Chapter 2
Trigonometric Generalisations

In this chapter we introduce the p-trigonometric functions, for $1 < p < \infty$, and establish their fundamental properties. These functions generalise the familiar trigonometric functions, coincide with them when $p = 2$, and otherwise have important similarities to and differences from their classical counterparts. As will be shown later, they play an important part in both the theory of the p-Laplacian and that of the Hardy operator. Particular attention is paid to the basis properties of the analogues of the sine functions in the context of Lebesgue spaces.

2.1 The Functions $\sin_p$ and $\cos_p$

Let $1 < p < \infty$ and define a (differentiable) function $F_p : [0,1] \to \mathbb{R}$ by

$$F_p(x) = \int_0^x (1 - t^p)^{-1/p} dt. \tag{2.1}$$

Plainly $F_2 = \text{arc}\sin$. Since F_p is strictly increasing it has an inverse which, by analogy with the case $p = 2$, we denote by $\sin_p$. This is defined on the interval $[0, \pi_p/2]$, where

$$\pi_p = 2 \int_0^1 (1 - t^p)^{-1/p} dt. \tag{2.2}$$

Thus $\sin_p$ is strictly increasing on $[0, \pi_p/2]$, $\sin_p(0) = 0$ and $\sin_p(\pi_p/2) = 1$. We extend $\sin_p$ to $[0, \pi_p]$ by defining

$$\sin_p(x) = \sin_p(\pi_p - x) \text{ for } x \in [\pi_p/2, \pi_p]; \tag{2.3}$$

further extension to $[-\pi_p, \pi_p]$ is made by oddness; and finally $\sin_p$ is extended to the whole of $\mathbb{R}$ by $2\pi_p$-periodicity. It is clear that this extension is continuously differentiable on $\mathbb{R}$.

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A function $\cos_p : \mathbb{R} \to \mathbb{R}$ is defined by the prescription

$$\cos_p(x) = \frac{d}{dx}\sin_p(x),\ x \in \mathbb{R}. \tag{2.4}$$

Evidently $\cos_p$ is even, $2\pi_p$-periodic and odd about $\pi_p/2$. If $x \in [0, \pi_p/2]$ and we put $y = \sin_p(x)$, then

$$\cos_p(x) = (1 - y^p)^{1/p} = (1 - (\sin_p(x))^p)^{1/p}. \tag{2.5}$$

Thus $\cos_p$ is strictly decreasing on $[0, \pi_p/2]$, $\cos_p(0) = 1$ and $\cos_p(\pi_p/2) = 0$. Also

$$\left|\sin_p x\right|^p + \left|\cos_p x\right|^p = 1; \tag{2.6}$$

this is immediate if $x \in [0, \pi_p/2]$, but it holds for all $x \in \mathbb{R}$ in view of symmetry and periodicity. Note that the analogy between these p-functions and the classical trigonometric functions is not complete. For example, while the extended $\sin_p$ function belongs to $C^1(\mathbb{R})$, it is far from being real analytic on $\mathbb{R}$ if $p \neq 2$. To see this, observe that with the aid of (2.6) its second derivative at $x \in [0, \pi_p/2)$ can be shown to be $-h(\sin_p x)$, where

$$h(y) = (1 - y^p)^{\frac{2}{p}-1} y^{p-1},$$

and so is not continuous at $\pi_p/2$ if $2 < p < \infty$. Nevertheless, $\sin_p$ is real analytic on $[0, \pi_p/2)$. Figure 2.1 below gives the graphs of $\sin_p$ and $\cos_p$ for $p = 1.2$ and $p = 6$.

To calculate π_p we make the change of variable $t = s^{1/p}$ in the formula above for π_p. Then

$$\pi_p/2 = p^{-1}\int_0^1 (1-s)^{-1/p} s^{1/p-1} ds = p^{-1}B(1-1/p, 1/p) = p^{-1}\Gamma(1-1/p)\Gamma(1/p),$$

