Chapter 2
Hyperbolic Numbers

Abstract Complex numbers can be considered as a two components quantity, as the plane vectors. Following Gauss complex numbers are also used for representing vectors in Euclidean plane. As a difference with vectors the multiplication of two complex numbers is yet a complex number. By means of this property complex numbers can be generalized and hyperbolic numbers that have properties corresponding to Lorentz group of two-dimensional Special Relativity are introduced.

Keywords Complex numbers · Gauss-Argand · Generalization of complex numbers · Hyperbolic numbers · Space-time geometry · Lorentz group

Complex numbers represent one of the most intriguing and emblematic discoveries in the history of science. Even if they were introduced for an important but restricted mathematical purpose, they came into prominence in many branches of mathematics and applied sciences. This association with applied sciences generated a synergistic effect: applied sciences gave relevance to complex numbers and complex numbers allowed formalizing practical problems. A similar effect can be found today in the "system of hyperbolic numbers", which has acquired the meaning and importance as the *Mathematics of Special Relativity*, as shown in this book.

Let us recall some points from the history of complex numbers and their generalization.

Complex numbers are today introduced with the purpose of extending the field of real numbers and for having always two solutions for the second degree equations and, as an important applicative example, we recall the Gauss *Fundamental theorem of algebra* stating that "all the algebraic equations of degree N has N real or imaginary roots". Further Gauss has shown that complex and real numbers are adequate for obtaining all the solutions for any degree equation.

Coming back to complex numbers we now recall how their introduction has a practical reason. Actually they were introduced in the 16th century for solving a

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mathematical paradox: to give a sense to the real solutions of cubic equations that appear as the sum of square roots of negative quantities (see Sect. 2.6.1). Really the goal of mathematical equations was to solve practical problems, in particular geometrical problems, and if the solutions were square roots of negative quantities, as can happen for the second degree equations, it simply meant that the problem does not have solutions. Therefore it was unexplainable that the real solutions of a problem were given by some "imaginary quantities" as the square roots of negative numbers.

Their introduction was thorny and the square roots of negative quantities are still called *imaginary numbers* and contain the symbol "i" which satisfies the relation $\mathrm{i}^2 = -1$. *Complex numbers* are those given by the symbolic sum of one real and one imaginary number $z = x + \mathbf{i}y$. This sum is a symbolic one because it does not represent the usual sum of "homogeneous quantities", rather a "two components quantity" written as $z = \mathbf{1}x + \mathrm{i}y$, where $\mathbf{1}$ and i identify the two components.

Today we know another two-component quantity: the plane vector, which we write $\mathbf{v} = \mathbf{i}x + \mathbf{j}y$, where $\mathbf{i}$ and $\mathbf{j}$ represent two unit vectors indicating the coordinate axes in a Cartesian representation. Despite there being no a priori indication that a complex number could represent a vector on a Cartesian plane, complex numbers were the first representation of two-component quantities on a Cartesian (or Gauss–Argand) plane (see Fig. 2.1), and they are also used for representing vectors in a Euclidean Cartesian plane.

Now we can ask: what are the reasons that allow complex numbers to represent plane vectors? The answer to this question has allowed us to formalizing the geometry and trigonometry of Special Relativity space–time.

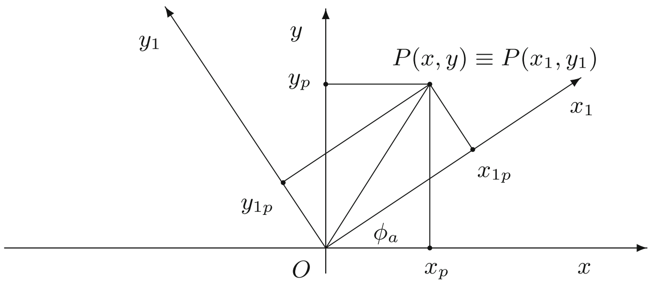
2.1 The Geometry Associated with Complex Numbers

Let us now recall the properties that allow us to use complex numbers for representing plane vectors. The first property derives from the invariant of complex numbers the modulus, indicated with $|z|$, and given by $|z| = \sqrt{(x + \mathrm{i}y)(x - \mathrm{i}y)} \equiv \sqrt{x^2 + y^2}$.

An important property of the modulus is: given two complex numbers z_1, z_2, we have $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

If we represent the complex number $x + \mathrm{i}y$ as a point $P \equiv (x, y)$ of the Gauss–Argand plane (Fig. 2.1), the quantity $\sqrt{x^2 + y^2}$ represents the distance of P from the coordinates origin. This quantity is invariant with respect to translations and rotations of the coordinate axes. Now if in $z = x + \mathrm{i}y$, we give to 1 and i the same meaning of $\mathbf{i}, \mathbf{j}$ in the vectors representation, $|z|$ is the modulus of the vector.

In addition another relevant property allows complex numbers representing plane vectors and the related linear algebra. Actually let us consider the product of a complex constant, $a = a_r + \mathrm{i}a_i$ by a complex number:



$$z = x + \mathrm{i}\,y\,;\ z_1 = x_1 + \mathrm{i}\,y_1\,;\ z_1 = e^{-\mathrm{i}\,\phi_a}\,z$$

$$\overline{OP} = \sqrt{x_p^2 + y_p^2} = |z| \equiv \sqrt{x_{1p}^2 + y_{1p}^2} = |z_1|$$

Fig. 2.1 Gauss representation of complex numbers. The square roots of negative numbers are called "imaginary" and are preceded by the symbol "i", which satisfies the relation $\mathrm{i}^2 = -1$. The expressions $z = x + \mathbf{i}y$, given by the symbolic sum of a real and an imaginary number are called *complex numbers*. We call this sum "symbolic" since it does not represent the usual sum between homogeneous quantities, rather it is a "two component quantity", written as: $z = \mathbf{1}x + \mathbf{i}y$, where **1** and **i** identify the two components. Gauss represented these numbers on a Cartesian plane x, y, associating with the complex number the point P with abscissa x and ordinate y. This "strange representation" can derive from the fact that probably Gauss noted that the product between $z = x + \mathrm{i}y$ and the particular number, called the *complex conjugate* $\bar{z} = x - \mathrm{i}y$ is the real number given by $z \cdot \bar{z} = (x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 + y^2$, that also represent the Euclidean distance of P from the coordinate origin. The square root of this quantity, written as $|z|$, is called the modulus and is characteristic of the complex number. The modulus satisfies the relation: given two complex numbers z_1, z_2, we have $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$, from which another link with Euclidean geometry follows. Actually let us consider the complex constant $a = \cos\phi_a + \mathrm{i}\sin\phi_a$, that can be written (2.4) as $a = e^{\mathrm{i}\phi_a}$, therefore $|a| = e^{\mathrm{i}\phi_a} \cdot e^{-\mathrm{i}\phi_a} = 1$. By considering the product $z_1 = az$, we have $|z_1| = |a||z| = |z|$. This transformation preserves the modulus. Developing the transformation we have: $x_1 + \mathrm{i}y_1 = (\cos\phi_a + \mathrm{i}\sin\phi_a)(x + \mathrm{i}y) = x\cos\phi_a - y\sin\phi_a + \mathrm{i}(x\sin\phi_a + y\cos\phi_a)$. Making equal the real and the imaginary terms, we obtain the relations between the coordinates of the point P after the rotation of the segment $\overline{OP}$ around the axes origin O of an angle ϕ_a. These same expressions represent the transformation of the coordinates of a point in the Cartesian plane, when the reference axes are rotated by the angle $-\phi_a$. We also have $\mathrm{i} = \exp[\mathrm{i}\pi/2]$, then the axes x and y are, automatically orthogonal. These properties show an "unimaginable" correspondence between complex numbers and Euclidean geometry

