Chapter 2
Solution concepts

For more than five decades, vector optimization has been a subject of intensive research. A common notation for a vector optimization problem is

$$\operatorname*{Min}_{x \in S} f(x), \tag{VOP}$$

where f is a vector-valued function and S is a feasible set. The central question of this chapter is the following:

What is a *solution to* (VOP)?

It is rather surprising that there is no standard answer to this fundamental question in textbooks on vector optimization. Luc (1988) states that (VOP) *"amounts to finding a point $x \in S$, called an optimal solution of"* (VOP), where $f(x)$ is required to be minimal in the set $f[S] := \{f(x) |\ x \in S\}$ for such a point x. Similarly, Jahn (2004, p. 105) writes that (VOP) *"is to be interpreted in the following way: Determine a minimal solution $x \in S$ which is defined as the inverse image of a minimal element $f(x)$ of the image set $f(S)$."* Ehrgott (2000) writes in the same situation that *"a solution $x \in S$ is called Pareto optimal"*, which means that the term solution seems to refer to a *feasible solution* rather than a solution to (VOP). In the recent textbook by Boţ *et al.* (2009) it is stated that (VOP) *"consists in determining the minimal [...] elements of the image set of S"* and that one is *"also interested in finding the so-called efficient [...] solutions to"* (VOP), where an efficient solution is what Luc called "optimal solution"[1]. It is also stated by Boţ *et al.* (2009) that *"in practice a decision maker is only interested to have a subset or even a single element"* of the set of efficient solutions.

Therefore, it is not clear whether a single efficient solution, a subset or even the set of all efficient solutions is a "solution to (VOP)". This dilemma is underlined by the following lines, taken from an online encyclopedia[2]:

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[1] It is not relevant in this discussion that there are different types of efficient solutions.

[2] Wikipedia, the free online encyclopedia, "Multiobjective optimization", english version, 2010-10-10

A. Löhne, *Vector Optimization with Infimum and Supremum*, Vector Optimization,
DOI 10.1007/978-3-642-18351-5_2, © Springer-Verlag Berlin Heidelberg 2011

"The solution to [a multiobjective optimization problem[3]] is a set of Pareto points. Pareto solutions are those for which improvement in one objective can only occur with the worsening of at least one other objective. Thus, instead of a unique solution to the problem (which is typically the case in traditional mathematical programming), the solution to a multiobjective problem is a (possibly infinite) set of Pareto points.

Even though this definition gives the precise statement that a solution to (VOP) is a set of efficient (or Pareto) points there is no further requirement to this set; a singleton set is therefore also a solution. For typical vector optimization problems, however, a single efficient point can be already obtained by solving a scalarized optimization problem. Only a fraction of the theory on vector optimization would be necessary for this reason.

The main idea of vector optimization is that a decision maker chooses an efficient solution from the set of all efficient solutions. This decision is supported by the solution to the vector optimization problem. This means, the problem must be solved prior to the decision.

We prepend this chapter two postulates.

(1) The goal of vector optimization is to provide a decision maker with a sufficient amount of information on the problem in terms of efficient elements.
(2) A solution concept for a vector optimization problem should provide a specification of the term "sufficient" in (1).

The second hypothesis consists of two aspects.

(a) Does the set of all efficient elements provide enough information?
(b) If so, are there proper subsets of the set of efficient elements that already contain enough information?

The first aspect (a) is a question of existence. The second question (b) is concerned with uniqueness, i.e., if the set of all efficient elements is the only choice, we can say the solution is unique.

Scalar optimization is of course a special case of vector optimization, so that a solution concept should reduce to the standard concept in this special case. To this end, let us first consider a general scalar optimization problem. Let X be a nonempty set and let $f : X \to \overline{\mathbb{R}}$ be a proper function on X, i.e., $f(x) \neq -\infty$ for all $x \in X$ and $f \not\equiv +\infty$. We denote by $S \subseteq X$ the set of feasible elements. Let us

$$\text{minimize } f : X \to \overline{\mathbb{R}} \text{ with respect to } \leq \text{ over } S. \tag{2.1}$$

The following statements are equivalent characterizations of $\bar{x} \in X$ being a solution to (2.1):

[3] The term "multiobjective optimization" is stands for optimization problems with more than one real-valued objective functions. These functions can be interpreted as a single vector-valued objective function.

(i) $\bar{x} \in S$ and $f(\bar{x}) \leq f(x)$ for all $x \in S$,
(ii) $\bar{x} \in S$ and $f(\bar{x}) \not> f(x)$ for all $x \in S$,
(iii) $\bar{x} \in S$ and $f(\bar{x}) = \inf_{x \in S} f(x)$.

Since in vector optimization the ordering relation is more complex than in scalar optimization, the latter conditions do not coincide any longer. While condition (i) is obviously too restrictive for vector optimization problems (utopia points), the common "solution concepts" in the literature are mainly based on (ii). There are several possibilities to interpret the relation $\not>$ ("not greater than"), which leads to a variety of different notions, such as efficient, weakly efficient and properly efficient elements. All these concepts don't take into account the infimum and supremum, which is quite important in scalar optimization. The usage of infimal sets in the literature is related to condition (iii), but the complete lattice has not been pointed out.

The solution concept for vector optimization problems, which is introduced in the next two sections, involves all the conditions (i), (ii) and (iii).

2.1 A solution concept for lattice-valued problems

A complete-lattice-valued optimization problem provides the abstract framework for solution concepts based on the *attainment of the infimum or supremum.*

Let $f : X \to Z$, where X is an arbitrary nonempty set and, unless otherwise indicated, $(Z, \leq)$ is a complete lattice. For a nonempty subset $S \subseteq X$, called *feasible set*, we consider the optimization problem

$$\text{minimize } f : X \to Z \text{ with respect to } \leq \text{ over } S. \tag{$\mathcal{L}$}$$

A standard concept is the following, where $(Z, \leq)$ is only supposed to be a partially ordered set in the following definition.

Definition 2.1. An element $\bar{x} \in S$ is called an *efficient solution* to $(\mathcal{L})$ if

$$[x \in S \quad \wedge \quad f(x) \leq f(\bar{x})] \quad \Longrightarrow \quad f(x) = f(\bar{x}).$$

The set of all efficient solutions to $(\mathcal{L})$ is denoted by $\operatorname{Eff}(\mathcal{L})$.

For $A \subseteq Z$ we denote by

$$\operatorname{Min} A := \{z \in A |\; (y \in A \wedge y \leq z) \Rightarrow y = z\}$$

the set of *minimal elements* of A. Using the notation

$$f[V] := \{f(x) |\; x \in V\},$$

we obtain

$$\operatorname{Min} f[S] = f[\operatorname{Eff}(\mathcal{L})].$$

It is demonstrated by the following two examples that the set $\operatorname{Eff}(\mathcal{L})$ without any further requirement is unsatisfactory as a solution concept for vector optimization problems.

Example 2.2. Let $X = Z = \mathbb{R}^2$ and let Z be partially ordered by the natural ordering cone $\mathbb{R}^2_+$. Let f be the identity map and

$$S = \left\{x \in \mathbb{R}^2 | \; x_1 > 0,\; x_2 > 0,\; x_1 + 10x_2 > 10\right\} \cup \left\{(13,0)^T\right\}.$$

We have $\operatorname{Eff}(\mathcal{L}) = \left\{(13,0)^T\right\}$. But the nonempty set $\operatorname{Eff}(\mathcal{L})$ does not yield a sufficient amount of information about the problem. From a practical point of view, for instance, the feasible, non-efficient point $(1,1)^T$ could be more interesting than the set of efficient solutions, see Figure 2.1.

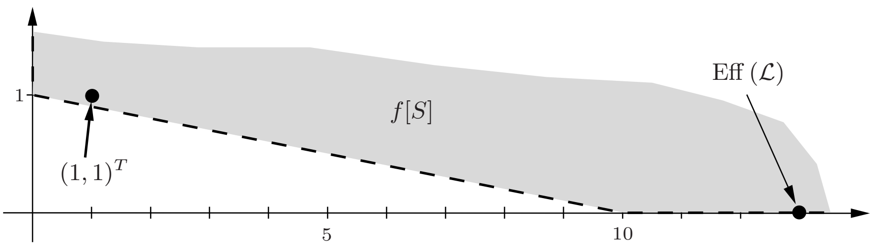

Fig. 2.1 Illustration of Example 2.2. The set of efficient points is not a useful solution concept.

On the other hand, there are vector optimization problems where it is already sufficient for the decision maker to know a proper subset of $\operatorname{Eff}(\mathcal{L})$.

Example 2.3. Let $X = Z = \mathbb{R}^2$, Z partially ordered by $\mathbb{R}^2_+$, and

$$S = \left\{x \in \mathbb{R}^2 | \; x_1 \geq 0,\; x_2 \geq 0,\; 2x_1 + x_2 \geq 2,\; x_1 + 2x_2 \geq 2\right\}.$$

The objective $f : X \to Z$ is given as $f(x) = (0, x_2)^T$. Then

$$\operatorname{Eff}(\mathcal{L}) = \left\{x \in \mathbb{R}^2 | \; x_1 \geq 2, x_2 = 0\right\}.$$

In typical applications the decision maker selects a point in the image $f[\operatorname{Eff}(\mathcal{L})]$ of $\operatorname{Eff}(\mathcal{L})$ with respect to f. We have $f[\operatorname{Eff}(\mathcal{L})] = \left\{(0,0)^T\right\}$. But, the same image is already obtained by any nonempty subset of $\operatorname{Eff}(\mathcal{L})$, see Figure 2.2.

Example 2.3 indicates that the condition $f[\bar{X}] = \operatorname{Min} f[S]$ could be one suitable requirement for a set $\bar{X} \subseteq S$ to be a solution. But additionally, the

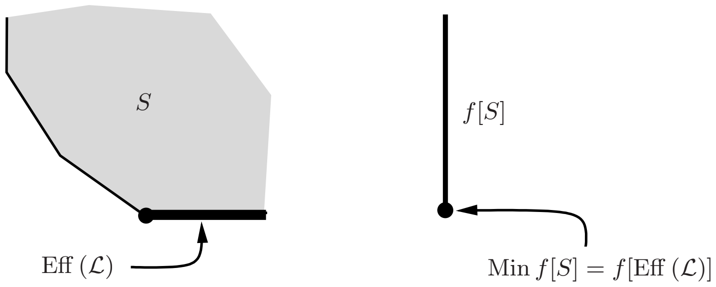

Fig. 2.2 Illustration of Example 2.3. Every nonempty subset of Eff $(\mathcal{L})$ generates the same image.

situation in Example 2.2 must be avoided. This would be possible by assuming the well-known domination property, which is recalled and discussed below. We choose, however, a weaker condition, which is connected with the *attainment of the infimum*. To ensure the existence of the infimum, we need to assume $(Z, \leq)$ to be a complete lattice.

The infimum of f over a set $S \subseteq X$ is defined by

$$\inf_{x \in S} f(x) := \inf \{f(x) |\ x \in S\} = \inf f[S].$$

Definition 2.4. Let $S \subseteq X$ and $\bar{x} \in X$. We say the *infimum of f over S is attained at $\bar{x}$* if

$$\bar{x} \in S \quad \wedge \quad f(\bar{x}) = \inf_{x \in S} f(x).$$

In case such an element $\bar{x}$ exists (does not exist), we say the infimum of f over S is (not) attained.

