Chapter 1
Dynamic Risk Measures

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Abstract This paper gives an overview of the theory of dynamic convex risk measures for random variables in discrete-time setting. We summarize robust representation results of conditional convex risk measures, and we characterize various time consistency properties of dynamic risk measures in terms of acceptance sets, penalty functions, and by supermartingale properties of risk processes and penalty functions.

Keywords Dynamic convex risk measure · Robust representation · Penalty function · Time consistency · Entropic risk measure

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1.1 Introduction

Risk measures are quantitative tools developed to determine minimum capital reserves that are required to be maintained by financial institutions in order to ensure their financial stability. An axiomatic analysis of risk assessment in terms of capital requirements was initiated by Artzner, Delbaen, Eber, and Heath [2, 3], who introduced coherent risk measures. Föllmer and Schied [23] and Frittelli and Rosazza

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Gianin [25] replaced subadditivity and positive homogeneity by convexity in the set of axioms and established the more general concept of a convex risk measure. Since then, convex and coherent risk measures and their applications have attracted a growing interest both in mathematical finance research and among practitioners.

One of the most appealing properties of a convex risk measure is its robustness against model uncertainty. Under some regularity condition, it can be represented as a suitably modified worst expected loss over a whole class of probabilistic models. This was initially observed in [3, 23, 25] in the static setting, where financial positions are described by random variables on some probability space, and a risk measure is a real-valued functional. For a comprehensive presentation of the theory of static coherent and convex risk measures, we refer to Delbaen [15] and Föllmer and Schied [24, Chap. 4].

A natural extension of a static risk measure is given by a conditional risk measure, which takes into account the information available at the time of risk assessment. As its static counterpart, a conditional convex risk measure can be represented as the worst conditional expected loss over a class of suitably penalized probability measures; see [6, 12, 18, 26, 29, 34, 37]. In the dynamical setting described by some filtered probability space, risk assessment is updated over the time in accordance with the new information. This leads to the notion of dynamic risk measure, which is a sequence of conditional risk measures adapted to the underlying filtration.

A crucial question in the dynamical framework is how risk evaluations at different times are interrelated. Several notions of time consistency were introduced and studied in the literature. One of today's most used notions is strong time consistency, which corresponds to the dynamic programming principle; see [4, 7, 12, 13, 16–18, 22, 26, 29] and references therein. As shown in [7, 16, 22], strong time consistency can be characterized by additivity of the acceptance sets and penalty functions, and also by a supermartingale property of the risk process and the penalty function process. Similar characterizations of the weaker notions of time consistency, so-called rejection and acceptance consistency, were given in [19, 33]. Rejection consistency, also called prudence in [33], seems to be a particularly suitable property from the point of view of a regulator, since it ensures that one always stays on the safe side when updating risk assessment. The weakest notions of time consistency considered in the literature are weak acceptance and weak rejection consistency, which require that if some position is accepted (or rejected) for any scenario tomorrow, it should be already accepted (or rejected) today; see [4, 9, 35, 41, 43].

As pointed out in [21, 28], risk assessment in the multiperiod setting should also account for uncertainty about the time value of money. This requires to consider entire cash flow processes rather than total amounts at terminal dates as risky objects, and it leads to a further extension of the notion of risk measure. Risk measures for processes were studied in [1, 4, 10–13, 27, 28, 34]. The new feature in this framework is that not only the amounts but also the timing of payments matters; cf. [1, 12, 13, 28]. However, as shown in [4] in the static and in [1] in the dynamical setting, risk measures for processes can be identified with risk measures for random variables on an appropriate product space. This allows a natural translation of results obtained in the framework of risk measures for random variables to the framework of processes; see [1].

The aim of this paper it to give an overview of the current theory of dynamic convex risk measures for random variables in discrete-time setting; the corresponding results for risk measures for processes are given in [1]. The paper is organized as follows. Section 1.2 recalls the definition of a conditional convex risk measure and its interpretation as the minimal capital requirement from [18]. Section 1.3 summarizes robust representation results from [8, 18, 22]. In Sect. 1.4 we first give an overview of different time consistency properties based on [40]. Then we focus on the strong notion of time consistency in Sect. 1.4.1, and we characterize it by supermartingale properties of risk processes and penalty functions. The results of this subsection are mainly based on [22], with the difference that here we give characterizations of time consistency also in terms of absolutely continuous probability measures, similar to [8]. In addition, we relate the martingale property of a risk process with the worst-case measure, and we provide explicit forms of the Doob and Riesz decompositions of the penalty function process. Section 1.4.2 generalizes [33, Sects. 2.4 and 2.5] and characterizes rejection and acceptance consistency in terms of acceptance sets, penalty functions, and, in case of rejection consistency, by a supermartingale property of risk processes and one-step penalty functions. Section 1.4.3 recalls characterizations of weak time consistency from [9, 41, 43], and Sect. 1.4.4 characterizes the recursive construction of time consistent risk measures suggested in [12, 13]. Finally, the dynamic entropic risk measure with a nonconstant risk aversion parameter is studied in Sect. 1.5.

1.2 Setup and Notation

Let $T \in \mathbb{N} \cup \{\infty\}$ be the time horizon, $\mathbb{T} := \{0, \dots, T\}$ for $T < \infty$, and $\mathbb{T} := \mathbb{N}_0$ for $T = \infty$. We consider a discrete-time setting given by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, P)$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F} = \mathcal{F}_T$ for $T < \infty$, and $\mathcal{F} = \sigma(\bigcup_{t\geq 0} \mathcal{F}_t)$ for $T = \infty$. For $t \in \mathbb{T}$, $L^\infty_t := L^\infty(\Omega, \mathcal{F}_t, P)$ is the space of all essentially bounded $\mathcal{F}_t$-measurable random variables, and $L^\infty := L^\infty(\Omega, \mathcal{F}_T, P)$. All equalities and inequalities between random variables and between sets are understood to hold P-almost surely, unless stated otherwise. We denote by $\mathcal{M}_1(P)$ (resp. by $\mathcal{M}^e(P)$) the set of all probability measures on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to P (resp. equivalent to P).

In this work we consider risk measures defined on the set L^∞, which is understood as the set of discounted terminal values of financial positions. In the dynamical setting, a conditional risk measure ρ_t assigns to each terminal payoff X an $\mathcal{F}_t$-measurable random variable $\rho_t(X)$ that quantifies the risk of the position X given the information $\mathcal{F}_t$. A rigorous definition of a conditional convex risk measure was given in [18, Definition 2].

Definition 1.1 A map $\rho_t : L^\infty \to L^\infty_t$ is called a *conditional convex risk measure* if it satisfies the following properties for all $X, Y \in L^\infty$:

(i) Conditional cash invariance: For all $m_t \in L^\infty_t$,

$$\rho_t(X + m_t) = \rho_t(X) - m_t;$$

(ii) Monotonicity: $X \le Y \Rightarrow \rho_t(X) \ge \rho_t(Y)$;
(iii) Conditional convexity: for all $\lambda \in L_t^\infty$, $0 \le \lambda \le 1$,

$$\rho_t\big(\lambda X + (1-\lambda)Y\big) \le \lambda\rho_t(X) + (1-\lambda)\rho_t(Y);$$

(iv) Normalization: $\rho_t(0) = 0$.

A conditional convex risk measure is called a *conditional coherent risk measure* if it has in addition the following property:

(v) Conditional positive homogeneity: for all $\lambda \in L_t^\infty$, $\lambda \ge 0$,

$$\rho_t(\lambda X) = \lambda\rho_t(X).$$

In the dynamical framework one can also analyze risk assessment for cumulated cash flow *processes* rather than just for terminal payoffs, i.e., one can consider a risk measure that accounts not only for the amounts but also for the timing of payments. Such risk measures were studied in [1, 10–13, 27, 28]. As shown in [4] in the static and in [1] in the dynamical setting, convex risk measures for processes can be identified with convex risk measures for random variables on an appropriate product space. This allows one to extend results obtained in our present setting to the framework of processes; cf. [1].

If ρ_t is a conditional convex risk measure, the function $\phi_t := -\rho_t$ defines a conditional monetary utility function in the sense of [12, 13]. The term "monetary" refers to conditional cash invariance of the utility function, the only property in Definition 1.1 that does not come from the classical utility theory. Conditional cash invariance is a natural request in view of the interpretation of ρ_t as a conditional capital requirement. In order to formalize this aspect, we first recall the notion of the *acceptance set* of a conditional convex risk measure ρ_t:

$$\mathcal{A}_t := \big\{X \in L^\infty \;\big|\; \rho_t(X) \le 0\big\}.$$

The following properties of the acceptance set were given in [18, Proposition 3].

Proposition 1.2 *The acceptance set $\mathcal{A}_t$ of a conditional convex risk measure ρ_t is*

1. *conditionally convex, i.e., $\alpha X + (1-\alpha)Y \in \mathcal{A}_t$ for all $X, Y \in \mathcal{A}_t$ and $\mathcal{F}_t$-measurable α such that* $0 \le \alpha \le 1$;
2. *solid, i.e., $Y \in \mathcal{A}_t$ whenever $Y \ge X$ for some $X \in \mathcal{A}_t$*;
3. *such that* $0 \in \mathcal{A}_t$ *and* $\operatorname{ess\,inf}\{X \in L_t^\infty \mid X \in \mathcal{A}_t\} = 0$.

Moreover, ρ_t is uniquely determined through its acceptance set, since

$$\rho_t(X) = \operatorname{ess\,inf}\big\{Y \in L_t^\infty \;\big|\; X + Y \in \mathcal{A}_t\big\}. \tag{1.1}$$

Conversely, if some set $\mathcal{A}_t \subseteq L^\infty$ satisfies conditions (1)–(3), *then the functional $\rho_t : L^\infty \to L_t^\infty$ defined via* (1.1) *is a conditional convex risk measure.*

Proof Properties (1)–(3) of the acceptance set follow easily from properties (i)–(iv) in Definition 1.1. To prove (1.1), note that by cash invariance $\rho_t(X) + X \in \mathcal{A}_t$ for all X, and this implies "$\geq$" in (1.1). On the other hand, for all $Z \in \{Y \in L_t^\infty \mid X + Y \in \mathcal{A}_t\}$, we have

$$0 \geq \rho_t(Z + X) = \rho_t(X) - Z,$$

and thus $\rho_t(X) \leq \operatorname{ess\,inf}\{Y \in L_t^\infty \mid X + Y \in \mathcal{A}_t\}$.

For the proof of the last part of the assertion, we refer to [18, Proposition 3]. □

Due to (1.1), the value $\rho_t(X)$ can be viewed as the minimal conditional capital requirement needed to be added to the position X in order to make it acceptable at time t. Moreover, (1.1) can be used to define risk measures; cf. Example 1.8.

1.3 Robust Representation

As observed in [3, 24, 25] in the static setting, the axiomatic properties of a convex risk measure yield, under some regularity condition, a representation of the minimal capital requirement as a suitably modified worst expected loss over a whole class of probabilistic models. In the dynamical setting, such robust representations of conditional coherent risk measures were obtained in [6, 8, 18, 22, 29, 37] for random variables and in [12, 34] for stochastic processes. In this section we mainly summarize the results from [8, 18, 22].

The alternative probability measures in a robust representation of a risk measure ρ_t contribute to the risk evaluation to a different degree. To formalize this aspect, we use the notion of the minimal penalty function $\alpha_t^{\min}$, defined for each $Q \in \mathcal{M}_1(P)$ as

$$\alpha_t^{\min}(Q) = Q\text{-}\operatorname*{ess\,sup}_{X \in \mathcal{A}_t} E_Q[-X|\mathcal{F}_t]. \tag{1.2}$$

The following property of the minimal penalty function is a standard result that will be used in the proof of Theorem 1.4.

Lemma 1.3 *For $Q \in \mathcal{M}_1(P)$ and $0 \leq s \leq t$,*

$$E_Q\big[\alpha_t^{\min}(Q)\big|\mathcal{F}_s\big] = Q\text{-}\operatorname*{ess\,sup}_{Y \in \mathcal{A}_t} E_Q[-Y|\mathcal{F}_s] \quad Q\text{-a.s.}$$

and in particular

$$E_Q\big[\alpha_t^{\min}(Q)\big] = \sup_{Y \in \mathcal{A}_t} E_Q[-Y].$$

Proof First we claim that the set

$$\big\{E_Q[-X|\mathcal{F}_t] \,\big|\, X \in \mathcal{A}_t\big\}$$

is directed upward for any $Q \in \mathcal{M}_1(P)$. Indeed, for $X, Y \in \mathcal{A}_t$, we can define $Z := XI_A + YI_{A^c}$, where $A := \{E_Q[-X|\mathcal{F}_t] \geq E_Q[-Y|\mathcal{F}_t]\} \in \mathcal{F}_t$. Conditional convexity of ρ_t implies that $Z \in \mathcal{A}_t$, and by definition of Z,

$$E_Q[-Z|\mathcal{F}_t] = \max\bigl(E_Q[-X|\mathcal{F}_t], E_Q[-Y|\mathcal{F}_t]\bigr) \quad Q\text{-a.s.}$$

Hence, there exists a sequence $(X_n^Q)_{n\in\mathbb{N}}$ in $\mathcal{A}_t$ such that

$$\alpha_t^{\min}(Q) = \lim_n E_Q\bigl[-X_n^Q \big| \mathcal{F}_t\bigr] \quad Q\text{-a.s.}, \tag{1.3}$$

and by monotone convergence we get

$$\begin{aligned} E_Q\bigl[\alpha_t^{\min}(Q) \big| \mathcal{F}_s\bigr] &= \lim_n E_Q\bigl[E_Q\bigl[-X_n^Q \big| \mathcal{F}_t\bigr] \big| \mathcal{F}_s\bigr] \\ &\leq Q\text{-}\operatorname*{ess\,sup}_{Y\in\mathcal{A}_t} E_Q[-Y|\mathcal{F}_s] \quad Q\text{-a.s.} \end{aligned}$$

The converse inequality follows directly from the definition of $\alpha_t^{\min}(Q)$. □

The following theorem relates robust representations to some continuity properties of conditional convex risk measures. It combines [18, Theorem 1] with [22, Corollary 2.4]; similar results can be found in [6, 12, 29].