where B is the Beta function. Hence

$$\pi_p = \frac{2\pi}{p\sin(\pi/p)}. \tag{2.7}$$

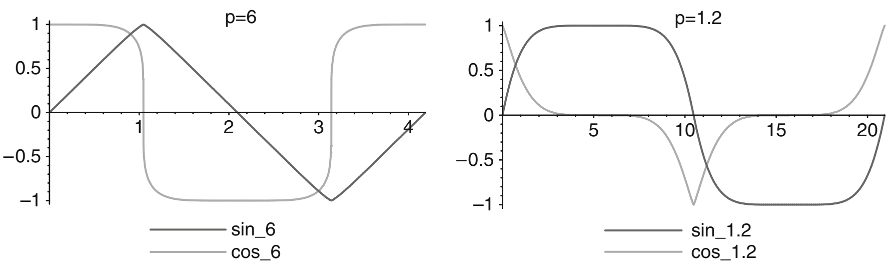

Fig. 2.1 $\sin_6, \cos_6$ and $\sin_{1.2}, \cos_{1.2}$

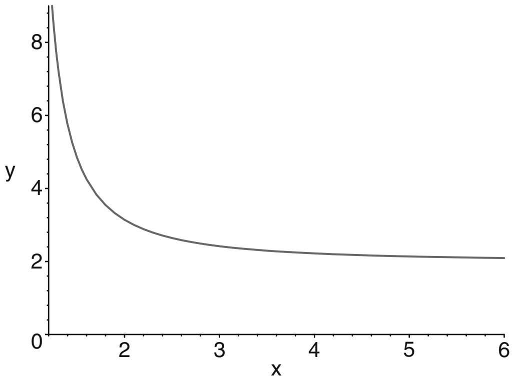

Fig. 2.2 $y = \pi_p$

Note that $\pi_2 = \pi$ and

$$p\pi_p = 2\Gamma(1/p')\Gamma(1/p) = p'\pi_{p'}. \tag{2.8}$$

Using (2.7) and (2.8) we see that π_p decreases as p increases, with

$$\lim_{p\to 1} \pi_p = \infty,\ \lim_{p\to\infty} \pi_p = 2,\ \lim_{p\to 1}(p-1)\pi_p = \lim_{p\to 1} \pi_{p'} = 2. \tag{2.9}$$

The dependence of π_p on p is illustrated by Fig. 2.2.

An analogue of the tangent function is obtained by defining

$$\tan_p x = \frac{\sin_p x}{\cos_p x} \tag{2.10}$$

for those values of x at which $\cos_p x \neq 0$. This means that $\tan_p x$ is defined for all $x \in \mathbb{R}$ except for the points $(k + 1/2)\pi_p$ $(k \in \mathbb{Z})$. Plainly $\tan_p$ is odd and π_p-periodic; also $\tan_p 0 = 0$. Some idea of the dependence of $\tan_p$ on p is provided by Fig. 2.3, in which the graph of this function is given for $p = 1.2$ and $p = 6$.

Use of (2.6) shows that on $(-\pi_p/2, \pi_p/2)$, $\tan_p$ has derivative $1 + \left|\tan_p x\right|^p$; and so if the inverse of $\tan_p$ on this interval is denoted by A, it follows that

$$A'(t) = 1/(1 + |t|^p),\ t \in \mathbb{R}.$$

When $p = 2$, $A(t)$ is simply arctan t, giving a direct connection with an angle. To provide a similar geometric interpretation when $p \neq 2$ we follow Elbert [57] and endow the plane $\mathbb{R}^2$ with the l_p metric, so that the distance between points (x_1, x_2) and (y_1, y_2) of $\mathbb{R}^2$ is

$$\{|x_1 - y_1|^p + |x_2 - y_2|^p\}^{1/p}.$$

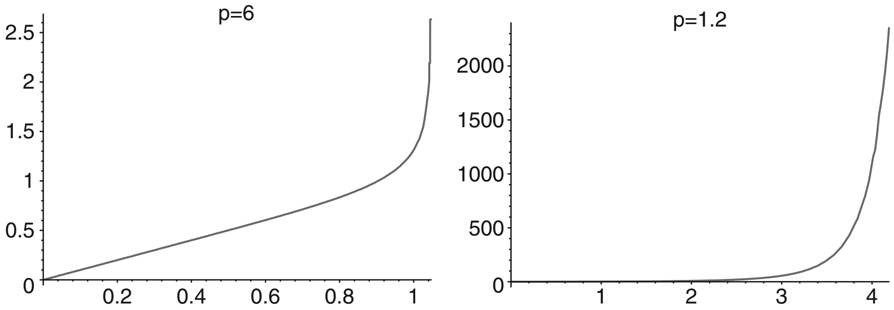

Fig. 2.3 $y = \tan_6(x),\ [0, \pi_6/2)$ $\quad y = \tan_{1.2}(x),\ [0, \pi_{1.2}/2)$

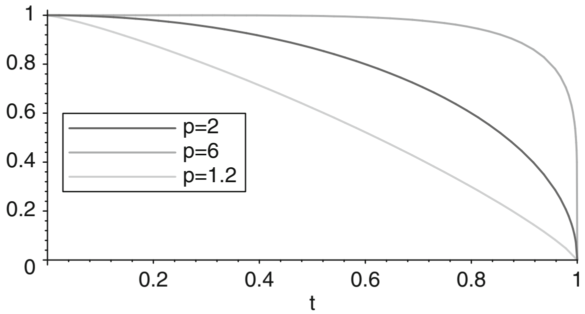

Fig. 2.4 The first quadrant of S_1 for $p = 2, 6, 1.2$

Given $R > 0$, when $1 < p < \infty$ the curve in $\mathbb{R}^2$ defined by $|x|^p + |y|^p = R^p$ will be called the p-circle with radius R, or the unit p-circle S_p if $R = 1$. The first quadrant of S_p is illustrated for $p = 1.2, 2, 4$ in Fig. 2.4.