The attainment of the infimum is an important concept in optimization. In vector optimization it is, however, very hard to fulfill as the following example shows.

Example 2.5. Let $X = \mathbb{R}^2$, $Z = \mathbb{R}^2 \cup \{\pm\infty\}$, $\mathbb{R}^2$ partially ordered by the cone $\mathbb{R}^2_+$. The ordering is denoted by $\leq$ and extended to Z by setting $-\infty \leq z \leq +\infty$ for all $z \in Z$. Then, $(Z, \leq)$ is a complete lattice. Let

$$S = \left\{x \in \mathbb{R}^2 |\ x_1 \geq 0,\ x_2 \geq 0,\ 2x_1 + x_2 \geq 2,\ x_1 + 2x_2 \geq 2\right\}$$

and let f be the identity map. Then the infimum of f over S is not attained. Indeed, we have $\inf_{x \in S} f(x) = \{0, 0\}^T$, but there is no $\bar{x} \in S$ with $f(\bar{x}) = \{0, 0\}^T$, see Figure 2.3.

A further aspect can be observed in the previous example. We enforce that the infimum is attained in a single vector. In vector optimization we intend

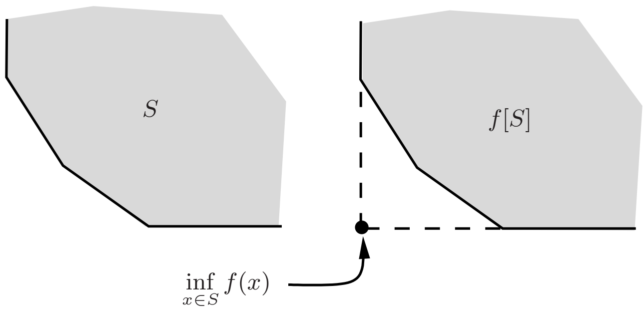

Fig. 2.3 Illustration of Example 2.5. The infimum is not attained.

to present the decision maker all or at least a representative choice of efficient vectors. Therefore, we expect a solution to be a set of feasible vectors.

This requirement is taken into account by a concept that we call *canonical extension.*

Definition 2.6. The *canonical extension* of the objective function $f : X \to Z$ in the complete-lattice-valued optimization problem $(\mathcal{L})$ is the function

$$F : 2^X \to Z, \quad F(A) := \inf_{x\in A} f(x).$$

Of course, we have $f(x) = F(\{x\})$ for all $x \in X$. Working with the canonical extension F instead of f, we make the following two observations: First, we see that attainment of the infimum is easier to realize. The second difference is that the infimum is attained in a set rather than in a single element of X.

We now give a characterization of the attainment of the infimum of the canonical extension F in terms of the given function f.

Proposition 2.7. *Let $S \subseteq X$. The following statements are equivalent.*

(i) *The infimum of F over 2^S is attained at $\bar{X}$, i.e.,*

$$\bar{X} \in 2^S \quad \wedge \quad F(\bar{X}) = \inf_{A\in 2^S} F(A).$$

(ii) $\bar{X} \subseteq S \quad \wedge \quad \inf_{x\in \bar{X}} f(x) = \inf_{x\in S} f(x).$

Proof. It remains to prove the equality

$$\inf_{A\in 2^S} F(A) = \inf_{x\in S} f(x). \tag{2.2}$$

For all $x \in S$ we have

$$\inf_{A\in 2^S} F(A) \le F(\{x\}) = f(x).$$

The infimum over $x \in S$ yields $\leq$ in (2.2). For all $A \subseteq S$ we have

$$F(A) = \inf_{x \in A} f(x) \geq \inf_{x \in S} f(x).$$

Taking the infimum over all $A \in 2^S$ we get $\geq$ in (2.2). $\square$

Next we define a solution concept for the complete-lattice-valued problem $(\mathcal{L})$.

Definition 2.8. A nonempty set $\bar{X}$ with $f[\bar{X}] = \operatorname{Min} f[S]$ is called a *solution* to $(\mathcal{L})$ if the infimum of the canonical extension F over 2^S is attained in $\bar{X}$.

In terms of f a solution can be characterized as follows.

Corollary 2.9. *A nonempty set $\bar{X}$ is a solution to $(\mathcal{L})$ if and only if the following conditions hold:*

(i) $\bar{X} \subseteq S$,

(ii) $f[\bar{X}] = \operatorname{Min} f[S]$,

(iii) $\inf\limits_{x \in \bar{X}} f(x) = \inf\limits_{x \in S} f(x)$.

Proof. Follows from Proposition 2.7. $\square$

It can easily be seen that, if a solution to $(\mathcal{L})$ exists, then $\operatorname{Eff}(\mathcal{L})$ is a solution to $(\mathcal{L})$. Of course, if $\operatorname{Eff}(\mathcal{L})$ is a solution to $(\mathcal{L})$, every subset $\bar{X}$ of $\operatorname{Eff}(\mathcal{L})$ with $f[\bar{X}] = \operatorname{Min} f[S]$ is a solution to $(\mathcal{L})$, too.

Definition 2.10. If $\bar{X} = \operatorname{Eff}(\mathcal{L})$ is the only solution to $(\mathcal{L})$, we say $\bar{X}$ is a *unique* solution.

Example 2.11. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}^2$ partially ordered by the cone $\mathbb{R}^2_+$ and $Z = \overline{Y}$. Then, $(Z, \leq)$ is a complete lattice. Let

$$S = \left\{ x \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0,\ 2x_1 + x_2 \geq 2,\ x_1 + 2x_2 \geq 2 \right\}$$

and let f be the identity map. Then, $\bar{X} := \operatorname{Eff}(\mathcal{L}) = \operatorname{Min} f[S]$ is the unique solution to $(\mathcal{L})$. For the same problem with the choice $f(x) := (0, x_2)^T$, we get $\operatorname{Eff}(\mathcal{L}) = \left\{ x \in \mathbb{R}^2 \mid x_1 \geq 2, x_2 = 0 \right\}$. But, every nonempty subset of $\operatorname{Eff}(\mathcal{L})$ is also a solution. Thus the solution is not unique in this case. Both cases are illustrated in Figure 2.4. Note that this example is not based on a useful solution concept for vector optimization, because the current complete lattice is not suitable.

Let us consider Problem $(\mathcal{L})$ for the special case $(Z, \leq) = (\overline{\mathbb{R}}, \leq)$.

Proposition 2.12. *Let $(Z, \leq) = (\overline{\mathbb{R}}, \leq)$. For a nonempty set $\bar{X}$, the following is equivalent:*

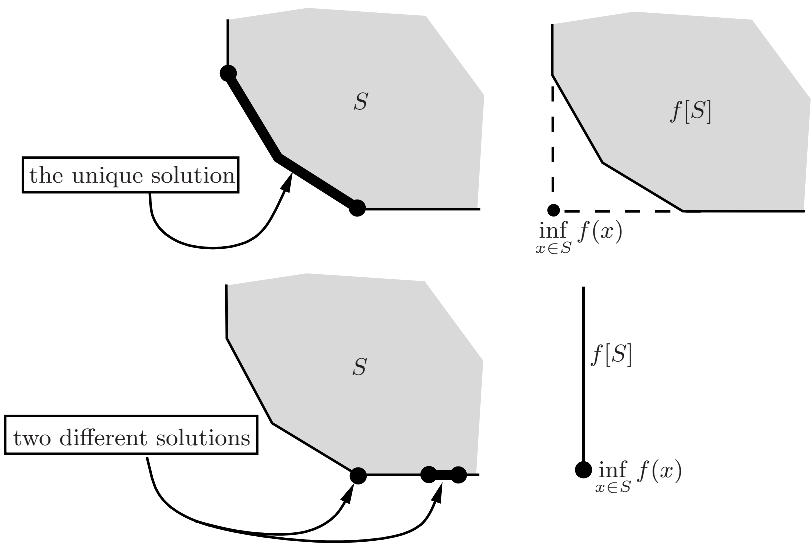

Fig. 2.4 Illustration of Example 2.11. Unique and non-unique solutions.

(i) $f[\bar{X}] = \operatorname{Min} f[S]$,
(ii) $\forall \bar{x} \in \bar{X} :\ \{f(\bar{x})\} = \operatorname{Min} f[S]$.

Proof. (i) $\Rightarrow$ (ii). This follows from the fact that $\operatorname{Min} f[S]$ is a singleton set, which is a consequence of $\leq$ being a total ordering in $\overline{\mathbb{R}}$ (i.e. arbitrary elements y^1, y^2 satisfy either $y^1 \leq y^2$ or $y^2 \leq y^1$).

(ii) $\Rightarrow$ (i). By (ii), f is constant on $\bar{X}$. Hence we have $\{f(\bar{x})\} = f[\bar{X}]$ for all $\bar{x} \in \bar{X}$. □

We next show the connection between a solution to the complete-lattice-valued problem $(\mathcal{L})$ for the case $(Z, \leq) = (\overline{\mathbb{R}}, \leq)$ and solutions to the classical extended real-valued optimization problem (2.1).

Theorem 2.13. *Consider Problem* $(\mathcal{L})$ *for the special case* $(Z, \leq) = (\overline{\mathbb{R}}, \leq)$ *and the corresponding real-valued optimization problem* (2.1). *For a nonempty set* $\bar{X}$*, the following is equivalent:*

(i) $\bar{X}$ *is a solution to* $(\mathcal{L})$*,*
(ii) $\bar{x}$ *is a solution to* (2.1) *for every* $\bar{x} \in \bar{X}$*.*

An element $\bar{x}$ *is a unique solution to* (2.1) *if and only if* $\{\bar{x}\}$ *is a unique solution to* $(\mathcal{L})$*.*

Proof. (i) is equivalent to

$$\bar{X} \subseteq S \quad \wedge \quad \inf_{x \in \bar{X}} f(x) = \inf_{x \in S} f(x) \quad \wedge \quad f[\bar{X}] = \operatorname{Min} f[S].$$

By Proposition 2.12, this is equivalent to

$$\forall \bar{x} \in \bar{X}: \qquad \bar{x} \in S \quad \wedge \quad f(\bar{x}) = \inf_{x \in S} f(x) \quad \wedge \quad \{f(\bar{x})\} = \operatorname{Min} f[S],$$

which is an alternative way to express (ii). $\square$

In Example 2.2 (where a complete lattice Z is obtained by extending $\mathbb{R}^2$ by two elements $\pm\infty$), $\operatorname{Eff}(\mathcal{L})$ is not a solution to $(\mathcal{L})$; whence a solution does not exist. A natural condition ensuring that $\operatorname{Eff}(\mathcal{L})$ is a solution is the well-known *domination property* (see e.g. Dolecki and Malivert, 1993).

Definition 2.14. Let $(Z, \leq)$ be a partially ordered set. We say that the *domination property* holds for Problem $(\mathcal{L})$ if

$$\forall x \in S, \ \ \exists \bar{x} \in \operatorname{Eff}(\mathcal{L}): \ \ f(\bar{x}) \leq f(x). \tag{2.3}$$

Proposition 2.15. *The set* $\operatorname{Eff}(\mathcal{L})$ *is a solution to* $(\mathcal{L})$ *if the domination property holds.*

Proof. Set $\bar{X} := \operatorname{Eff}(\mathcal{L})$. Of course, we have $\bar{X} \in 2^S$. According to Proposition 2.7, the attainment of the infimum of the canonical extension F over 2^S in $\bar{X}$ is equivalent to

$$\inf_{x \in \bar{X}} f(x) = \inf_{x \in S} f(x). \tag{2.4}$$

From (2.3) we get $\inf_{x \in \bar{X}} f(x) \leq \inf_{x \in S} f(x)$ and the opposite inequality in (2.4) follows from $\bar{X} \subseteq S$. $\square$

The domination property is not necessary for the existence of a solution. An example is given below (Example 2.23).