Theorem 1.4 *For a conditional convex risk measure ρ_t, the following are equivalent*:

1. *ρ_t has a robust representation*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t}\bigl(E_Q[-X|\mathcal{F}_t] - \alpha_t(Q)\bigr), \quad X \in L^\infty, \tag{1.4}$$

where

$$\mathcal{Q}_t := \bigl\{Q \in \mathcal{M}_1(P) \bigm| Q = P|_{\mathcal{F}_t}\bigr\},$$

and α_t is a map from $\mathcal{Q}_t$ to the set of $\mathcal{F}_t$-measurable random variables with values in $\mathbb{R} \cup \{+\infty\}$ such that $\operatorname{ess\,sup}_{Q\in\mathcal{Q}_t}(-\alpha_t(Q)) = 0$.

2. *ρ_t has the robust representation in terms of the minimal penalty function*, *i.e.*,

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t}\bigl(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\bigr), \quad X \in L^\infty, \tag{1.5}$$

where $\alpha_t^{\min}$ is given in (1.2).

3. *ρ_t has the robust representation*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t^f}\bigl(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\bigr) \quad P\text{-a.s.}, \quad X \in L^\infty, \tag{1.6}$$

where

$$\mathcal{Q}_t^f := \bigl\{Q \in \mathcal{M}_1(P) \bigm| Q = P|_{\mathcal{F}_t},\ E_Q\bigl[\alpha_t^{\min}(Q)\bigr] < \infty\bigr\}.$$

4. ρ_t *has the "Fatou-property"*: *for any bounded sequence* $(X_n)_{n\in\mathbb{N}}$ *which converges* P*-a.s. to some* X,

$$\rho_t(X) \le \liminf_{n\to\infty} \rho_t(X_n) \quad P\text{-a.s.}$$

5. ρ_t *is continuous from above*, *i.e.*,

$$X_n \searrow X \quad P\text{-a.s} \quad \Longrightarrow \quad \rho_t(X_n) \nearrow \rho_t(X) \quad P\text{-a.s}$$

for any sequence $(X_n)_n \subseteq L^\infty$ *and* $X \in L^\infty$.

Proof (3) ⇒ (1) and (2) ⇒ (1) are obvious. (1) ⇒ (4): Dominated convergence implies that $E_Q[X_n|\mathcal{F}_t] \to E_Q[X|\mathcal{F}_t]$ for each $Q \in \mathcal{Q}_t$, and $\liminf_{n\to\infty} \rho_t(X_n) \ge \rho_t(X)$ follows by using the robust representation of ρ_t as in the unconditional setting, see, e.g., [24, Lemma 4.20].

(4) ⇒ (5): Monotonicity implies $\limsup_{n\to\infty} \rho_t(X_n) \le \rho_t(X)$, and $\liminf_{n\to\infty} \rho_t(X_n) \ge \rho_t(X)$ follows by (4).

(5) ⇒ (2): The inequality

$$\rho_t(X) \ge \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t} \big(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \big) \tag{1.7}$$

follows from the definition of $\alpha_t^{\min}$. In order to prove the equality, we will show that

$$E_P\big[\rho_t(X)\big] \le E_P\Big[\operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t} \big(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \big) \Big].$$

To this end, consider the map $\rho^P : L^\infty \to \mathbb{R}$ defined by $\rho^P(X) := E_P[\rho_t(X)]$. It is easy to check that ρ^P is a convex risk measure which is continuous from above. Hence [24, Theorem 4.31] implies that ρ^P has the robust representation

$$\rho^P(X) = \sup_{Q\in\mathcal{M}_1(P)} \big(E_Q[-X] - \alpha(Q) \big), \quad X \in L^\infty,$$

where the penalty function $\alpha(Q)$ is given by

$$\alpha(Q) = \sup_{X\in L^\infty : \rho^P(X)\le 0} E_Q[-X].$$

Next we will prove that $Q \in \mathcal{Q}_t$ if $\alpha(Q) < \infty$. Indeed, let $A \in \mathcal{F}_t$ and $\lambda > 0$. Then

$$-\lambda P[A] = E_P\big[\rho_t(\lambda I_A)\big] = \rho^P(\lambda I_A) \ge E_Q[-\lambda I_A] - \alpha(Q),$$

so

$$P[A] \le Q[A] + \frac{1}{\lambda}\alpha(Q) \quad \text{for all } \lambda > 0,$$

and hence $P[A] \le Q[A]$ if $\alpha(Q) < \infty$. The same reasoning with $\lambda < 0$ implies $P[A] \ge Q[A]$, and thus $P = Q$ on $\mathcal{F}_t$ if $\alpha(Q) < \infty$. By Lemma 1.3, we have for every $Q \in \mathcal{Q}_t$,

$$E_P\left[\alpha_t^{\min}(Q)\right] = \sup_{Y \in \mathcal{A}_t} E_P[-Y].$$

Since $\rho^P(Y) \le 0$ for all $Y \in \mathcal{A}_t$, this implies

$$E_P\left[\alpha_t^{\min}(Q)\right] \le \alpha(Q)$$

for all $Q \in \mathcal{Q}_t$, by definition of the penalty function $\alpha(Q)$.

Finally we obtain

$$\begin{aligned}
E_P\left[\rho_t(X)\right] = \rho^P(X) &= \sup_{Q \in \mathcal{M}_1(P), \alpha(Q) < \infty} \bigl(E_Q[-X] - \alpha(Q)\bigr) \\
&\le \sup_{Q \in \mathcal{Q}_t, E_P[\alpha_t^{\min}(Q)] < \infty} \bigl(E_Q[-X] - \alpha(Q)\bigr) \\
&\le \sup_{Q \in \mathcal{Q}_t, E_P[\alpha_t^{\min}(Q)] < \infty} E_P\left[E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\right] \\
&\le E_P\Bigl[\operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t, E_P[\alpha_t^{\min}(Q)] < \infty} \bigl(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\bigr)\Bigr] \\
&\le E_P\Bigl[\operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} \bigl(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\bigr)\Bigr], \qquad (1.8)
\end{aligned}$$

proving (1.5).

(5) $\Rightarrow$ (3) The inequality

$$\rho_t(X) \ge \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^f} \bigl(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\bigr)$$

follows from (1.7) since $\mathcal{Q}_t^f \subseteq \mathcal{Q}_t$, and (1.8) proves the equality. □

Remark 1.5 The penalty function $\alpha_t^{\min}(Q)$ is minimal in the sense that any other function α_t in a robust representation (1.4) of ρ_t satisfies

$$\alpha_t^{\min}(Q) \le \alpha_t(Q) \quad P\text{-a.s.}$$

for all $Q \in \mathcal{Q}_t$. An alternative formula for the minimal penalty function is given by

$$\alpha_t^{\min}(Q) = \operatorname*{ess\,sup}_{X \in L^\infty} \bigl(E_Q[-X|\mathcal{F}_t] - \rho_t(X)\bigr) \quad \text{for all } Q \in \mathcal{Q}_t.$$

This follows as in the unconditional case; see, e.g., [24, Theorem 4.15, Remark 4.16].

In the *coherent* case the penalty function $\alpha_t^{\min}(Q)$ can only take values 0 or ∞ due to positive homogeneity of ρ_t. Thus representation (1.12) takes the following form.

Corollary 1.6 *A conditional coherent risk measure ρ_t is continuous from above if and only if it is representable in the form*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t^0} E_Q[-X|\mathcal{F}_t], \qquad X \in L^\infty, \tag{1.9}$$

where

$$\mathcal{Q}_t^0 := \{Q \in \mathcal{Q}_t \mid \alpha_t^{\min}(Q) = 0 \ Q\text{-a.s.}\}.$$

Remark 1.7 Another characterization of a conditional convex risk measure ρ_t that is equivalent to properties (1)–(5) of Theorem 1.4 is the following: The acceptance set $\mathcal{A}_t$ is weak*-closed, i.e., it is closed in L^∞ with respect to the topology $\sigma(L^\infty, L^1(\Omega, \mathcal{F}, P))$. This equivalence was shown in [12] in the context of risk measures for processes and in [29] for risk measures for random variables. Though in [29] a slightly different definition of a conditional risk measure is used, the reasoning given there works just the same in our case; cf. [29, Theorem 3.16].

Example 1.8 A class of examples of conditional convex risk measures can be obtained by considering a conditional robust version of a *shortfall risk* introduced in [24, Sect. 4.9]. To this end, let $l_t : \mathbb{R} \to \mathbb{R}$ be a convex and strictly increasing loss function, and let $\mathcal{R}_t$ be some convex subset of $\mathcal{Q}_t$. Then the set

$$\mathcal{A}_t := \{X \in L^\infty \mid E_Q[l_t(-X)|\mathcal{F}_t] \le l_t(0) \ \forall Q \in \mathcal{R}_t\} \tag{1.10}$$

satisfies properties (1)–(3) of Proposition 1.2 and thus induces a conditional convex risk measure. Such risk measures were introduced and studied in [41, Sect. 5], where they are called *conditional robust shortfall risk measures*.

A conditional robust shortfall risk measure is continuous from above by Remark 1.7. Indeed, if $(X_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{A}_t$ converging to some X, then $X \in \mathcal{A}_t$ due to Lebesgue convergence theorem, and thus the set $\mathcal{A}_t$ is weak*-closed by Krein–Šmulian theorem; cf., e.g., [24, Theorem A.63, Lemma A.64]. Moreover, if $P \in \mathcal{R}_t$ (or if there exists $Q^* \approx P$ such that $Q^* \in \mathcal{R}_t$), then the set of equivalent probability measures is dense in $\mathcal{R}_t$, and representation (1.10) can be written as

$$\mathcal{A}_t = \{X \in L^\infty \mid E_Q[l_t(-X)|\mathcal{F}_t] \le l_t(0) \ \forall Q \in \mathcal{R}_t^e\}, \tag{1.11}$$

where $\mathcal{R}_t^e$ denotes the set of all $Q \in \mathcal{M}^e(P)$ such that the corresponding $\mathcal{F}_t$-normalized measure $\tilde{Q}$ defined by $\frac{d\tilde{Q}}{dP} := \frac{Z_T}{Z_t}$ belongs to $\mathcal{R}_t$. Here Z_s denotes the density of Q with respect to P on $\mathcal{F}_s$, $s \in \mathbb{T}$.

Example 1.9 If one takes $\mathcal{R}_t = \{P\}$ and the exponential loss function $l_t(x) = \exp(\gamma_t x) - 1$ with $\gamma_t > 0$ in the previous example, one obtains the well-known *conditional entropic risk measure*

$$\rho_t(X) = \frac{1}{\gamma_t} \log E\big[\exp(-\gamma_t X)\big|\mathcal{F}_t\big], \quad X \in L^\infty.$$

The entropic risk measure was introduced in [24] in the static setting; in the dynamical setting it appeared in [5, 12, 13, 18, 22, 31]. We characterize the dynamic entropic risk measure in Sect. 1.5 in a slightly more general setting, where the risk aversion parameter γ_t might be random.

Example 1.10 Example 1.8 with a linear loss function $l_t(x) = x$ and

$$\mathcal{R}_t := \left\{ Q \in \mathcal{Q}_t \,\middle|\, \frac{dQ}{dP} \le \lambda_t^{-1} \right\}$$

for some $\lambda_t \in L_t^\infty$, $0 < \lambda_t \le 1$, yields an important example of a conditional coherent risk measure, the *conditional Average Value-at-Risk*

$$AV@R_{t,\lambda_t}(X) := \operatorname{ess\,sup}\big\{ E_Q[-X|\mathcal{F}_t] \,\big|\, Q \in \mathcal{R}_t \big\}.$$

Static Average Value-at-Risk was introduced in [3] as a valid alternative to the widely used yet criticized Value-at-Risk. The conditional version of Average Value-at-Risk appeared in [4] and was also studied in [19, 42].

For the characterization of time consistency in Sect. 1.4, we will need a robust representation of a conditional convex risk measure ρ_t under any measure $Q \in \mathcal{M}_1(P)$, where possibly $Q \notin \mathcal{Q}_t$. Such representation can be obtained as in Theorem 1.4 by considering ρ_t as a risk measure under Q, as shown in the next corollary. This result is a version of [8, Proposition 1].

Corollary 1.11 *A conditional convex risk measure ρ_t is continuous from above if and only if it has the robust representations*

$$\rho_t(X) = \underset{R \in \mathcal{Q}_t(Q)}{Q\text{-}\operatorname{ess\,sup}} \big(E_R[-X|\mathcal{F}_t] - \alpha_t^{\min}(R) \big) \tag{1.12}$$

$$= \underset{R \in \mathcal{Q}_t^f(Q)}{Q\text{-}\operatorname{ess\,sup}} \big(E_R[-X|\mathcal{F}_t] - \alpha_t^{\min}(R) \big) \quad Q\text{-a.s.}, \quad X \in L^\infty, \tag{1.13}$$

for all $Q \in \mathcal{M}_1(P)$, where

$$\mathcal{Q}_t(Q) = \big\{ R \in \mathcal{M}_1(P) \,\big|\, R = Q|_{\mathcal{F}_t} \big\}$$

and

$$\mathcal{Q}_t^f(Q) = \big\{ R \in \mathcal{M}_1(P) \,\big|\, R = Q|_{\mathcal{F}_t},\, E_R\big[\alpha_t^{\min}(R)\big] < \infty \big\}.$$

Proof To show that continuity from above implies representation (1.12), we can replace P by a probability measure $Q \in \mathcal{M}_1(P)$ and repeat all the reasoning of the proof of (5) ⇒ (2) in Theorem 1.4. In this case we consider the static convex risk measure

$$\rho^Q(X) = E_Q\big[\rho_t(X)\big] = \sup_{R\in\mathcal{M}_1(P)} \big(E_R[-X] - \alpha(R)\big), \quad X \in L^\infty,$$

instead of ρ^P. The proof of (1.13) follows in the same way from [22, Corollary 2.4]. Conversely, continuity from above follows from Theorem 1.4 since representation (1.12) holds under P. □

Remark 1.12 One can easily see that the set $\mathcal{Q}_t$ in representations (1.4) and (1.5) can be replaced by $\mathcal{P}_t := \{Q \in \mathcal{M}_1(P) \mid Q \approx P \text{ on } \mathcal{F}_t\}$. Moreover, representation (1.4) is also equivalent to