$$z_1 \equiv x_1 + \mathrm{i}y_1 = az \equiv (a_r + \mathrm{i}a_i)(x + \mathrm{i}y). \tag{2.1}$$

By considering another constant $b = b_r + \mathrm{i}b_i$, we have $z_2 = bz_1 \equiv baz \equiv cz$. Since c, i.e., the result of product between a and b is yet a complex constant, the product of z for a complex constant is a group (see Appendix A.4), called *multiplicative group* [1, Chap. 3]. Now we note that (2.1), is equivalent to the expression of linear algebra

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a_r & -a_i \\ a_i & a_r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.2}$$

In particular, the complex number plays the role of both a vector and an operator (matrix). Actually, the constant a is written in matrix form (like the operators in linear algebra), while z is represented as a column vector.

Now we look for the geometrical meaning of this multiplicative group, beginning with a different representation of complex numbers that starts from the famous Euler formula

$$\exp[\mathrm{i}\phi] = \cos\phi + \mathrm{i}\sin\phi. \tag{2.3}$$

This formula is very important for the following of the book and for this reason we think advisable to recall its first demonstration given by Euler. Actually Euler applied to a complex quantity the series development that is true for the real exponential function and realizes that the real and imaginary terms correspond to the series expansion of cosine and sine functions:

$$\begin{aligned}\exp[\mathrm{i}\phi] &= \sum_{l=0}^{\infty}\frac{(\mathrm{i}\phi)^{l}}{l!} = \sum_{l=0}^{\infty}\frac{(\mathrm{i}\phi)^{2l}}{(2l)!} + \sum_{l=0}^{\infty}\frac{(\mathrm{i}\phi)^{2l+1}}{(2l+1)!}\\ &= \sum_{l=0}^{\infty}(-1)^{l}\frac{(\phi)^{2l}}{(2l)!} + \mathrm{i}\sum_{l=0}^{\infty}(-1)^{l}\frac{(\phi)^{2l+1}}{(2l+1)!} = \cos\phi + \mathrm{i}\sin\phi.\end{aligned} \tag{2.4}$$

Actually, in Euler's time the theory of power series was not sufficiently developed. Therefore it was not known that the displacement of terms, necessary for bringing together the real and imaginary terms, is possible only for absolutely convergent series, a property that the series (2.4) holds. Therefore his procedure is today considered mathematically correct. From (2.4) it follows

$$\exp[-\mathrm{i}\phi] = \cos\phi - \mathrm{i}\sin\phi. \tag{2.5}$$

By multiplying (2.3) · (2.5) we obtain, in an algebraic way, the well known trigonometric relation

$$\begin{aligned}1 &\equiv \exp[\mathrm{i}\phi]\cdot\exp[-\mathrm{i}\phi] \equiv (\cos\phi + \mathrm{i}\sin\phi)(\cos\phi - \mathrm{i}\sin\phi)\\ &= \cos^2\phi + \sin^2\phi.\end{aligned} \tag{2.6}$$

Summing and subtracting (2.4) and (2.5) we obtain a formal relation between the trigonometric functions and the exponential of an imaginary quantity

$$\cos\phi = \frac{\exp[\mathrm{i}\phi] + \exp[-\mathrm{i}\phi]}{2}; \quad \sin\phi = \frac{\exp[\mathrm{i}\phi] - \exp[-\mathrm{i}\phi]}{2\mathrm{i}}. \tag{2.7}$$

We have called (2.7) a formal relations since we cannot give a meaning to the exponential of an imaginary quantity. In any case we see in the following its extension and its relevance.

The introduction of the exponential function of imaginary quantities allows us to introduce the exponential transformation

$$x + \mathrm{i}y = \exp[\rho' + \mathrm{i}\phi] \equiv \exp[\rho'](\cos\phi + \mathrm{i}\sin\phi), \tag{2.8}$$

and setting $\exp[\rho'] = \rho$ we obtain the *polar transformation*

$$x + \mathrm{i}y = \rho\exp[\mathrm{i}\phi] \equiv \rho(\cos\phi + \mathrm{i}\sin\phi). \tag{2.9}$$

$\rho = \sqrt{x^2 + y^2} = |z|$ is called *radial coordinate*, and $\phi = \tan^{-1}[y/x]$*angular coordinate*. If we write the constant a of (2.1) in polar form,

$$a \equiv (a_r + \mathrm{i}a_i) = \rho_a(\cos\phi_a + \mathrm{i}\sin\phi_a),$$

where $\rho_a = \sqrt{a_r^2 + a_i^2}$; $\phi_a = \tan^{-1}[a_i/a_r]$, (2.2) becomes

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \rho_a \begin{pmatrix} \cos\phi_a & -\sin\phi_a \\ \sin\phi_a & \cos\phi_a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \equiv \rho_a \begin{pmatrix} x\cos\phi_a - y\sin\phi_a \\ x\sin\phi_a + y\cos\phi_a \end{pmatrix}. \tag{2.10}$$

We see that the constant a plays the role of an operator representing a homogeneous dilatation ρ_a (homothety) and the transformation for the coordinates of a point P in a rotation, of an angle ϕ_a, around the coordinates origin. Or, changing $\phi_a \to -\phi_a$, for an orthogonal-axis rotation.

If $\rho_a = 1$, and if we add another constant $b = b_r + \mathrm{i}b_i$, then $z_1 = az + b$ gives the permissible vector transformations in a Euclidean plane. For these transformations we have $|z_1| = |a||z| = |z|$, i.e., the modulus of complex numbers or vectors (or the length of a segment) is invariant.

