggle, as was to be expected from an address which was slightly contemptuous of women workers (see the excerpt quoted on page 6), which did not prevent it from ending with a very catholic "love each other"...

[95] Charles de la Vallée Poussin is well-known, as a mathematician, because of his contribution to the prime number theorem. He would be the first president of the new International Mathematical Union, the structure that would organise the "International" congress of Mathematicians at Strasbourg in 1920. A picture of

an "expression of sympathy after the cruel acts of violence Belgium suffered" [témoignage de sympathie après les cruelles violences dont la Belgique a été victime], as he says in the preface of his book [1916]. His passing through Paris is mentioned in a letter from Lebesgue [1991, p. 330] to Borel: Lebesgue worried of the audience of de la Vallée Poussin's lectures (very few mathematicians were present in Paris, because of the war), which he would have to attend... to learn what the Lebesgue integral is.

On March 13th 1916, the Academy of Sciences elected Charles de la Vallée Poussin as a corespondent member, in replacement of Felix Klein. It is hard to not see in this election an answer to the

> It is not true that our troops brutally destroyed Louvain

in the Manifesto of the Ninety-Three.

At that time, the list of members of the SMF was published in the *Bulletin*, with the words:

> Because of the present war, the Council of the French Mathematical Society decided to suspend the relations of the Society with its members who belong to enemy nations; as a consequence, their names do not show up on the list below[96].

After the war, this statement would be replaced by the following:

> In its session of January 14th 1920, considering that the relations of the Society with those among its members who belong to enemy nations were suspended during the war, the Assembly of the French Mathematical Society decided that these relations could only be resumed after a formal request from the above mentioned members, which request would be submitted to the vote of the Council; consequently, the names of these members do not show up in the list above[97].

This text would still appear in the *Bulletin*, with the list of members, until January 1930.

Let us add that French people were not alone in being violently anti-German after the war. See for instance the excerpts of articles that appeared in the British journal *Nature* and that are quoted in [Dauben 1980].

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de la Vallée Poussin, with Julia, taken during the (more) international congress of 1928 can be found on page 210.

96 En raison de l'état de guerre actuel, le Conseil de la Société mathématique de France a décidé de suspendre les relations de la Société avec ceux de ses membres qui appartiennent aux nations ennemies; en conséquence, les noms de ces membres ne figurent pas sur la liste ci-dessous.

97 Dans la séance du 14 janvier 1920, l'Assemblée générale de la Société mathématique de France, considérant que les relations de la Société avec ceux de ses membres qui appartiennent aux nations ennemies ont été suspendues pendant la guerre, a décidé que ces relations ne pourraient être reprises qu'à la suite d'une demande formelle des membres susvisés, demande qui serait soumise au vote du Conseil; en conséquence, les noms de ces membres ne figurent pas sur la liste ci-dessous.

It should nevertheless be noticed that, delivering in his turn the speech during the annual session on January 18$^{\text{th}}$ 1916, Jordan who mentioned, of course, German barbarity[98]... but relatively briefly,

> Without speaking of the glorious success of our armies, the crimes that our enemies multiply are the omen of their defeat. They dare to speak of freedom, of liberation, while on each of their borders an oppressed nation moans; while whole populations are deported to slavery, and they prepare to enlist them by force in their armies. Who could believe in the final success of what they have undertaken, which seeks to erase twenty centuries of Christianity to take us back to the regime of the Babylonian monarchies. It is in vain that they will appeal to their "Old German God", the bloody idol forged by their pride. We leave them this God. Ours does not know old age and is not the prerogative of a people; but this is a King of justice, and with his help we shall overcome [Jordan 1916][99].

did not then forget, when he recalled the Academicians who had deceased that year, to devote a few lines to pay tribute to the German mathematician "who died full of years", Richard Dedekind, correspondent of the Academy since 1900 and foreign associate since 1910 (and who had not signed the Manifesto of the Ninety-Three)[100].

Seven of the award-winners of the Prizes awarded during this public session, among whom was René Gateaux, were killed at the front.

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[98] According to [Sirinelli 1992], even in the 20's, students applying for the Grandes Écoles would have to treat subjects like:

> Should war be waged in a humane or in a barbaric way?
> How should a people that uses science as the arm of barbarity be judged?