2.2 A solution concept for vector optimization

A vector optimization problem is now transformed such that it becomes a special case of the complete-lattice-valued problem $(\mathcal{L})$. One can infer from the examples in the previous section that the choice of a suitable complete lattice $(Z, \leq)$ is rather essential. Originally, the image space of a vector optimization problem is a partially ordered vector space $(Y, \leq)$. In some cases, Y can be extended to a complete lattice by setting $Z := Y \cup \{\pm\infty\}$, where the ordering is extended in the usual way by setting $-\infty \leq z \leq +\infty$ for all z. We already mentioned the two drawbacks of this procedure. On the one hand, in many (even finite dimensional) cases we do not obtain a complete lattice in this way, see Example 1.9. On the other hand, even if a complete lattice is acquired in this way, our solution concept is unsatisfactory with this choice of Z. This is demonstrated by the following example.

Example 2.16. Let $X = \mathbb{R}^2$, $(Z, \leq)$ the complete lattice from Example 2.5, f the identity map and

$$S = \left\{x \in \mathbb{R}^2 |\ x_1 > 0,\ x_2 > 0,\ x_1 + x_2 > 1\right\} \\ \cup\ \{x \in \mathbb{R}^2 |\ x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \geq 2\}.$$

Then $\bar{X} := \left\{(0,2)^T, (2,0)^T\right\}$ is a solution. This is unsatisfactory from the viewpoint of vector optimization, because this set does not contain enough information, see Figure 2.5.

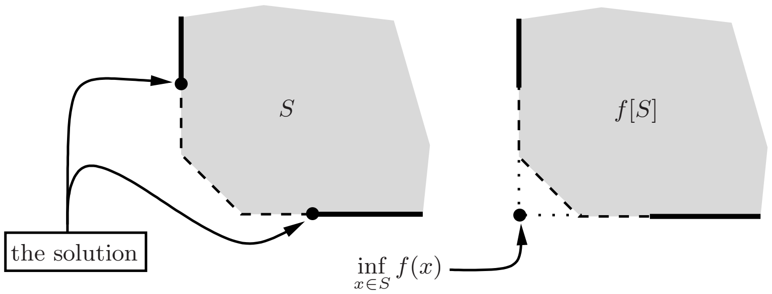

Fig. 2.5 Illustration of Example 2.16. The extended vector space $\mathbb{R}^2 \cup \{\pm\infty\}$ is a complete lattice, but not suitable for vector optimization.

The loophole is the usage of the complete lattice $\mathcal{I}$ of self-infimal subsets of $\overline{Y}$ instead of the space $\overline{Y}$ as the image space. The space $\mathcal{I}$ was introduced in Section 1.5. Recall further that we denote by $\operatorname{Inf} A$ the infimal set of a set $A \subseteq \overline{Y}$, see Section 1.4. We can identify a vector y in Y by the element $\operatorname{Inf}\{y\}$ of $\mathcal{I}$. In this way the ordering relation in $\mathcal{I}$ is an extension of the ordering relation in $\overline{Y}$. Note that the partial ordering on Y is generated by a pointed, convex cone C with $\emptyset \neq \operatorname{int} C \neq Y$, which is involved in the definition of infimal sets. The new image space $\mathcal{I}$ is a complete lattice even if $\overline{Y}$ is not a complete lattice. Moreover, an infimum is now an element of $\mathcal{I}$, which contains more information than a single vector. In particular, an infimum contains the information which is required by a solution concept based on the above postulates.

Let X be a nonempty set and $S \subseteq X$. Let $\overline{Y}$ be an extended partially ordered topological vector space, let the ordering cone C of Y be closed and let $\emptyset \neq \operatorname{int} C \neq Y$. Note that C is automatically pointed and convex, compare the remark after Definition 1.27. We consider the vector optimization problem

$$\text{minimize } f : X \to \overline{Y} \text{ with respect to } \leq_C \text{ over } S. \tag{V}$$

We assign to (V) a corresponding $\mathcal{I}$-valued-problem, i.e., a problem of type $(\mathcal{L})$, where the complete lattice $(Z, \leq) = (\mathcal{I}, \preccurlyeq)$ is used. Note that $(\mathcal{I}, \preccurlyeq)$

is defined with respect to the ordering cone C of the vector optimization problem.

Given a function $f : X \to \overline{Y}$, we set

$$\bar{f} : X \to \mathcal{I}, \quad \bar{f}(x) := \operatorname{Inf}\{f(x)\}$$

and we assign to (V) the problem

$$\text{minimize } \bar{f} : X \to \mathcal{I} \text{ with respect to } \preccurlyeq \text{ over } S. \tag{$\mathcal{V}$}$$

Problem ($\mathcal{V}$) is said to be the *lattice extension*, or more precisely the *$\mathcal{I}$-extension*, of the vector optimization problem (V). This terminology can be motivated by the fact that the lattice extension of the vector optimization problem allows us to handle the problem in the framework of complete lattices. The ordering relation of the original objective space $\overline{Y}$ is extended to the complete lattice $\mathcal{I}$ as shown in the following proposition. Note that this extension is the reason for the assumption of C being closed.

Proposition 2.17. *For all $x, v \in X$ we have*

$$f(x) \leq_C f(v) \qquad \Longleftrightarrow \qquad \bar{f}(x) \preccurlyeq \bar{f}(v).$$

Proof. Let $\operatorname{Inf}\{y\} \preccurlyeq \operatorname{Inf}\{z\}$, then $\operatorname{Cl}_+\{y\} \supseteq \operatorname{Cl}_+\{z\}$. By Proposition 1.40, we get $z \in \operatorname{cl}(\{z\}+C) \subseteq \operatorname{cl}(\{y\}+C)$. Since C is closed, we obtain $z \in \{y\}+C$. This means $y \leq_C z$. The opposite inclusion is obvious. □

We next see that both problems (V) and ($\mathcal{V}$) are related as they have the same efficient solutions.

Proposition 2.18. *A feasible element $\bar{x} \in S$ is an efficient solution to the vector optimization problem* (V) *if and only if it is an efficient solution to its lattice extension* ($\mathcal{V}$).

Proof. This is a direct consequence of Proposition 2.17. □

Proposition 2.19. *The domination property holds for the vector optimization problem* (V) *if and only it holds for its lattice extension* ($\mathcal{V}$).

Proof. Follows from Proposition 2.17. □

We now define a solution concept for the vector optimization problem (V).

Definition 2.20. A nonempty set $\bar{X} \subseteq X$ is called a *solution* to the vector optimization problem (V) if $\bar{X}$ is a solution to its lattice extension ($\mathcal{V}$).

The next theorem provides a characterization of a solution to the vector optimization problem (V) by standard notations.

Theorem 2.21. *A nonempty set $\bar{X} \subseteq X$ is a solution to the vector optimization problem* (V) *if and only if the following three conditions are satisfied:*

(i) $\bar{X} \subseteq S$,
(ii) $f[\bar{X}] = \operatorname{Min} f[S]$,
(iii) $\operatorname{Inf} f[\bar{X}] = \operatorname{Inf} f[S]$.

Proof. This is a direct consequence of Proposition 2.7 and Theorem 1.54. □

Example 2.22. Consider the vector optimization problem (V) with a linear objective function f and a polyhedral convex feasible set S. Then, the set $\operatorname{Eff}(\mathcal{L})$ is a solution whenever it is nonempty. As shown in (Hamel *et al.*, 2004, Lemma 2.1) (note that the cone has to be pointed there) the domination property is fulfilled in this case. Thus Proposition 2.15 yields that $\operatorname{Eff}(\mathcal{L})$ is a solution.

Example 2.23. Consider the vector optimization problem (V) with $f : \mathbb{R}^2 \to \mathbb{R}^2$ being the identity map, let $C = \mathbb{R}^2_+$ and

$$S = \{x \in \mathbb{R}^2 |\; x_1 \geq 0,\; x_2 \geq 0,\; x_1 + x_2 \geq 1\} \setminus \{(0,1)^T\}.$$

Then $\bar{X} := \operatorname{Eff}(\mathcal{L}) = \{\lambda\,(0,1)^T + (1-\lambda)\,(1,0)^T |\; 0 \leq \lambda < 1\}$ is a solution, but the domination property is not satisfied.

As we will see in Chapter 3 the solution concept of Definition 2.8 is also relevant for problems which are not a lattice extension of a given vector optimization problem. There we consider a set-valued dual problem of a given vector optimization problem. In special cases, the values of the dual objective map are self-infimal hyperplanes.

Another lattice extension will be of interest in this work. The $\mathcal{I}$-valued extension $\bar{f} : X \to \mathcal{I}$ of a vectorial objective (as introduced above) is actually $\mathcal{I}_{\mathrm{co}}$-valued, see Section 1.6. Therefore, we also consider the lattice extension

$$\text{minimize } \bar{f} : X \to \mathcal{I}_{\mathrm{co}} \text{ with respect to } \preccurlyeq \text{ over } S. \qquad (\mathcal{V}_{\mathrm{co}})$$

If $\bar{f}$ is regarded to be $\mathcal{I}_{\mathrm{co}}$-valued, we have a different infimum and thus a different solution concept. Problem $(\mathcal{V}_{\mathrm{co}})$ is said to be the *convex lattice extension*, or more precisely, the $\mathcal{I}_{\mathrm{co}}$*-extension* of the vector optimization problem (V).

Definition 2.24. A nonempty set $\bar{X} \subseteq X$ is called a *convexity solution* or $\mathcal{I}_{\mathrm{co}}$*-solution* to the vector optimization problem (V) if $\bar{X}$ is a solution to the corresponding convex lattice extension $(\mathcal{V}_{\mathrm{co}})$.

Convexity solutions can be characterized as follows.

Theorem 2.25. *A nonempty set $\bar{X} \subseteq X$ is a convexity solution to the vector optimization problem* (V) *if and only if the following three conditions are satisfied:*

(i) $\bar{X} \subseteq S$,

(ii) $f[\bar{X}] = \operatorname{Min} f[S]$,
(iii) $\operatorname{Inf} \operatorname{co} f[\bar{X}] = \operatorname{Inf} \operatorname{co} f[S]$.

Proof. This is a direct consequence of Proposition 2.7 and Theorem 1.63. □

Proposition 2.26. *Every solution to* (V) *is also a convexity solution to* (V).

Proof. This follows from the fact that, by Proposition 1.60, $\operatorname{Inf} f[\bar{X}] = \operatorname{Inf} f[S]$ implies $\operatorname{Inf} \operatorname{co} f[\bar{X}] = \operatorname{Inf} \operatorname{co} f[S]$. □

Example 2.27. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}^2$ partially ordered by the cone $\mathbb{R}^2_+$ and $Z = \overline{Y}$. Then, $(Z, \leq)$ is a complete lattice. Let

$$\begin{aligned} S = &\left\{x \in \mathbb{R}^2 \mid x_1 > 0,\ x_2 > 0,\ 2x_1 + x_2 > 2,\ x_1 + 2x_2 > 2\right\} \\ &\cup \left\{(0,2)^T, (2,0)^T, \left(\frac{2}{3}, \frac{2}{3}\right)^T\right\} \end{aligned}$$

and let f be the identity map. Then, $\bar{X} = \left\{(0,2)^T, (2,0)^T, \left(\frac{2}{3}, \frac{2}{3}\right)^T\right\}$ is a convexity solution but not a solution to (V), see Figure 2.6.