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{M}_1(P)} \big(E_Q[-X|\mathcal{F}_t] - \hat{\alpha}_t(Q)\big), \quad X \in L^\infty,$$

where the conditional expectation under $Q \in \mathcal{M}_1(P)$ is defined under P as

$$E_Q[X|\mathcal{F}_t] := \frac{E_P[Z_T X|\mathcal{F}_t]}{Z_t} I_{\{Z_t>0\}}$$

with $Z_s := \frac{dQ}{dP}|_{\mathcal{F}_s}$, $s \in \mathbb{T}$, and the extended penalty function $\hat{\alpha}_t$ is given by

$$\hat{\alpha}_t(Q) = \begin{cases} \alpha_t(Q) & \text{on } \{Z_t > 0\}, \\ +\infty & \text{otherwise.} \end{cases}$$

As observed, e.g., in [12, Remark 3.13], the minimal penalty function has the local property. In our context it means that for any $Q^1, Q^2 \in \mathcal{Q}_t(Q)$ with the corresponding density processes Z^1 and Z^2 with respect to P and for any $A \in \mathcal{F}_t$, the probability measure R defined via $\frac{dR}{dP} := I_A Z_T^1 + I_{A^c} Z_T^2$ has the penalty function value

$$\alpha_t^{\min}(R) = I_A \alpha_t^{\min}\big(Q^1\big) + I_{A^c}\alpha_t^{\min}\big(Q^2\big) \quad Q\text{-a.s.}$$

In particular, $R \in \mathcal{Q}_t^f(Q)$ if $Q^1, Q^2 \in \mathcal{Q}_t^f(Q)$. Standard arguments (cf., e.g., [18, Lemma 1]) imply then that the set

$$\big\{E_R[-X|\mathcal{F}_t] - \alpha_t^{\min}(R) \bigm| R \in \mathcal{Q}_t^f(Q)\big\}$$

is directed upward, and thus

$$E_Q\big[\rho_t(X)\big|\mathcal{F}_s\big] = Q\text{-}\operatorname*{ess\,sup}_{R\in\mathcal{Q}_t^f(Q)} \big(E_R[-X|\mathcal{F}_s] - E_R\big[\alpha_t^{\min}(R)\big|\mathcal{F}_s\big]\big) \tag{1.14}$$

for all $Q \in \mathcal{M}_1(P)$, $X \in L^\infty(\Omega, \mathcal{F}, P)$ and $0 \le s \le t$.

1.4 Time Consistency Properties

In the dynamical setting, risk assessment of a financial position is updated when new information is released. This leads to the notion of a dynamic risk measure.

Definition 1.13 A sequence $(\rho_t)_{t\in\mathbb{T}}$ is called a *dynamic convex risk measure* if ρ_t is a conditional convex risk measure for each $t \in \mathbb{T}$.

A key question in the dynamical setting is how the conditional risk assessments at different times are interrelated. This question has led to several notions of time consistency discussed in the literature. A unifying view was suggested in [40].

Definition 1.14 Assume that $(\rho_t)_{t\in\mathbb{T}}$ is a dynamic convex risk measure and let $\mathcal{Y}_t$ be a subset of L^∞ such that $0 \in \mathcal{Y}_t$ and $\mathcal{Y}_t + \mathbb{R} = \mathcal{Y}_t$ for each $t \in \mathbb{T}$. Then $(\rho_t)_{t\in\mathbb{T}}$ is called *acceptance (resp. rejection) consistent with respect to* $(\mathcal{Y}_t)_{t\in\mathbb{T}}$ if for all $t \in \mathbb{T}$ such that $t < T$ and for any $X \in L^\infty$ and $Y \in \mathcal{Y}_{t+1}$, the following condition holds:

$$\rho_{t+1}(X) \le \rho_{t+1}(Y) \quad (\text{resp.} \ge) \quad \Longrightarrow \quad \rho_t(X) \le \rho_t(Y) \quad (\text{resp.} \ge). \tag{1.15}$$

The idea is that the degree of time consistency is determined by a sequence of benchmark sets $(\mathcal{Y}_t)_{t\in\mathbb{T}}$: if a financial position at some future time is always preferable to some element of the benchmark set, then it should also be preferable today. The bigger the benchmark set, the stronger is the resulting notion of time consistency. In the following we focus on three cases.

Definition 1.15 We call a dynamic convex risk measure $(\rho_t)_{t\in\mathbb{T}}$

1. *strongly time consistent* if it is either acceptance consistent or rejection consistent with respect to $\mathcal{Y}_t = L^\infty$ for all t in the sense of Definition 1.14;
2. *middle acceptance (resp. middle rejection) consistent* if for all t, we have $\mathcal{Y}_t = L^\infty_t$ in Definition 1.14;
3. *weakly acceptance (resp. weakly rejection) consistent* if for all t, we have $\mathcal{Y}_t = \mathbb{R}$ in Definition 1.14.

Note that there is no difference between rejection consistency and acceptance consistency with respect to L^∞, since the role of X and Y is symmetric in that case. Obviously strong time consistency implies both middle rejection and middle acceptance consistency, and middle rejection (resp. middle acceptance) consistency implies weak rejection (resp. weak acceptance) consistency. In the rest of the paper we drop the terms "middle" and "strong" in order to simplify the terminology.

1.4.1 Time Consistency

Time consistency has been studied extensively in the recent work on dynamic risk measures, see [4, 8, 9, 12, 13, 16–18, 22, 29, 33, 34] and the references therein. In the next proposition we recall some equivalent characterizations of time consistency.

Proposition 1.16 *A dynamic convex risk measure $(\rho_t)_{t\in\mathbb{T}}$ is time consistent if and only if any of the following conditions holds*:

1. *for all $t \in \mathbb{T}$ such that $t < T$ and for all $X, Y \in L^\infty$*,
$$\rho_{t+1}(X) \le \rho_{t+1}(Y) \quad P\text{-}a.s \quad \Longrightarrow \quad \rho_t(X) \le \rho_t(Y) \quad P\text{-}a.s.; \tag{1.16}$$
2. *for all $t \in \mathbb{T}$ such that $t < T$ and for all $X, Y \in L^\infty$*,
$$\rho_{t+1}(X) = \rho_{t+1}(Y) \quad P\text{-}a.s \quad \Longrightarrow \quad \rho_t(X) = \rho_t(Y) \quad P\text{-}a.s.; \tag{1.17}$$
3. *$(\rho_t)_{t\in\mathbb{T}}$ is recursive, i.e.*,
$$\rho_t = \rho_t(-\rho_{t+s}) \quad P\text{-}a.s.$$
for all $t, s \ge 0$ such that $t, t+s \in \mathbb{T}$.

Proof It is obvious that time consistency implies condition (1.16) and that (1.16) implies (1.17). By cash invariance we have $\rho_{t+1}(-\rho_{t+1}(X)) = \rho_{t+1}(X)$, and hence one-step recursiveness follows from (1.17). We prove that one-step recursiveness implies recursiveness by induction on s. For $s = 1$, the claim is true for all t. Assume that the induction hypothesis holds for each t and all $k \le s$ for some $s \ge 1$. Then we obtain
$$\begin{aligned} \rho_t\big(-\rho_{t+s+1}(X)\big) &= \rho_t\big(-\rho_{t+s}\big(-\rho_{t+s+1}(X)\big)\big) \\ &= \rho_t\big(-\rho_{t+s}(X)\big) \\ &= \rho_t(X), \end{aligned}$$
where we have applied the induction hypothesis to the random variable $-\rho_{t+s+1}(X)$. Hence the claim follows. Finally, due to monotonicity, recursiveness implies time consistency. □

Remark 1.17 The recursivity property (3) of Proposition 1.16 corresponds to the dynamic programming principle, and it is crucial for many applications. In continuous time and in Brownian setting, it allows one to relate time consistent dynamic risk measures to the solutions of a certain type of backward stochastic differential equations, so-called *g-expectations*; cf. [20, 26, 32, 38]. Indeed, as shown in [38, Proposition 19], a conditional g-expectation defines a time consistent dynamic convex risk measure on $L^2(P)$ if the BSDE generator g is convex (and satisfies the usual assumptions ensuring existence of a solution). Conversely, as shown in [38, Proposition 20], if $(\rho_t)_{t\in[0,T]}$ is a strictly monotone time consistent dynamic convex risk measure in Brownian setting and if ρ_0 satisfies a certain boundedness condition, then (ρ_t) can be identified as a conditional g-expectation. This relation allows one in particular to characterize penalty functions of time consistent dynamic convex risk measures in Brownian setting; cf. [17].

If we restrict a conditional convex risk measure ρ_t to the space L^∞_{t+s} for some $s \geq 0$, the corresponding acceptance set is given by

$$\mathcal{A}_{t,t+s} := \left\{ X \in L^\infty_{t+s} \,\middle|\, \rho_t(X) \leq 0 \; P\text{-a.s.} \right\},$$

and the minimal penalty function by

$$\alpha^{\min}_{t,t+s}(Q) := Q\text{-}\operatorname*{ess\,sup}_{X \in \mathcal{A}_{t,t+s}} E_Q[-X|\mathcal{F}_t], \quad Q \in \mathcal{M}_1(P). \tag{1.18}$$

The following lemma recalls equivalent characterizations of recursive inequalities in terms of acceptance sets from [22, Lemma 4.6]; property (1.19) was shown in [16].

Lemma 1.18 *Let $(\rho_t)_{t \in \mathbb{T}}$ be a dynamic convex risk measure. Then the following equivalences hold for all s, t such that $t, t+s \in \mathbb{T}$ and all $X \in L^\infty$:*

$$X \in \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \iff -\rho_{t+s}(X) \in \mathcal{A}_{t,t+s}, \tag{1.19}$$

$$\mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \iff \rho_t(-\rho_{t+s}) \leq \rho_t \quad P\text{-a.s.}, \tag{1.20}$$

$$\mathcal{A}_t \supseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \iff \rho_t(-\rho_{t+s}) \geq \rho_t \quad P\text{-a.s.} \tag{1.21}$$

Proof To prove "$\Rightarrow$" in (1.19), let $X = X_{t,t+s} + X_{t+s}$ with $X_{t,t+s} \in \mathcal{A}_{t,t+s}$ and $X_{t+s} \in \mathcal{A}_{t+s}$. Then

$$\rho_{t+s}(X) = \rho_{t+s}(X_{t+s}) - X_{t,t+s} \leq -X_{t,t+s}$$

by cash invariance, and monotonicity implies

$$\rho_t\big(-\rho_{t+s}(X)\big) \leq \rho_t(X_{t,t+s}) \leq 0.$$

The converse direction follows immediately from $X = X + \rho_{t+s}(X) - \rho_{t+s}(X)$ and $X + \rho_{t+s}(X) \in \mathcal{A}_{t+s}$ for all $X \in L^\infty$.

In order to show "$\Rightarrow$" in (1.20), fix $X \in L^\infty$. Since $X + \rho_t(X) \in \mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$, we obtain

$$\rho_{t+s}(X) - \rho_t(X) = \rho_{t+s}\big(X + \rho_t(X)\big) \in -\mathcal{A}_{t,t+s},$$

by (1.19) and cash invariance. Hence,

$$\rho_t\big(-\rho_{t+s}(X)\big) - \rho_t(X) = \rho_t\Big(-\big(\rho_{t+s}(X) - \rho_t(X)\big)\Big) \leq 0 \quad P\text{-a.s.}$$

To prove "$\Leftarrow$", let $X \in \mathcal{A}_t$. Then $-\rho_{t+s}(X) \in \mathcal{A}_{t,t+s}$ by the right-hand side of (1.20), and hence $X \in \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$ by (1.19).

Now let $X \in L^\infty$ and assume $\mathcal{A}_t \supseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$. Then

$$\begin{aligned}\rho_t\big(-\rho_{t+s}(X)\big) + X &= \rho_t\big(-\rho_{t+s}(X)\big) - \rho_{t+s}(X) + \rho_{t+s}(X) + X \\ &\in \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \subseteq \mathcal{A}_t.\end{aligned}$$

Hence,

$$\rho_t(X) - \rho_t\big(-\rho_{t+s}(X)\big) = \rho_t\big(X + \rho_t\big(-\rho_{t+s}(X)\big)\big) \le 0$$

by cash invariance, and this proves "⇒" in (1.21). For the converse direction, let $X \in \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$. Since $-\rho_{t+s}(X) \in \mathcal{A}_{t,t+s}$ by (1.19), we obtain

$$\rho_t(X) \le \rho_t\big(-\rho_{t+s}(X)\big) \le 0,$$

and hence, $X \in \mathcal{A}_t$. □

We also have the following relation between acceptance sets and penalty functions; cf. [33, Lemma 2.2.5].

Lemma 1.19 *Let $(\rho_t)_{t\in\mathbb{T}}$ be a dynamic convex risk measures. Then the following implications hold for all t, s such that $t, t+s \in \mathbb{T}$ and for all $Q \in \mathcal{M}_1(P)$:*

$$\mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \quad \Longrightarrow \quad \alpha_t^{\min}(Q) \le \alpha_{t,t+s}^{\min}(Q) + E_Q\big[\alpha_{t+s}^{\min}(Q)\big|\mathcal{F}_t\big] \quad Q\text{-a.s.},$$

$$\mathcal{A}_t \supseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \quad \Longrightarrow \quad \alpha_t^{\min}(Q) \ge \alpha_{t,t+s}^{\min}(Q) + E_Q\big[\alpha_{t+s}^{\min}(Q)\big|\mathcal{F}_t\big] \quad Q\text{-a.s.}$$

Proof Straightforward from the definition of the minimal penalty function and Lemma 1.3. □

The following theorem gives equivalent characterizations of time consistency in terms of acceptance sets, penalty functions, and a supermartingale property of the risk process.