[99] Sans parler des glorieux succès de nos armées, les crimes multipliés de nos ennemis sont le présage de leur défaite. Ils osent parler de liberté, d'affranchissement, lorsque sur chacune de leurs frontières gémit une nation opprimée; lorsque des populations entières sont déportées en esclavage, et qu'ils s'apprêtent à les enrôler de force dans leurs armées. Qui pourrait croire au succès final de leur entreprise, qui prétend effacer vingt siècles de christianisme pour nous ramener au régime des monarchies de Babylone. Ils invoqueront en vain leur "Vieux Dieu Allemand", sanglante idole que s'est forgée leur orgueil. Nous leur laissons ce Dieu-là. Le nôtre ne connaît pas la vieillesse et n'est pas l'apanage d'un peuple; mais c'est un Roi de justice, et avec son aide nous vaincrons.

[100] In the middle of the war, on May 12$^{\text{th}}$ 1917, the German mathematician David Hilbert (who had not signed the Manifesto either) delivered a speech on Gaston Darboux (who had died two months earlier) to the Academy of Sciences of Göttingen, this speech would be translated into French and appear in [Hilbert 1920], a warm academic speech, the conclusion of which was about the influence that Darboux' ideas had on Felix Klein's efforts (regarding teaching mathematics) and about his role in the International association of scientific academies. The publication by*Acta mathematica* was obviously not by pure chance—regarding the role Mittag-Leffler wanted "his" journal to play after the war, see [Dauben 1980].

I.4 Iteration, a few definitions and notation

This is a good place to return, at last, to mathematics. In the following chapters, we shall try to explain this mathematics while telling chronologically the story of the publication of Fatou's and Julia's works on iteration.

Let us start by reminding the readers of a few definitions and by establishing the notation. We are dealing with holomorphic functions defined on open subsets of $\mathbf{C}$, and most often with rational fractions, namely with analytic functions on the Riemann sphere $\widehat{\mathbf{C}} = \mathbf{P}_1 = \mathbf{P}_1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$, although many results still hold true in the case of an entire function having an essential singularity at ∞. As the authors did it at that time, we shall often speak of rational fractions from $\mathbf{C}$ to $\mathbf{C}$, considering the point at infinity as an ordinary point and not worrying about the poles of the fractions under consideration. Such a fraction will in general be denoted by R and k will be its degree as a map from $\mathbf{P}_1$ to $\mathbf{P}_1$, that is, the number of points in $R^{-1}(a)$ (counted with multiplicity) for $a \in \mathbf{P}_1$, or

$$k = \sup(\deg P, \deg Q) \text{ if } R = \frac{P}{Q} \text{ is irreducible}$$

(P and Q are polynomials). In the example where we apply Newton's method to find the roots of a polynomial f of degree k, we iterate the rational fraction

$$R(z) = z - \frac{f(z)}{f'(z)}$$

which also has degree k.

Almost always (but not always, to iterate an isomorphism is also tempting, think of a rotation...), we assume that $k \geq 2$. The n^{th} iterate $R \circ R \circ \cdots \circ R$ (n times) of R is denoted R^n. This is a rational fraction of degree k^n.

The periodic points for R are the fixed points for some R^n. The period of a periodic point a is the smallest integer n such that $R^n(a) = a$. The n distinct points $a, R(a), \ldots, R^{n-1}(a)$ constitute what is called a periodic cycle.

The *multiplier* of the fixed point z is the derivative of R at z. Note that, if

$$z_1 = R(z), \ldots, z_{n-1} = R^{n-1}(z),$$

then

$$(R^n)'(z) = R'(z_{n-1}) \cdots R'(z_1) R'(z),$$

so that, if z is periodic of period n, we have

$$(R^n)'(z) = (R^n)'(z_1) = \cdots = (R^n)'(z_{n-1}) :$$

all the points of a cycle have the same multiplier.

Some fixed points of R or of its iterates will be[101] called *repelling*, they are the ones for which

$$R^n(z) = z \text{ and } |(R^n)'(z)| > 1.$$

Following Julia, let us denote by E the (countable) set of all the repelling points and by E' its derived set, that is, the set of its accumulation points. We do not discuss here the question of whether E indeed contains some points, for this see Remark III.1.2.

Attracting fixed points will be those whose multiplier has absolute value strictly less than 1. Since the work of Kœnigs[102] around 1880, it has been known that every attracting fixed point has a neighbourhood on which the sequence R^n converges to a constant (equal to the fixed point).

Indifferent fixed points are those such that the absolute value of their multiplier is 1.

