Chapter 1
Hyperfinite Dirichlet Forms

The interplay between methods from functional analysis and the theory of stochastic processes is one of the most important and exciting aspects of mathematical physics today. It is a highly technical and sophisticated theory based on decades of research in both areas. Numerous papers have been written on the standard theory of Dirichlet forms. Apart from the articles and monographs cited below, other notable contributions to the area include: Albeverio and Bernabei [5], Albeverio, Kondratiev, and Röckner [32], Albeverio and Kondratiev [33], Albeverio and Ma [39], Albeverio, Rüdiger, and Wu [54], Bliedtner [94], Bouleau [98], Bouleau and Hirsch [99], Chen et al. [112], Chen, Ma, and Röckner [116], Eberle [149], Exner [154], Fabes, Fukushima, Gross, Kenig, Röckner, and Stroock [155], Fitzsimmons and Kuwae [172], Fukushima [177,179,180], Fukushima and Tanaka [185], Fukushima and Ying [188,189], Gesztesy et al. [191,192], Grothaus et al. [198], Hesse et al. [208], Jacob [218–220], Jacob and Moroz [221], Jacob and Schilling [222], Jost et al. [225], Kassmann [232], Kim et al. [240], Kumagai and Sturm [248], Le Jan [258], Liskevich and Röckner [265], Ma and Röckner [272, 273], Ma et al. [274], Mosco [283], Okura [292], Oshima [294,295], D.W. Robinson [312], Röckner and Wang [317], Röckner and Zhang [319], Schmuland and Sun [329], Shiozawa and Takeda [331], da Silva et al. [332], Stannat [336,338], Stroock [340], Sturm [343], Takeda [346,347], Wu [363], and Yosida [364].

In this monograph, we present the theory of Dirichlet forms from a unified vantage point, using nonstandard analysis, thus viewing the continuum of the time line as a discrete lattice of infinitesimal spacing. This approach is close in spirit to the discrete classical formulation of Dirichlet space theory in A. Beurling and J. Deny's seminal article [87].

The discrete setup in this monograph permits to study the diffusion and the jump part by essentially the same methods. This setting being independent of special topological properties of the state space, it is also considerably less technical than other approaches. Thus, the theory has found its natural setting and no longer depends on choosing particular topological spaces; in particular, it is valid for both finite and infinite dimensional spaces.

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Whilst Albeverio et al. [25], Chap. 5, only discussed symmetric hyperfinite Dirichlet forms and related Markov chains (refer to [165, 166] also), we shall extend the theory to the nonsymmetric case. We shall try to follow as much as possible the path suggested by the work on the symmetric case.

An important sub-class of Markov process are Feller processes with stationary and independent increments (*Lévy processes*), and in recent years, these processes have attracted a lot of interest, including from nonstandard analysts. Initiated by T. Lindstrøm [263], a number of articles have been devoted to the investigation of *hyperfinite Lévy processes.* Chapter 5 of this monograph is a detailed exposition of Lindstrøm's theory [263] and its subsequent continuation by Albeverio and Herzberg [14]. The book ends with an expository summary (without proofs) of the model theory of stochastic processes as developed by H.J. Keisler and his coauthors, who formulated and proved the "universality" of hyperfinite adapted probability spaces in a rigorous manner, and a short description of recent fundamental results about the definability of nonstandard universes.

Meanwhile, our purpose in the first chapter is to develop a general theory of hyperfinite quadratic forms. We shall set the scene in Sect. 1.1. Sections 1.2 and 1.3 will study the domains of symmetric parts, the standard parts and resolvents. We shall discuss the property of weak coercive quadratic forms in Sects. 1.4 and 1.7. In Sect. 1.5, we shall study Markov forms and begin the analysis of associated Markov chains and get the basic Beurling–Deny formula. We discuss the hyperfinite lifting theory of standard Dirichlet forms in Sect. 1.6.

1.1 Hyperfinite Quadratic Forms

We shall develop a hyperfinite theory of nonnegative quadratic forms on infinite dimensional spaces. It is well-known that in the Hilbert space case the theory of closed forms of this kind is equivalent to the theory of nonnegative operators. In fact, there is a natural correspondence between forms $E(\cdot,\cdot)$ and operators A given by $E(u,u) = \langle Au, u\rangle$, where $\langle\cdot,\cdot\rangle$ is the scalar product in the Hilbert space. We have chosen to present the theory in terms of forms and not operators for two reasons: partly because forms are real-valued, and this makes it simpler to take standard parts, but also because in most of our applications, the form is what is naturally given.