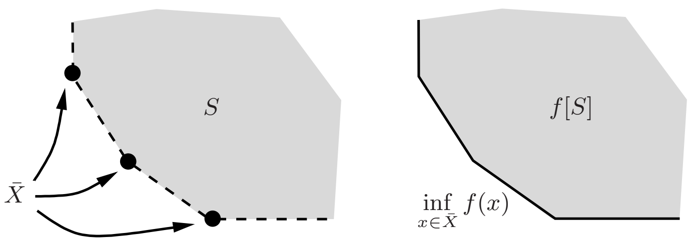

Fig. 2.6 Illustration of Example 2.27. The set $\bar{X}$ is a convexity solution but not a solution. The infimum on the right refers to the complete lattice $\mathcal{I}_{\mathrm{co}}$. It coincides with $\inf_{x \in S} f(x)$.

This example looks somewhat artificial. Convexity solutions will play a role in Section 2.5, where we introduce *mild solutions* by relaxing the condition $f[\bar{X}] = \operatorname{Min} f[S]$. *Mild convexity solutions* will naturally occur in linear vector optimization problems.

2.3 Semicontinuity concepts

Lower semicontinuity of the objective function is typically required as an assumption for the existence of minimal solutions. This section provides a summary of different notions of lower semicontinuity for functions with values in Z, where Z is a partially ordered set and sometimes even a complete lattice. We are mainly interested in the general case without any a priori topology on Z. However, we also consider the special case where $Z = \overline{Y} := Y \cup \{\pm\infty\}$ is the extension of a partially ordered topological vector space Y. In particular, we examine $\overline{Y}$- and $\mathcal{F}$-valued functions. Note that $(\mathcal{F}, \supseteq)$ is isomorphic and isotone to $(\mathcal{I}, \preccurlyeq)$, see Proposition 1.52.

If $f : X \to Z = \overline{Y}$ is a function from a topological space X into the extended real numbers $\overline{\mathbb{R}}$, i.e., $Y = \mathbb{R}$ and $Z = \overline{\mathbb{R}}$, then the following five properties are equivalent characterizations of lower semicontinuity of f.

(a) For all $z \in Z$ the level sets $L_f(z) := \{x \in X |\ f(x) \leq z\}$ are closed.
(b) For all $y \in Y$ the level sets $L_f(y)$ are closed.
(c) For all $\bar{x} \in X$,

$$f(\bar{x}) \leq \sup_{U \in \mathcal{U}(\bar{x})} \inf_{x \in U} f(x) =: \liminf_{x \to \bar{x}} f(x)$$

holds true, where $\mathcal{U}(\bar{x})$ is a neighborhood base of $\bar{x}$ (that is, for every neighborhood U of $\bar{x}$, there exists some $\bar{U} \in \mathcal{U}(\bar{x})$ such that $\bar{U} \subseteq U$).
(d) For every $\bar{x} \in X$, every $\bar{y} \in Y$ with $\bar{y} \leq f(\bar{x})$ and every neighborhood V of $\bar{y}$ there is some neighborhood U of $\bar{x}$ such that

$$\forall x \in U,\ \exists y \in V : f(x) \geq y.$$

(e) The epigraph of f, $\operatorname{epi} f := \{(x, y) \in X \times Y |\ f(x) \leq y\}$, is closed.

For more general instances of Z these five properties do not coincide any longer. If Z is merely a complete lattice without any additional structure, then only the properties (a) and (c) are applicable.

Definition 2.28. Let X be a topological space, and let $(Y, \leq)$ and $(Z, \leq)$ be partially ordered sets. A function $f : X \to Z$ is called *level closed* if property (a) holds. $f : X \to \overline{Y}$ is called *weakly level closed* if property (b) holds. In case Z is a complete lattice, a function $f : X \to Z$ is called *lattice lower semi-continuous (lattice-l.s.c.)* if property (c) holds. In case Y is a partially ordered topological space $f : X \to \overline{Y}$ is called *topologically l.s.c.* if property (d) holds and *epi-closed* if (e) holds.

In the following we investigate the relationships between these properties. First we clarify the connection between the two notions that do not require further structural assumptions for the image space Z in addition to the lattice property.

Proposition 2.29. *Let X be a topological space and $(Z, \leq)$ a complete lattice. If a function $f : X \to Z$ is lattice-l.s.c., then it is level closed.*

Proof. Assume that f is lattice-l.s.c. but not level closed, i.e., there is some $\bar{z} \in Z$ such that $L_f(\bar{z})$ is not closed. Then there is some $\bar{x} \in X$ with $\bar{x} \notin L_f(\bar{z})$ such that for all $U \in \mathcal{U}(\bar{x})$ there exists some $x \in U$ with $x \in L_f(\bar{z})$. This implies

$$\sup_{U \in \mathcal{U}(\bar{x})} \inf_{x \in U} f(x) \leq \bar{z}.$$

Since f is lattice-l.s.c., we conclude $f(\bar{x}) \leq \bar{z}$. But this means $\bar{x} \in L_f(\bar{z})$, a contradiction. □

The converse is generally not true as the following example shows.

Example 2.30. Let $X = \mathbb{R}$ and $Z = \overline{\mathbb{R}^2}$, where $\mathbb{R}^2$ is partially ordered by $\mathbb{R}^2_+$. The function $f : X \to Z$ defined by

$$f(x) = \begin{cases} (1,0)^T & \text{if } x \geq 0 \\ (0,-1/x)^T & \text{if } x < 0 \end{cases}$$

is level closed since

$$\begin{aligned} L_f(y) &= \{x \in X |\, f(x) \leq y\} \\ &= \begin{cases} [0,+\infty) & \text{if } y_2 = 0,\ y_1 \geq 1 \\ (-\infty,-1/y_2] \cup [0,+\infty) & \text{if } y_2 > 0,\ y_1 \geq 1 \\ (-\infty,-1/y_2] & \text{if } y_2 > 0,\ 0 \leq y_1 < 1 \\ \mathbb{R} & \text{if } y = +\infty \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

But f is not lattice-l.s.c. at $\bar{x} = 0$. Indeed, if we take the set of open ε-intervals as a neighborhood base of $\bar{x} = 0$, i.e., $\mathcal{U}(0) = \{(-\varepsilon,+\varepsilon) |\, \varepsilon > 0\}$, we obtain for every $U = (-\varepsilon,+\varepsilon) \in \mathcal{U}(0)$,

$$\inf_{x \in U} f(x) = \inf\left\{(0,1/\varepsilon)^T,(1,0)^T\right\} = (0,0)^T.$$

We conclude

$$\sup_{U \in \mathcal{U}(0)} \inf_{x \in U} f(x) = (0,0)^T \not\geq (1,0)^T = f(0)$$

i.e., the condition of f being lattice-l.s.c. at $\bar{x} = 0$ is violated.

The following relations between epi-closedness, weak level closedness and level closedness follow immediately from the definitions.

Proposition 2.31. *Let X be a topological space, $(Y,\leq)$ a partially ordered topological space and $f : X \to \overline{Y}$. The following statements hold.*

(i) *If f is epi-closed, then it is weakly level closed.*
(ii) *If f is level closed, then it is weakly level closed.*

The converse implications are only true under additional assumptions.

Proposition 2.32. *Let X be a topological space.*

(i) *Assume that $(Y, \leq_C)$ is a topological vector space ordered by a pointed closed convex cone C with nonempty interior. If a function $f : X \to \overline{Y}$ is weakly level closed, then it is epi-closed.*
(ii) *Assume that $(Y, \leq)$ is a partially ordered set having no least element. If a function $f : X \to \overline{Y}$ is weakly level closed, then it is level closed.*

Proof. (i) Assume that f is weakly level closed, i.e., for every $\bar{x} \in X$ and $y \in Y$ we have

$$[\forall U \in \mathcal{U}(\bar{x}),\ \exists x \in U : f(x) \leq_C y] \quad \Longrightarrow \quad f(\bar{x}) \leq_C y. \tag{2.5}$$

In order to prove that f is epi-closed we assume that $(\bar{x}, \bar{y}) \in \operatorname{cl}(\operatorname{epi} f)$, i.e.,

$$\forall U \in \mathcal{U}(\bar{x}),\ \forall V \in \mathcal{V},\ \exists x \in U,\ \exists y \in V :\ f(x) \leq_C \bar{y} + y, \tag{2.6}$$

where $\mathcal{V}$ denotes a neighborhood base of 0 in Y. We have to show that $f(\bar{x}) \leq_C \bar{y}$. Since for every $c \in \operatorname{int} C$ there is some $V \in \mathcal{V}$ with $V \subseteq c - C$ we obtain from (2.6),

$$\forall c \in \operatorname{int} C,\ \forall U \in \mathcal{U}(\bar{x}),\ \exists x \in U :\ f(x) \leq_C \bar{y} + c.$$

Now, (2.5) implies that $f(\bar{x}) \leq_C \bar{y} + c$ holds for all $c \in \operatorname{int} C$. Therefore, we have $f(\bar{x}) \leq_C \bar{y}$ as C is closed.

(ii) It remains to show that $L_f(+\infty)$ and $L_f(-\infty)$ are closed. $L_f(+\infty) = X$ is closed by definition. Since Y has no least element, for $z \in \overline{Y}$ we have $z = -\infty$ if and only if $z \leq y$ for all $y \in Y$. Hence

$$L_f(-\infty) = \bigcap_{y \in Y} L_f(y)$$

is a closed set as well. □

In general, there is no inclusion between the sets of lattice-l.s.c., topologically l.s.c. and epi-closed functions (Gerritse, 1997, appendix). Some of the inclusions are valid under additional assumptions (Penot and Théra, 1982; Gerritse, 1997; Ait Mansour *et al.*, 2007). In this context we only mention the following two results.

Proposition 2.33. *Let X be a topological space and let $(Y, \leq)$ be a partially ordered topological space that has no greatest element. If the ordering of Y is closed (i.e., the set $G := \{(z, y) \in Y \times Y |\ z \leq y\}$ is closed) then every topologically l.s.c. function $f : X \to \overline{Y}$ is epi-closed.*

Proof. In order to prove that $\operatorname{epi} f$ is closed we take a pair $(\bar{x}, \bar{y}) \in (X \times Y) \setminus (\operatorname{epi} f)$ and show that there are neighborhoods $U \in \mathcal{U}(\bar{x})$ and $W \in \mathcal{V}(\bar{y})$ such that $(U \times W) \cap (\operatorname{epi} f) = \emptyset$.

If $(\bar{x}, \bar{y}) \in (X \times Y) \setminus (\operatorname{epi} f)$, then $f(\bar{x}) \neq -\infty$ and $(f(\bar{x}), \bar{y}) \notin G$. Let $\hat{y} \in Y$ be chosen such that $\hat{y} \leq f(\bar{x})$ and $(\hat{y}, \bar{y}) \notin G$. Such an element $\hat{y}$ always exists. Indeed, if $f(\bar{x}) \in Y$, we can use $\hat{y} = f(\bar{x})$. On the other hand, assuming that no such $\hat{y}$ exists in the case $f(\bar{x}) = +\infty$, we obtain that $\bar{y}$ is the greatest element of Y, a contradiction.