Since $\left|\sin_p t\right|^p + \left|\cos_p t\right|^p = 1$, the p-circle of radius R may be parametrised by

$$x = R\cos_p t,\ y = \ R\sin_p t\ (0 \le t \le 2\pi_p), \tag{2.11}$$

just as in the familiar case in which $p = 2$. Let $P_1 = (\cos_p t, \sin_p t) \in I_p$ for some $t \in (0, 2\pi_p)$; we shall refer to t as the angle between the ray OP_1 (where $O = (0,0)$) and the positive x_1-axis. Now put

$$C_p(t) = \int_t^{\pi_p/2} \sin_p s\, ds$$

and let C be the curve $\left\{(C_p(t), \sin_p t) : t \in [0, 2\pi_p]\right\}$. The arc length of that part of C between $P_0 = (C_p(0), 0)$ and $P_2 = (C_p(t), \sin_p t)$, measured by means of the l_p metric on $\mathbb{R}^2$, is

$$\int_0^t \left\{\left|C_p'(s)\right|^p + \left|\cos_p s\right|^p\right\}^{1/p} ds = \int_0^t \left\{\left|\sin_p s\right|^p + \left|\cos_p s\right|^p\right\}^{1/p} ds = t.$$

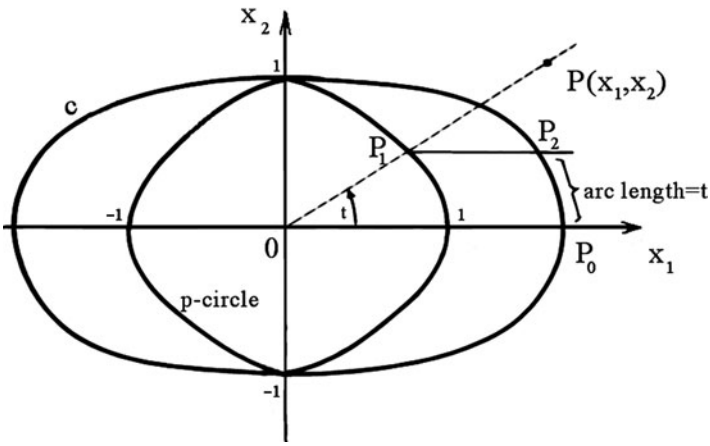

Fig. 2.5 Angles

This enables us to explain our method of measuring angles as follows. The ray OP (where $P = (x_1, x_2)$) meets the unit p-circle at $P_1 = (\cos_p t, \sin_p t)$; the line through P_1 parallel to the x_1-axis meets C in the same quadrant of the plane at P_2 : see Fig. 2.5 (based on [57]).

Then the signed length of the arc P_0P_2, namely t, is our measure of the angle $P_0\widehat{O}P$: such a procedure corresponds to what is done when $p = 2$. Note also that $x_2/x_1 = \sin_p t/\cos_p t = \tan_p t$, so that the arc length $t = A(x_2/x_1)$. This enables us to introduce polar coordinates ρ and θ in $\mathbb{R}^2$ by

$$\rho = (|x_1|^p + |x_2|^p)^{1/p}, \ \theta = A(x_2/x_1).$$

Next we record some basic facts about derivatives of the p-trigonometric functions. They follow immediately from the definitions and (2.6).

Proposition 2.1. *For all* $x \in [0, \pi_p/2)$,

$$\frac{d}{dx}\cos_p x = -\sin_p^{p-1} x \cos_p^{2-p} x, \ \frac{d}{dx}\tan_p x = 1 + \tan_p^p x,$$

$$\frac{d}{dx}\cos_p^{p-1} x = -(p-1)\sin_p^{p-1} x, \ \frac{d}{dx}\sin_p^{p-1} x = (p-1)\sin_p^{p-2} x \cos_p x.$$

Some elementary identities are provided in the proposition below.

Proposition 2.2. *For all* $y \in [0, 1]$,

$$\cos_p^{-1} y = \sin_p^{-1}(1 - y^p)^{1/p}, \ \sin_p^{-1} y = \cos_p^{-1}(1 - y^p)^{1/p}$$

and

$$\frac{2}{\pi_p}\sin_p^{-1} y^{1/p} + \frac{2}{\pi_{p'}}\sin_{p'}^{-1}(1-y^p)^{1/p'} = 1,\ \cos_p^p(\pi_p y/2) = \sin_{p'}^{p'}(\pi_{p'}(1-y)/2).$$

Proof. The first two claims follow directly from (2.6). For the third, note that

$$\sin_{p'}^{-1}(1-y^p)^{1/p'} = \int_0^{(1-y^p)^{1/p'}} (1-t^{p'})^{-1/p'}dt,$$

and that the change of variable $s = (1-t^{p'})^{1/p}$ transforms this integral into

$$\frac{p}{p'}\int_y^1 (1-s^p)^{-1/p}ds = \frac{p}{p'}\left(\frac{\pi_p}{2} - \sin_p^{-1} y\right) = \frac{\pi_{p'}}{\pi_p}\left(\frac{\pi_p}{2} - \sin_p^{-1} y\right),$$

the final step following from (2.8). To obtain the fourth identity, write

$$\cos_p^p(\pi_p y/2) = 1 - \sin_p^p(\pi_p y/2) := 1 - x$$

and observe that in view of the third identity,

$$y = \frac{2}{\pi_p}\sin_p^{-1} x^{1/p} = 1 - \frac{2}{\pi_{p'}}\sin_{p'}^{-1}(1-x)^{1/p'},$$

which gives

$$1-x = \sin_{p'}^{p'}(\pi_{p'}(1-y)/2). \qquad \square$$

It is also convenient to have more refined extensions of the trigonometric functions. To obtain these, suppose first that $p,q \in (1,\infty)$ and put

$$\pi_{p,q} = 2\int_0^1 (1-t^q)^{-1/p}dt. \tag{2.12}$$

This coincides with π_p when $p = q$. Use of the substitution $s = t^q$ shows that

$$\pi_{p,q} = 2q^{-1}\int_0^1 (1-s)^{-1/p}s^{1/q-1}ds = 2q^{-1}B(1/p', 1/q). \tag{2.13}$$

