Thermodynamically Consistent Systems of Hyperbolic Equations

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Abstract A discussion of the author's advanced viewpoint on the underlying principles for construction of Godunov's scheme is presented. The application of these principles in problems of elastic and elastoplastic deformations is outlined. The presentation is based on extensive numerical simulations performed for both an analysis of the solution convergence details with decreasing mesh step size (for equations of Fluid Dynamics) and the motivation of modeling an elastoplastic media by an "effective elastic deformation" and the employment of Maxwell's viscosities.

Being engaged for 58 years in gas dynamics problems and in more general ones of continuum mechanics, also developing algorithms for numerical simulation of unsteady phenomena, I understood the need to directly construct discrete models for a media composed of elementary microcells rather than to rely only on commonly used differential equations available in monographs and textbooks. In this way, one should aim at obtaining such a behavior of the macro objects (composed of the microcells) that would realistically enough resemble the phenomena in the actual media to be approximated by our model.

In fact, exactly this kind of modeling had been performed in "Godunov's scheme" in 1953–1954 (published in 1959), though the above interpretation of the scheme was elicited some later. In the current lecture, I would like to deliver my up-to-date understanding of the scheme and describe its extension to problems of elastoplastic deformations, which has been developed together with my colleagues. This lecture can be viewed as a sequel of the one given at Michigan University in May 1997 and published in Novosibirsk [5]. A condensed version of the paper [5] came out in English in J. Comp. Phys. [6], whereas a translation of the whole paper into English was published by INRIA in 2008 [7].

The overdetermination of equations governing smooth solutions turned out to be of crucial importance. At the same time, for modeling discontinuities of solutions

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(shock waves), one needs to replace an equation by an inequality (the entropy rise), whereas for the smooth solutions the overdetermination remains valid.

The lecture is focused on the points as follows:

1. The structure of fluid dynamics equations as well as a number of other systems of mathematical physics, which are noticeably diversified. These equations are typically overdetermined with respect to the smooth solutions, whereas the overdetermination used to disappear with respect to the discontinuous ones [1, 3, 4]. In the second case, an equation (the entropy conservation law) turns into an inequality, as the discontinuity relations should often take into consideration dissipative phenomena inside the microscopic zone being modeled by the discontinuity.
2. The symmetric hyperbolicity of the systems at hand (for the smooth portions of solutions). This hyperbolicity is usually a corollary of thermodynamics laws [2, 9, 10].
3. A detailed discussion of the original Godunov's scheme structure on the basis of the ideas introduced in items 1 and 2. Feasible simplifications, e.g., a linearization of the used Riemann problem, implicit versions of the scheme, and costs to be paid for these modernizations [11].
4. Thermodynamically consistent modeling the equations of elasticity and the ones of elastoplastic deformations using a Maxwell scheme; the modeling has been developed up to a difference realization based on the same principles as those in gasdynamics [8, 10].
5. A review of conclusions inferred from the analysis of the solution convergence with mesh refinement; a discussion of the convergence concept itself. The weak convergence (with mesh refinement) of approximate solutions of the gas dynamics equations modeled with the standard Godunov's scheme has been studied. Though this scheme is of the first order in the step size h for smooth solutions, numerical simulations have shown that, when modeling discontinuous solutions, the order of accuracy can be estimated by $h^{0.8}$ for principal conservation laws, and by $h^{0.5}$ for the entropy. It turned out that this accuracy is attained in both cases: when one considers exact solutions of the Riemann problem, and when the problem is solved approximately. We refer the reader to the paper [11] for a detailed description of the above mentioned numerical simulations.

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