Since $(\hat{y}, \bar{y}) \notin G$ and G is closed there exist neighborhoods V of $\hat{y}$ and W of $\bar{y}$ such that $(V \times W) \cap G = \emptyset$. Since f is topologically l.s.c. there exists a neighborhood $U \in \mathcal{U}(\bar{x})$ such that

$$\forall x \in U,\ \exists y \in V:\ y \leq f(x). \tag{2.7}$$

This implies $(U \times W) \cap (\operatorname{epi} f) = \emptyset$. Otherwise there would exist $\hat{x} \in U, \hat{y} \in W$ with $f(\hat{x}) \leq \hat{y}$. By (2.7) there would exist $y \in V$ with $y \leq f(\hat{x})$. Hence we obtain $y \leq \hat{y}$, which contradicts $(V \times W) \cap G = \emptyset$. □

Let us summarize the connections between the different notions. If Y is a partially ordered topological space with a closed ordering that has no greatest element such that $\overline{Y}$ is a complete lattice, then for functions $f : X \to \overline{Y}$ the concept of weak level closedness is the weakest one. It is equivalent to level closedness if Y has no least element. Moreover, epi-closedness, level closedness and weak level closedness coincide and the first two concepts are stronger than the last three.

We next study the relationship between the different concepts for functions with values in the complete lattice $(Z, \leq) = (\mathcal{F}, \supseteq)$, where $\mathcal{F} := \mathcal{F}_C(Y)$ is the space of upper closed subsets of a partially ordered topological vector space Y with an ordering cone C such that $\emptyset \neq \operatorname{int} C \neq Y$. Moreover, the ordering cone C is supposed to be closed. By Proposition 1.52, the corresponding results for the space $\mathcal{I}$ follow immediately.

In Propositions 2.34 and 2.35 the assumption $\emptyset \neq \operatorname{int} C \neq Y$ could be relaxed so that C is only required to be proper. In this case we would need a new definition of the upper closure, because our definition involves the interior of C. For this purpose the condition in Proposition 1.40 could be used.

We only consider the notions of (weakly) level closedness and lattice-semicontinuity in the case of $\mathcal{F}$-valued functions. A topology for $\mathcal{F}$ is not considered, but we investigate the connections to semi-continuity notions based on the topology of the underlying topological vector space Y. If we identify a function $f : X \to \mathcal{F}$ with a corresponding multivalued map $f : X \rightrightarrows Y$, the set

$$\operatorname{gr} f := \{(x, y) \in X \times Y \mid y \in f(x)\},$$

is called the *graph* of f.

Proposition 2.34. *A function $f : X \to \mathcal{F}$ is lattice-l.s.c. if and only if* $\operatorname{gr} f$ *is closed.*

Proof. We have

$$\sup_{U\in\mathcal{U}(\bar{x})}\inf_{x\in U} f(x) = \bigcap_{U\in\mathcal{U}(\bar{x})} \operatorname{cl} \bigcup_{x\in U} f(x).$$

It follows that

$$\bar{y} \in \sup_{U\in\mathcal{U}(\bar{x})}\inf_{x\in U} f(x)$$

is equivalent to

$$\forall U \in \mathcal{U}(\bar{x}),\ \forall V \in \mathcal{V}(\bar{y}),\ \exists x \in U,\ \exists y \in V :\ y \in f(x), \tag{2.8}$$

where $\mathcal{V}(\bar{y})$ denotes a neighborhood base of $\bar{y}$ in Y. Consequently f is lattice-l.s.c. if and only if for all $(\bar{x}, \bar{y})$ satisfying (2.8) one has $\bar{y} \in f(\bar{x})$. But, this is equivalent to $\operatorname{gr} f$ being closed. □

Proposition 2.35. *A function $f : X \to \mathcal{F}$ is level closed if and only if for all $y \in Y$ the sets $\{x \in X |\, y \in f(x)\}$ are closed.*

Proof. If f is level closed then the sets $L_f(\operatorname{Cl}_+\{y\})$ are closed for all $y \in Y$. We have $y \in f(x)$ if and only if $\operatorname{Cl}_+\{y\} \subseteq \operatorname{Cl}_+ f(x) = f(x)$. Thus the "only if"-part follows. The "if"-part follows from

$$L_f(A) = \bigcap_{y\in A} \{x \in X |\, y \in f(x)\}$$

and the fact that the intersection of closed sets is closed. □

Corollary 2.36. *Let $f : X \to \overline{Y}$ be an extended vector-valued function and $\tilde{f} : X \to \mathcal{F}$ its $\mathcal{F}$-valued extension, defined by $\tilde{f}(x) := \operatorname{Cl}_+\{f(x)\}$. Then $\tilde{f}$ is level closed if and only if f is weakly level closed.*

Proof. By Proposition 2.35, $\tilde{f}$ is level closed if and only if for all $y \in Y$ the sets $\{x \in X |\, y \in \tilde{f}(x)\}$ are closed. Similarly to Proposition 2.17, we have $y \in \tilde{f}(x)$ if and only if $y \geq_C f(x)$, where we use that C is closed. Thus the statement follows. □

By Proposition 2.29, every lattice-l.s.c. function is also level closed. For functions with values in $\mathcal{F}$ the converse implication also holds. As seen in Example 2.30 this is generally not true.

Proposition 2.37. *A function $f : X \to \mathcal{F}$ is lattice-l.s.c. if and only if it is level closed.*

Proof. Assume that f is level closed. We show that $\operatorname{gr} f$ is closed. By Proposition 2.34, this implies that f is lattice-l.s.c.. Assume that $(\bar{x}, \bar{y}) \in X \times Y$ is given such that for all $U \in \mathcal{U}(\bar{x}), V \in \mathcal{V}(\bar{y})$ there exist $x \in U, y \in V$ with $y \in f(x)$. We have to show that $\bar{y} \in f(\bar{x})$.

Take $z \in \{\bar{y}\} + \operatorname{int} C$ arbitrarily. Then there exists some neighborhood $\tilde{V} \in \mathcal{V}(\bar{y})$ such that $y \leq_C z$, i.e., $z \in \operatorname{Cl}_+\{y\}$ holds for all $y \in \tilde{V}$. Thus, for all $U \in \mathcal{U}(\bar{x})$ there exist some $x \in U$ and some $y \in \tilde{V}$ with $y \in f(x)$, hence $z \in \mathrm{Cl}_+ \{y\} \subseteq f(x)$. By Proposition 2.35 we get

$$\bar{x} \in \mathrm{cl}\, \{x \in X|\, z \in f(x)\} = \{x \in X|\, z \in f(x)\}\,.$$

Thus we have $\bar{y}+\mathrm{int}\, C \subseteq f(\bar{x})$ and consequently $\bar{y} \in \mathrm{cl}\,(\bar{y}+\mathrm{int}\, C) \subseteq f(\bar{x})$. □

We next formulate a sufficient condition for the domination property of the general optimization problem ($\mathcal{L}$). As in the classical Weierstrass theorem, the assumptions are lower semicontinuity of f and compactness of the feasible set. The appropriate semicontinuity condition for the function f in the general case is level closedness.

Proposition 2.38. *Let X be a compact topological space, $(Z, \leq)$ be a partially ordered set and $f : X \to Z$ a level closed function. Then the domination property holds, i.e., for every $x \in X$ there exists a minimal element $y \in f[X]$ with $y \leq f(x)$.*

Proof. We have to show that for every $x \in X$ the set $\{y \in f[X]|\, y \leq f(x)\} = f[L_f(f(x))]$ has minimal elements. Because of Zorn's lemma it suffices to show that every chain in $f[L_f(f(x))]$ has a lower bound in $f[L_f(f(x))]$. Since every lower bound (in $f[X]$) of a subset W of $f[L_f(f(x))]$ is obviously in $f[L_f(f(x))]$, it is sufficient to prove that every chain in $f[X]$ has a lower bound.

Let W be a chain in $f[X]$. A subset W of $f[X]$ has a lower bound in $f[X]$ if and only if the set

$$\{x \in X|\, \forall w \in W : f(x) \leq w\} = \bigcap_{w \in W} L_f(w)$$

is nonempty. If B is a finite subset of W then $\bigcap_{b \in B} L_f(b)$ is nonempty since every finite chain in $f[X]$ has a least element and hence a lower bound. Since all the sets $L_f(w)$ are closed, X being compact implies that $\bigcap_{w \in W} L_f(w)$ is nonempty, too. Hence W has a lower bound. □

For special cases of the complete lattice Z, the semicontinuity assumption in the latter result can be replaced by other concepts. For the case $(Z, \leq) = (\mathcal{I}, \preccurlyeq)$ this is pointed out in the next section. As a consequence we obtain the existence of solutions based on a variety of different semicontinuity notions. This matches the situation in scalar optimization.

2.4 A vectorial Weierstrass theorem

The results of the previous section can be applied to the vector optimization problems (V) and its lattice extension ($\mathcal{V}$) in order to obtain conditions for the existence of solutions.

Let X be a topological space and $S \subseteq X$. Moreover, let $\overline{Y}$ be an extended partially ordered topological vector space, let the ordering cone C of Y be closed and let $\emptyset \neq \operatorname{int} C \neq Y$. We consider the vector optimization problem (V) as introduced in Section 2.2 as well as its lattice extension ($\mathcal{V}$). The semicontinuity concept required for the existence result can be characterized in terms of the objective function $f : X \to \overline{Y}$ of (V) and in terms of the objective function $\bar{f} : X \to \mathcal{I}$ of the lattice extension ($\mathcal{V}$) of (V), where $\bar{f}$ is defined by f as

$$\bar{f}(x) := \operatorname{Inf}\{f(x)\}. \tag{2.9}$$

Theorem 2.39. *For a function $f : X \to \overline{Y}$ and the corresponding function $\bar{f} : X \to \mathcal{I}$ according to* (2.9)*, the following statements are equivalent:*

(i) f *is epi-closed, i.e., the epigraph of* f *is closed;*
(ii) f *is level closed, i.e.,* f *has closed level sets for all levels in* $\overline{Y}$*;*
(iii) f *is weakly level closed, i.e.,* f *has closed level sets for all levels in* Y*;*
(iv) $\bar{f}$ *is level closed, i.e.,* $\bar{f}$ *has closed level sets for all levels in* $\mathcal{I}$*;*
(v) $\bar{f}$ *is lattice-l.s.c., i.e., for all* $\bar{x} \in X$ *one has* $\bar{f}(\bar{x}) \preccurlyeq \liminf_{x \to \bar{x}} \bar{f}(x)$*.*

Proof. The equivalence of (i), (ii) and (iii) follows directly from Proposition 2.31 and Proposition 2.32. The equivalence of (iii), (iv) and (v) follows from Corollary 2.36, Proposition 2.37, the fact (see Proposition 1.52) that a function $g : X \to \mathcal{F}$ is level closed (lattice l.s.c.) if and only if $j \circ g : X \to \mathcal{I}$ is level closed (lattice l.s.c.) and the fact that for the $\mathcal{I}$-valued extension $\bar{f}$ and the $\mathcal{F}$-valued extension $\tilde{f}$ of a function $f : X \to \overline{Y}$, $\bar{f} = j \circ \tilde{f}$ holds true. □

Applying Proposition 2.38 we can formulate the following existence result for a solution to a vector optimization problem. The result is a vectorial analogue of the famous Weierstrass theorem.