Then, *the additive and unitary multiplicative groups of complex numbers are equivalent to the Euclidean groups of rotations and translations*, *which depends on the three parameters* ϕ_a, b_r, b_i *and*, *as shown in Fig.* 2.1, *complex numbers can be used for describing plane-vector algebra*. Now we can ask if other systems of numbers have similar properties. For inquiring into this possibility we begin by comparing the algebraic properties of sum and product for complex numbers with the ones of plane vectors:

As the sum is concerned it is defined in the same way, for both complex numbers and vectors, as the sum of the components, i.e., given the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we have

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

In the definition of the product there is a relevant difference:

1. the product between two complex numbers is the same as for real numbers just by adding the rule $i^2 = -1$: we have $z_1z_2 \equiv (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$. Therefore *the product of two complex numbers is yet a complex number.*
2. As the plane vectors are concerned, their product is not a vector but rather it is a new quantities derived from physics. In particular the *scalar* and the *vector* products are defined.

So we can summarize: the product between vectors is a new quantity while the product between complex numbers is yet a complex number: *complex numbers are a group also with respect to the product operation*.

Now we recall how this property allows us to generalize the complex numbers [2]. This research for generalization can look as opposed to Gauss theorem that stated it is not necessary the introduction of new number systems more than real and complex numbers, but Gauss referred to solutions of algebraic equations that was the purpose of the introduction of complex numbers. Differently we are now looking for new uses of complex numbers, anyway we see that these systems are related with the kind of roots of the second degree equations.

2.2 Generalization of Complex Numbers

Let us consider a two components quantity written as the complex numbers $z_1 = x_1 + uy_1$ and $z_2 = x_2 + uy_2$, where u represents a general versor[1] for which we have not, a priori, defined the multiplication rule, i.e., the meaning of u^2 and, as a consequence, of all the powers of u. For the product we have

$$z_3 \equiv z_1 z_2 \equiv (x_1 + uy_1)(x_2 + uy_2) = x_1 x_2 + u(x_1 y_2 + x_2 y_1) + u^2 y_1 y_2, \qquad (2.11)$$

we say that z is *a generalized complex numbers, if the result of this multiplication is a number of the same kind*

$$z_3 = q_1(x_1, x_2, y_1, y_2) + uq_2(x_1, x_2, y_1, y_2), \qquad (2.12)$$

where q_1, q_2 are quadratic forms as function of the components.

We obtain this result setting u^2 as a linear combination of 1 and of the versor u: $u^2 = \alpha + u\beta;\ \alpha, \beta \in \mathbf{R}$ [4]. Actually with this position (2.11) becomes

$$z_3 = x_1 x_2 + \alpha y_1 y_2 + u(x_1 y_2 + x_2 y_1 + \beta y_1 y_2) \qquad (2.13)$$

In this way the generalized complex numbers are a group with respect to the product. These numbers are also indicated by

$$\{z = x + uy;\ u^2 = \alpha + u\beta;\ x, y, \alpha, \beta \in \mathbf{R},\ u \notin \mathbf{R}\}, \qquad (2.14)$$

[1] The name *versor* has been firstly introduced by Hamilton for the unitary vectors of his quaternions [3]. This name derives from the property of the imaginary unity “i” since, as can be seen from Euler formulas, multiplying by “i” is equivalent to “rotate”, in a Cartesian representation, the complex number of $\pi/2$. Since this property also holds for the hyperbolic numbers that we are going to introduce, we use this name that also states the difference with the unitary vectors of linear algebra.

In the theory of hypercomplex numbers [1, Chap. 2], the constants α, β are called structure constants and, as we see in this two-dimensional example, *from their values derive the properties of the three systems of two-dimensional numbers*.

It is known that complex numbers are considered as an extension of real numbers, as regarding the division. Actually as for real numbers it is ever possible except for the null element $x = 0, y = 0$. Now we see how the general complex numbers can be classified by means of their property about the division.

Actually for the division, given a number $a + ub$, one has to look for a number $z = x + uy$ such that

$$(a + ub)(x + uy) = 1. \tag{2.15}$$

If (2.15) is satisfied, the inverse z of $a + ub$ exists and we can divide any number by $a + ub$, by multiplying it by z. Thanks to the multiplication rule (2.14), (2.15) is equivalent to the real system obtained by equating the coefficients of the versors 1 and u

$$\begin{aligned} ax + \alpha by &= 1, \\ bx + (a + \beta b)y &= 0, \end{aligned} \tag{2.16}$$

As it is known from the theory of linear systems, (2.16) has a solution if the determinant of the coefficients, given by

$$D = a^2 + \beta ab - \alpha b^2 \equiv \left(a + \frac{\beta}{2}b\right)^2 - \left(\alpha + \frac{\beta^2}{4}\right)b^2, \tag{2.17}$$

is different from zero. Actually if $D = 0$ the associated homogeneous system $ax + \alpha by = 0, bx + (a + \beta b)y = 0$ admits non-null solutions. These numbers for which the product between two non-null numbers $a + ub$ and $x + uy$, is zero are called divisors of zero [1, Chap. 2]. The origin of this name derives from the exposed considerations: actually, for these numbers we can formally write

$$a + ub = \frac{0}{x + uy},$$

therefore dividing zero for $x + uy$ we obtain the finite quantity $a + ub$.

For studying this property (2.17), in the last passage, has been divided into two terms: now we inquire into the second one, by setting

$$\Delta = \beta^2 + 4\alpha \tag{2.18}$$

we see that the sign of the real quantity Δ determines the possibility of executing the division between two numbers. Actually, let us consider in the (β, α) plane the parabola, obtained by setting $\Delta = 0$

$$\alpha = -\beta^2/4, \tag{2.19}$$

that divides the plane into three regions (see Fig. 2.2). In these regions we have $\Delta > 0, \Delta = 0, \Delta < 0$.