From (2.12) it is easy to see that $\pi_{p,q}$ decreases as either p or q increases, the other being held constant, and that

$$\lim_{p\to\infty} \pi_{p,q} = 2\ (1 < q < \infty),\ \lim_{q\to\infty} \pi_{p,q} = 2\ (1 < p < \infty). \tag{2.14}$$

By analogy with the case $p = q$ we define $\sin_{p,q}$ on the interval $[0, \pi_{p,q}/2]$ to be the inverse of the strictly increasing function $F_{p,q} : [0,1] \to [0, \pi_{p,q}/2]$ given by

$$F_{p,q}(x) = \int_0^x (1-t^q)^{-1/p} dt. \tag{2.15}$$

This is then extended to all of the real line by the same processes involving symmetry and $2\pi_{p,q}$-periodicity as for the case $p = q$. The function $\cos_{p,q}$ is defined to be the derivative of $\sin_{p,q}$, and it follows easily that for all $x \in \mathbb{R}$,

$$\left|\sin_{p,q} x\right|^q + \left|\cos_{p,q} x\right|^p = 1. \tag{2.16}$$

So far we have supposed that $p, q \in (1,\infty)$, but with natural interpretations of the integrals involved the extreme values 1 and ∞ can be allowed. This gives

$$\pi_{p,q} = \begin{cases} 2p', & \text{if } 1 \le p \le \infty, q = 1, \\ 2, & \text{if } 1 \le p \le \infty, q = \infty, \\ \infty, & \text{if } p = 1, 1 \le q < \infty, \\ 2, & \text{if } p = \infty, 1 \le q \le \infty. \end{cases} \tag{2.17}$$

Corresponding values of $\sin_{p,q}$ and $\cos_{p,q}$ are given by

$$\sin_{p,q} x = \begin{cases} 1-(1-x/p')^{p'}, & \text{if } 1 < p \le \infty, q = 1, \\ x, & \text{if } 1 \le p \le \infty, q = \infty, \\ x, & \text{if } p = \infty, 1 \le q \le \infty, \end{cases} \tag{2.18}$$

and

$$\cos_{p,q} x = \begin{cases} (1-x/p')^{1/(p-1)}, & \text{if } 1 < p \le \infty, q = 1, \\ 1, & \text{if } 1 \le p \le \infty, q = \infty, \\ 1, & \text{if } p = \infty, 1 \le q \le \infty. \end{cases} \tag{2.19}$$

When $p = 1$ these functions can be expressed in terms of elementary functions only when q is rational, in general. Thus

$$\sin_{1,1} x = 1 - e^{-x}, \;\; \cos_{1,1} x = e^{-x}, \;\; \sin_{1,2} x = \tanh x, \;\; \cos_{1,2} x = (\cosh x)^{-2}. \tag{2.20}$$

Note that the area A (measured in the usual way) enclosed by the p-circle $|x|^p + |y|^p = 1$ is given by

$$A = 2p^{-1}(\Gamma(1/p))^2/\Gamma(2/p) = \pi_{p',p}. \tag{2.21}$$

To establish this, note that

$$A = 4\int\int dxdy,$$

where the integration is over all those non-negative values of x and y such that $x^p + y^p \le 1$. The change of variable $x = w^{1/p}$, $y = z^{1/p}$ shows that

$$A = 4p^{-2}\int\int w^{1/p-1} z^{1/p-1} dwdz,$$

where now the integration is taken over the set $w \geq 0, z \geq 0, w+z \leq 1$. By a result of Dirichlet (see [121], 12.5),

$$A = \frac{4(\Gamma(1/p))^2}{p^2\Gamma(2/p)} \int_0^1 \tau^{2/p-1} d\tau,$$

from which (2.21) follows.

Moreover,

$$\int_0^1 (\sin_{p,q}(\pi_{p,q}x/2))^q\, dx = p'/(p'+q) \text{ if } p,q \in (1,\infty). \tag{2.22}$$

To establish this, observe that, with the above integral denoted by I,

$$I = \frac{2}{\pi_{p,q}} \int_0^{\pi_{p,q}/2} (\sin_{p,q} y)^q\, dy,$$

so that the substitution $z = \sin_{p,q} y$ gives

$$\begin{aligned} I &= \frac{2}{\pi_{p,q}} \int_0^1 z^q (1-z^q)^{-1/p} dz = \frac{2}{q\pi_{p,q}} \int_0^1 t^{1/q}(1-t)^{-1/p} dt \\ &= \frac{2}{\pi_{p,q}} B(1/p', 1+1/q) = \frac{\Gamma(1/p'+1/q)}{\Gamma(1/q)} = \frac{p'}{q+p'}. \end{aligned}$$

Since $\left|\cos_{p,q} x\right|^p = 1 - \left|\sin_{p,q} x\right|^q$ we also have

$$\int_0^1 (\cos_{p,q}(\pi_{p,q}x/2))^p\, dx = q/(p'+q) \text{ if } p,q \in (1,\infty). \tag{2.23}$$