If ∞ is a fixed point, we take it to 0 by $w = 1/z$, this conjugates R to

$$S(w) = \frac{1}{R\left(\dfrac{1}{w}\right)},$$

so that the multiplier is

$$s = S'(0) = \lim_{w \to 0} \frac{S(w)}{w} = \lim_{|z| \to \infty} \frac{z}{R(z)}.$$

Thus for instance, if R is a polynomial of degree $k \geq 2$, the point ∞ is always an attracting fixed point (it is even *super-attracting*, that is, $s = 0$).

Assume that a is an attracting fixed point of R. As $|R'(a)| < 1$, there exist real numbers $r > 0$ and $\sigma < 1$ such that

$$|z| < r \Rightarrow |R(z) - a| < \sigma \, |z - a| \, .$$

---

[101] We use the future here because the terminology will be invented as the story goes along.

[102] Gabriel Kœnigs (1858–1931) would be elected to the Academy of Sciences, in the Section of mechanics on March 18th 1918. Regarding him and his mathematical connection with Darboux, see [Alexander 1995]. He would be the secretary of the IMU from 1920 to his death in 1931, a position he is supposed to have not taken very seriously (according to [Lehto 1998]). He would also be, with Paul Appell, Émile Borel, Jacques Hadamard, Jean Perrin and a few others, one of the members of the Honour committee of the Rationalist union [Union rationaliste], founded in 1930 with Langevin as vice-president. Neither of the obituaries [de Launay 1931 ; Buhl 1931a] mentions either the IMU or the Rationalist union. However, it is not absolutely true that nothing can be found in the archives of the Academy of Sciences on his activity at the IMU: in a letter to Villat (Villat collection 61J) dated January 19th 1920, Kœnigs wrote that he had given the statutes of the IMU to the printer.

Thus

$$|R^n(z) - a| \leq \sigma^n\, |z - a|$$

and the sequence R^n converges uniformly to the constant function equal to a on the disc.

The set of points z of the plane such that $R^n(z)$ tends to a is the attraction basin (or domain) of the fixed point a. This basin may have infinitely many connected components. The one containing a is the immediate basin (or domain) of a. This notion is easily generalised to the case of an attracting cycle.

Let us mention, since the word will show up, that what we call a fixed point was often called, at that time, a "double" point.

Example I.4.1. The first example is, of course, the simplest one, that of the function $R(z) = z^2$. The fixed points of R are 0 and ∞ and are attracting (even super-attracting). In the figure, the iterates of the points in the (grey) disc converge to 0, those of the outer (white) points converge to ∞.

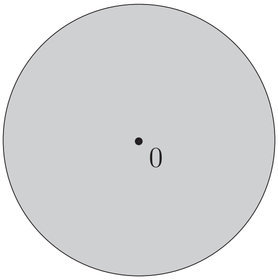

Fig. I.1. The set E' for $z \mapsto z^2$

The fixed points of R^n are the solutions of the equation

$$z^{2^n} - z = 0.$$

The origin is a (super-attracting) fixed point. If $n \geq 2$, all the other solutions are repelling fixed points, hence

$$E = \left\{ z \in \mathbf{C} \mid z^{2^n - 1} = 1 \right\} \subset S^1.$$

This set is dense in the circle, so that $E' = S^1$ is the whole circle. It divides $\mathbf{P}_1(\mathbf{C})$ into two connected components (attraction basins or domains), the sequence R^n converges to a constant function on each component (the constant is equal to the attracting point which the component contains).

This example was well known by all the "iterators" in history, long before the point at which we began our account. It was explained by our authors, for instance in [Julia 1918f, p. 103] and [Montel 1927, p. 228]. It is related to Newton's method: the polynomial $R(z) = z^2$ is conjugated with

$$S(w) = \frac{1}{2}\left(w + \frac{1}{w}\right) \text{ by } z = \frac{w-1}{w+1}$$

and S is the rational fraction

$$S(w) = w - \frac{w^2 - 1}{2w}$$

given by Newton's method for finding the roots of $w^2 - 1$. As the study of R shows, the iterates of the points in the half-plane $\mathrm{Re}(w) > 0$ converge to the root $+1$, those of the half-plane $\mathrm{Re}(w) < 0$ to the root -1... an ideal situation, which is specific to degree 2. As was shown by Schröder[103] in a paper [Schröder 1871] we shall have the opportunity to mention again, if α and β are the (distinct) roots of a degree 2 polynomial, the bisector of the segment $\alpha\beta$ divides $\mathbf{C}$ into two open half-planes and Newton's method converges to one of the roots, whatever point in the half-plane of this root we start from.