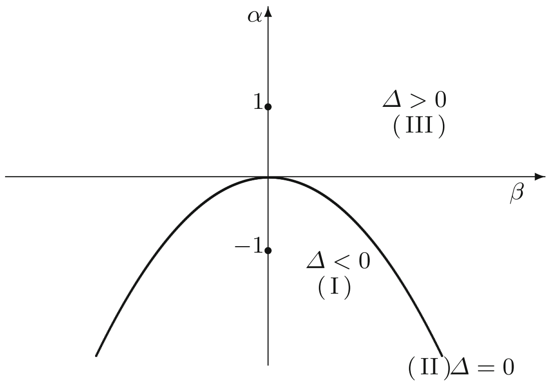


Fig. 2.2 The three types of two-dimensional algebras. Complex numbers can be considered as a two components quantity and are used to representing plane vectors. On the other hand, together with this correspondence (Sect. 2.1) there is a relevant difference between complex numbers and vectors: the definition of the product. Actually for complex numbers, just adding the rule $i^2 = -1$, it is an extension of the product between real numbers, i.e., the result is a complex number. As the vectors are concerned, their product is not a vector but a new quantity introduced from physics. In particular the *scalar product* and the *vector product* are defined. *With respect to multiplication complex numbers are a group, vectors are not*. Thanks to this property complex numbers can be generalized and two other two-dimensional systems of numbers are introduced [4, 5]. Actually let us consider a two-components quantity we write as the complex numbers: $z = x + uy$, where u represents a generic versor. If we request that the product between two numbers z_1 and z_2 is a number of the same kind, i.e., z is a multiplicative group, u^2 must be defined as a linear combination of the versors 1 and u of the number z, i.e., we must set $u^2 = \alpha + u\beta$; $\alpha, \beta \in \mathbf{R}$ (Sect. 2.2). The values of α, β determine the properties of the system of numbers. In particular, by considering in the (β, α) plane the parabola $\alpha = -\beta^2/4$: the position of the point $P \equiv (\beta, \alpha)$ with respect to parabola, determines if the division, except for $x = y = 0$, is possible. This property allows us to classify the two-dimensional numbers into three types: (I) Inside the parabola $(\Delta < 0)$ we call these systems elliptic numbers. In particular for $\beta = 0, \alpha = -1$ we have the complex numbers. The division is ever possible. (II) On the parabola $(\Delta = 0)$, we call these systems parabolic numbers. The division is not possible for the numbers $2x + \beta y = 0$. (III) Outside the parabola $(\Delta > 0)$, we call these systems hyperbolic numbers. The division is not possible for the numbers $2x + (\beta \pm 2\sqrt{\Delta})y = 0$. All these three systems have a geometrical or physical relevance. Actually complex numbers can represent Euclidean geometry, the group of parabolic numbers is equivalent to Galileo's group of classical dynamics [6] and hyperbolic numbers, as we at length see in this book, represent the Lorentz's group of Special Relativity

The position of a point $P \equiv (\beta, \alpha)$ with respect to parabola, determines three types of systems and we have

Theorem 2.1 *We can classify the general two-dimensional numbers into three classes according to the position of point $P \equiv (\beta, \alpha)$ with respect to parabola* (2.19).
The numbers of the same type have also in common:

1. *the characteristic property of the modulus, i.e., the definition of distance that relates the system of numbers with a geometry*;
2. *the topological properties of the representative plane.*

Proof By referring to Fig. 2.2, we have

1. If $P \in (I), \Delta < 0$ and D, as the sum of two squares, is never negative and it is equal to 0 just for $a = b = 0$. Therefore any non-null element has an inverse and, as a consequence, division is possible for any non-null number. These systems are called **elliptic numbers**.
2. If P is on the parabola, $D = (a + \beta b/2)^2$ is zero if a, b are on the straight line $a + \beta b/2 = 0$. Each of them admits divisors of zero satisfying $x + (\beta/2)y = 0$. Division is possible for all the other numbers. These systems are called **parabolic numbers**.
3. If $P \in (III)$, the system (2.16) has solutions for a, b on the straight lines $a + (\beta \pm \sqrt{\Delta})b/2 = 0$. Each of them admits divisors of zero satisfying $x + (\beta \pm \sqrt{\Delta})/2y = 0$. Division is possible for all the other numbers.
   These systems are called **hyperbolic numbers**.
   The divisor of zero determines the topology of the representative plane: this plane is divided in four sectors. □

We can say that *the types of the general two-dimensional systems derive from the kinds of solutions of the second degree equation* (2.17) *in a/b, obtained by setting $D = 0$.*

Let us now see the derivation of the nouns for the systems. Actually, let us consider the conic with equation

$$x^2 + \beta xy - \alpha y^2 = 0, \tag{2.20}$$

obtained from (2.17), by considering a, b as variables in Cartesian plane. According to whether $\Delta \equiv \beta^2 + 4\alpha$ is <0, $=0$, >0, the curve is an ellipse, a parabola or a hyperbola. This is the reason for the names used for the three types of the general two-dimensional numbers. For the three cases, we define the **canonical systems** by setting $\beta = 0$ and

1. $\alpha \equiv u^2 = -1$. This is the case of the ordinary complex numbers. For these numbers we set, as usual $u \Rightarrow \mathrm{i}$.
2. $\alpha \equiv u^2 = 0$.
3. $\alpha \equiv u^2 = 1$. This system is related to the pseudo-Euclidean (space–time) geometry, as we see in this book. For this system we set $u \Rightarrow \mathrm{h}$. In this case the divisors of zero satisfy $y = \pm x$. In a Cartesian representation they are represented by the axes bisectors.

One can verify that any system can be obtained, from its canonical system, by a linear transformation of the versors (with the inverse transformation for the variables), i.e., it is isomorphic to the canonical system.

2.2.1 Definition of the Modulus

We note that the left-hand side of (2.20), for $\beta = 0, \alpha = -1$ represents the squared modulus of complex numbers, i.e., the real and invariant quantity obtained by multiplying $z \cdot \bar{z}$. Now we see how this quantity can be defined for a generic algebra. Actually looking at the last term of (2.17), we note that it can be written as $\Delta = 4\alpha + \beta^2 \equiv (2u - \beta)^2$: in this way D is the difference between two squared terms and can be written as the product of two linear terms that by substituting $a, b \Rightarrow x, y$ becomes $(x + uy)(x + \beta y - uy) \in \Re$. Therefore *for the generic algebra we define the number* $\bar{z} = x + (\beta - u)y$, *that multiplied for* $x + uy$*gives a real quantity, as the complex conjugate of* $z = x + uy$.

We conclude this generalization of complex numbers by recalling the definition of [5, p. 11][2]: ◁ The fact that the most general complex numbers can be added, subtracted and multiplied, all the usual laws of these operations being conserved, but that it is not always possible divide one by another, is expressed by saying that such numbers form a ring. ▷

The representation of Euclidean geometry by means of complex numbers and the equivalence, from the algebraic point of view, between complex and hyperbolic numbers let us suppose that also the hyperbolic numbers, can be associated with a geometry. Now we see that their geometry is the one of special relativity.

2.3 Lorentz Transformations and Space–Time Geometry

We briefly recall how Lorentz transformations of Special Relativity were established.

For some decades, at the end of 19th century the Newton dynamics and gravitational theory together with Maxwell equations of electro-magnetic field were considered adequate for a complete description of physical world: the mechanics and gravitation law formalize the motions on the Earth and of celestial bodies, Maxwell equations, besides the technical and scientific relevance, also explain the light propagation.

Actually these two theories and the effort to put them in a same logical frame, brought about the starting ideas for the "scientific revolutions" of 20th century, that are today considered very far of being concluded.

We begin by setting out their different mathematical nature and the role the time holds.

1. The Newton dynamics equations give the bodies positions as functions of time. The time acts as a parameter.

[2] We use ◁…▷ to identify material that reports the original author's words or is a literal translation.

2. The Maxwell equations allow us to calculate the electric and magnetic fields, from static and moving charges. These fields depend in an equivalent way on space coordinates and time.