As shown in [92], it is interesting to compute the length $L_{p'}$ of the unit p'-circle, measured by means of the l_p metric on the plane. This is

$$L_{p'} = 4\int_0^{\pi_{p'}/2} \left(\left|x'(t)\right|^p + \left|y'(t)\right|^p\right)^{1/p} dt,$$

where $x(t) = \cos_{p'} t$ and $y(t) = \sin_{p'} t$. Routine computations plus the use of (2.6) (with p replaced by p') show that

$$L_{p'} = 4\int_0^1 (1-z^{p'})^{-1/p} dz = 2\pi_{p,p'} = \frac{4\,(\Gamma(1/p'))^2}{p'\Gamma(2/p)}.$$

In [92] it is observed that the p'-circle has an isoperimetric property, namely that among all closed curves with the same p-length, the p'-circle encloses the largest area. Since the area A enclosed by the p'-circle $|x|^{p'} + |y|^{p'} = R^{p'}$ is $\pi_{p,p'}R^2$ and the p-length of this p'-circle is $2\pi_{p,p'}R$, we have the isoperimetric inequality

$$L_{p'}^2 \geq 4\pi_{p,p'}A,$$

which reduces to the more familiar $L_2^2 \geq 4\pi A$ when $p = 2$.

As might be expected, there are connections between the generalised trigonometric functions we have been discussing and some functions from classical analysis. For example, consider the incomplete Beta function $I(\cdot;a,b)$, defined for any positive a and b by

$$I(x;a,b) = \frac{1}{B(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}dt, \; x \in [0,1];$$

see, for example, [1, 26.5.1]. The change of variable $u = t^q$ in (2.15) shows that

$$F_{p,q}(x) = q^{-1}\int_0^{x^q} u^{-1/q'}(1-u)^{-1/p}du = q^{-1}B(1/q,1/p')I(x^q;1/q,1/p'),$$

and so, by (2.13),

$$\sin_{p,q}^{-1}(x) = F_{p,q}(x) = \frac{1}{2}\pi_{p,q}I(x^q;1/q,1/p'), \; x \in [0,1]. \tag{2.24}$$

Moreover, since the incomplete Beta function is related to the hypergeometric function F by

$$I(x;a,b) = \frac{x^a}{aB(a,b)}F(a,1-b;a+1;x)$$

(see [1, 6.6.2]), we have

$$\sin_{p,q}^{-1}(x) = xF(1/q,1/p;1+1/q;x^q), \; x \in [0,1]. \tag{2.25}$$

Since

$$I(x;a,b) = \frac{x^a(1-x)^b}{aB(a,b)}\left\{1+\sum_{n=0}^{\infty}\frac{B(a+1,n+1)}{B(a+b,n+1)}x^{n+1}\right\}, \; x \in (0,1),$$

(see, for example, [1, 26.5.9]), we have

$$\sin_{p,q}^{-1}(x) = x(1-x^q)^{1/p'}\left\{1+\sum_{n=0}^{\infty}\frac{B(1+1/q,n+1)}{B(1/q+1/p',n+1)}x^{q(n+1)}\right\}, \; x \in (0,1). \tag{2.26}$$

We can also use the well-known fact that

$$F(a,b;c;x) = \sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)}\frac{x^n}{n!}$$

to obtain the expansion

$$\sin_{p,q}^{-1}(x) = x\sum_{n=0}^{\infty}\frac{\Gamma(n+1/p)}{(qn+1)\Gamma(1/p)}\frac{x^{nq}}{n!},\ x\in(0,1). \tag{2.27}$$

From (2.27) it is possible to obtain a series expansion for $\sin_{p,q}(x)$ in the form $x\sum_{n=0}^{\infty} a_n x^{qn}$, but we leave this delightful task to the intrepid reader, who is urged to show that if $x\in[0,\pi_p/2)$, then

$$\sin_p x = x - \frac{1}{p(p+1)}x^{p+1} - \frac{(p^2-2p-1)}{2p^2(p+1)(2p+1)}x^{2p+1} + \dots .$$

Finally, we consider various integrals involving the p-trigonometric functions.

Proposition 2.3. *For all* $x\in(0,\pi_p/2)$,

$$\int \cos_p x dx = \sin_p x,\ p\int \cos_p^p x dx = (p-1)x + \sin_p x \cos_p^{p-1} x,$$

$$(p-1)\int \sin_p^{p-1} x dx = -\cos_p^{p-1} x,\ \int \tan_p^p x dx = \tan_p x - x$$

and

$$\int \sin_p x dx = \frac{1}{2}\sin_p^2 x F(1/p, 2/p; 1+2/p; \sin_p^p x).$$

Proof. Apart from the last integral, these follow directly from the definitions. To obtain the final result, make the substitution $u=\sin_p x$, note that

$$\int \sin_p x dx = \int u(1-u^p)^{-1/p}du = \int u\sum_{n=0}^{\infty}\frac{\Gamma(n+1/p)}{\Gamma(1/p)}\frac{u^{pn}}{n!}du,$$

integrate, and then write the resulting series in terms of the hypergeometric function. □

For definite integrals we note the following elementary results.