Example I.4.2. The second example is one of those that show up in Pierre Fatou's Note [1906d] in 1906. This is the rational fraction

$$Z = R(z) = \frac{z^2}{z^2 + 2}.$$

Since more recent work of Douady, Hubbard, and others, popularised the examples of quadratic polynomials, it is nicer for today's readers to use the polynomial

$$V = P(v) = v^2 + 2$$

which is conjugated to R by $v = 2/z$ (and by $V = 2/Z$, as one would have said at that time).

The set E' is then a Cantor set (one of the first examples of Cantor sets in dimension 2) which is sometimes called a "dust" and rather hard to visualise. It looks like the one in the figure, which corresponds to $z \mapsto z^2 + 1$ but that Arnaud Chéritat suggested that I use because it is a little bit more visible.

---

[103] The reduction of the resolution of a degree-2 equation to the iteration of $z \mapsto z^2$ can also be found in the note [Cayley 1890] of Cayley... the conclusion of which is the sentence

> I hope to apply this theory to the case of a cubic equation but the computations in this case are far more difficult. [J'espère appliquer cette théorie au cas d'une équation cubique mais les calculs sont beaucoup plus difficiles.]

As we shall see, quite apart from computation, the situation is much more intricate in higher degrees. See for instance Figure II.4.

Fig. I.2. The set E' for $z \mapsto z^2 + 1$

The terminology "Cantor set" was not as commonly used at that time as it is nowadays. Fatou proved that this set is perfect, everywhere discontinuous, and invariant under R. The iterates of all the points of $\mathbf{C}$, except those in E', converge to the fixed point. To be quite precise, Fatou did not define (in his Note [1906d]) the set we call E' (and that he denotes E) exactly as we just did (derived set of the set of repelling points), but rather as the limit of a sequence of sets, each included in the following one. The point 0 is an attracting fixed point, it thus attracts all sufficiently small discs. Let us choose a large enough $r > 0$, such that the disc of radius r contains all the critical points of the inverse mapping of R. The inverse image of the circle C of radius r then comprises of k disjoint closed curves such that the restriction of R to each of them is a diffeomorphism onto C. Fatou calls E_n the set of points z such that

$$|z| > r,\ |R(z)| > r, \dots,$$
$$\dots,\ \left|R^{n-1}(z)\right| > r,\ |R^n(z)| \geq r \text{ and } \left|R^{n+1}(z)\right| < r$$

(it would seem more natural to put-non strict inequalities everywhere). See Figure I.3, in which the sets E_n for $n = 0, 1, 2$ and $k = 2$ are very schematically depicted. We then have $E_{n+1} \subset E_n$ and it is easy to check that $\cap_n E_n$ is the set E' we are interested in. We note here that Fatou, as a specialist of the Lebesgue integral, was familiar with infinite intersections.

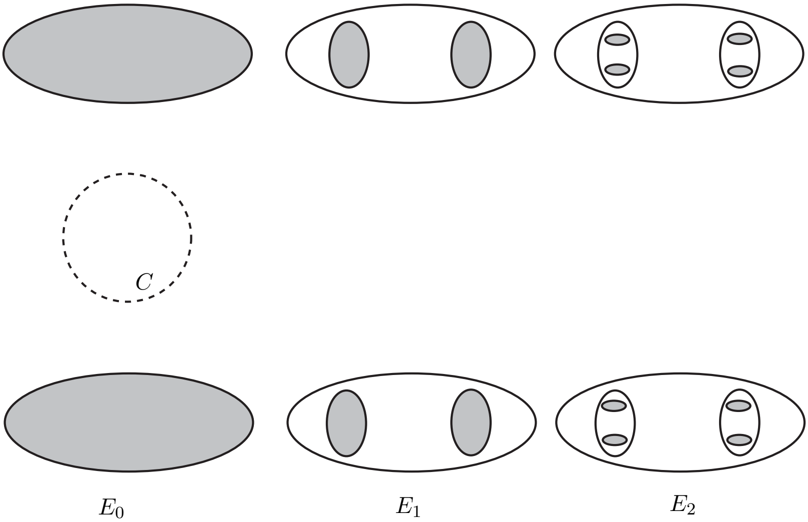

Fig. I.3. A schematic view of the sets E_0, E_1, E_2

Example I.4.3. In his Note [1906d], Fatou also considered[104] the polynomial

$$R(z) = \frac{z + z^2}{2}.$$

The fixed points are 0 and ∞ (attracting) and 1 (repelling). The domains of convergence to 0 and ∞ are separated by lines which are not analytic.