From a mathematical point of view the Newton equations are ordinary differential equations, Maxwell equations are a partial differential system.

Moreover besides these mathematical differences there were theoretical considerations and experimental results, as the Michelson and Morley experiment, that we directly recall, that stated that Newton and Maxwell equations are not equivalent also for a physical point of view.

The result of this debate was that Poincaré and Einstein, in the same year (1905), looked for the variable transformations that leaved the same expressions of Maxwell equations when one considers two reference systems in uniform relative motion. This requirement is the same as the invariance of Newton dynamics equations with respect to Galileo's group.

Both the scientists obtained the today known *Lorentz transformations* of special relativity.

Since Maxwell equations depend in an equivalent way from both time and space variables, also the transformations depend in an equivalent way on these variables.

For practical purposes the Lorentz transformations, notwithstanding can be considered as elementary from a mathematical point of view, have represented, for the connexion between space and time, a "revolution" with respect to settled philosophical concepts about "*time*".

The Poincaré and Einstein works reflect their professionalism and their interpretation of the results are complementary:

- Poincaré, one of the most important and encyclopedic mathematician at the turn of the century, associated these transformations with group theory (today known as Lorentz–Poincaré's group).
- Einstein, young physicist, had the cheek to extend the transformation laws relating space with time, obtained for Maxwell equations, to dynamics equations. This extension, together with the paper about photoelectric effect published by Einstein in the same year, was the basis for the most important scientific results of 20th century. Actually the results of both these works entail the equivalence between waves and corpuscles.

Now we briefly recall Einstein's formulation, who gives to the obtained transformations the physical meaning today accepted, in particular the extension to Newton dynamics of the obtained relation between space and time.

Einstein was able to obtain in a straightforward way and by means of elementary mathematics the today named Lorentz transformations, starting from the two postulates

1. all inertial reference frames must be equivalent
2. light's velocity is constant in all inertial systems.

The first postulate is the same stated by Galileo and applied to the laws of dynamics, that starts from the principle that by means of physical experiments we cannot detect the state of relative uniform motion.

The second one takes into account the results of experiments carried out by Michelson and Morley. These experiments have shown that *the speed of the light is the same in all inertial systems* and is independent of the direction of the motion of the reference frame relative to the ray of light. Actually also this experimental result, in contrast with the traditional physics, stimulated the search for new theories which could explain it.

For the formalization of the problem let us consider a reference system (t, x) and another (t_1, x_1) in motion with constant speed v_1 with respect to the first one. In this description t represents the time multiplied by light's velocity ($c = 1$) and v_1 the speed divided by c. The obtained transformations are

$$x_1 = \frac{x + v_1 t}{\sqrt{1 - v_1^2}}, \quad t_1 = \frac{v_1 x + t}{\sqrt{1 - v_1^2}}. \tag{2.21}$$

From these equations we note two relevant differences with respect to classical dynamics

1. both the length x and the time t depend on the speed of the reference frame in which are measured,
2. the square root of the quantity $t^2 - x^2 = t_1^2 - x_1^2$ is invariant and is called proper time.

The dependence of time on the speed of the reference system originated the "twin paradox". This problem is exhaustively formalized in Chap. 6.

Now let us consider a third system (t_2, x_2) in motion with speed v_2, with respect to system (t_1, x_1). The transformation equations from the first system and this one are again (2.21) with the substitution $v_1 \to v_T$ where v_T is given by

$$v_T = \frac{v_1 \pm v_2}{1 \pm v_1 v_2}, \tag{2.22}$$

where the $+$ and $-$ signs refer if v_1 and v_2 have the same or different directions. Therefore, as recalled in Appendix A.4, the relations (2.21) represent a group, i.e., two repeated transformations have the same expression as (2.21) with a speed v_T that is a function of the speeds (parameters) of the component transformations. From (2.22) it follows that if v_1 or v_2 are equal to 1 (one reference system have the speed of the light), we have $v_T = 1$. *The light's velocity is a limiting speed.*

Therefore a line in the t, x plane can represent the motion of a body only if the tangent lines have an angular coefficient $dx/dt \equiv v < 1$. These curves (or lines) are called time-like. In a similar way the lines that have an angular coefficient $dx/dt \equiv v > 1$ are called space-like and if $dx/dt \equiv v = 1$ are called light-like.

From the considerations of Appendix A it follows:

we can introduce a geometry that, with respect to the distance of Euclidean geometry, has as invariant the square root of the quantity $t^2 - x^2$, *i.e., the proper time*.

For this geometry the transformations (2.21) represent the motions.

This geometry is called pseudo-Euclidean or Minkowskian geometry.

2.4 The Geometry Associated with Hyperbolic Numbers

Let us now apply to hyperbolic numbers the same considerations that allow one to associate complex numbers with Euclidean geometry. Let us consider the system of hyperbolic numbers defined as

$$\{z = x + \mathrm{h}\, y; \mathrm{h}^2 = 1; x, y \in \mathbf{R}, \mathrm{h} \notin \mathbf{R}\}.$$

As for complex numbers, we call $\tilde{z} = x - \mathrm{h}\, y$ the hyperbolic conjugate[3] and define the modulus as $|z| = \sqrt{|z\,\tilde{z}|} \equiv \sqrt{|x^2 - y^2|}$. Therefore, by giving to one variable the physical meaning of *time*, the modulus can be recognized as the invariant of the two-dimensional special relativity (proper time). Now we see that *hyperbolic geometry is equivalent with the* "*geometry of special relativity*". Let us now see its properties (Fig. 2.3).

As for complex numbers, given two hyperbolic numbers, z_1, z_2, we have $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

Let us now consider the *multiplicative group* given by the product of z for a hyperbolic constant

$$z_1 = az \equiv (a_r + ha_h)(x + \mathrm{h}\, y), \tag{2.23}$$

this group can be expressed in vector–matrix form by

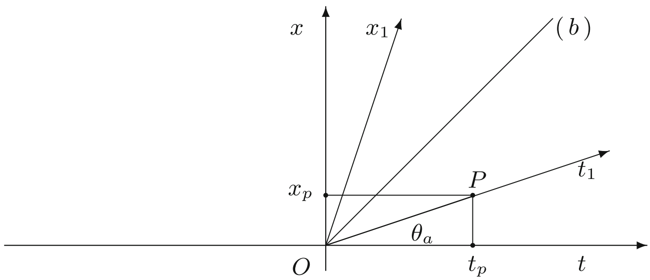
$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a_r & a_h \\ a_h & a_r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.24}$$

This result is the same as (2.2): the multiplicative constant is an operator that acts on the vector (x, y).

As for complex numbers we can introduce the Hyperbolic exponential function and hyperbolic polar transformation.