Proposition 2.4. *Let* $k,l>0$. *Then*

$$\int_0^{\pi_p/2}\sin_p^k x dx = \frac{1}{p}B\left(\frac{k+1}{p},\frac{1}{p'}\right),\ \int_0^{\pi_p/2}\cos_p^k x dx = \frac{1}{p}B\left(\frac{1}{p},1+\frac{k-1}{p}\right)$$

and

$$\int_0^{\pi_p/2}\sin_p^k x \cos_p^l x dx = \frac{1}{p}B\left(\frac{k+1}{p},1+\frac{l-1}{p}\right).$$

These follow directly by making natural substitutions: for example, in the first integral we put $y=\sin_p x$ and then $t=y^p$. The conditions on k and l can be weakened: in the first and third equality the condition on k can be weakened to $k>-1$, while in the remaining cases the conditions $k,l>1-p$ will do.

To illustrate the utility of Proposition 2.4 we give a result concerning the Catalan constant G, defined to be

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}.$$

This constant plays a prominent rôle in various combinatorial identities. From the power series representation (2.27) of $\sin_p^{-1} x$ we have

$$x = \sin_p x \sum_{n=0}^{\infty} \frac{\Gamma(n+1/p)}{(np+1)\Gamma(1/p)} \frac{(\sin_p x)^{np}}{n!},\ 0 < x < \frac{\pi_p}{2}.$$

Hence, with the aid of the first part of Proposition 2.4, we have

$$\int_0^{\pi_p/2} \frac{x}{\sin_p x} dx = \frac{\pi_p}{2} \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+1/p)}{n!\Gamma(1/p)} \right)^2 \frac{1}{np+1}.$$

It is known that (see, for example, [63], 1.7.4)

$$\int_0^{\pi/2} \frac{x}{\sin x} dx = 2G.$$

Thus the Catalan constant is expressible as

$$G = \frac{\pi}{4} \sum_{n=0}^{\infty} \left(\frac{(2n)!}{(n!)^2 2^{2n}} \right)^2 \frac{1}{2n+1}.$$

We refer to [20, 39, 89, 90] for further information and additional references concerning these functions and their applications. A fascinating account of early work on generalisations of trigonometric functions is given by Lindqvist and Peetre in [93].

2.2 Basis Properties

We have already remarked in 1.1.1 that $(\sin(n\pi\cdot))_{n\in\mathbb{N}}$ is a basis in $L_q(0,1)$ for any $q \in (1,\infty)$. It is natural to ask whether the functions $\sin_p(n\pi\cdot)$ have a similar property: the answer, given in [9], is that they do, at least if p is not too close to 1, and we now give an account of this result. For simplicity the action will take place in $L_q(0,1)$ rather than $L_q(a,b)$, and for this reason we introduce the functions $f_{n,p}$ defined by

$$f_{n,p}(t) = \sin_p(n\pi_p t)\ (n \in \mathbb{N}, 1 < p < \infty, t \in \mathbb{R}). \tag{2.28}$$

When $p = 2$ these functions are simply the usual sine functions, and we write

$$e_n(t) = f_{n,2}(t) = \sin(n\pi t). \tag{2.29}$$

Since each $f_{n,p}$ is continuous on $[0,1]$ it has a Fourier sine expansion:

$$f_{n,p}(t) = \sum_{k=1}^{\infty} \widehat{f_{n,p}}(k)\sin(k\pi t),\ \widehat{f_{n,p}}(k) = 2\int_0^1 f_{n,p}(t)\sin(k\pi t)dt. \tag{2.30}$$

From the symmetry of $f_{1,p}$ about $t = 1/2$ it follows that $\widehat{f_{1,p}}(k) = 0$ when k is even and that

$$\begin{aligned}\widehat{f_{n,p}}(k) &= 2\int_0^1 f_{1,p}(nt)\sin(k\pi t)dt = 2\sum_{m=1}^{\infty}\widehat{f_{1,p}}(m)\int_0^1 \sin(k\pi t)\sin(mn\pi t)dt \\ &= \begin{cases} \widehat{f_{1,p}}(m) & \text{if } mn = k \text{ for some odd } m, \\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{2.31}$$

For brevity put $\tau_m(p) = \widehat{f_{1,p}}(m)$. As all the Fourier coefficients of the $f_{n,p}$ may be expressed in terms of the $\tau_m(p)$, we concentrate on the behaviour of these numbers, beginning with their decay properties as $m \to \infty$. For even m, $\tau_m(p) = 0$. If m is odd, integration by parts and the substitution $s = \cos_p(\pi_p t)$ show that

$$\begin{aligned}\tau_m(p) &= 4\int_0^{1/2} f_{1,p}(t)\sin(m\pi t)dt = \frac{4\pi_p}{m\pi}\int_0^{1/2}\cos_p(\pi_p t)\cos(m\pi t)dt \\ &= -\frac{4\pi_p}{m^2\pi^2}\int_0^{1/2}\sin(m\pi t)\frac{d}{dt}\cos_p(\pi_p t)dt \\ &= \frac{4\pi_p}{m^2\pi^2}\int_0^1 \sin\left(\frac{m\pi}{\pi_p}\cos_p^{-1}s\right)ds.\end{aligned} \tag{2.32}$$

In a similar way we have, for odd m,

$$\tau_m(p) = \frac{4}{m\pi}\int_0^1 \cos\left(\frac{m\pi}{\pi_p}\sin_p^{-1}s\right)ds. \tag{2.33}$$

From (2.32) we obtain the estimate

$$|\tau_m(p)| \le 4\pi_p/(\pi m)^2 \ (m \text{ odd}). \tag{2.34}$$

Next we consider the dependence of $\sin_p(n\pi_p t)$ on p.