For modern readers: the function is conjugated with

$$r(u) = u^2 + \frac{1}{8},$$

the polynomial has a unique attracting fixed point and the set E' is a slightly deformed circle (with dimples), a not very differentiable Jordan curve which has no tangent at any point. See Figure I.4[105].

[104] In the notice he would write in 1921, Fatou would say:

> I also showed cases in which the substitution has two attracting double points the respective domains of which are connected, simply connected and separated by a non-analytic curve, [J'ai également indiqué des cas où la substitution présente deux points doubles attractifs dont les domaines respectifs sont d'un seul tenant simplement connexe et séparés par une courbe non analytique,]

And Hadamard, in the report we have already quoted:

> Starting from this simple case however, the strangest singularities show up.

Fatou file, archives of the Academy of Sciences.

[105] In this figure, as in Figure I.1 and in most of the other figures in this book, the set E' is the boundary of the grey part.

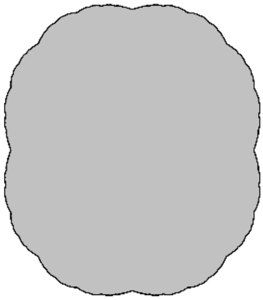

Fig. I.4. The set E' for $R(z) = \dfrac{z + z^2}{2}$

In addition to what we shall cite here, see also Douady's talk [1983] at the Bourbaki seminar, a very beautiful modern introduction to the subject (although already old), and the more recent paper of Yoccoz [1999] and books of Milnor [2006a], Berteloot and Mayer [2001] and of Tan Lei [2000].

I.5 Normal families

One of the main objects investigated in the work of Fatou and Julia which we are discussing here is what is nowadays called the "Julia set" of the function R which is defined, since Fatou [1920a], as the set of points at which the sequence of iterates R^n of R is not normal. The idea to use the notion of a normal sequence, or family, appeared, as we shall see, while Fatou and Julia had already begun to work on the iteration of rational fractions. It stimulated their work, which leaped forward.

Let us remind the readers that a family of holomorphic functions on an open subset $U \subset \mathbf{P}_1$ taking their values in $\mathbf{P}_1$ is said to be *normal*, a notion that was invented by Paul Montel at the beginning of the century (to make things simpler, we refer here to a later book [Montel 1927, p. 32]), if, from any infinite sequence of functions in this family, a sub-sequence that converges uniformly on U can be extracted (in modern terms, this is a relatively compact subset of the space $\mathcal{O}(U)$ of holomorphic maps from U to $\mathbf{P}_1$, endowed with the compact open topology)[106]. In this case, the limit is automatically a

[106] As it was well known at the time of our story, sets of continuous functions (contrary to sets of points in $\mathbf{R}^n$) do not have the property that the compact sets are the closed bounded subsets. The notion of equicontinuity of Arzelà and Ascoli gives the compactness. It was the work of Montel in complex analysis that showed

holomorphic function (that may be constantly equal to ∞). A family is normal at a point if there exists a neighbourhood of the point on which it is normal.

Examples I.5.1.

(1) We start these examples with a counter-example: consider the function $f(z) = e^z$ and the sequence (f_n) defined by $f_n(z) = f(nz)$. At a purely imaginary point iy_0, the sequence (f_n) is not normal: on any disc centred at this point, we have

$$\lim_{n\to+\infty} |f_n(z)| = \begin{cases} 0 & \text{if } \operatorname{Re}(z) < 0 \\ 1 & \text{if } \operatorname{Re}(z) = 0 \\ +\infty & \text{if } \operatorname{Re}(z) > 0, \end{cases}$$

so that no sub-sequence of (f_n) can converge uniformly on such a disc.

(2) Back to iteration. At an attracting fixed point a of the rational fraction R, the sequence of iterates R^n is a normal family, since it converges uniformly, on a disc centred at a, to the constant function equal to a.

(3) In Example I.4.1, an open disc centred at a point of the circle contains points at which $R^n(z)$ converges to ∞ and others at which it converges to 0. Thus the family R^n is not normal at any point of the circle (and it is normal everywhere else). In this example, it can be seen that the set E', defined as the derived set of the set of repelling fixed points of the R^n (being perfect, it is also its closure), happens to coincide with the set of points at which the sequence R^n is not normal. We shall see that this property is fairly general.