In hyperbolic geometry these transformations play the same important role as the corresponding complex ones in Euclidean geometry. In [1, Chap. 7] the functions of a hyperbolic variable are introduced and analogies and differences with respect to functions of a complex variable are pointed out. Here we define the exponential function of a hyperbolic number following the method that Euler used for introducing the complex exponential with his famous formula (2.3). Actually going on as

[3] Here and in the following we use the symbol $\tilde{\cdot}$ for indicating the hyperbolic conjugate.



$$z = t + \mathrm{h}\, x\,;\;\; z_1 = t_1 + \mathrm{h}\, x_1\,;\;\; z_1 = e^{-\mathrm{h}\,\theta_a}\, z \;\Rightarrow |z| = |z_1|$$

$$\overline{OP} = \sqrt{t_p^2 - x_p^2} = |z| = |z_1| = \sqrt{t_{1p}^2 - x_{1p}^2} = t_{1p} < t_p$$

Fig. 2.3 Geometry in pseudo-Euclidean plane The Euclidean geometry is defined by the invariance of geometrical figures with respect to their rotations and translations. Or, in a Cartesian representation, with respect to the reference axes rotations. These same two criterion can be applied to Lorentz transformations of special relativity. Following the Newton dynamics, the motion of a body is represented in a Cartesian reference frame x, y, by a curve expressed as function of a parameter t, to which we can give the physical meaning of *time*. From special relativity, formalized by (2.21), the "time" is "equivalent" to space then, in a representation on a plane, one reference axis must represent the time. Therefore, in this plane, a curve $x = x(t)$ represents the motion of a body. In particular a straight line $x = \beta t, \beta < 1$ represents a uniform motion. As we do for Euclidean geometry, represented in a Cartesian plane (Fig. 2.1), we can consider a second reference frame t_1, x_1 for which the relation between the old and new variables, corresponding to the Euclidean (2.10), is given by (2.21). In this transformation, as it has been shown for the first time by Minkowski after whom the space–time geometry is named, the transformed axes are not yet orthogonal. As it is shown in the figure, they go in a symmetric way toward the axes bisector (b) $t = x$. As we see in Fig. 3.2 these axes are yet "orthogonal in the hyperbolic geometry". Minkowski, by means of this geometrical representation, studied the dependence of t_1, x_1 on the body's velocity. The second possibility is the one studied in this book, i.e., to stay in a representative Cartesian plane by applying in this plane the geometry that leaves as invariant the "space–time" distance. We observe that while the axes rotation allow us to study just the uniform motions, represented by (2.21), this second approach allows us to quantify the "relativistic effect" for every motion. Actually for all the lines in t, x plane that represent a motion ($\beta < 1$), its "relativistic length" is the *proper time*. These lengths can be evaluated for all the lines and then the differences or the ratios between the respective proper times. In particular (see Chap. 6) for uniform and uniformly accelerated motions, and all their compositions, these calculations are performed by elementary methods (Figs. 6.1–6.5); for a general motion by means of differential geometry (Fig. 6.6.)

for (2.4), taking into account that the even powers of h are equal to 1 and the odd powers to h, we can recognize that the real and hyperbolic parts of the series developments correspond to the hyperbolic trigonometric functions, and we have

$$\begin{aligned} \exp[\mathrm{h}\,\theta] &= \sum_{l=0}^{\infty} \frac{(\mathrm{h}\,\theta)^{l}}{l!} = \sum_{l=0}^{\infty} \frac{(\mathrm{h}\,\theta)^{2l}}{(2l)!} + \sum_{l=0}^{\infty} \frac{(\mathrm{h}\,\theta)^{2l+1}}{(2l+1)!} \\ &= \sum_{l=0}^{\infty} \frac{(\theta)^{2l}}{(2l)!} + \mathrm{h} \sum_{l=0}^{\infty} \frac{(\theta)^{2l+1}}{(2l+1)!} = \cosh\theta + \mathrm{h}\sinh\theta. \end{aligned} \tag{2.25}$$

In particular $\cosh\theta = 1 +$ even powers, then $\cosh\theta > 1$.

By means of exponential function we introduce the exponential transformation

$$z \equiv x + \mathrm{h}\, y = \exp[\rho' + \mathrm{h}\,\theta] \equiv \exp[\rho'](\cosh\theta + \mathrm{h}\sinh\theta), \tag{2.26}$$

and setting $0 < \exp[\rho'] = \rho$ we obtain the *hyperbolic polar transformation*

$$z \equiv x + \mathrm{h}\, y = \rho \exp[\mathrm{h}\,\theta] \equiv \rho(\cosh\theta + \mathrm{h}\sinh\theta). \tag{2.27}$$

Where, as for complex numbers, ρ is called *radial coordinate*, and θ is called *angular coordinate*.

By comparing the real and the hyperbolic parts we can obtain, as for the polar transformation $\rho = \sqrt{x^2 - y^2} \equiv |z|$ and $\theta = \tanh^{-1}(y/x)$. From (2.27) we have

$$\tilde{z} \equiv x - \mathrm{h}\, y = \rho \exp[-\mathrm{h}\,\theta] \equiv \rho(\cosh\theta - \mathrm{h}\sinh\theta). \tag{2.28}$$

Setting $\rho = 1$ and multiplying (2.27) by (2.28) we obtain the relevant relations between the hyperbolic trigonometric functions

$$\begin{aligned} 1 &\equiv \exp[\mathrm{h}\,\theta]\cdot\exp[-\mathrm{h}\,\theta] \equiv (\cosh\theta + \mathrm{h}\sinh\theta)(\cosh\theta - \mathrm{h}\sinh\theta) \\ &= \cosh^2\theta - \sinh^2\theta. \end{aligned} \tag{2.29}$$

From this relation and the previous one $(\cosh\theta > 1)$, it follows $\cosh\theta > \sinh\theta$, therefore the polar representation (2.27) holds for $x > y, x > 0$. In Chap. 3 we see how it is possible to extend it for representing points in the whole x, y plane.

Let us come back to the multiplicative groups (2.23) and, as for complex numbers, let us write the constant

$$a \equiv a_r + \mathrm{h}\, a_h = \rho_a(\cosh\theta_a + \mathrm{h}\sinh\theta_a) \tag{2.30}$$

in hyperbolic polar form, (2.24) becomes

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \rho_a \begin{pmatrix} \cosh\theta_a & \sinh\theta_a \\ \sinh\theta_a & \cosh\theta_a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \equiv \rho_a \begin{pmatrix} x\cosh\theta_a + y\sinh\theta_a \\ x\sinh\theta_a + y\cosh\theta_a \end{pmatrix}. \tag{2.31}$$

By considering, as for complex numbers, the transformation with $\rho_a = 1$ the modulus, i.e., the square root of the quantity $(x^2 - y^2)$, equivalent to proper time, is invariant. These transformations shall be called hyperbolic rotations and, we are going to see, they represent the Lorentz transformations of Special Relativity.