Proposition 2.5. *Suppose that* $1 < p < q < \infty$. *Then the function* f *defined by*

$$f(t) = \frac{\sin_q^{-1}(t)}{\sin_p^{-1}(t)}$$

is strictly decreasing on $(0,1)$.

Proof. Let

$$g(t) = \frac{(1-t^q)^{1/q}}{(1-t^p)^{1/p}} \ (0 < t < 1).$$

For all $t \in (0,1)$,

$$g'(t) = g(t)\left\{\frac{-t^{q-1}}{1-t^q} + \frac{t^{p-1}}{1-t^p}\right\} = \frac{(t^p - t^q)g(t)}{t(1-t^q)(1-t^p)} > 0.$$

Put

$$G(t) = \sin_p^{-1}(t) - g(t)\sin_q^{-1}(t)$$

and observe that

$$G'(t) = -(\sin_q^{-1} t)g'(t) < 0 \text{ in } (0,1).$$

Hence $G(t) < 0$ in $(0,1)$, so that

$$f'(t) = \frac{G(t)}{(\sin_q^{-1} t)^2 (1-t^q)^{1/q}} < 0 \text{ in } (0,1). \qquad \square$$

From this we immediately have

Corollary 2.1. *(i) If* $1 < p < q < \infty$, *then*

$$1 > \frac{\sin_q^{-1}(t)}{\sin_p^{-1}(t)} \geq \frac{\pi_q}{\pi_p} \textit{ in } (0,1].$$

(ii) If $1 < p \leq q < \infty$, *then*

$$\sin_p^{-1}(t) \geq \sin_q^{-1}(t) \textit{ and } \frac{1}{\pi_q}\sin_q^{-1}(t) \geq \frac{1}{\pi_p}\sin_p^{-1}(t) \textit{ in } [0,1].$$

(iii) If $1 < p \leq q < \infty$, *then*

$$\sin_p(\pi_p t) \geq \sin_q(\pi_q t) \textit{ in } [0,1/2].$$

The following analogue of the classical Jordan inequality will also be useful.

Proposition 2.6. *Let* $1 < p < \infty$. *For all* $\theta \in (0, \pi_p/2]$,

$$\frac{2}{\pi_p} \leq \frac{\sin_p \theta}{\theta} < 1.$$

Proof. Change of variable shows that

$$\sin_p^{-1} x = x\int_0^1 (1 - x^p s^p)^{-1/p} ds,$$

and so

$$\theta = (\sin_p \theta) \int_0^1 (1 - (\sin_p \theta)^p s^p)^{-1/p} ds.$$

Since

$$1 \leq \int_0^1 (1 - (\sin_p \theta)^p s^p)^{-1/p} ds \leq \frac{\pi_p}{2}$$

for all $\theta \in (0, \pi_p/2]$, the result follows. □

Corollary 2.2. *For all $p \in (1,\infty)$ and all $t \in (0,1/2)$,*

$$\sin_p(\pi_p t) > 2t.$$

Proof. By Proposition 2.6, $\sin_p \theta > 2\theta/\pi_p$ if $0 < \theta < \pi_p/2$. Now put $\theta = \pi_p t$. □

Given any function f on $[0,1]$, we extend it to a function $\widetilde{f}$ on $\mathbb{R}_+ := [0,\infty)$ by setting

$$\widetilde{f}(t) = -\widetilde{f}(2k - t) \text{ for } t \in [k, k+1], k \in \mathbb{N}. \tag{2.35}$$

With this understanding, we define maps $M_m : L_q(0,1) \to L_q(0,1)$ $(1 < q < \infty)$ by

$$M_m g(t) = \widetilde{g}(mt),\ m \in \mathbb{N},\ t \in (0,1). \tag{2.36}$$

Note that $M_m e_n = e_{mn}$.

Lemma 2.1. *For all $m \in \mathbb{N}$ and all $q \in (1,\infty)$ the map $M_m : L_q(0,1) \to L_q(0,1)$ is isometric and linear.*

Proof. Let $g \in L_q(0,1)$. Then

$$\begin{aligned} \int_0^1 |M_m g(t)|^q \, dt &= m^{-1} \int_0^m |\widetilde{g}(s)|^q \, ds = m^{-1} \sum_{k=1}^m \int_{k-1}^k |\widetilde{g}(s)|^q \, ds \\ &= m^{-1} \sum_{k=1}^m \int_{k-1}^k |g(s)|^q \, ds = \int_0^1 |g(s)|^q \, ds. \end{aligned}$$ □

The maps M_m are introduced because they help to construct a linear homeomorphism T of $L_q(0,1)$ onto itself that maps each e_n to $f_{n,p}$: once this is done it will follow from general considerations that the $f_{n,p}$ form a basis of $L_q(0,1)$. The map T is defined by

$$Tg(t) = \sum_{m=1}^{\infty} \tau_m M_m g(t). \tag{2.37}$$

Lemma 2.2. *Let $p, q \in (1,\infty)$. The map T is a bounded linear map of $L_q(0,1)$ to itself with $\|T\| \leq \pi_p/2$. For all $n \in \mathbb{N}$, $Te_n = f_{n,p}$.*