Theorem 1.20 *Let $(\rho_t)_{t\in\mathbb{T}}$ be a dynamic convex risk measure such that each ρ_t is continuous from above. Then the following conditions are equivalent:*

1. *$(\rho_t)_{t\in\mathbb{T}}$ is time consistent.*
2. *$\mathcal{A}_t = \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$ for all t, s such that $t, t+s \in \mathbb{T}$.*
3. *$\alpha_t^{\min}(Q) = \alpha_{t,t+s}^{\min}(Q) + E_Q[\alpha_{t+s}^{\min}(Q)|\mathcal{F}_t]$ Q-a.s. for all t, s such that $t, t+s \in \mathbb{T}$ and all $Q \in \mathcal{M}_1(P)$.*
4. *For all $X \in L^\infty(\Omega, \mathcal{F}, P)$ and all t, s such that $t, t+s \in \mathbb{T}$ and all $Q \in \mathcal{M}_1(P)$, we have*

$$E_Q\big[\rho_{t+s}(X) + \alpha_{t+s}^{\min}(Q)\big|\mathcal{F}_t\big] \le \rho_t(X) + \alpha_t^{\min}(Q) \quad Q\text{-a.s.}$$

The equivalence of properties (1) and (2) of Theorem 1.20 was proved in [16]. Characterizations of time consistency in terms of penalty functions as in (3) of Theorem 1.20 appeared in [7, 8, 13, 22]; similar results for risk measures for processes were given in [12, 13]. In [7, 8] property (3) is called *cocycle property*. The supermartingale property as in (4) of Theorem 1.20 was obtained in [22]; cf. also [8] for continuous-time setting.

Proof The proof of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) follows from Lemmas 1.18 and 1.19. To prove (3) $\Rightarrow$ (4), fix $Q \in \mathcal{M}_1(P)$. By (1.14) we have

$$E_Q\big[\rho_{t+s}(X)\big|\mathcal{F}_t\big] = \underset{R\in\mathcal{Q}^f_{t+s}(Q)}{Q\text{-ess}\sup}\big(E_R[-X|\mathcal{F}_t] - E_R\big[\alpha^{\min}_{t+s}(R)\big|\mathcal{F}_t\big]\big).$$

On the set $\{\alpha^{\min}_t(Q) = \infty\}$ property (4) holds trivially. On the set $\{\alpha^{\min}_t(Q) < \infty\}$ property (3) implies $E_Q[\alpha^{\min}_{t+s}(Q)|\mathcal{F}_t] < \infty$ and $\alpha^{\min}_{t,t+s}(Q) < \infty$; then for $R \in \mathcal{Q}^f_{t+s}(Q)$,

$$\alpha^{\min}_t(R) = \alpha^{\min}_{t,t+s}(Q) + E_R\big[\alpha^{\min}_{t+s}(R)\big|\mathcal{F}_t\big] < \infty \quad Q\text{-a.s.}$$

Thus,

$$E_Q\big[\rho_{t+s}(X) + \alpha^{\min}_{t+s}(Q)\big|\mathcal{F}_t\big] = \underset{R\in\mathcal{Q}^f_{t+s}(Q)}{Q\text{-ess}\sup}\big(E_R[-X|\mathcal{F}_t] - \alpha^{\min}_t(R)\big) + \alpha^{\min}_t(Q)$$

on $\{\alpha^{\min}_t(Q) < \infty\}$. Moreover, since $\mathcal{Q}^f_{t+s}(Q) \subseteq \mathcal{Q}_t(Q)$, (1.12) implies

$$\begin{aligned} E_Q\big[\rho_{t+s}(X) + \alpha^{\min}_{t+s}(Q)\big|\mathcal{F}_t\big] &\le \underset{R\in\mathcal{Q}_t(Q)}{Q\text{-ess}\sup}\big(E_R[-X|\mathcal{F}_t] - \alpha^{\min}_t(R)\big) + \alpha^{\min}_t(Q) \\ &= \rho_t(X) + \alpha^{\min}_t(Q) \quad Q\text{-a.s.} \end{aligned}$$

It remains to prove (4) $\Rightarrow$ (1). To this end, fix $Q \in \mathcal{Q}^f_t$ and $X, Y \in L^\infty$ such that $\rho_{t+1}(X) \le \rho_{t+1}(Y)$. Note that $E_Q[\alpha_{t+s}(Q)] < \infty$ due to (4), and hence $Q \in \mathcal{Q}^f_{t+s}(Q)$. Using (4) and representation (1.13) for ρ_{t+s} under Q, we obtain

$$\begin{aligned} \rho_t(Y) + \alpha^{\min}_t(Q) &\ge E_Q\big[\rho_{t+1}(Y) + \alpha^{\min}_{t+1}(Q)\big|\mathcal{F}_t\big] \\ &\ge E_Q\big[\rho_{t+1}(X) + \alpha^{\min}_{t+1}(Q)\big|\mathcal{F}_t\big] \\ &\ge E_Q\big[E_Q[-X|\mathcal{F}_{t+1}] - \alpha^{\min}_{t+1}(Q) + \alpha^{\min}_{t+1}(Q)\big|\mathcal{F}_t\big] \\ &= E_Q[-X|\mathcal{F}_t]. \end{aligned}$$

Hence representation (1.6) yields $\rho_t(Y) \ge \rho_t(X)$, and time consistency follows from Proposition 1.16. □

Properties (3) and (4) of Theorem 1.20 imply in particular supermartingale properties of penalty function processes and risk processes. This allows one to apply martingale theory for characterization of the dynamics of these processes, as we do in Propositions 1.21 and 1.24; cf. also [8, 16, 17, 22, 33].

Proposition 1.21 *Let* $(\rho_t)_{t\in\mathbb{T}}$ *be a time consistent dynamic convex risk measure such that each* ρ_t *is continuous from above. Then the process*

$$V^Q_t(X) := \rho_t(X) + \alpha^{\min}_t(Q), \quad t \in \mathbb{T},$$

is a Q-supermartingale for all $X \in L^\infty$ and all $Q \in \mathcal{Q}_0$, where

$$\mathcal{Q}_0 := \left\{ Q \in \mathcal{M}_1(P) \,\middle|\, \alpha_0^{\min}(Q) < \infty \right\}.$$

Moreover, $(V_t^Q(X))_{t\in\mathbb{T}}$ is a Q-martingale if $Q \in \mathcal{Q}_0$ is a "worst-case" measure for X at time 0, i.e., if the supremum in the robust representation of $\rho_0(X)$ is attained at Q:

$$\rho_0(X) = E_Q[-X] - \alpha_0^{\min}(Q).$$

In this case Q is a "worst-case" measure for X at any time t, i.e.,

$$\rho_t(X) = E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \quad Q\text{-a.s.} \quad \text{for all } t \in \mathbb{T}.$$

The converse holds if $T < \infty$ or $\lim_{t\to\infty} \rho_t(X) = -X$ P-a.s. (what is called asymptotic precision in [22]): If $(V_t^Q(X))_{t\in\mathbb{T}}$ is a Q-martingale, then $Q \in \mathcal{Q}_0$ is a "worst-case" measure for X at any time $t \in \mathbb{T}$.

Proof The supermartingale property of $(V_t^Q(X))_{t\in\mathbb{T}}$ under each $Q \in \mathcal{Q}_0$ follows directly from properties (3) and (4) of Theorem 1.20. To prove the remaining part of the claim, fix $Q \in \mathcal{Q}_0$ and $X \in L^\infty$. If Q is a "worst-case" measure for X at time 0, the process

$$U_t(X) := V_t^Q(X) - E_Q[-X|\mathcal{F}_t], \quad t \in \mathbb{T},$$

is a nonnegative Q-supermartingale beginning at 0. Indeed, the supermartingale property follows from that of $(V_t^Q(X))_{t\in\mathbb{T}}$, and nonnegativity follows from representation (1.13), since $Q \in \mathcal{Q}_t^f(Q)$. Thus, $U_t = 0$ Q-a.s. for all t, and this proves the "if" part of the claim. To prove the converse direction, note that if $(V_t^Q(X))_{t\in\mathbb{T}}$ is a Q-martingale and $\rho_T(X) = -X$ (resp. $\lim_{t\to\infty} \rho_t(X) = -X$ P-a.s.), the process $U(X)$ is a Q-martingale ending at 0 (resp. converging to 0 in $L^1(Q)$), and thus $U_t(X) = 0$ Q-a.s. for all $t \in \mathbb{T}$. □

Remark 1.22 The fact that a worst-case measure for X at time 0, if it exists, remains a worst-case measure for X at any time $t \in \mathbb{T}$ was also shown in [13, Theorem 3.9] for a time consistent dynamic risk measure in finite time horizon without using the supermartingale property from Proposition 1.21.

Remark 1.23 In difference to [22, Theorem 4.5], without the additional assumption that the set

$$\mathcal{Q}^* := \left\{ Q \in \mathcal{M}^e(P) \,\middle|\, \alpha_0^{\min}(Q) < \infty \right\} \tag{1.22}$$

is nonempty, the supermartingale property of $(V_t^Q(X))_{t\in\mathbb{T}}$ for all $X \in L^\infty$ and all $Q \in \mathcal{Q}^*$ is not sufficient to prove time consistency. In this case we also do not have the robust representation of ρ_t in terms of the set $\mathcal{Q}^*$.

The process $(\alpha_t^{\min}(Q))_{t\in\mathbb{T}}$ is a Q-supermartingale for all $Q \in \mathcal{Q}_0$ due to Property (3) of Theorem 1.20. The next proposition provides explicit forms of its Doob and Riesz decompositions; cf. also [33, Proposition 2.3.2].

Proposition 1.24 *Let $(\rho_t)_{t\in\mathbb{T}}$ be a time consistent dynamic convex risk measure such that each ρ_t is continuous from above. Then for each $Q \in \mathcal{Q}_0$, the process $(\alpha_t^{\min}(Q))_{t\in\mathbb{T}}$ is a nonnegative Q-supermartingale with the Riesz decomposition*

$$\alpha_t^{\min}(Q) = Z_t^Q + M_t^Q \quad Q\text{-a.s.}, \quad t \in \mathbb{T},$$

where

$$Z_t^Q := E_Q\left[\sum_{k=t}^{T-1} \alpha_{k,k+1}^{\min}(Q)\middle|\mathcal{F}_t\right] \quad Q\text{-a.s.}, \quad t \in \mathbb{T},$$

is a Q-potential, and

$$M_t^Q := \begin{cases} 0 & \text{if } T < \infty, \\ \lim_{s\to\infty} E_Q[\alpha_s(Q)|\mathcal{F}_t] & \text{if } T = \infty, \end{cases} \quad Q\text{-a.s.}, \quad t \in \mathbb{T},$$

is a nonnegative Q-martingale.

Moreover, the Doob decomposition of $(\alpha_t^{\min}(Q))_{t\in\mathbb{T}}$ is given by

$$\alpha_t^{\min}(Q) = E_Q\left[\sum_{k=0}^{T-1} \alpha_{k,k+1}^{\min}(Q)\middle|\mathcal{F}_t\right] + M_t^Q - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q), \quad t \in \mathbb{T},$$

with the Q-martingale

$$E_Q\left[\sum_{k=0}^{T-1} \alpha_{k,k+1}^{\min}(Q)\middle|\mathcal{F}_t\right] + M_t^Q, \quad t \in \mathbb{T},$$

and the nondecreasing predictable process $(\sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q))_{t\in\mathbb{T}}$.

Proof We fix $Q \in \mathcal{M}_1(P)$ and applying property (3) of Theorem 1.20 step by step, we obtain

$$\alpha_t^{\min}(Q) = E_Q\left[\sum_{k=t}^{t+s-1} \alpha_{k,k+1}^{\min}(Q)\middle|\mathcal{F}_t\right] + E_Q\left[\alpha_{t+s}^{\min}(Q)\middle|\mathcal{F}_t\right] \quad Q\text{-a.s.} \tag{1.23}$$

for all t, s such that $t, t+s \in \mathbb{T}$. If $T < \infty$, the Doob and Riesz decompositions follow immediately from (1.23), since $\alpha_T(Q) = 0$ Q-a.s. If $T = \infty$, by monotonicity there exists the limit

$$Z_t^Q = \lim_{s\to\infty} E_Q\left[\sum_{k=t}^{s} \alpha_{k,k+1}^{\min}(Q)\middle|\mathcal{F}_t\right] = E_Q\left[\sum_{k=t}^{\infty} \alpha_{k,k+1}^{\min}(Q)\middle|\mathcal{F}_t\right] \quad Q\text{-a.s.}$$

for all $t \in \mathbb{T}$, where we have used the monotone convergence theorem for the second equality. Equality (1.23) implies then that there exists

$$M_t^Q = \lim_{s \to \infty} E_Q\big[\alpha_{t+s}^{\min}(Q)\big|\mathcal{F}_t\big] \quad Q\text{-a.s.}, \quad t \in \mathbb{T},$$

and

$$\alpha_t^{\min}(Q) = Z_t^Q + M_t^Q \quad Q\text{-a.s.}$$

for all $t \in \mathbb{T}$.

The process $(Z_t^Q)_{t\in\mathbb{T}}$ is a nonnegative Q-supermartingale. Indeed,

$$E_Q\big[Z_t^Q\big] \leq E_Q\left[\sum_{k=0}^{\infty} \alpha_{k,k+1}^{\min}(Q)\right] \leq \alpha_0^{\min}(Q) < \infty \tag{1.24}$$

and $E_Q[Z_{t+1}^Q|\mathcal{F}_t] \leq Z_t^Q$ Q-a.s. for all $t \in \mathbb{T}$ by definition. Moreover, monotone convergence implies

$$\lim_{t\to\infty} E_Q\big[Z_t^Q\big] = E_Q\left[\lim_{t\to\infty} \sum_{k=t}^{\infty} \alpha_{k,k+1}^{\min}(Q)\right] = 0 \quad Q\text{-a.s.},$$

since $\sum_{k=0}^{\infty} \alpha_{k,k+1}^{\min}(Q) < \infty$ Q-a.s. by (1.24). Hence the process $(Z_t^Q)_{t\in\mathbb{T}}$ is a Q-potential.

The process $(M_t^Q)_{t\in\mathbb{T}}$ is a nonnegative Q-martingale, since

$$E_Q\big[M_t^Q\big] \leq E_Q\big[\alpha_t^{\min}(Q)\big] \leq \alpha_0^{\min}(Q) < \infty$$

and

$$\begin{aligned} E_Q\big[M_{t+1}^Q - M_t^Q\big|\mathcal{F}_t\big] &= E_Q\big[\alpha_{t+1}^{\min}(Q)\big|\mathcal{F}_t\big] - \alpha_t^{\min}(Q) - E_Q\big[Z_{t+1}^Q - Z_t^Q\big|\mathcal{F}_t\big] \\ &= \alpha_{t,t+1}^{\min}(Q) - \alpha_{t,t+1}^{\min}(Q) = 0 \quad Q\text{-a.s.} \end{aligned}$$

for all $t \in \mathbb{T}$ by property (3) of Theorem 1.20 and the definition of $(Z_t^Q)_{t\in\mathbb{T}}$.