2.4.1 Hyperbolic Rotations as Lorentz Transformations

Let us write a space–time vector as a hyperbolic variable,[4] $w = t + \mathrm{h}\,x$ and consider a unitary hyperbolic constant $a = a_r + \mathrm{h}\,a_\mathrm{h}; a_r^2 - a_h^2 = 1$. If we give to the components of constant a the physical meaning given to the variables, a_r corresponds to time and a_h to a space variable. Therefore $a_\mathrm{h}/a_r \equiv x/t$ has the meaning of a velocity v. If a represents a physical motion $(v<1)$, it must be $a_r > a_h$ (with this position a is a time-like constant). Setting a in polar form (2.30), we have

$$\begin{aligned} a_r + \mathrm{h}\,a_\mathrm{h} &\equiv \exp[\mathrm{h}\,\theta_a] \equiv \cosh\theta_a + \mathrm{h}\sinh\theta_a \\ \text{where} \quad \theta_a &= \tanh^{-1}[a_\mathrm{h}/a_r] \equiv \tanh^{-1}[v]. \end{aligned} \tag{2.32}$$

Transformation (2.31) becomes

$$t_1 + \mathrm{h}\,x_1 = t\cosh\theta_a + x\sinh\theta_a + \mathrm{h}(t\sinh\theta_a + x\cosh\theta_a). \tag{2.33}$$

By considering as equal the coefficients of versors "1" and "h", as we do in complex analysis, we get the Lorentz transformation of two-dimensional special relativity [7]. Actually, from the second of (2.32) we have $\tanh\theta_a = v$, and, by means of the relation (2.29), we have

$$\sinh\theta_a = \frac{v}{\sqrt{1-v^2}}, \quad \cosh\theta_a = \frac{1}{\sqrt{1-v^2}}. \tag{2.34}$$

These relations allow us to verify that (2.33) are the same as (2.21).

In addition the composition (2.22) of speeds of two motions is given by the sum of the hyperbolic angles corresponding to the two speeds (see 4.24).

We also have

Theorem 2.2 *The Lorentz transformation is equivalent to a "hyperbolic rotation".*

Proof By writing the hyperbolic variable $t + \mathrm{h}\,x$ in exponential form (2.26)

$$t + \mathrm{h}\,x = \rho\exp[\mathrm{h}\,\theta],$$

the Lorentz transformation (2.33), becomes

$$t_1 + \mathrm{h}\,x_1 = a(t + \mathrm{h}\,x) \equiv \rho\exp[\mathrm{h}(\theta + \theta_a)]. \tag{2.35}$$

From this expression we see that the Lorentz transformation is equivalent to a "hyperbolic rotation" of the angle θ_a, of the $t + \mathrm{h}\,x$ variable. □

---

[4] In all the problems which refer to Special Relativity (in particular in Chap. 6) we change the symbols by indicating the variables with letters reflecting their physical meaning $x, y \Rightarrow t, x$, i.e., t is a normalized time variable (light velocity $c = 1$) and x a space variable.

This correspondence allows us to call hyperbolic and pseudo-Euclidean the representative plane of space–time (Minkowski's) geometry and trigonometry. We note that to writing the Lorentz transformations by means of hyperbolic trigonometric functions, is normally achieved by following a number of "formal" steps, i.e., by introducing an "imaginary" time $t' = it$ which makes the Lorentz invariant $(x^2 - t^2)$ equivalent to the Euclidean invariant $(x^2 + y^2)$, and by introducing the hyperbolic trigonometric functions through their equivalence with circular functions of an imaginary angle. We stress that this procedure is essentially formal, while the approach based on hyperbolic numbers leads to *a direct description of Lorentz transformation explainable as a result of symmetry* (*or invariants*) *preservation*: the Lorentz invariant (space–time "distance") is the invariant of hyperbolic numbers and *the unimodular multiplicative group of hyperbolic numbers represents the Lorentz transformations*, *as the unimodular multiplicative group of complex numbers represents the rotations in a Euclidean plane*.[5]

Therefore we conclude:

For the description of the physical world the hyperbolic numbers have the same relevance of complex numbers.

And, following Beltrami (see Sect. A.3), we can say: results that seem contradictory with respect to Euclidean geometry are compatible with another geometry as simple and relevant as the Euclidean one.

2.5 Conclusions

The association of hyperbolic numbers with the two-dimensional Lorentz's group of Special Relativity makes hyperbolic numbers relevant for physics and stimulate us to find their application in the same way as complex numbers are applied to Euclidean plane geometry.

In Chap. 4 we see that it is possible go over with respect to this project. Actually we show that the link between complex numbers and Euclidean geometry allow us to formalize, in a Cartesian plane, the trigonometric functions as a direct consequence of Euclid's rotation group (Sect. 4.1.1). This result allows us to show that all the trigonometry theorems can be obtained by an analytical method, as mathematical identities, instead of the usual method of Euclidean geometry and trigonometry for which theorems are demonstrated by means of the axiomatic-deductive method and geometrical observations. Afterward, taking into account that hyperbolic numbers have the same algebraic properties of complex numbers, these approaches to Euclidean geometry and trigonometry are extended to the space–time and, by means of hyperbolic numbers, the theorems are demonstrated

[5] Within the limits of our knowledge, the first description of Special Relativity, directly by these numbers was introduced by I. M. Yaglom [6].

through an algebraic method that replaces the absence, in space–time plane, of the intuitive Euclidean observations.

In this way, we obtain *the complete formalization of space–time geometry*, *by means of the axiomatic-deductive method*, *starting from experimental axioms*, *thus equivalent to Euclid's geometry construction*.

Therefore the problems in Minkowski space–time are solved as we usually do in the Euclidean Cartesian plane.

2.6 Appendix

2.6.1 Cubic Equation and Introduction of Complex Numbers

All the third degree equations were reduced to

$$x^3 + px + q = 0 \tag{2.36}$$

by the mathematicians Nicolò Fontana (named Tartaglia) and Girolamo Cardano (*Ars magna*, Nurberg, 1545) and they found the solution

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}. \tag{2.37}$$

This equation has three real roots if

$$\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 < 0.$$

This negative quantity appears under a square root, then a paradox grows that the solution of a geometrical problem is obtained by means of quantity that does not have a geometrical meaning.

Today we say that these solutions are the sum of two complex conjugate quantities, therefore the final result is real.

Raffaele Bombelli, at the end of 16th century has introduced (*Algebra*, Bologna, 1572), complex numbers for the solution of cubic equations, by formulating, practically in modern form, the four operations with complex numbers and introducing the expression that today we write $a + \mathrm{i}b$.[6] These numbers have been called *imaginary* by Descartes and this name is also the actual one. The way

[6] Bombelli writes: ◁ … also if this introduction can appear as an extravagant idea and I considered it, for some time, as sophistical rather than true, I have found the demonstration that it works well in the operations. ▷

for the modern formalization has required more than two centuries and has been completed by Euler and Gauss by:

- the introduction of the imaginary unity i,
- to call *complex numbers* the binomial $a + \mathrm{i}b$
- to introduce the *functions of a complex variable*, for relevant physical (Euler: the motion of fluids) and geometrical (Gauss: conformal mapping) applications.