Theorem 2.40. *If one of the equivalent characterizations of lower semicontinuity in the preceding theorem is satisfied for the objective function $f : X \to \overline{Y}$ of* (V) *and if S is a compact subset of X, then there exists a solution to* (V).

Proof. This is a direct consequence of Proposition 2.15, Proposition 2.38 and Theorem 2.39. □

It is remarkable that $\bar{f} : X \to \mathcal{I}$ being lattice-l.s.c. is an adequate semicontinuity assumption for a vectorial Weierstrass existence result. The condition that $f : X \to \overline{Y}$ is lattice-l.s.c. is usually (if it is well-defined at all) too strong and not satisfiable.

2.5 Mild solutions

For a solution $\bar{X}$ to the complete-lattice-valued optimization problem ($\mathcal{L}$) as defined in Section 2.1, the condition $f[\bar{X}] = \operatorname{Min} f[S]$ is part of the definition. This requirement can be by several reasons too strong. Relaxing this condition, we obtain an alternative solution concept.

Definition 2.41. A nonempty set $\hat{X}$ with $f[\hat{X}] \subseteq \operatorname{Min} f[S]$ is called a *mild solution* to $(\mathcal{L})$ if the infimum of the canonical extension F over 2^S is attained at $\hat{X}$.

The idea of a mild solution can be explained as follows. A mild solution $\hat{X}$ is allowed to be a smaller set than a solution. However, as the attainment of the infimum is required, the set $\hat{X}$ cannot become arbitrarily small. This ensures that $\hat{X}$ contains a sufficient amount of information. Of course, every solution to $(\mathcal{L})$ is also a mild solution to $(\mathcal{L})$. But a mild solution can be a proper subset of a solution.

Theorem 2.42. *If a mild solution to* $(\mathcal{L})$ *exists, then there exists a solution to* $(\mathcal{L})$*.*

Proof. Let $\hat{X}$ be a mild solution to $(\mathcal{L})$. Set $\bar{X} := \operatorname{Eff}(\mathcal{L})$, then $S \supseteq \bar{X} \supseteq \hat{X} \neq \emptyset$. Since $\inf_{x \in S} f(x) = \inf_{x \in \hat{X}} f(x)$, we get $\inf_{x \in S} f(x) = \inf_{x \in \bar{X}} f(x)$. Thus $\bar{X}$ is a solution to $(\mathcal{L})$. □

We now consider the vector optimization problem (V) as defined in Section 2.2.

Definition 2.43. A set $\hat{X}$ is called *mild solution* to the vector optimization problem (V) if it is a mild solution to its lattice extension $(\mathcal{V})$.

For the special case of a vector optimization problem, we have the following characterization of a mild solution.

Theorem 2.44. *Assume that a solution to* (V) *exists. A set* $\hat{X} \subseteq S$ *is a mild solution to* (V) *if and only if*

$$f[\hat{X}] \subseteq \operatorname{Min} f[S] \subseteq \operatorname{Inf} f[\hat{X}]. \tag{2.10}$$

Proof. If $\{x \in \hat{X} |\ f(x) = -\infty\} \neq \emptyset$, then

$$\{-\infty\} = \operatorname{Min} f[S] = \operatorname{Inf} f[S] = \operatorname{Inf} f[\hat{X}].$$

Hence $\hat{X}$ is a mild solution if and only if $f[\hat{X}] = \{-\infty\}$.

In case that $f(x) = +\infty$ for all $x \in S$ we have

$$\{+\infty\} = f[\hat{X}] = \operatorname{Min} f[S] = \operatorname{Inf} f[S] = \operatorname{Inf} f[\hat{X}]$$

for every nonempty subset $\hat{X} \subseteq S$. Therefore, every nonempty subset $\hat{X} \subseteq S$ is a mild solution.

We can assume that $f[S] \subseteq Y$ because otherwise we have

$$\operatorname{Min} f[S] = \operatorname{Min}(f[S] \setminus \{+\infty\}) \quad \text{and} \quad \operatorname{Inf} f[S] = \operatorname{Inf}(f[S] \setminus \{+\infty\}).$$

If $\hat{X}$ is a mild solution to (V), we have

$$\emptyset \neq \hat{X} \subseteq S \quad \wedge \quad f[\hat{X}] \subseteq \operatorname{Min} f[S] \quad \wedge \quad \operatorname{Inf} f[\hat{X}] = \operatorname{Inf} f[S].$$

It remains to show $\operatorname{Min} f[S] \subseteq \operatorname{Inf} f[\hat{X}]$. Let $y \in \operatorname{Min} f[S]$, i.e.,

$$y \in f[S] \qquad \wedge \qquad y \notin f[S] + C \setminus \{0\}.$$

It follows

$$\{y\} + \operatorname{int} C \subseteq f[S] + \operatorname{int} C \qquad \wedge \qquad y \notin f[S] + \operatorname{int} C.$$

We have $\emptyset \neq \operatorname{Cl}_+ f[S] \neq Y$. By Corollary 1.48 (ii) we get $y \in \operatorname{Inf} f[S] = \operatorname{Inf} f[\hat{X}]$.

Let $\bar{X}$ be a solution to (V) and let (2.10) be satisfied. It follows $f[\hat{X}] \subseteq f[\bar{X}] \subseteq \operatorname{Inf} f[\hat{X}]$. From Corollary 1.49 (i), we get $\operatorname{Cl}_+ f[\bar{X}] = \operatorname{Cl}_+ f[\hat{X}]$. Proposition 1.52 yields $\operatorname{Inf} f[\bar{X}] = \operatorname{Inf} f[\hat{X}]$. Hence $\hat{X}$ is a mild solution to (V). □

We next focus on a relationship to properly efficient solutions (e.g. Luc, 1988; Göpfert *et al.*, 2003; Jahn, 2004). The famous theorem by Arrow *et al.* (1953) and related results state that, under certain assumptions, the set of properly minimal vectors is a dense subset of the set of minimal vectors. In the literature, there are many density results for different types of proper efficiency (e.g. Borwein, 1980; Jahn, 1988; Ferro, 1999; Fu, 1996; Göpfert *et al.*, 2003). The following theorem shows that the set of proper efficient solutions is just an instance of a mild solution, whenever (under certain assumptions) a corresponding density result holds.

Theorem 2.45. *Assume that a solution to* (V) *exists. Let* $\hat{X} \subseteq S$ *be a set such that* $f[\hat{X}] \subseteq Y$ *and*

$$f[\hat{X}] \subseteq \operatorname{Min} f[S] \subseteq \operatorname{cl} f[\hat{X}]. \tag{2.11}$$

Then $\hat{X}$ *is a mild solution to* (V).

Proof. Let $\bar{X}$ be a solution to (V). Then $\operatorname{Min} f[S]$ is nonempty, hence $\operatorname{cl} f[\hat{X}]$ is nonempty and thus $\hat{X}$ is nonempty, too. We have

$$f[\hat{X}] \subseteq \operatorname{Min} f[S] = f[\bar{X}] \subseteq \operatorname{cl} f[\hat{X}].$$

Using Corollary 1.49 (i) and the fact $\operatorname{Cl}_+ \operatorname{cl} f[\hat{X}] = \operatorname{Cl}_+ f[\hat{X}]$, we get $\operatorname{Cl}_+ f[\bar{X}] = \operatorname{Cl}_+ f[\hat{X}]$. Proposition 1.52 yields $\operatorname{Inf} f[\bar{X}] = \operatorname{Inf} f[\hat{X}]$. Hence $\hat{X}$ is a mild solution to (V). □

In general, (2.11) does not hold for a mild solution $\hat{X}$ to (V).

Example 2.46. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}^2$ partially ordered by $C = \mathbb{R}^2_+$, f the identity map and

$$S = \left\{x \in \mathbb{R}^2 |\ x_1 > 0,\ x_2 > 0,\ x_1 + x_2 \geq 1\right\} \cup \{x \in \mathbb{R}^2 |\ x_1 = 0, x_2 \geq 2\}.$$

Then $\hat{X} := \{\lambda\,(0,1)^T + (1-\lambda)\,(1,0)^T \mid\ \lambda \in (0,1)\}$ is a mild solution. But $(0,2)^T \in \operatorname{Min} f[S] \setminus \operatorname{cl} f[\hat{X}]$, hence (2.11) is violated, see Figure 2.7.

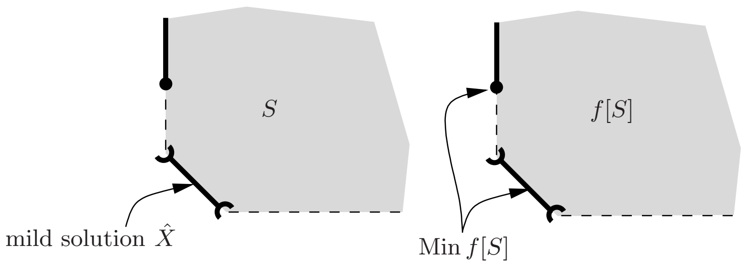

Fig. 2.7 Illustration of Example 2.46. The mild solution $\hat{X}$ does not satisfy the density condition (2.11).

If we additionally assume that $f[S]+C$ is closed and Y is a finite dimensional space, say $Y = \mathbb{R}^q$, we obtain that a mild solution satisfies (2.11). For instance, if S is a polyhedral convex set, C is polyhedral and f linear (see Chapter 4), then $f[S]+C$ is closed (Rockafellar, 1972, Theorem 19.3). Also, the assumptions of the Weierstrass existence result, Theorem 2.40, imply that $f[S]+C$ is closed (this follows from epi f being closed and S compact).

Theorem 2.47. *Let $Y = \mathbb{R}^q$. If $\hat{X}$ is a mild solution to* (V)*, $f[S] \subseteq \mathbb{R}^q$ and $f[S]+C$ is closed, then*

$$f[\hat{X}] \subseteq \operatorname{Min} f[S] \subseteq \operatorname{cl} f[\hat{X}].$$

Proof. It remains to show the second inclusion. Let $y \in \operatorname{Min} f[S]$, i.e.,

$$y \in f[S] \subseteq \operatorname{cl}\,(f[S]+C) = \operatorname{Cl}_+ f[S].$$

and (take into account that the cone C is pointed and convex and $f[S]+C$ is closed)

$$\begin{aligned} y \notin f[S] + C\setminus\{0\} &= (f[S]+C) + C\setminus\{0\} \\ &= \operatorname{cl}\,(f[S]+C) + C\setminus\{0\} = \operatorname{Cl}_+ f[S] + C\setminus\{0\}. \end{aligned}$$

This yields $y \in \operatorname{Min}\operatorname{Cl}_+ f[S]$. As $\hat{X}$ is a mild solution, we have $\operatorname{Inf} f[\hat{X}] = \operatorname{Inf} f[S]$. Proposition 1.52 implies $\operatorname{Cl}_+ f[\hat{X}] = \operatorname{Cl}_+ f[S]$. Thus we have $y \in \operatorname{Min}\operatorname{Cl}_+ f[\hat{X}]$.