The Doob decomposition follows straightforward from the Riesz decomposition. □

Remark 1.25 It was shown in [22, Theorem 5.4] that the martingale M^Q in the Riesz decomposition of $(\alpha_t^{\min}(Q))_{t\in\mathbb{T}}$ vanishes if and only if $\lim_{t\to\infty} \rho_t(X) \geq -X$ P-a.s., i.e., the dynamic risk measure $(\rho_t)_{t\in\mathbb{T}}$ is asymptotically safe. This is not always the case; see [22, Example 5.5].

For a *coherent* risk measure, we have

$$\mathcal{Q}_t^f(Q) = \mathcal{Q}_t^0(Q) := \big\{R \in \mathcal{M}^1(P) \,\big|\, R = Q|_{\mathcal{F}_t},\ \alpha_t^{\min}(R) = 0\ Q\text{-a.s.}\big\}.$$

In order to give an equivalent characterization of property (3) of Theorem 1.20 in the coherent case, we introduce the sets

$$\mathcal{Q}^0_{t,t+s}(Q)=\left\{R\ll P|_{\mathcal{F}_{t+s}}\;\middle|\;R=Q|_{\mathcal{F}_t},\ \alpha^{\min}_{t,t+s}(R)=0\ Q\text{-a.s.}\right\}$$

for all $t,s\ge 0$ such that $t,t+s\in\mathbb{T}$. For $Q^1\in\mathcal{Q}^0_{t,t+s}(Q)$ and $Q^2\in\mathcal{Q}^0_{t+s}(Q)$, we denote by $Q^1\oplus^{t+s}Q^2$ the pasting of Q^1 and Q^2 in $t+s$ via Ω, i.e., the measure $\widetilde{Q}$ defined via

$$\widetilde{Q}(A)=E_{Q^1}\left[E_{Q^2}[I_A|\mathcal{F}_{t+s}]\right],\quad A\in\mathcal{F}. \tag{1.25}$$

The relation between stability under pasting and time consistency of coherent risk measures that can be represented in terms of equivalent probability measures was studied in [4, 16, 22, 29]. In our present setting, Theorem 1.20 applied to a coherent risk measure takes the following form.

Corollary 1.26 *Let $(\rho_t)_{t\in\mathbb{T}}$ be a dynamic coherent risk measure such that each ρ_t is continuous from above. Then the following conditions are equivalent*:

1. *$(\rho_t)_{t\in\mathbb{T}}$ is time consistent.*
2. *For all $Q\in\mathcal{M}_1(P)$ and all t,s such that $t,t+s\in\mathbb{T}$,*

$$\mathcal{Q}^0_t(Q)=\left\{Q^1\oplus^{t+s}Q^2\;\middle|\;Q^1\in\mathcal{Q}^0_{t,t+s}(Q),\ Q^2\in\mathcal{Q}^0_{t+s}\left(Q^1\right)\right\}.$$

3. *For all $Q\in\mathcal{M}_1(P)$ such that $\alpha^{\min}_t(Q)=0$ Q-a.s.,*

$$E_Q\left[\rho_{t+s}(X)\middle|\mathcal{F}_t\right]\le\rho_t(X)\quad\textit{and}\quad\alpha^{\min}_{t+s}(Q)=0\quad Q\textit{-a.s.}$$

   for all $X\in L^\infty(\Omega,\mathcal{F},P)$ and for all t,s such that $t,t+s\in\mathbb{T}$.

Proof (1) $\Rightarrow$ (2): Time consistency implies property (3) of Theorem 1.20, and we will show that this implies property (2) of Corollary 1.26. Fix $Q\in\mathcal{M}_1(P)$. To prove "$\supseteq$", let $Q^1\in\mathcal{Q}^0_t(Q)$, $Q^2\in\mathcal{Q}^0_{t+s}(Q^1)$, and consider $\widetilde{Q}$ defined as in (1.25). Note that $\widetilde{Q}=Q^1$ on $\mathcal{F}_{t+s}$ and

$$E_{\widetilde{Q}}[X|\mathcal{F}_{t+s}]=E_{Q^2}[X|\mathcal{F}_{t+s}]\quad Q^1\text{-a.s.}\quad\text{for all }X\in L^\infty(\Omega,\mathcal{F},P).$$

Hence, using (3) of Theorem 1.20, we obtain

$$\begin{aligned}\alpha^{\min}_t(\widetilde{Q})&=\alpha^{\min}_{t,t+s}(\widetilde{Q})+E_{\widetilde{Q}}\left[\alpha^{\min}_{t+s}(\widetilde{Q})\middle|\mathcal{F}_t\right]\\&=\alpha^{\min}_{t,t+s}\left(Q^1\right)+E_{Q^1}\left[\alpha^{\min}_{t+s}\left(Q^2\right)\middle|\mathcal{F}_t\right]=0\quad Q\text{-a.s.,}\end{aligned}$$

and thus $\widetilde{Q}\in\mathcal{Q}^0_t(Q)$. Conversely, for every $\widetilde{Q}\in\mathcal{Q}^0_t(Q)$, we have $\alpha^{\min}_{t+s}(\widetilde{Q})=\alpha^{\min}_{t,t+s}(\widetilde{Q})=0$ $\widetilde{Q}$-a.s. by (3) of Theorem 1.20, and $\widetilde{Q}=\widetilde{Q}\oplus\widetilde{Q}$. This proves "$\subseteq$".

(2) $\Rightarrow$ (3): Let $R\in\mathcal{M}_1(P)$ with $\alpha^{\min}_t(R)=0$ R-a.s.. Then $R\in\mathcal{Q}^0_t(R)$, and thus $R=Q^1\oplus^{t+s}Q^2$ for some $Q^1\in\mathcal{Q}^0_{t,t+s}(R)$ and $Q^2\in\mathcal{Q}^0_{t+s}(Q^1)$. This implies

$R = Q^1$ on $\mathcal{F}_{t+s}$ and

$$E_R[X|\mathcal{F}_{t+s}] = E_{Q^2}[X|\mathcal{F}_{t+s}] \quad R\text{-a.s.}$$

Hence $\alpha^{\min}_{t,t+s}(R) = \alpha^{\min}_{t,t+s}(Q^1) = 0$, R-a.s., and $\alpha^{\min}_{t+s}(R) = \alpha^{\min}_{t+s}(Q^2) = 0$ R-a.s. To prove inequality (3), note that due to (1.14),

$$\begin{aligned} E_R\big[\rho_{t+s}(X)\big|\mathcal{F}_t\big] &= R\text{-}\operatorname*{ess\,sup}_{Q\in\mathcal{Q}^0_{t+s}(R)} E_Q[-X|\mathcal{F}_t] \\ &\le R\text{-}\operatorname*{ess\,sup}_{Q\in\mathcal{Q}^0_t(R)} E_Q[-X|\mathcal{F}_t] = \rho_t(X) \quad R\text{-a.s.,} \end{aligned}$$

where we have used that the pasting of $R|_{\mathcal{F}_{t+s}}$ and Q belongs to $\mathcal{Q}^0_t(R)$.

(3) $\Rightarrow$ (1): Obviously, property (3) of Corollary 1.26 implies property (4) of Theorem 1.20 and thus time consistency. $\square$

1.4.2 Rejection and Acceptance Consistency

Rejection and acceptance consistency were introduced and studied in [19, 33, 40, 41]. These properties can be characterized via recursive inequalities as stated in the next proposition; see [40, Theorem 3.1.5] and [19, Proposition 3.5].

Proposition 1.27 *A dynamic convex risk measure $(\rho_t)_{t\in\mathbb{T}}$ is rejection (resp. acceptance) consistent if and only if for all $t \in \mathbb{T}$ such that $t < T$,*

$$\rho_t(-\rho_{t+1}) \le \rho_t \quad (\textit{resp.} \ge) \quad P\textit{-a.s.} \tag{1.26}$$

Proof We argue for the case of rejection consistency; the case of acceptance consistency follows in the same manner. Assume first that $(\rho_t)_{t\in\mathbb{T}}$ satisfies (1.26) and let $X \in L^\infty$ and $Y \in L^\infty(\mathcal{F}_{t+1})$ such that $\rho_{t+1}(X) \ge \rho_{t+1}(Y)$. Using cash invariance, (1.26), and monotonicity, we obtain

$$\rho_t(X) \ge \rho_t\big(-\rho_{t+1}(X)\big) \ge \rho_t\big(-\rho_{t+1}(Y)\big) = \rho_t(Y).$$

The converse implication follows due to cash invariance by applying (1.15) to $Y = -\rho_{t+1}(X)$. $\square$

Remark 1.28 As shown in [19, Proposition 3.9], for a dynamic *coherent* risk measure, weak acceptance consistency and acceptance consistency are equivalent. Indeed, let $(\rho_t)_{t\in\mathbb{T}}$ be a coherent dynamic risk measure that is weakly acceptance consistent. Then

$$\rho_t(X) \le \rho_t\big(X + \rho_{t+1}(X)\big) + \rho_t\big(-\rho_{t+1}(X)\big) \quad \forall X \in L^\infty$$

due to subadditivity. Since $\rho_{t+1}(X + \rho_{t+1}(X)) = 0$, weak acceptance consistency implies $\rho_t(X + \rho_{t+1}(X)) \le 0$, and thus $\rho_t(X) \le \rho_t(-\rho_{t+1}(X))$ for all t and all $X \in L^\infty$.

Example 1.29 One obtains acceptance-consistent dynamic risk measures by taking suprema over families of time consistent dynamic risk measures. Indeed, if $\mathcal{R}$ is a collection of time consistent dynamic convex risk measures, then

$$\widehat{\rho}_t(X) := \operatorname*{ess\,sup}_{\rho \in \mathcal{R}} \rho_t(X), \quad t \in \mathbb{T},\ X \in L^\infty,$$

defines a dynamic convex risk measure. Moreover, monotonicity of $(\widehat{\rho}_t)$ and time consistency of (ρ_t) imply $\widehat{\rho}_t(X) \le \widehat{\rho}_t(-\widehat{\rho}_{t+1}(X))$ for all t, i.e., $(\widehat{\rho}_t)_{t\in\mathbb{T}}$ is acceptance consistent. This was noted in [36, Lemma 7.1].

Rejection consistency can be characterized as follows.

Proposition 1.30 *A dynamic convex risk measure $(\rho_t)_{t\in\mathbb{T}}$ is rejection consistent if and only if any of the following conditions holds*:

1. *For all $t \in \mathbb{T}$ such that $t < T$ and all $X \in L^\infty$,*
$$\rho_t(X) - \rho_{t+1}(X) \in \mathcal{A}_{t,t+1}; \tag{1.27}$$
2. *For all $t \in \mathbb{T}$ such that $t < T$ and all $X \in \mathcal{A}_t$, we have $-\rho_{t+1}(X) \in \mathcal{A}_t$.*

Proof Since

$$\rho_t\big(-\rho_{t+1}(X)\big) = \rho_t\big(\rho_t(X) - \rho_{t+1}(X)\big) + \rho_t(X)$$

by cash invariance, (1.27) implies rejection consistency, and obviously rejection consistency implies condition (2). If (2) holds, then for any $X \in L^\infty$,

$$\rho_t\big(\rho_t(X) - \rho_{t+1}(X)\big) = \rho_t\big(-\rho_{t+1}\big(X + \rho_t(X)\big)\big) \le 0,$$

due to cash invariance and the fact that $X + \rho_t(X) \in \mathcal{A}_t$. □

Property (1.27) was introduced in [33] under the name *prudence*. It means that the adjustment $\rho_{t+1}(X) - \rho_t(X)$ of the minimal capital requirement for X at time $t+1$ is acceptable at time t. In other words, one stays on the safe side at each period of time by making capital reserves according to a rejection consistent dynamic risk measure.

Similar to time consistency, rejection and acceptance consistency can be characterized in terms of acceptance sets and penalty functions.

Theorem 1.31 *Let $(\rho_t)_{t\in\mathbb{T}}$ be a dynamic convex risk measure such that each ρ_t is continuous from above. Then the following properties are equivalent*:

1. *$(\rho_t)_{t\in\mathbb{T}}$ is rejection consistent (resp. acceptance consistent).*
2. *The inclusion*
$$\mathcal{A}_t \subseteq \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1} \quad \textit{resp.} \quad \mathcal{A}_t \supseteq \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}$$
holds for all $t \in \mathbb{T}$ such that $t < T$.

3. *The inequality*

$$\alpha_t^{\min}(Q) \leq (\textit{resp.} \geq)\ \alpha_{t,t+1}^{\min}(Q) + E_Q\big[\alpha_{t+1}^{\min}(Q)\big|\mathcal{F}_t\big] \quad Q\text{-}a.s.$$

holds for all $t \in \mathbb{T}$ *such that* $t < T$ *and all* $Q \in \mathcal{M}_1(P)$.

Proof Equivalence of (1) and (2) was proved in Proposition 1.27 and Lemma 1.18, and the proof of (2) $\Rightarrow$ (3) is given in Lemma 1.19.