(4) The repelling fixed points of R (and its iterates) are always points at which the sequence of iterates is not normal: if $R(a) = a$ with $|R'(a)| > 1$, since $(R^n)'(a) = R'(a)^n$, the sequence of derivatives of the R^n has no sub-sequence that converges in a neighbourhood of a, hence R^n has no uniformly convergent sub-sequence.

(5) Application to Fuchsian and Kleinian groups. One can also consider the set of elements of a discrete group Γ of Möbius transformations

$$z \longmapsto \frac{az+b}{cz+d}, \quad ad - bc \neq 0$$

(a Kleinian group) as a family of holomorphic maps from $\mathbf{P}_1(\mathbf{C})$ to $\mathbf{P}_1(\mathbf{C})$. The set of points where this family is not normal is called the limit set of Γ, this is also the set of the accumulation points of the fixed points of the group. For instance, if the group is that of the Möbius transformations R such that

$$\frac{R(z)-\alpha}{R(z)-\beta} = k^n \frac{z-\alpha}{z-\beta} \text{ with } k \neq 1,$$

the limit set consists of the two fixed points α and β. See the book (by Fatou) [Appell et al. 1930, § 31–33]. The fact that this situation is analogous with

how this notion could be useful. The readers will certainly have understood that we allowed ourselves an anachronism: the word “compact”.

that of iteration was already mentioned in Fatou's 1906 work on iteration[107], it would also be noticed by Julia. We shall come back to this (on page 112 and in § IV.5.b).

The property that a family is normal at a point is clearly a local property. The fundamental result on normal families is the following theorem [Montel 1927, p. 21]:

Theorem. *If holomorphic functions defined on an open set U are uniformly bounded on U (with a uniform bound), they constitute a normal family.*

In modern terms: in the space $\mathcal{O}(U)$, the compact sets are the closed and bounded subsets.

As an application of the conformal mapping theorem, Montel [1912] also showed the following normality criterion:

Theorem (of Montel). *A family $\mathcal{F}$ of analytic functions on an open set U which has only two exceptional values*[108] *is normal.*

This is because the functions in the family $\mathcal{F}$ take their values in the complement of two points in $\mathbf{C}$, an open set which is conformally equivalent to a bounded open subset. This theorem allows us, for instance, to prove the celebrated theorem of Picard (first proved in 1879, see [Picard 1879]).

Theorem (Picard's first theorem). *A non-constant entire function takes all values, except possibly one.*

Let us assume, Montel says, that f avoids two values. We want to apply a theorem about sequences of functions to the investigation of the single function f. We must thus construct a sequence of functions, the properties of this sequence giving some information on the properties of the function. The sequence (f_n) defined by

$$f_n(z) = f(nz)$$

takes the same values as f, so that it is normal (according to Montel's theorem). Hence it is bounded on a disc, hence f is bounded on $\mathbf{C}$, and it must be constant, thanks to Liouville's theorem.

There is a whole host of Picard theorems, to the proof of which Montel's theorem applies, for instance the following.

---

107 Poincaré [1883] himself considered the limit set, defined as the set of points at which the action of the group is not properly discontinuous (without the notion of a normal family). This is the line L that we have already met in Note 14 and that will show up again in § IV.5.b.

108 Let us remind the readers that what is called an "exceptional value" is a complex number a that is not a value, more precisely:

$$\forall f \in \mathcal{F}, \quad a \notin f(U).$$

Theorem (Picard's second theorem). *Let f be a holomorphic function on the punctured disc*

$$D_0 = \{z \in \mathbf{C} \mid 0 < |z| < \varepsilon\}.$$

If f has an essential singularity at 0*, it takes all complex values, except maybe one.*

The argument is analogous, using this time the sequence (f_n), which, if f avoided two values, would be uniformly bounded (but then f would have a removable singularity at 0) or would tend to the constant function equal to ∞ (then f would have a pole at 0), thus 0 would not be an essential singularity[109].

Montel's theorem, as one can see, establishes a parallel between the investigation of the points at which a family is not normal and the investigation of the essential singular points of a single function[110]. See also §§IV.3 and VI.3. For a complete panorama of the theorems of Picard and their variants, see [Segal 2008].