2.6.2 Geometrical and Classical Definition of Hyperbolic Angles

2.6.2.1 Geometrical Meaning of Hyperbolic Angle

We have seen that the circular trigonometric functions could be introduced, in a formal way, by means of Euler's formula (2.7). In a similar way the hyperbolic trigonometric functions can be introduced. Actually by summing and subtracting (2.27) and (2.28) for $\rho = 1$ we obtain a formal relation between the hyperbolic trigonometric functions and exponential function of a hyperbolic quantity

$$\cosh\theta \equiv x = \frac{\exp[\mathrm{h}\,\theta] + \exp[-\mathrm{h}\,\theta]}{2}; \quad \sinh\theta \equiv y = \frac{\exp[\mathrm{h}\,\theta] - \exp[-\mathrm{h}\,\theta]}{2\mathrm{h}}. \tag{2.38}$$

We can check that also from these relations (2.29) follows. From (2.29) we can obtain the geometrical interpretation of hyperbolic trigonometric functions. Actually let us consider in the x, y Cartesian plane, the curve

$$x = \cosh\theta, \quad y = \sinh\theta, \tag{2.39}$$

as function of the parameter θ.

This curve, taking into account that $\cosh\theta > \sinh\theta$ and $\cosh\theta > 1$ represents the right arm of the unitary equilateral hyperbola $x^2 - y^2 = 1$.

Therefore, by analogy with the circular angles defined on the unitary circle, we can call θ the hyperbolic angle and define $\cosh\theta$ and $\sinh\theta$ as the abscissa and the ordinate of the hyperbola point defined by θ, respectively (see Fig. 2.4).

Now we see another correspondence with the circular trigonometric functions. Actually we know that trigonometric angles, measured by radiants, are equal to twice the area of the circular sectors they identify. The same is true for hyperbolic angles. We have

Theorem 2.3 *The hyperbolic angle θ is twice the area of the sector OVP* (*Fig*. 2.4)

Proof The area of the sector *OVP* is given by the difference between the areas of triangle *OHP* and *VHP*. Therefore, by means of (2.39), we have

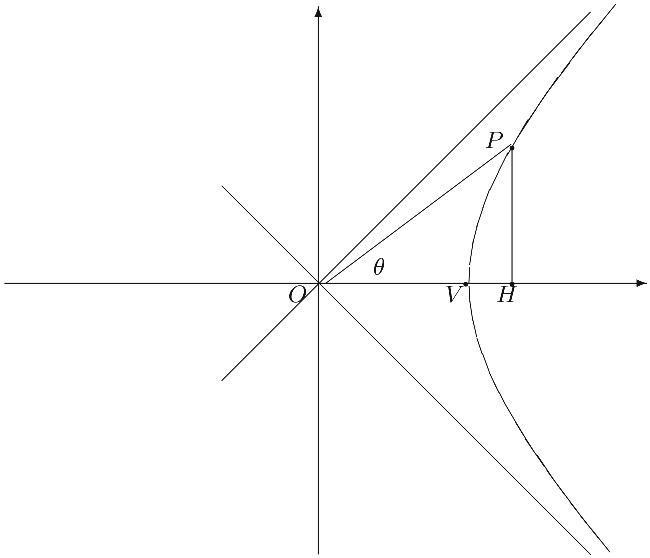


Fig. 2.4 Geometrical definition of hyperbolic angles. The trigonometric circular functions are defined by means of goniometric circle. In a similar way the hyperbolic trigonometric functions can be defined by means of the unitary equilateral hyperbola. Actually let us consider the right arm of hyperbola $x^2 - y^2 = 1$ and define an angle θ corresponding to half-line OP so that $\cosh\theta = \overline{OH}, \sinh\theta = \overline{HP}$. In Appendix 2.6.2 we see that, as for circular angles measured in radiants, also to hyperbolic trigonometric angles θ we can give the geometrical meaning of an area $\theta = 2\text{area}(OVP)$. In Chap. 4, we also see that this area has the same value measured in both "hyperbolic" or "Euclidean" way

$$
\begin{aligned}
\text{area}(OHP) &= \frac{1}{2}\sinh\theta\cosh\theta = \frac{\sinh 2\theta}{4} \\
\text{area}(VHP) &= \int_0^\theta y\,dx \equiv \int_0^\theta \sinh^2\theta\,d\theta = \frac{\sinh 2\theta}{4} - \frac{\theta}{2} \\
\text{area}(OVP) &\equiv \text{area}(OHP) - \text{area}(VHP) = \frac{\theta}{2}.
\end{aligned}
\tag{2.40}
$$

The integral is solved by means of (4.28). □

Now we see how this definition allows one to introduce the classical hyperbolic trigonometric functions.

2.6.2.2 Classical Definition of Hyperbolic Trigonometric Functions

Let us consider the equation of hyperbola in Cartesian coordinates $x^2 - y^2 \equiv (x+y)(x-y) = 1$ that can also be written

$$y = \pm\sqrt{x^2 - 1} \tag{2.41}$$

$$x - y = \frac{1}{x + y} \tag{2.42}$$

we have

$$\text{area}(OHP) = \frac{xy}{2} \equiv \frac{x\sqrt{x^2 - 1}}{2} \tag{2.43}$$

$$\begin{aligned} \text{area}(VHP) &= \int_1^x y dx \equiv \int_1^x \sqrt{x^2 - 1} dx \equiv \frac{x\sqrt{x^2 - 1} - \ln(x + \sqrt{x^2 - 1})}{2} \\ &\equiv \frac{x\sqrt{x^2 - 1} - \ln(x + y)}{2} \end{aligned} \tag{2.44}$$

$$\text{area}(OVP) \equiv \text{area}(OHP) - \text{area}(VHP) = \frac{\ln(x + y)}{2}. \tag{2.45}$$

Comparing (2.40) with (2.45) we have

$$\theta = \ln(x + y) \Rightarrow x + y = \exp[\theta], \tag{2.46}$$

and from (2.42)

$$\ln(x - y) = -\ln(x + y) \Rightarrow \ln(x - y) = -\theta \Rightarrow x - y = \exp[-\theta]. \tag{2.47}$$

Summing and subtracting (2.46) and (2.47), the classical definition follows

$$\cosh\theta \equiv x = \frac{\exp[\theta] + \exp[-\theta]}{2}, \quad \sinh\theta \equiv y = \frac{\exp[\theta] - \exp[-\theta]}{2}. \tag{2.48}$$

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