Let us show that property (3) implies property (1). We argue for the case of rejection consistency; the case of acceptance consistency follows in the same manner. We fix $t \in \mathbb{T}$ such that $t < T$ and consider the risk measure

$$\widetilde{\rho}_t(X) := \rho_t\big(-\rho_{t+1}(X)\big), \quad X \in L^\infty.$$

It is easily seen that $\widetilde{\rho}_t$ is a conditional convex risk measure that is continuous from above. Moreover, the dynamic risk measure $(\widetilde{\rho}_t, \rho_{t+1})$ is time consistent by definition, and thus it fulfills properties (2) and (3) of Theorem 1.20. We denote by $\widetilde{\mathcal{A}}_t$ and $\widetilde{\mathcal{A}}_{t,t+1}$ the acceptance sets of the risk measure $\widetilde{\rho}_t$, and by $\widetilde{\alpha}_t^{\min}$ its penalty function. Since

$$\widetilde{\rho}_t(X) = \rho_t\big(-\rho_{t+1}(X)\big) = \rho_t(X)$$

for all $X \in L_{t+1}^\infty$, we have $\widetilde{\mathcal{A}}_{t,t+1} = \mathcal{A}_{t,t+1}$, and thus

$$\widetilde{\mathcal{A}}_t = \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}$$

by (2) of Theorem 1.20. Lemma 1.19 and property (3) then imply

$$\widetilde{\alpha}_t^{\min}(Q) = \alpha_{t,t+1}^{\min}(Q) + E_Q\big[\alpha_{t+1}^{\min}(Q)\big|\mathcal{F}_t\big] \geq \alpha_t^{\min}(Q)$$

for all $Q \in \mathcal{Q}_t$. Thus,

$$\rho_t(X) \geq \widetilde{\rho}_t(X) = \rho_t\big(-\rho_{t+1}(X)\big)$$

for all $X \in L^\infty$, due to representation (1.6). □

Remark 1.32 Similarly to Corollary 1.26, condition (3) of Theorem 1.31 can be restated for a dynamic *coherent* risk measure $(\rho_t)_{t\in\mathbb{T}}$ as follows:

$$\mathcal{Q}_t^0(Q) \supseteq \big\{Q^1 \oplus^{t+1} Q^2 \,\big|\, Q^1 \in \mathcal{Q}_{t,t+1}^0(Q),\ Q^2 \in \mathcal{Q}_{t+1}^0\big(Q^1\big)\big\} \quad (\text{resp.} \subseteq)$$

for all $t \in \mathbb{T}$ such that $t < T$ and all $Q \in \mathcal{M}_1(P)$.

The following proposition provides an additional equivalent characterization of rejection consistency that can be viewed as an analogon of the supermartingale property (4) of Theorem 1.20.

Proposition 1.33 *Let $(\rho_t)_{t\in\mathbb{T}}$ be a dynamic convex risk measure such that each ρ_t is continuous from above. Then $(\rho_t)_{t\in\mathbb{T}}$ is rejection consistent if and only if the inequality*

$$E_Q\big[\rho_{t+1}(X)\big|\mathcal{F}_t\big] \le \rho_t(X) + \alpha^{\min}_{t,t+1}(Q) \quad Q\text{-a.s.} \tag{1.28}$$

holds for all $Q \in \mathcal{M}_1(P)$ and all $t \in \mathbb{T}$ such that $t < T$. In this case the process

$$U_t^Q(X) := \rho_t(X) - \sum_{k=0}^{t-1} \alpha^{\min}_{k,k+1}(Q), \quad t \in \mathbb{T},$$

is a Q-supermartingale for all $X \in L^\infty$ and all $Q \in \mathcal{Q}^f$, where

$$\mathcal{Q}^f := \left\{ Q \in \mathcal{M}_1(P) \,\middle|\, E_Q\left[\sum_{k=0}^{t} \alpha^{\min}_{k,k+1}(Q)\right] < \infty \ \forall t \in \mathbb{T} \right\}.$$

The proof of Proposition 1.33 is a special case of Theorem 1.35, which involves the notion of sustainability; cf. [33].

Definition 1.34 Let $(\rho_t)_{t\in\mathbb{T}}$ be a dynamic convex risk measure. We call a bounded adapted process $X = (X_t)_{t\in\mathbb{T}}$ *sustainable with respect to the risk measure* $(\rho_t)_{t\in\mathbb{T}}$ if

$$\rho_t(X_t - X_{t+1}) \le 0 \quad \text{for all } t \in \mathbb{T} \text{ such that } t < T.$$

Consider X to be a cumulative investment process. If it is sustainable, then for all $t \in \mathbb{T}$, the adjustment $X_{t+1} - X_t$ is acceptable with respect to ρ_t.

The next theorem characterizes sustainable processes in terms of a supermartingale inequality; it is a generalization of [33, Corollary 2.4.10].

Theorem 1.35 *Let $(\rho_t)_{t\in\mathbb{T}}$ be a dynamic convex risk measure such that each ρ_t is continuous from above*, *and let $(X_t)_{t\in\mathbb{T}}$ be a bounded adapted process. Then the following properties are equivalent*:

1. *The process $(X_t)_{t\in\mathbb{T}}$ is sustainable with respect to the risk measure $(\rho_t)_{t\in\mathbb{T}}$.*
2. *For all $Q \in \mathcal{M}_1(P)$ and all $t \in \mathbb{T}, t \ge 1$, we have*

$$E_Q[X_t|\mathcal{F}_{t-1}] \le X_{t-1} + \alpha^{\min}_{t-1,t}(Q) \quad Q\text{-a.s.}. \tag{1.29}$$

Proof The proof of (1) $\Rightarrow$ (2) follows directly from the definition of sustainability and the definition of the minimal penalty function.

To prove (2) $\Rightarrow$ (1), let $(X_t)_{t\in\mathbb{T}}$ be a bounded adapted process such that (1.29) holds. In order to prove

$$X_t - X_{t-1} =: A_t \in -\mathcal{A}_{t-1,t} \quad \text{for all } t \in \mathbb{T}, t \ge 1,$$

suppose by way of contradiction that $A_t \notin -\mathcal{A}_{t-1,t}$. Since the set $\mathcal{A}_{t-1,t}$ is convex and weak*-closed due to Remark 1.7, the Hahn–Banach separation theorem (see, e.g., [24, Theorem A.56]) ensures the existence of $Z \in L^1(\Omega, \mathcal{F}_t, P)$ such that

$$a := \sup_{X \in \mathcal{A}_{t-1,t}} E\big[Z(-X)\big] < E[ZA_t] =: b < \infty. \tag{1.30}$$

Since $\lambda I_{\{Z<0\}} \in \mathcal{A}_{t-1,t}$ for every $\lambda \geq 0$, (1.30) implies $Z \geq 0$ P-a.s., and in particular $E[Z] > 0$. Define the probability measure $Q \in \mathcal{M}_1(P)$ via $\frac{dQ}{dP} := \frac{Z}{E[Z]}$ and note that, due to Lemma 1.3 and (1.30), we have

$$E_Q\big[\alpha^{\min}_{t-1,t}(Q)\big] = \sup_{X \in \mathcal{A}_{t-1,t}} E_Q\big[(-X)\big] = \sup_{X \in \mathcal{A}_{t-1,t}} E\big[Z(-X)\big]\frac{1}{E[Z]} = \frac{a}{E[Z]} < \infty. \tag{1.31}$$

Moreover, (1.30) and (1.31) imply

$$\begin{aligned} E_Q\big[\big(X_t - X_{t-1} - \alpha^{\min}_{t-1,t}(Q)\big)\big] &= E[Z]\big(E[ZA_t] - E_Q\big[\alpha^{\min}_{t-1,t}(Q)\big]\big) \\ &= E[Z](b-a) > 0, \end{aligned}$$

which cannot be true if (1.29) holds under Q. □

Remark 1.36 In particular, property (2) of Theorem 1.35 implies that the process

$$X_t - \sum_{k=0}^{t-1} \alpha^{\min}_{k,k+1}(Q), \quad t \in \mathbb{T},$$

is a Q-supermartingale for all $Q \in \mathcal{Q}^f$ if X is sustainable with respect to (ρ_t). As shown in [33, Theorem 2.4.6, Corollary 2.4.8], this supermartingale property is equivalent to the sustainability of X under some additional assumptions.

1.4.3 Weak Time Consistency

In this section we characterize the weak notions of time consistency from Definition 1.15. Due to cash invariance, they can be restated as follows: A dynamic convex risk measure $(\rho_t)_{t\in\mathbb{T}}$ is weakly acceptance (resp. weakly rejection) consistent if and only if

$$\rho_{t+1}(X) \leq 0 \quad (\text{resp.} \geq) \quad \Longrightarrow \quad \rho_t(X) \leq 0 \quad (\text{resp.} \geq)$$

for any $X \in L^\infty$ and for all $t \in \mathbb{T}$ such that $t < T$. This means that if some position is accepted (or rejected) for any scenario tomorrow, it should be already accepted (or rejected) today. In this form, weak acceptance consistency was introduced in [4]. Both weak acceptance and weak rejection consistency appeared in [35, 40, 41, 43].

Weak acceptance consistency was characterized in terms of acceptance sets in [41, Corollary 3.6] and in terms of a supermartingale property of penalty functions in [9, Lemma 3.17]. We summarize these characterizations in our present setting in the next proposition.

Proposition 1.37 *Let $(\rho_t)_{t\in\mathbb{T}}$ be a dynamic convex risk measure such that each ρ_t is continuous from above. Then the following properties are equivalent*:

1. $(\rho_t)_{t\in\mathbb{T}}$ *is weakly acceptance consistent.*
2. $\mathcal{A}_{t+1} \subseteq \mathcal{A}_t$ *for all* $t \in \mathbb{T}$ *such that* $t < T$.
3. *The inequality*
$$E_Q\big[\alpha_{t+1}^{\min}(Q)\big|F_t\big] \le \alpha_t^{\min}(Q) \quad Q\text{-}a.s. \tag{1.32}$$
holds for all $Q \in \mathcal{M}_1(P)$ *and all* $t \in \mathbb{T}$ *such that* $t < T$. *In particular,* $(\alpha_t^{\min}(Q))_{t\in\mathbb{T}}$ *is a* Q*-supermartingale for all* $Q \in \mathcal{Q}_0$.

Proof The equivalence of (1) and (2) follows directly from the definition of weak acceptance consistency. Property (2) implies (3), since by Lemma 1.3

$$\begin{aligned}E_Q\big[\alpha_{t+1}^{\min}(Q)\big|F_t\big] &= Q\text{-}\operatorname*{ess\,sup}_{X_{t+1}\in\mathcal{A}_{t+1}} E_Q[-X_{t+1}|\mathcal{F}_t]\\ &\le Q\text{-}\operatorname*{ess\,sup}_{X\in\mathcal{A}_t} E_Q[-X|\mathcal{F}_t] = \alpha_t^{\min}(Q) \quad Q\text{-a.s.}\end{aligned}$$

for all $Q \in \mathcal{M}_1(P)$.

To prove that (3) implies (2), we fix $X \in \mathcal{A}_{t+1}$ and note that

$$E_Q[-X|\mathcal{F}_{t+1}] \le \alpha_{t+1}^{\min}(Q) \quad Q\text{-a.s.} \quad \text{for all } Q \in \mathcal{M}_1(P)$$

by the definition of the minimal penalty function. Using (1.32), we obtain

$$E_Q[-X|\mathcal{F}_t] \le E_Q\big[\alpha_{t+1}^{\min}(Q)\big|F_t\big] \le \alpha_t^{\min}(Q) \quad Q\text{-a.s.}$$

for all $Q \in \mathcal{M}_1(P)$ and in particular for $Q \in \mathcal{Q}_t^f(P)$. Thus, $\rho_t(X) \le 0$ by (1.6). □

Example 1.38 Consider a dynamic risk measure $(\rho_t)_{t\in\mathbb{T}}$, where each ρ_t is a conditional robust shortfall risk measure as defined in Example 1.8.

1. If $\mathcal{R}_t = \{P\}$ and $l_t = l_0$ for all t, then it is easy to see that $(\rho_t)_{t\in\mathbb{T}}$ is both weakly acceptance and weakly rejection consistent; see, e.g., [43], [39, Example 3.6], [41, Remark 5.3]. However, $(\rho_t)_{t\in\mathbb{T}}$ is in general not time consistent, as illustrated in [39, Example 3.7].
2. Assume that $l_t = l_0$ and that we have representation (1.11) in terms of equivalent probability measures for all t. Then $(\rho_t)_{t\in\mathbb{T}}$ is weakly acceptance consistent if $\mathcal{R}_t^e \subseteq \mathcal{R}_{t+1}^e$ for all t. This was noted in [41, Corollary 5.4] and follows directly from Proposition 1.37, since $A_{t+1} \subseteq \mathcal{A}_t$ for all t in this case.

   This applies in particular to dynamic Average Value-at-Risk $(AV@R_{t,\lambda_t})_{t\in\mathbb{T}}$ from Example 1.10. Indeed, in this case, $P \in \mathcal{R}_t$ for all t, and thus representation (1.11) holds. Condition $\mathcal{R}_t^e \subseteq \mathcal{R}_{t+1}^e$ is satisfied if

$$\lambda_{t+1} \le \lambda_t \operatorname*{ess\,inf}_{Q\in\mathcal{R}_t} E\left[\frac{dQ}{dP}\,\middle|\,\mathcal{F}_{t+1}\right] \quad \forall t \in \mathbb{T}.$$

Thus, $(AV@R_{t,\lambda_t})_{t\in\mathbb{T}}$ is weakly acceptance consistent in this case, and it is even acceptance consistent due to Remark 1.28. A dynamic Average Value-at-Risk with constant parameter λ is in general neither weakly acceptance nor weakly rejection consistent, see, e.g., [4, 35].

3. Consider the case where we have representation (1.11) and $\mathcal{R}_t^e = \mathcal{R}_0^e$ for all t. Assume further that all loss functions l_t are twice continuously differentiable, and let $\gamma_t := \frac{l_t''}{l_t'}$ denote the corresponding Arrow–Pratt coefficient of risk aversion. Then $(\rho_t)_{t\in\mathbb{T}}$ is weakly acceptance consistent if $\gamma_t \le \gamma_{t+1}$ for all $t \in \mathbb{T}$. This was shown in [41, Corollary 5.5].

1.4.4 A Recursive Construction

In this section we assume that the time horizon T is finite. Then one can define a time consistent dynamic convex risk measure $(\widetilde{\rho}_t)_{t=0,\dots,T}$ in a recursive way, starting with an arbitrary dynamic convex risk measure $(\rho_t)_{t=0,\dots,T}$, via

$$\begin{aligned} \widetilde{\rho}_T(X) &:= \rho_T(X) = -X, \\ \widetilde{\rho}_t(X) &:= \rho_t\big(-\widetilde{\rho}_{t+1}(X)\big), \quad t = 0, \dots, T-1, \ X \in L^\infty. \end{aligned} \tag{1.33}$$

The recursive construction (1.33) was introduced in [12, Sect. 4.2], and also studied in [13, 19]. It is easy to see that $(\widetilde{\rho}_t)_{t=0,\dots,T}$ is indeed a time consistent dynamic convex risk measure, and each $\widetilde{\rho}_t$ is continuous from above if each ρ_t has this property.