Proof. From (2.31), (2.34) and Lemma 2.1 we see that

$$\|T\| \leq \sum_{m=1}^{\infty} \frac{4\pi_p}{(2m-1)^2\pi^2} = \pi_p/2.$$

A second application of (2.31) shows that

$$Te_n = \sum_{m=1}^{\infty} \tau_m e_{mn} = \sum_{m=1}^{\infty} \widehat{f_{1,p}}(m) e_{mn} = \sum_{k=1}^{\infty} \widehat{f_{n,p}}(k) e_k = f_{n,p}. \qquad \square$$

Lemma 2.3. *There exists $p_0 \in (1,2)$ such that if $p > p_0$, then for all $q \in (1,\infty)$, $T : L_q(0,1) \to L_q(0,1)$ has a bounded inverse.*

Proof. Since M_1 is the identity map id, we have from (2.31) and Lemma 2.1 that

$$\|T - \tau_1 id\| \leq \sum_{j=1}^{\infty} \left|\tau_{2j+1}(p)\right|,$$

and so the invertibility of T will follow from Theorem II.1.2 of [123] if we can show that

$$\sum_{j=1}^{\infty} \left|\tau_{2j+1}(p)\right| < |\tau_1(p)|. \tag{2.38}$$

From (2.34) we have, for all $p \in (1,\infty)$,

$$\sum_{j=1}^{\infty} \left|\tau_{2j+1}(p)\right| \leq \frac{4\pi_p}{\pi^2}\left(\frac{\pi^2}{8} - 1\right). \tag{2.39}$$

To estimate $|\tau_1(p)|$, note that by Corollary 2.2,

$$\tau_1(p) = 4\int_0^{1/2} \sin_p(\pi_p t)\sin(\pi t)dt > 4\int_0^{1/2} 2t\sin(\pi t)dt = 8/\pi^2,$$

from which (2.38) follows if $2 \leq p < \infty$ since $\pi_p \leq \pi$.

If $1 < p < 2$, then the monotonic dependence of $\sin_p(\pi_p t)$ on p given by Corollary 2.1 (iii) shows that

$$\tau_1(p) > 4\int_0^{1/2} \sin^2(\pi t)dt = 1.$$

Now define p_0 by

$$\pi_{p_0} = \frac{\pi^2}{4} \Big/ \left(\frac{\pi^2}{8} - 1\right).$$

Then if $p > p_0$,

$$\frac{4\pi_p}{\pi^2}\left(\frac{\pi^2}{8}-1\right)<1,$$

and again we have (2.38).

We summarise these results in the following theorem.

Theorem 2.1. *The map T is a homeomorphism of $L_q(0,1)$ onto itself for every $q \in (1,\infty)$ if $p_0 < p < \infty$, where p_0 is defined by the equation*

$$\pi_{p_0} = \frac{2\pi^2}{\pi^2-8}. \tag{2.40}$$

Remark 2.1. Numerical solution of (2.40) shows that p_0 is approximately equal to 1.05.

Theorem 2.2. *Let $p \in (p_0,\infty)$ and $q \in (1,\infty)$. Then the family $(f_{n,p})_{n\in\mathbb{N}}$ forms a Schauder basis of $L_q(0,1)$ and a Riesz basis of $L_2(0,1)$.*

Proof. Since the e_n form a basis of $L_q(0,1)$ and T is a linear homeomorphism of $L_q(0,1)$ onto itself with $Te_n = f_{p,n}$ $(n \in \mathbb{N})$, it follows from [73], p. 75 or [114], Theorem 3.1, p. 20 that the $f_{n,p}$ form a Schauder basis of $L_q(0,1)$. When $q = 2$ the argument is similar and follows [67], Sect. VI.2. □

The condition $p > p_0 > 1$ in this theorem arises from the techniques used in the proof: a discussion of this is given in [20]. Whether the result remains true for all $p > 1$ appears to be unknown at the moment.

Notes

Note 2.1. As the literature contains various different definitions of the $\sin_p$ and $\cos_p$ functions, confusion about the nature of such functions is possible. Our choice was largely motivated by the wish to have available the identity $\left|\sin_p x\right|^p + \left|\cos_p x\right|^p = 1$, while other authors attached greater importance to different properties. Power series expansions for his versions of $\sin_p$, $\cos_p$ and $\tan_p$ are given by Linqvist [90]; see also the detailed work in this direction on related functions by Peetre [104]. No sensible addition formulae (e.g. for $\sin_p(x+y)$) seem to be known. Further details of properties of p-trigonometric functions are given in [20].

Note 2.2. The only work on the basis properties of the $\sin_p$ functions of which we are aware is that of [9]. Our treatment gives the modification of their proof presented in [20], which in particular seals a gap in the proof of Corollary 2.1(iii) given in [9].

Completeness properties of certain function sequences of the form $\{f(nx)\}_{n\in\mathbb{N}}$ have been investigated by Bourgin ([16]; see also [17]) in an L_2 setting and by Szász [119] in the context of L_r. However, these papers require properties, such as orthogonality or specified behaviour of the Fourier coefficients of f, that are not available when $f = \sin_p$.

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