It remains to show that $\operatorname{Min}\operatorname{Cl}_+ f[\hat{X}] \subseteq \operatorname{cl} f[\hat{X}]$. Assuming the contrary, there exists some $y \in \operatorname{Cl}_+ f[\hat{X}] = \operatorname{cl}\,(f[\hat{X}]+C)$ such that $y \notin \operatorname{cl} f[\hat{X}]$ and

$$\big(y - C\setminus\{0\}\big) \cap \operatorname{cl}\,(f[\hat{X}]+C) = \emptyset. \tag{2.12}$$

Let (b_n) and (c_n) be sequences, respectively, in $f[\hat{X}]$ and C such that $b_n + c_n \to y$. There is no subsequence of c_n that converges to 0, because otherwise we get the contradiction $y \in \operatorname{cl} f[\hat{X}]$. Hence there exists $n_0 \in \mathbb{N}$ and $\alpha > 0$ such that $\|c_n\| \geq \alpha$ for all $n \geq n_0$. There is a subsequence $(c_n)_{n\in M}$ (M an infinite subset of $\{n \in \mathbb{N} |\; n \geq n_0\}$) such that

$$\tilde{c}_n := \frac{\alpha c_n}{\|c_n\|} \xrightarrow{M} \tilde{c} \in C \setminus \{0\}.$$

It follows

$$b_n + \left(1 - \frac{\alpha}{\|c_n\|}\right) c_n = b_n + c_n - \tilde{c}_n \xrightarrow{M} y - \tilde{c}.$$

We obtain $y - \tilde{c} \in \operatorname{cl}(f[\hat{X}] + C)$ which contradicts (2.12). □

In Section 2.2 we introduced convexity solutions to (V). To this end the complete lattice $\mathcal{I}$ is replaced by the complete lattice $\mathcal{I}_{\text{co}}$. We proceed in the same way and introduce *mild convexity solutions* to (V).

Definition 2.48. A nonempty set $\hat{X} \subseteq X$ is called a *mild convexity solution* or *mild $\mathcal{I}_{\text{co}}$-solution* to the vector optimization problem (V) if $\hat{X}$ is a mild solution to the corresponding convex lattice extension ($\mathcal{V}_{\text{co}}$).

Parallel to Theorem 2.25, mild convexity solutions can be characterized in terms of the vectorial objective function f.

Theorem 2.49. *A set $\hat{X} \subseteq X$ is a mild convexity solution to the vector optimization problem* (V) *if and only if the following three conditions are satisfied:*

(i) $\hat{X} \subseteq S$,
(ii) $f[\hat{X}] \subseteq \operatorname{Min} f[S]$,
(iii) $\operatorname{Inf} \operatorname{co} f[\hat{X}] = \operatorname{Inf} \operatorname{co} f[S]$.

Proof. This follows in the same way as Theorem 2.25. □

Corollary 2.50. *Every convexity solution to* (V) *is also a mild convexity solution to* (V).

Proof. This follows from Theorem 2.25 and Theorem 2.49. □

Corollary 2.51. *Every mild solution to* (V) *is also a mild convexity solution to* (V).

Proof. This follows from the fact that, by Proposition 1.60, $\operatorname{Inf} f[\hat{X}] = \operatorname{Inf} f[S]$ implies $\operatorname{Inf} \operatorname{co} f[\hat{X}] = \operatorname{Inf} \operatorname{co} f[S]$. □

The different solution concepts to (V) are compared in Figure 2.8.

The next example illustrates a mild convexity solution to a linear vector optimization problem. An essential advantage of mild convexity solutions is

$$
\begin{tikzcd}
 & \boxed{\text{$\bar{X}$ is a solution to (V)}} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\boxed{\text{$\bar{X}$ is a mild solution to (V)}} \arrow[dr, Rightarrow] & & \boxed{\text{$\bar{X}$ is a convexity solution to (V)}} \arrow[dl, Rightarrow] \\
 & \boxed{\text{$\bar{X}$ is a mild convexity solution to (V)}} &
\end{tikzcd}
$$

Fig. 2.8 Connections between different solution concepts to (V)

that finite sets sometimes are sufficient. In Chapter 4 we consider a modification of this concept in order to ensure that a "solution" to a linear vector optimization problem can always be a finite set. To this end we have to involve directions of the feasible set.

Example 2.52. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}^2$ partially ordered by the cone $\mathbb{R}^2_+$. Consider Problem (V) with

$$S = \left\{x \in \mathbb{R}^2 \middle|\ x_1 \geq 0,\ x_2 \geq 0,\ 2x_1 + x_2 \geq 2,\ x_1 + 2x_2 \geq 2\right\}$$

and let f be the identity map. Then

$$\hat{X} := \left\{(0,2)^T, (2,0)^T, \left(\frac{2}{3}, \frac{2}{3}\right)^T\right\}$$

is a mild convexity solution to (V), see Figure 2.9.

2.6 Maximization problems and saddle points

Saddle points play a crucial role in duality theory. The goal of this section is to introduce saddle points in the framework of complete-lattice-valued optimization problems. As a consequence we obtain a corresponding saddle point notion for vector optimization problems, which differs from those in the literature. It is necessary to consider minimization and maximization problems simultaneously and it should be initially clarified how the solution concepts apply in this case.

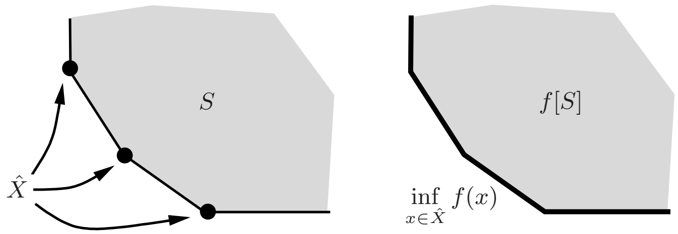

Fig. 2.9 Illustration of Example 2.52. On the left we see a mild convexity solution $\hat{X}$. On the right, the infimum of f over $\hat{X}$ with respect to the complete lattice $\mathcal{I}_{\mathrm{co}}$ is shown. It coincides with $\inf_{x\in S} f(x)$.

Let V be a nonempty set, $T \subseteq V$ and let $(Z, \leq)$ be a complete lattice. Parallel to the minimization problem $(\mathcal{L})$ introduced in Section 2.1, we consider the complete-lattice-valued maximization problem

$$\text{maximize } g : V \to Z \text{ with respect to } \leq \text{ over } T. \qquad (\mathcal{L}_{\max})$$

The canonical extension of the function $g : V \to Z$ in the complete-lattice-valued maximization problem $(\mathcal{L}_{\max})$ is the function

$$G : 2^V \to Z, \quad G(B) := \sup_{v\in B} g(v).$$

The set of *maximal elements* of a set $B \subseteq Z$ is defined by

$$\operatorname{Max} B := \{z \in B |\ (y \in B \wedge y \geq z) \Rightarrow y = z\}.$$

A solution to $(\mathcal{L}_{\max})$ can now be defined in the same way as for Problem $(\mathcal{L})$ in Definition 2.8.

Definition 2.53. A nonempty set $\bar{V}$ with $g[\bar{V}] = \operatorname{Max} g[T]$ is called a *solution* to $(\mathcal{L}_{\max})$ if the supremum of the canonical extension G over 2^T is attained in $\bar{V}$.

In terms of g a solution can be characterized as follows.

Corollary 2.54. *A nonempty set $\bar{V}$ is a solution to $(\mathcal{L}_{\max})$ if and only if the following conditions hold:*

(i) $\bar{V} \subseteq T$,
(ii) $g[\bar{V}] = \operatorname{Max} g[T]$,
(iii) $\sup_{v\in\bar{V}} g(v) = \sup_{v\in T} g(v)$.

Proof. This follows from an analogous result to Proposition 2.7. $\square$

Let X also be a nonempty set. We consider a function $l : X \times V \to Z$ depending on two variables, where we minimize with respect to the first variable and we maximize with respect to the second one. It turns out to be useful to distinguish between two types of canonical extensions for a function l depending on two variables. The function

$$L_l : 2^X \times 2^V \to Z, \quad L_l(\bar{X}, \bar{V}) := \sup_{v \in \bar{V}} \inf_{x \in \bar{X}} l(x, v)$$

is called the *lower canonical extension* of $l : X \times V \to Z$, and

$$L_u : 2^X \times 2^V \to Z, \quad L_u(\bar{X}, \bar{V}) := \inf_{x \in \bar{X}} \sup_{v \in \bar{V}} l(x, v)$$

is called the *upper canonical extension* of $l : X \times V \to Z$. This notion can be motivated by the fact that for all $(\bar{X}, \bar{V}) \in 2^X \times 2^V$ one has

$$L_l(\bar{X}, \bar{V}) \leq L_u(\bar{X}, \bar{V}),$$

which is an easy consequence of Z being a complete lattice.

Denoting by $+\infty$ and $-\infty$, respectively, the largest and the smallest element in Z, we set

$$S := \left\{ x \in X \,\middle|\, \sup_{v \in V} l(x, v) \neq +\infty \right\}$$

and

$$T := \left\{ v \in V \,\middle|\, \inf_{x \in X} l(x, v) \neq -\infty \right\}.$$

Let $p : X \to Z$ and $d : V \to Z$ be two functions such that

$$\forall x \in S : \; p(x) = \sup_{v \in V} l(x, v),$$

$$\forall v \in T : \; d(u) = \inf_{x \in X} l(x, v).$$

We assign to $l : X \times V \to Z$ the pair of dual optimization problems

$$\text{minimize } p : X \to Z \text{ with respect to } \leq \text{ over } S \subseteq X, \tag{2.13}$$

$$\text{maximize } d : V \to Z \text{ with respect to } \leq \text{ over } T \subseteq V. \tag{2.14}$$

Problem (2.13) corresponds to minimize $l : X \times V \to Z$ with respect to the first variable and likewise, Problem (2.14) corresponds to maximize $l : X \times V \to Z$ with respect to the second variable. Note that weak duality relation always holds, that is

$$\inf_{x \in S} p(x) = \inf_{x \in X} \sup_{v \in V} l(x, v) \leq \sup_{v \in V} \inf_{x \in X} l(x, v) = \sup_{v \in T} d(v).$$

According to our solution concept we propose the following notion of a saddle point for complete-lattice-valued problems.

Definition 2.55. Let X, V be two nonempty sets, $(Z, \leq)$ a complete lattice and let a function $l : X \times V \to Z$ be given. An element $(\bar{X}, \bar{V}) \in 2^S \times 2^T$, where $\bar{X} \neq \emptyset$ and $\bar{V} \neq \emptyset$, is called a *saddle point* of l if the following conditions are satisfied:

(i) $p[\bar{X}] = \operatorname{Min} p[S]$,
(ii) $d[\bar{V}] = \operatorname{Max} d[T]$,
(iii) $\forall A \in 2^X, \forall B \in 2^V : L_u(\bar{X}, B) \leq L_u(\bar{X}, \bar{V}) = L_l(\bar{X}, \bar{V}) \leq L_l(A, \bar{V})$.

Condition (iii) in the latter definition is a generalization of the well-known saddle point condition for an extended real-valued function, i.e., $(\bar{x}, \bar{v}) \in X \times V$ with $l(\bar{x}, \bar{v}) \in \mathbb{R}$ is a saddle point of $l : X \times V \to \overline{\mathbb{R}}$ if

$$\forall a \in X, \forall b \in V : \; l(\bar{x}, b) \leq l(\bar{x}, \bar{v}) \leq l(a, \bar{v}). \tag{2.15}$$

Note that in the extended real-valued case, $(\bar{x}, \bar{v}) \in S \times T$ implies $l(\bar{x}, \bar{v}) \in \mathbb{R}$. Vice versa, (2.15) and $l(\bar{x}, \bar{v}) \in \mathbb{R}$ implies $(\bar{x}, \bar{v}) \in S \times T$. Note further that condition (2.15) implies

$$\operatorname{Min} p[S] = \{p(\bar{x})\} \qquad \text{and} \qquad \operatorname{Max} d[T] = \{d(\bar{v})\}.$$

Consequently, conditions like (i) and (ii) of Definition 2.55 do not occur in the scalar case.