Remark 1.39 If the original dynamic convex risk measure $(\rho_t)_{t=0,\dots,T}$ is rejection (resp. acceptance) consistent, then the time consistent dynamic convex risk measure $(\widetilde{\rho}_t)_{t=0,\dots,T}$ defined via (1.33) lies below (resp. above) $(\rho_t)_{t=0,\dots,T}$, i.e.,

$$\widetilde{\rho}_t(X) \le (\text{resp.} \ge)\ \rho_t(X) \quad \text{for all } t = 0, \dots, T \text{ and all } X \in L^\infty.$$

This can be easily proved by backward induction using Proposition 1.27, monotonicity, and (1.33). Moreover, as shown in [19, Theorem 3.10] in the case of rejection consistency, $(\widetilde{\rho}_t)_{t=0,\dots,T}$ is the biggest time consistent dynamic convex risk measure that lies below $(\rho_t)_{t=0,\dots,T}$.

For all $X \in L^\infty$, the process $(\widetilde{\rho}_t(X))_{t=0,\dots,T}$ has the following properties: $\widetilde{\rho}_T(X) \ge -X$, and

$$\rho_t\big(\widetilde{\rho}_t(X) - \widetilde{\rho}_{t+1}(X)\big) = -\widetilde{\rho}_t(X) + \rho_t\big(-\widetilde{\rho}_{t+1}(X)\big) = 0 \quad \forall t = 0, \dots, T-1, \tag{1.34}$$

by definition and cash invariance. In other words, the process $(\widetilde{\rho}_t(X))_{t=0,\dots,T}$ covers the final loss $-X$ and is sustainable with respect to the original risk measure $(\rho_t)_{t=0,\dots,T}$. The next proposition shows that $(\widetilde{\rho}_t(X))_{t=0,\dots,T}$ is in fact the smallest process with both these properties. This result is a generalization of [33, Proposition 2.5.2] and, in the coherent case, related to [16, Theorem 6.4].

Proposition 1.40 *Let $(\rho_t)_{t=0,\dots,T}$ be a dynamic convex risk measure such that each ρ_t is continuous from above. Then, for each $X \in L^\infty$, the risk process $(\widetilde{\rho}_t(X))_{t=0,\dots,T}$ defined via (1.33) is the smallest bounded adapted process $(U_t)_{t=0,\dots,T}$ such that $(U_t)_{t=0,\dots,T}$ is sustainable with respect to $(\rho_t)_{t=0,\dots,T}$ and $U_T \ge -X$.*

Proof We have already seen that $\widetilde{\rho}_T(X) \ge -X$ and $(\widetilde{\rho}_t(X))_{t=0,\dots,T}$ is sustainable with respect to $(\rho_t)_{t=0,\dots,T}$ due to (1.34). Now let $(U_t)_{t=0,\dots,T}$ be another bounded adapted process with both these properties. We will show by backward induction that

$$U_t \ge \widetilde{\rho}_t(X) \quad P\text{-a.s.} \quad \forall t = 0, \dots, T. \tag{1.35}$$

Indeed, we have

$$U_T \ge -X = \widetilde{\rho}_T(X) \quad P\text{-a.s.}$$

If (1.35) holds for $t+1$, Theorem 1.35 yields for all $Q \in \mathcal{Q}_t^f$:

$$\begin{aligned} U_t &\ge E_Q\big[U_{t+1} - \alpha_{t,t+1}^{\min}(Q)\big|\mathcal{F}_t\big] \\ &\ge E_Q\big[\widetilde{\rho}_{t+1}(X) - \alpha_{t,t+1}^{\min}(Q)\big|\mathcal{F}_t\big] \quad P\text{-a.s.} \end{aligned}$$

Thus,

$$\begin{aligned} U_t &\ge \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t^f}\Big(E_Q\big[\widetilde{\rho}_{t+1}(X)\big|\mathcal{F}_t\big] - \alpha_{t,t+1}^{\min}(Q)\Big) \\ &= \rho_t\big(-\widetilde{\rho}_{t+1}(X)\big) = \widetilde{\rho}_t(X) \quad P\text{-a.s.}, \end{aligned}$$

where we have used representation (1.6). This proves (1.35). □

The recursive construction (1.33) can be used to construct a time consistent dynamic Average Value-at-Risk, as shown in the next example.

Example 1.41 It is well known that dynamic Average Value-at-Risk $(AV@R_{t,\lambda_t})_{t=0,\dots,T}$ (cf. Example 1.10) is not time consistent; see, e.g., [4, 14, 35]. Moreover, since $\alpha_0^{\min}(P) = 0$ in this case, the set $\mathcal{Q}^*$ in (1.22) is not empty, and [22, Corollary 4.12] implies that there exists no time consistent dynamic convex risk measure $(\rho_t)_{t\in\mathbb{T}}$ such that each ρ_t is continuous from above and $\rho_0 = AV@R_{0,\lambda_0}$. However, for $T < \infty$, the recursive construction (1.33) can be applied to $(AV@R_{t,\lambda_t})_{t=0,\dots,T}$ in order to modify it to a time consistent dynamic coherent risk measure $(\tilde{\rho}_t)_{t=0,\dots,T}$. This modified risk measure takes the form

$$\tilde{\rho}_t(X) = \operatorname*{ess\,sup}\left\{E_Q[-X|\mathcal{F}_t]\ \middle|\ Q \in \mathcal{Q}_t, \frac{Z_{s+1}^Q}{Z_s^Q} \le \lambda_s^{-1}, s = t, \dots, T-1\right\}$$

$$= \operatorname{ess\,sup}\left\{ E\left[-X \prod_{s=t+1}^{T} L_s \middle| \mathcal{F}_t \right] \middle| L_s \in L_s^\infty, 0 \le L_s \le \lambda_s^{-1}, \right.$$
$$\left. E[L_s|\mathcal{F}_{s-1}] = 1, s = t+1, \dots, T \right\}$$

for all $t = 0, \dots, T-1$, where $Z_t^Q = \frac{dQ}{dP}|_{\mathcal{F}_t}$. This was shown, e.g., in [13, Example 3.3.1].

1.5 The Dynamic Entropic Risk Measure

In this section we study time consistency properties of the dynamic entropic risk measure

$$\rho_t(X) = \frac{1}{\gamma_t} \log E\big[\exp(-\gamma_t X)\big|\mathcal{F}_t\big], \quad t \in \mathbb{T},\ X \in L^\infty, \tag{1.36}$$

where the risk aversion parameter γ_t is random and satisfies $\gamma_t > 0$ P-a.s. and $\gamma_t, \frac{1}{\gamma_t} \in L_t^\infty$ for all $t \in \mathbb{T}$; cf. also Example 1.9.

It is well known (see, e.g., [18, 22]) that the conditional entropic risk measure ρ_t has the robust representation (1.5) with the minimal penalty function α_t given by

$$\alpha_t(Q) = \frac{1}{\gamma_t} H_t(Q|P), \quad Q \in \mathcal{Q}_t,$$

where $H_t(Q|P)$ denotes the conditional relative entropy of Q with respect to P at time t:

$$H_t(Q|P) = E_Q\left[\log \frac{dQ}{dP} \middle| \mathcal{F}_t\right], \quad Q \in \mathcal{Q}_t.$$

The dynamic entropic risk measure with constant risk aversion parameter $\gamma_t = \gamma_0 \in \mathbb{R}$ for all t was studied in [12, 13, 18, 22]. It plays a particular role, as explained in the following remark.

Remark 1.42 Kupper and Schachermayer [30] showed that the entropic risk measure with constant risk aversion parameter $\gamma_0 \in [0, \infty]$ is the only time consistent dynamic convex risk measure $(\rho_t)_{t \in \mathbb{N}_0}$ such that ρ_0 is law invariant.

In this section we consider an *adapted* risk aversion process $(\gamma_t)_{t \in \mathbb{T}}$ that depends both on time and on the available information. As shown in the next proposition, the process $(\gamma_t)_{t \in \mathbb{T}}$ determines time consistency properties of the corresponding dynamic entropic risk measure. This result corresponds to [33, Proposition 4.1.4] and generalizes [19, Proposition 3.13].

Proposition 1.43 *Let $(\rho_t)_{t\in\mathbb{T}}$ be the dynamic entropic risk measure with risk aversion given by an adapted process $(\gamma_t)_{t\in\mathbb{T}}$ such that $\gamma_t > 0$ P-a.s. and $\gamma_t, 1/\gamma_t \in L_t^\infty$. Then the following assertions hold*:

1. $(\rho_t)_{t\in\mathbb{T}}$ *is rejection consistent if $\gamma_t \geq \gamma_{t+1}$ P-a.s. for all $t \in \mathbb{T}$, $t < T$;*
2. $(\rho_t)_{t\in\mathbb{T}}$ *is acceptance consistent if $\gamma_t \leq \gamma_{t+1}$ P-a.s. for all $t \in \mathbb{T}$, $t < T$;*
3. $(\rho_t)_{t\in\mathbb{T}}$ *is time consistent if $\gamma_t = \gamma_0 \in \mathbb{R}$ P-a.s. for all $t \in \mathbb{T}$.*

Moreover, assertions (1), (2), *and* (3) *hold with "if and only if" if $\gamma_t \in \mathbb{R}$ for all t, or if the filtration $(\mathcal{F}_t)_{t\in\mathbb{T}}$ is rich enough in the sense that for all t and for all $B \in \mathcal{F}_t$ such that $P[B] > 0$, there exists $A \subset B$ such that $A \notin \mathcal{F}_t$ and $P[A] > 0$.*

Proof Fix $t \in \mathbb{T}$ and $X \in L^\infty$. Then

$$\begin{aligned}\rho_t\big(-\rho_{t+1}(X)\big) &= \frac{1}{\gamma_t}\log\left(E\left[\exp\left(\frac{\gamma_t}{\gamma_{t+1}}\log\big(E\big[\exp(-\gamma_{t+1}X)\big|\mathcal{F}_{t+1}\big]\big)\right)\Big|\mathcal{F}_t\right]\right)\\ &= \frac{1}{\gamma_t}\log\Big(E\Big[E\big[\exp(-\gamma_{t+1}X)\big|\mathcal{F}_{t+1}\big]^{\frac{\gamma_t}{\gamma_{t+1}}}\Big|\mathcal{F}_t\Big]\Big).\end{aligned}$$

Thus, $\rho_t(-\rho_{t+1}) = \rho_t$ if $\gamma_t = \gamma_{t+1}$, and this proves time consistency. Rejection (resp. acceptance) consistency follows by the generalized Jensen inequality that will be proved in Lemma 1.44. We apply this inequality at time $t+1$ to the bounded random variable $Y := \exp(-\gamma_{t+1}X)$ and the $\mathcal{B}((0,\infty)) \otimes \mathcal{F}_{t+1}$-measurable function

$$u : (0,\infty) \times \Omega \to \mathbb{R}, \quad u(x,\omega) := x^{\frac{\gamma_t(\omega)}{\gamma_{t+1}(\omega)}}.$$

Note that $u(\cdot,\omega)$ is convex if $\gamma_t(\omega) \geq \gamma_{t+1}(\omega)$ and concave if $\gamma_t(\omega) \leq \gamma_{t+1}(\omega)$. Moreover, $u(X,\cdot) \in L^\infty$ for all $X \in L^\infty$, and $u(\cdot,\omega)$ is differentiable on $(0,\infty)$ with

$$\big|u'(x,\cdot)\big| = \frac{\gamma_t}{\gamma_{t+1}} x^{\frac{\gamma_t}{\gamma_{t+1}}-1} \leq a x^b \quad P\text{-a.s.}$$

for some $a, b \in \mathbb{R}$ if $\gamma_t \geq \gamma_{t+1}$, due to our assumption $\frac{\gamma_t}{\gamma_{t+1}} \in L^\infty$. On the other hand, for $\gamma_t \leq \gamma_{t+1}$, we obtain

$$\big|u'(x,\cdot)\big| = \frac{\gamma_t}{\gamma_{t+1}} x^{\frac{\gamma_t}{\gamma_{t+1}}-1} \leq a\frac{1}{x^c} \quad P\text{-a.s.}$$

for some $a, c \in \mathbb{R}$. Thus the assumptions of Lemma 1.44 are satisfied, and we obtain

$$\rho_t(-\rho_{t+1}) \leq \rho_t \quad \text{if } \gamma_t \geq \gamma_{t+1} \quad P\text{-a.s. for all } t \in \mathbb{T} \text{ such that } t < T$$

and

$$\rho_t(-\rho_{t+1}) \geq \rho_t \quad \text{if } \gamma_t \leq \gamma_{t+1} \ P\text{-a.s. for all } t \in \mathbb{T} \text{ such that } t < T.$$

The "only if" direction for constant γ_t follows by the classical Jensen inequality.