Coming back to iteration itself. It was not as the set of points at which the sequence is not normal that the Julia set was defined, at least by Julia, but starting from the set E of repelling periodic points of R. See above and next chapter.

[109] This is a good place to mention that the short article [Dieudonné 1990] devoted to Montel by the *Dictionary of scientific biography* (DSB) is a very efficient introduction to normal families.

[110] Here is a comment made by Émile Borel (probably in 1937, at the latest in 1947) on this application of normal families [Borel 1966, p. 46]:

> [...] in this application, it is necessary to tackle the study of a single function by a method, the principle of which introduces an infinity of functions. In all the other applications of the theory of normal families, an infinite sequence of functions is naturally present. Here it was necessary to construct it from scratch. Paul Montel was able to deduce from a unique function a family of functions in which the properties of the initial function are collectively reflected. [dans cette application, il faut aborder l'étude d'une fonction unique par une méthode dont le principe même introduit une infinité de fonctions. Dans toutes les autres applications de la théorie des familles normales, une suite infinie de fonctions se présente naturellement. Ici, il a été nécessaire de la construire de toutes pièces. Paul Montel est arrivé à déduire d'une fonction unique une famille de fonctions sur laquelle se reflètent collectivement les propriétés de la fonction initiale.]

Among the infinite sequences of functions that appear naturally, Borel was certainly thinking of the family of the R^n from iteration theory.

I.6 Relation to functional equations

One may wonder why so many of the works devoted to the question of iteration which we have cited contain the words "functional equations" in their title. We shall come back later to the reappearance of functional equations in Fatou's titles (see Note 55 in Chapter II).

The fact remains that the investigation of some functional equations is naturally related to that of iteration. It was in a paper [Schröder 1871] on iteration that Schröder[111] considered the question of replacing the function R by a conjugate $S = \psi^{-1} \circ R \circ \psi$ to give it a simpler form, trying in particular

$$\psi(sw) = R(\psi(w))$$

or, equivalently,

$$F(R(z)) = sF(z) \text{ with } w = F(z) \text{ and } F = \psi^{-1}.$$

This has been called, since that paper, a "Schröder equation" (R and s are given and we look for F).

The work of Kœnigs—which we have already qualified as a prehistory of the subject—deals with functional equations. The connection with iteration is very clearly explained in Paul Montel's book [1927]. Let us recall briefly what he explains there. We try to solve Schröder's equation

$$F(R(z)) = sF(z)$$

where F is the unknown, a holomorphic function we have to determine, defined on an open set we also have to determine.

Let us assume that $a \in \mathbf{C}$ is an attracting (but not super-attracting) fixed point[112] of R, which is simple in the sense that we have, in a neighbourhood of a,

$$R(z) = a + s(z - a) + \sum_{k \geq 0} a_k (z - a)^{k+2},$$

hence $s = R'(a)$ is the multiplier of R at a, and our assumption that a is attracting but not super-attracting says that $0 < |s| < 1$. For $r > 0$ small enough, we have, on the disc $D(a, r)$

$$|R(z) - a| < \sigma |z - a| \text{ for some } \sigma < 1$$

and also

$$|R(z) - a - s(z - a)| < A |z - a|^2 \text{ for some } A > 0.$$

[111] Ernst Schröder (1841–1902) was a prolific German mathematician, influenced by Cantor, and, in an area far from iteration, the author of a proof of the so-called Cantor-Bernstein Theorem (there exists a bijection from A to B if and only if there exists an injection from A into B and an injection from B into A). Other aspects of the resolution of Schröder's equation can be found in [Alexander 1994].

[112] We assume here that the fixed point is not the point at infinity.

We set

$$F_n(z) = \frac{R^n(z) - a}{s^n}$$

and prove that the sequence of holomorphic functions F_n converges to a holomorphic function F on $D(a, r)$ that is a solution of Schröder's equation. Put $z_0 = z$ and $z_n = R(z_{n-1})$ for $n \geq 1$. We then have

$$F_n(z) = \frac{R(z_{n-1}) - a}{s^n} = (z - a) \prod_{k=0}^{n-1} \frac{R(z_k) - a}{s(z_k - a)}.$$

The function F_n being written as a product, the convergence of the sequence follows from that of the infinite product of the terms $(R(z_k) - a)/s(z_k - a)$, which in turn is equivalent to that of the series with general term

$$\frac{R(z_k) - a}{s(z_k - a)} - 1 = \frac{R(z_k) - a - s(z_k - a)}{s(z_k - a)}.$$