In our general setting, $(\bar{X}, \bar{V}) \in 2^S \times 2^T$ implies the following two conditions:

$$\forall a \in \bar{X} : \; L_u(\{a\}, \bar{V}) \neq +\infty \tag{2.16}$$

$$\forall b \in \bar{V} : \; L_l(\bar{X}, \{b\}) \neq -\infty. \tag{2.17}$$

Vice versa, if (iii) in Definition 2.55 holds, (2.16) $\wedge$ (2.17) implies $(\bar{X}, \bar{V}) \in 2^S \times 2^T$.

The following equivalent characterization of condition (iii) in Definition 2.55 is useful.

Lemma 2.56. *For nonempty sets $\bar{X} \subseteq X$ and $\bar{V} \subseteq V$, statement (iii) in Definition 2.55 is equivalent to*

$$\sup_{v \in \bar{V}} d(v) = \inf_{x \in \bar{X}} p(x). \tag{2.18}$$

Proof. From (iii) in Definition 2.55, we get

$$L_u(\bar{X}, V) \leq L_l(X, \bar{V})$$

and hence

$$\inf_{x \in \bar{X}} \sup_{v \in V} l(x, v) \leq \sup_{v \in \bar{V}} \inf_{x \in X} l(x, v).$$

Moreover, we have

$$\sup_{v\in\bar V}\inf_{x\in X} l(x,v) \le \inf_{x\in X}\sup_{v\in\bar V} l(x,v) \le \inf_{x\in\bar X}\sup_{v\in V} l(x,v).$$

This means that (2.18) is obtained from (iii) in Definition 2.55.

Now, let (2.18) be satisfied. It follows that

$$\forall A\in 2^X,\ \forall B\in 2^V:\quad \inf_{x\in\bar X}\sup_{v\in B} l(x,v) \le \sup_{v\in\bar V}\inf_{x\in A} l(x,v).$$

In particular, this implies

$$\forall A\in 2^X:\quad L_u(\bar X,\bar V)\le L_l(A,\bar V),$$

$$\forall B\in 2^V:\quad L_u(\bar X,B)\le L_l(\bar X,\bar V),$$

$$L_u(\bar X,\bar V)\le L_l(\bar X,\bar V).$$

Moreover, we have

$$L_l(\bar X,\bar V)\le L_u(\bar X,\bar V).$$

The last four statements imply statement (iii) in Definition 2.55. $\square$

We are now able to relate saddle points to solutions of (2.13) and (2.14).

Theorem 2.57. *The following statements are equivalent:*

(i) $\bar X$ *is a solution to* (2.13), $\bar V$ *is a solution to* (2.14) *and*

$$\sup_{v\in T} d(v) = \inf_{x\in S} p(x);$$

(ii) $(\bar X,\bar V)$ *is a saddle point of* l.

Proof. Condition (i) can be equivalently expressed as

(a) $\operatorname{Min} p[S] = p[\bar X],\quad \emptyset\ne\bar X,\quad \bar X\subseteq S,$

(b) $\inf_{x\in\bar X} p(x) = \inf_{x\in S} p(x),$

(c) $\operatorname{Max} d[T] = d[\bar V],\quad \emptyset\ne\bar V,\quad \bar V\subseteq T,$

(d) $\sup_{v\in\bar V} d(v) = \sup_{v\in T} d(v),$

(e) $\sup_{v\in T} d(v) = \inf_{x\in S} p(x).$

In view of Lemma 2.56 it remains to show that (b)∧(d)∧(e) is equivalent to (2.18) in the present situation. Of course, (b)∧(d)∧(e) implies (2.18). On the other hand, since $\bar X\subseteq S$ and $\bar V\subseteq T$, (2.18) implies that

$$\inf_{x\in\bar X} p(x) = \sup_{v\in\bar V} d(v) \le \sup_{v\in T} d(v) \le \inf_{x\in S} p(x) \le \inf_{x\in\bar X} p(x).$$

The last expression holds with equality. This yields (b)$\wedge$(d)$\wedge$(e). □

We next focus on the special case $(Z, \leq) = (\overline{\mathbb{R}}, \leq)$ and show that an ordinary saddle point is obtained.

Theorem 2.58. *For $l : X \times V \to \overline{\mathbb{R}}$ the following is equivalent.*

(i) $(\bar{X}, \bar{V})$ *is a saddle point of l in the sense of Definition 2.55.*
(ii) *Every* $(\bar{x}, \bar{v}) \in \bar{X} \times \bar{V}$ *is a saddle point of l in the classic sense, that is* $(\bar{x}, \bar{v}) \in X \times V$ *with* $l(\bar{x}, \bar{v}) \in \mathbb{R}$ *such that* (2.15) *holds.*

Proof. As discussed above, $(\bar{x}, \bar{v}) \in S \times T$ corresponds to $l(\bar{x}, \bar{v}) \in \mathbb{R}$ in the present situation. By Theorem 2.57 and Theorem 2.13, (i) is equivalent to

$$\forall \bar{x} \in \bar{X},\ \forall \bar{v} \in \bar{V} : \quad p(\bar{x}) = \inf_{x \in S} p(x) = \sup_{v \in T} d(v) = d(\bar{v}). \tag{2.19}$$

From the definition of p and d we get $p(\bar{x}) \geq l(\bar{x}, \bar{v}) \geq d(\bar{v})$ and (2.19) yields $p(\bar{x}) = l(\bar{x}, \bar{v}) = d(\bar{v})$ for all $\bar{x} \in \bar{X}$ and all $\bar{v} \in \bar{V}$. This implies (ii).

On the other hand, (ii) implies that for all $\bar{x} \in \bar{X}$ and all $\bar{v} \in \bar{V}$ one has

$$\begin{aligned} \inf_{x \in S} p(x) \leq p(\bar{x}) = \sup_{b \in V} l(\bar{x}, b) \leq l(\bar{x}, \bar{v}) \\ \leq \inf_{a \in X} l(a, \bar{v}) = d(\bar{v}) \leq \sup_{v \in T} d(v). \end{aligned}$$

Weak duality yields equality. This implies (2.19). □

Similarly to mild solutions we can define mild saddle points by relaxing the conditions (i) and (ii) in Definition 2.55.

Definition 2.59. Let X, V be two nonempty sets, $(Z, \leq)$ a complete lattice and let a function $l : X \times V \to Z$ be given. An element $(\hat{X}, \hat{V}) \in 2^S \times 2^T$, where $\hat{X} \neq \emptyset$ and $\hat{V} \neq \emptyset$, is called a *mild saddle point* of l if the following conditions are satisfied:

(i) $p[\hat{X}] \subseteq \operatorname{Min} p[S]$,
(ii) $d[\hat{V}] \subseteq \operatorname{Max} d[T]$,
(iii) $\forall A \in 2^X, \forall B \in 2^V : L_u(\hat{X}, B) \leq L_u(\hat{X}, \hat{V}) = L_l(\hat{X}, \hat{V}) \leq L_l(A, \hat{V})$.

A corresponding characterization follows immediately.

Theorem 2.60. *The following statements are equivalent:*

(i) $\hat{X}$ *is a mild solution to* (2.13), $\hat{V}$ *is a mild solution to* (2.14) *and*

$$\sup_{v \in T} d(v) = \inf_{x \in S} p(x);$$

(ii) $(\hat{X}, \hat{V})$ *is a mild saddle point of l.*

Proof. Similarly to the proof of Theorem 2.57. □

The notion of a saddle point introduced in this section can be used for arbitrary $\mathcal{I}$-valued problems. In case of a vector optimization problem we consider its lattice extension which yields an $\mathcal{I}$-valued problem. We obtain an $\mathcal{I}$-valued Lagrangian and an $\mathcal{I}$-valued dual problem. Thus, the saddle point notions introduced in this section easily apply to vector optimization.

2.7 Notes on the literature

In the framework of a mathematical optimization theory, the notion of an efficient element seems to be first used by Koopmans (1951), compare (Stadler, 1979), but the ideas can be traced back to the early works by Pareto and Edgeworth. Modifications of efficient solutions, such as weakly or properly efficient solutions, are commonly considered in the literature (Luc, 1988; Jahn, 1986, 2004; Ehrgott, 2000; Boţ *et al.*, 2009). The idea to compute a subset of the efficient solutions in order to present it to a decision maker is standard in the literature on vector optimization. Nevertheless, there is no unique and precise specification of such a subset, which is understood as a solution concept.

The solution concept for complete-lattice-valued problems in Section 2.1 and its application to vector optimization in Section 2.2 including the notion of a mild solution first appeared in (Heyde and Löhne, 2010). It should be mentioned that these ideas arose from several discussions about solution concepts for set-valued optimization problems between Andreas H. Hamel and the mentioned authors. The notion of (mild) convexity solutions and all the related results are new in this book.

Section 2.3 is a collection of results on semicontinuity concepts for set-valued maps which can be found similarly in the literature. The results and proofs in the presented form are taken from Heyde and Löhne (2010) and are due to the first author. Definition 2.28 follows the articles by Gerritse (1997) and Ait Mansour *et al.* (2007). Note that in (Ait Mansour *et al.*, 2007) the term "level closed" is used for property (b). We call a function level closed if all level sets are closed and we speak about weak level closedness if the weaker property (b) holds. The notions of lattice- and topological semicontinuity are introduced in (Gerritse, 1997). The definition of lattice semicontinuity coincides with that of Gerritse (1997). The definition of topological semicontinuity differs slightly from that in (Gerritse, 1997) since we do not require a topological structure on the whole set $\overline{Y}$. It coincides, however, with the concept denoted simply by lower semicontinuity in (Ait Mansour *et al.*, 2007). Note also that Gerritse (1997) deals with upper rather than lower semicontinuity. Proposition 2.33 is slightly different from (Penot and Théra, 1982, Proposition 1.3.a) but the proof follows essentially the lines of the one in (Penot and Théra, 1982). Note further that it was shown by Liu and Luo (1991, Theorem 3.6.) that every level closed function $f : X \to Z$

is lattice-l.s.c. if and only if Z is a completely distributive lattice (compare Proposition 2.29).

The existence result in Section 2.4, the notion of mild solutions as well as all related results in Section 2.5 are due to Heyde and Löhne (2010). There are other existence results in the literature; partially they are related to the domination property (e.g. Jahn, 1986, 2004; Luc, 1988; Sonntag and Zălinescu, 2000).

Saddle points for complete-lattice-valued problems as well as all related concepts and results in Section 2.6 seem to be new and arose from discussions with Andreas H. Hamel. In the literature (see e.g. Rödder, 1977; Luc, 1988; Tanaka, 1990, 1994; Li and Wang, 1994; Tan *et al.*, 1996; Li and Chen, 1997; Ehrgott and Wiecek, 2005b; Adán and Novo, 2005) there are other notions of saddle points for vector optimization problems which are not based on the structure of a complete lattice.

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