Now we assume that the sequence $(\rho_t)_{t\in\mathbb{T}}$ is rejection consistent and our assumption on the filtration $(\mathcal{F}_t)_{t\in\mathbb{T}}$ holds. We will show that the sequence $(\gamma_t)_{t\in\mathbb{T}}$ is decreasing in this case. Indeed, for $t \in \mathbb{T}$ such that $t < T$, consider $B := \{\gamma_t < \gamma_{t+1}\}$ and suppose that $P[B] > 0$. Our assumption on the filtration allows us to choose $A \subset B$ with $P[B] > P[A] > 0$ and $A \notin \mathcal{F}_{t+1}$. We define the random variable $X := -xI_A$ for some $x > 0$. Then

$$\begin{aligned}\rho_t\big(-\rho_{t+1}(X)\big) &= \frac{1}{\gamma_t}\log\bigg(E\bigg[\exp\bigg(\frac{\gamma_t}{\gamma_{t+1}}\log\big(E\big[\exp(\gamma_{t+1}xI_A)\big|\mathcal{F}_{t+1}\big]\big)\bigg)\bigg|\mathcal{F}_t\bigg]\bigg)\\ &= \frac{1}{\gamma_t}\log\bigg(E\bigg[\exp\bigg(\frac{\gamma_t}{\gamma_{t+1}}I_B\log\big(E\big[\exp(\gamma_{t+1}xI_A)\big|\mathcal{F}_{t+1}\big]\big)\bigg)\bigg|\mathcal{F}_t\bigg]\bigg),\end{aligned}$$

where we have used that $A \subset B$. Setting

$$Y := E\big[\exp(\gamma_{t+1}xI_A)\big|\mathcal{F}_{t+1}\big] = \exp(\gamma_{t+1}x)P[A|\mathcal{F}_{t+1}] + P\big[A^c\big|\mathcal{F}_{t+1}\big]$$

and bringing $\frac{\gamma_t}{\gamma_{t+1}}$ inside of the logarithm, we obtain

$$\rho_t\big(-\rho_{t+1}(X)\big) = \frac{1}{\gamma_t}\log\Big(E\Big[\exp\Big(I_B\log\Big(Y^{\frac{\gamma_t}{\gamma_{t+1}}I_B}\Big)\Big)\Big|\mathcal{F}_t\Big]\Big). \tag{1.37}$$

The function $x \mapsto x^{\gamma_t(\omega)/\gamma_{t+1}(\omega)}$ is strictly concave for almost each $\omega \in B$, and thus,

$$\begin{aligned}Y^{\frac{\gamma_t}{\gamma_{t+1}}} &= \big(\exp(\gamma_{t+1}x)P[A|\mathcal{F}_{t+1}] + \big(1 - P[A|\mathcal{F}_{t+1}]\big)\big)^{\frac{\gamma_t}{\gamma_{t+1}}}\\ &\geq \exp(\gamma_t x)P[A|\mathcal{F}_{t+1}] + \big(1 - P[A|\mathcal{F}_{t+1}]\big) \quad P\text{-a.s. on } B,\end{aligned} \tag{1.38}$$

with strict inequality on the set

$$C := \big\{P[A|\mathcal{F}_{t+1}] > 0\big\} \cap \big\{P[A|\mathcal{F}_{t+1}] < 1\big\} \cap B.$$

Our assumptions $P[A] > 0$, $A \subset B$, and $A \notin \mathcal{F}_{t+1}$ imply $P[C] > 0$, and using

$$\exp(\gamma_t x)P[A|\mathcal{F}_{t+1}] + \big(1 - P[A|\mathcal{F}_{t+1}]\big) = E\big[\exp(\gamma_t xI_A)\big|\mathcal{F}_{t+1}\big], \tag{1.39}$$

from (1.37), (1.38), and (1.39) we obtain

$$\rho_t\big(-\rho_{t+1}(X)\big) \geq \frac{1}{\gamma_t}\log\Big(E\Big[\exp\Big(I_B\log\Big(E[\exp(\gamma_t xI_A)|\mathcal{F}_{t+1}]\Big)\Big)\Big|\mathcal{F}_t\Big]\Big), \tag{1.40}$$

with the strict inequality on some set of positive probability due to strict monotonicity of the exponential and logarithmic functions. For the right-hand side of (1.40), we have

$$\begin{aligned}&\frac{1}{\gamma_t}\log\Big(E\Big[\exp\Big(I_B\log\Big(E\big[\exp(\gamma_t xI_A)\big|\mathcal{F}_{t+1}\big]\Big)\Big)\Big|\mathcal{F}_t\Big]\Big)\\ &\quad= \frac{1}{\gamma_t}\log\Big(E\Big[I_B E\big[\exp(\gamma_t xI_A)\big|\mathcal{F}_{t+1}\big] + I_{B^c}\Big|\mathcal{F}_t\Big]\Big)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\gamma_t} \log\bigl(E\bigl[\exp(\gamma_t x I_A)\big|\mathcal{F}_t\bigr]\bigr) \\
&= \rho_t(X),
\end{aligned}
$$

where we have used $A \subset B$ and $B \in \mathcal{F}_{t+1}$. This is a contradiction to rejection consistency of $(\rho_t)_{t\in\mathbb{T}}$, and we conclude that $\gamma_{t+1} \leq \gamma_t$ for all t. The proof in the case of acceptance consistency follows in the same manner. And since a time consistent dynamic risk measure is both acceptance and rejection consistent, we obtain $\gamma_{t+1} = \gamma_t$ for all t. □

The following lemma concludes the proof of Proposition 1.43.

Lemma 1.44 *Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathcal{F}_t \subseteq \mathcal{F}$ a σ-field. Let $I \subseteq \mathbb{R}$ be an open interval, and*

$$u : I \times \Omega \to \mathbb{R}$$

be a $\mathcal{B}(I) \otimes \mathcal{F}_t$-measurable function such that $u(\cdot, \omega)$ is convex (resp. concave) and finite on I for P-a.e. ω. Assume further that

$$\bigl|u'_+(x, \cdot)\bigr| \leq c(x) \quad P\text{-a.s. with some } c(x) \in \mathbb{R} \quad \text{for all } x \in I,$$

where $u'_+(\cdot, \omega)$ denotes the right-hand derivative of $u(\cdot, \omega)$. Let $X : \Omega \to [a, b]$, with $[a, b] \subseteq I$, be an $\mathcal{F}$-measurable bounded random variable such that $E[|u(X, \cdot)|] < \infty$. Then

$$E\bigl[u(X, \cdot)\big|\mathcal{F}_t\bigr] \geq u\bigl(E[X|\mathcal{F}_t], \cdot\bigr) \quad (\text{resp} \leq) \quad P\text{-a.s.}$$

Proof We will prove the assertion for the convex case; the concave one follows in the same manner. Fix $\omega \in \Omega$ such that $u(\cdot, \omega)$ is convex. Due to convexity, we obtain, for all $x_0 \in I$,

$$u(x, \omega) \geq u(x_0, \omega) + u'_+(x_0, \omega)(x - x_0) \quad \text{for all } x \in I.$$

Take $x_0 = E[X|\mathcal{F}_t](\omega)$ and $x = X(\omega)$. Then

$$u\bigl(X(\omega), \omega\bigr) \geq u\bigl(E[X|\mathcal{F}_t](\omega), \omega\bigr) + u'_+\bigl(E[X|\mathcal{F}_t](\omega), \omega\bigr)\bigl(X(\omega) - E[X|\mathcal{F}_t](\omega)\bigr) \tag{1.41}$$

for P-almost all $\omega \in \Omega$. Note further that the $\mathcal{B}(I) \otimes \mathcal{F}_t$-measurability of u implies the $\mathcal{B}(I) \otimes \mathcal{F}_t$-measurability of u_+. Thus,

$$\omega \to u\bigl(E[X|\mathcal{F}_t](\omega), \omega\bigr) \quad \text{and} \quad \omega \to u'_+\bigl(E[X|\mathcal{F}_t](\omega), \omega\bigr)$$

are $\mathcal{F}_t$-measurable random variables, and $\omega \to u(X(\omega), \omega)$ is $\mathcal{F}$-measurable. Moreover, due to our assumption on X, there are constants $a, b \in I$ such that $a \leq$

$E[X|\mathcal{F}_t] \le b$ P-a.s.. Since $u'_+(\cdot,\omega)$ is increasing by convexity, by using our assumption on the boundedness of u'_+ we obtain

$$-c(a) \le u'_+(a,\omega) \le u'_+\big(E[X|\mathcal{F}_t],\omega\big) \le u'_+(b,\omega) \le c(b),$$

i.e., $u'_+(E[X|\mathcal{F}_t],\cdot)$ is bounded. Since $E[|u(X,\cdot)|] < \infty$, we can build the conditional expectation on the both sides of (1.41), and we obtain

$$\begin{aligned} E\big[u(X,\cdot)\big|\mathcal{F}_t\big] &\ge E\big[u\big(E[X|\mathcal{F}_t],\cdot\big) + u'_+\big(E[X|\mathcal{F}_t],\cdot\big)\big(X - E[X|\mathcal{F}_t]\big)\big|\mathcal{F}_t\big] \\ &= E\big[u\big(E[X|\mathcal{F}_t],\cdot\big)\big|\mathcal{F}_t\big] \quad P\text{-a.s.}, \end{aligned}$$

where we have used the $\mathcal{F}_t$-measurability of $u(E[X|\mathcal{F}_t],\cdot)$ and of $u'_+(E[X|\mathcal{F}_t],\cdot)$ and the boundedness of $u'_+(E[X|\mathcal{F}_t],\cdot)$. This proves our claim. □

References

1. B. Acciaio, H. Föllmer, I. Penner, Risk assessment for uncertain cash flows: Model ambiguity, discounting ambiguity, and the role of bubbles (2010, submitted)
2. P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, Thinking coherently. Risk **10**, 68–71 (1997)
3. P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, Coherent measures of risk. Math. Finance **9**(3), 203–228 (1999)
4. P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, H. Ku, Coherent multiperiod risk adjusted values and Bellman's principle. Ann. Oper. Res. **152**, 5–22 (2007)
5. P. Barrieu, N. El Karoui, Optimal derivatives design under dynamic risk measures, in *Mathematics of Finance, Contemporary Mathematics (AMS Proceedings)* (2004), pp. 13–26
6. J. Bion-Nadal, Conditional risk measure and robust representation of convex conditional risk measures. CMAP preprint 557, Ecole Polytechnique Palaiseau (2004)
7. J. Bion-Nadal, Dynamic risk measures: time consistency and risk measures from BMO martingales. Finance Stoch. **12**(2), 219–244 (2008)
8. J. Bion-Nadal, Time consistent dynamic risk processes. Stoch. Process. Appl. **119**, 633–654 (2008)
9. C. Burgert, Darstellungssätze fuer statische und dynamische Risikomaße mit Anwendungen. Universität Freiburg (2005)
10. P. Cheridito, F. Delbaen, M. Kupper, Coherent and convex monetary risk measures for bounded càdlàg processes. Stoch. Process. Appl. **112**(1), 1–22 (2004)
11. P. Cheridito, F. Delbaen, M. Kupper, Coherent and convex monetary risk measures for unbounded càdlàg processes. Finance Stoch. **9**(3), 369–387 (2005)
12. P. Cheridito, F. Delbaen, M. Kupper, Dynamic monetary risk measures for bounded discrete-time processes. Electron. J. Probab. **11**(3), 57–106 (2006)
13. P. Cheridito, M. Kupper, Composition of time-consistent dynamic monetary risk measures in discrete time. Preprint (2006)
14. P. Cheridito, M. Stadje, Time-inconsistency of VaR and time-consistent alternatives. Finance Res. Lett. **6**, 40–46 (2009)
15. F. Delbaen, Coherent risk measures. Cattedra Galileiana, Scuola Normale Superiore, Classe di Scienze, Pisa (2000)
16. F. Delbaen, The structure of m-stable sets and in particular of the set of risk neutral measures, in *In memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX*. Lecture Notes in Math., vol. 1874 (Springer, Berlin, 2006), pp. 215–258
17. F. Delbaen, S. Peng, E. Rosazza Gianin, Representation of the penalty term of dynamic concave utilities. Finance Stoch. **14**(3), 449–472 (2010)

18. K. Detlefsen, G. Scandolo, Conditional dynamic convex risk measures. Finance Stoch. **9**(4), 539–561 (2005)
19. S. Drapeau, Dynamics of Optimized Certainty Equivalents and φ-Divergence. Humboldt-Universität zu Berlin (2006)
20. N. El Karoui, S. Peng, M.C. Quenez, Backward stochastic differential equations in finance. Math. Finance **7**(1), 1–71 (1997)
21. N. El Karoui, C. Ravanelli, Cash subadditive risk measures and interest rate ambiguity. Math. Finance **19**(4), 561–590 (2009)
22. H. Föllmer, I. Penner, Convex risk measures and the dynamics of their penalty functions. Stat. Decis. **24**(1), 61–96 (2006)
23. H. Föllmer, A. Schied, Convex measures of risk and trading constraints. Finance Stoch. **6**(4), 429–447 (2002)
24. H. Föllmer, A. Schied, Stochastic finance, in *An Introduction in Discrete Time*. De Gruyter Studies in Mathematics, vol. 27 (De Gruyter, Berlin, 2004)
25. M. Frittelli, E. Rosazza Gianin, Putting order in risk measures. J. Bank. Finance **26**(7), 1473–1486 (2002)
26. M. Frittelli, E. Rosazza Gianin, Dynamic convex risk measures, in *New Risk Measures in the 21st Century* (2004), pp. 227–248
27. M. Frittelli, G. Scandolo, Risk measures and capital requirements for processes. Math. Finance **16**(4), 589–612 (2006)
28. A. Jobert, L.C.G. Rogers, Valuations dynamic convex risk measures. Math. Finance **18**(1), 1–22 (2008)
29. S. Klöppel, M. Schweizer, Dynamic utility indifference valuation via convex risk measure. Working Paper N 209: National Centre of Competence in Research Financial Valuation and Risk Management (2005)
30. M. Kupper, W. Schachermayer, Representation results for law invariant time consistent functions. Math. Financ. Econ. **2**(3), 189–210 (2009)
31. M. Mania, M. Schweizer, Dynamic exponential utility indifference valuation. Ann. Appl. Probab. **15**(3), 2113–2143 (2005)
32. S. Peng, Backward SDE related g-expectation, in *Backward Stochastic Differential Equations*, Paris, 1995–1996. Pitman Res. Notes Math. Ser., vol. 364 (1995), pp. 141–159
33. I. Penner, Dynamic convex risk measures: time consistency, prudence, and sustainability. Humboldt-Universität zu Berlin (2007)
34. F. Riedel, Dynamic coherent risk measures. Stoch. Process. Appl. **112**(2), 185–200 (2004)
35. B. Roorda, J.M. Schumacher, Time consistency conditions for acceptability measures, with an application to Tail Value at Risk. Insur. Math. Econ. **40**(2), 209–230 (2007)
36. B. Roorda, J.M. Schumacher, When can a risk measure be updated consistently? (2010, submitted)
37. B. Roorda, J.M. Schumacher, J. Engwerda, Coherent acceptability measures in multiperiod models. Math. Finance **15**(4), 589–612 (2005)
38. E. Rosazza Gianin, Risk measures via g-expectations. Insur. Math. Econ. **39**(1), 19–34 (2006)
39. A. Schied, Optimal investments for risk- and ambiguity-averse preferences: a duality approach. Finance Stoch. **11**(1), 107–129 (2007)
40. S. Tutsch, Konsistente und konsequente dynamische Risikomaße und das Problem der Aktualisierung. Humboldt-Universität zu Berlin (2006)
41. S. Tutsch, Update rules for convex risk measures. Quant. Finance **8**, 833–843 (2008)
42. N. Vogelpoth, Some results on dynamic risk measures. Ludwig-Maximilians-Universität München (2006)
43. S. Weber, Distribution-invariant risk measures, information, and dynamic consistency. Math. Finance **16**(2), 419–441 (2006)

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