We have

$$\left| \frac{R(z_k) - a}{s(z_k - a)} - 1 \right| < \frac{A\,|z_k - a|^2}{|s|\,|z_k - a|} = \frac{A}{|s|}\,|z_k - a| = u_k.$$

But the series with general (positive) term u_k converges since we have

$$\frac{u_{k+1}}{u_k} = \left| \frac{z_{k+1} - a}{z_k - a} \right| = \left| \frac{R(z_k) - a}{z_k - a} \right| < \sigma < 1.$$

The sequence F_n indeed converges to a holomorphic function F and we have

$$F_n(R(z)) = \frac{R^{n+1}(z) - a}{s^n} = sF_{n+1}(z)$$

so that the limit F is indeed a solution of Schröder's equation.

Let us also observe that $F(a) = 0$ and $F'(a) = 1$, so that F is locally invertible in a neighbourhood of a. On the other hand, applying the result to a determination of R^{-1} near a (we have not used the fact that R is rational), we get the same result when $|s| > 1$. What we have obtained is a theorem of Kœnigs [1884 ; 1885], which we formulate here as a linearisation statement:

Theorem. *Let a be a fixed point of R with multiplier s (with $|s| \neq 0, 1$). There exists an invertible holomorphic function F, defined in a neighbourhood of a, such that*

$$F(a) = 0 \textit{ and } F \circ R \circ F^{-1}(w) = sw.$$

Remark. The inverse mapping θ (the existence of which was proved by Poincaré [1890] for $|s| > 1$) is thus a solution of

$$R(\theta(u)) = \theta(su) \text{ and } \theta(0) = a.$$

The function θ extends to the whole of $\mathbf{C}$, while F can be mutli-valued. See Examples II.1.2 and II.2.1 below.

In the case of a super-attracting fixed point (the $s = 0$ case), a theorem of Böttcher [1904a; 1904b; 1904c][113] shows that R is conjugated, in a neighbourhood of the fixed point, to a power function. The functional equation solved by Böttcher is

$$F(R(z)) = aF(z)^k.$$

This theorem is the essential tool in the study, for instance, of the attraction basin of the point at infinity when R is a polynomial.

The additive form of Schröder's equation,

$$F(R(z)) = R(z) + a$$

called Abel's equation[114], is useful in understanding what happens near indifferent fixed points (the fixed points whose multiplier s has absolute value 1), and would be investigated, in particular, by Fatou, in his study of what happens near parabolic fixed points (those whose multiplier is a root of unity).

[113] The articles of Böttcher (a Polish mathematician) that we cite here were published in Russian in a Kazan journal and it is not certain that our protagonists would have had the opportunity to read them. Fatou would write that

> The existence of this function [the function which conjugates R to a "power" function] seems to have been proven for the first time by M. Böttcher [Fatou 1919b, p. 189] [L'existence de cette fonction [celle qui conjugue R à la fonction puissance] semble avoir été démontrée pour la première fois par M. Böttcher]

but he would not cite Böttcher's papers. He might have known of their existence by the review published by the *Jahrbuch über die Fortschritte der Mathematik*. Unless if I am mistaken, Julia would mention Böttcher only in [Julia 1923, p. 145], also without a reference. I have not seen these papers myself and I cite them from the *Jahrbuch*. Ritt would mention and use "un théorème de L. Boettcher" in his Note [1918], and, moreover, he would publish a proof in his paper [1920]. It is possible that he, Ritt, read the papers: according to [Lorch 1951], he learned "to read mathematical Russian before that became fashionable".

[114] Abel devoted a few papers to functional equations. One of them [Abel 1881, Mémoire VI] was published in Volume II of his Complete Works in 1881. The paper was not published during Abel's lifetime. What might have been only a fragment appeared in Holmboe's edition (in Norwegian) of his works in 1839, but we shall never know, since some of Abel's manuscripts were later burned. It was then reproduced in the 1881 edition in French. To cut a long story short, the paper is called "Determination of a function by means of an equation with a single variable" and starts very abruptly with

> Given the function fx, find the function φx by the equation
>
> $$\varphi x + 1 = \varphi(fx).$$

The article is three and a half pages long (in which Abel transforms the functional equation to a difference equation) and the reasons why he was interested into this particular equation are not explained.

Abel's and Schröder's functional equations were the subject of several papers that are rather forgotten nowadays and that are too far from our subject to cite here. See [Alexander 1994, Chapter 2].

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