Let H be an internal, hyperfinite dimensional linear space[1] equipped with an inner product $\langle\cdot,\cdot\rangle$ generating a norm $||\cdot||$. Let ${}^*\mathbb{R}$ be the nonstandard

[1] The notions of hyperfinite dimensional linear space are given in Albeverio et al. [25].

real line[2]. We call a map $\mathcal{E} : H \times H \longrightarrow {}^*\mathbb{R}$ *nonnegative quadratic form* if and only if for all $\alpha \in {}^*\mathbb{R}, u, v, w \in H$,

$$\begin{aligned} \mathcal{E}(u,u) &\geq 0, \\ \mathcal{E}(\alpha u, v) &= \alpha \mathcal{E}(u,v), \\ \mathcal{E}(u, \alpha v) &= \alpha \mathcal{E}(u,v), \\ \mathcal{E}(u+v, w) &= \mathcal{E}(u,w) + \mathcal{E}(v,w), \\ \mathcal{E}(w, u+v) &= \mathcal{E}(w,u) + \mathcal{E}(w,v). \end{aligned}$$

Since $\mathcal{E}(\cdot,\cdot)$ is a nonnegative quadratic form on the hyperfinite dimensional space H, elementary linear algebra tells us that there is a unique nonnegative definite operator $A : H \longrightarrow H$ such that

$$\mathcal{E}(u,v) = \langle Au, v\rangle \quad \text{for all} \quad u, v \in H. \tag{1.1.1}$$

To see this, let ${}^*\mathbb{N}_0$ be the nonstandard integers[3]. Let $\{e_i \mid 1 \leq i \leq N\}$ be an orthonormal basis of $(H, \langle\cdot,\cdot\rangle)$ for an $N \in {}^*\mathbb{N}$. We put $Ae_i = \sum_{j=1}^{N} \mathcal{E}(e_i, e_j)e_j$. Then (1.1.1) follows immediately. Hence, A is given by the matrix $A = (\mathcal{E}(e_i,e_j))_{1\leq i,j\leq N}$, i.e.,

$$A = \begin{pmatrix} \mathcal{E}(e_1,e_1) & \mathcal{E}(e_1,e_2) & \dots & \mathcal{E}(e_1,e_N) \\ \mathcal{E}(e_2,e_1) & \mathcal{E}(e_2,e_2) & \dots & \mathcal{E}(e_2,e_N) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{E}(e_N,e_1) & \mathcal{E}(e_N,e_2) & \dots & \mathcal{E}(e_N,e_N) \end{pmatrix}. \tag{1.1.2}$$

Moreover, $\langle Au, u\rangle \geq 0$ for all $u \in H$. This means that A is a hyperfinite dimensional matrix (not necessarily symmetric). Let $\hat{A}$ be the *adjoint operator* of A, that is,

$$\mathcal{E}(u,v) = \langle u, \hat{A}v\rangle \quad \text{for all} \quad u, v \in H.$$

By (1.1.2), we have that $\hat{A}$ is the transpose of A. If $||A||$ and $||\hat{A}||$ are the operator norms of A and $\hat{A}$, respectively, we have $||A|| = ||\hat{A}||$. We fix an infinitesimal[4] Δt such that

[2] ${}^*\mathbb{R}$ is the standard notation for the nonstandard real line, refer to Appendix, Albeverio et al. [25], Cutland [125], Davis [135], Hurd [216], Hurd and Loeb [217], Lindstrøm [262], Stroyan and Bayod [341], and Stroyan and Luxemburg [342].

[3] ${}^*\mathbb{N}_0$ is the standard notation for the nonstandard integers, refer to Appendix, Albeverio et al. [25], Cutland [125], Davis [135], Hurd [216], Hurd and Loeb [217], Lindstrøm [262], Stroyan and Bayod [341], and Stroyan and Luxemburg [342].

[4] In the sense of nonstandard analysis, refer to Appendix, Albeverio et al. [25], Keisler [237, 238], Stroyan and Bayod [341], and Stroyan and Luxemburg [342].

$$0 < \Delta t \leq \frac{1}{||A||} = \frac{1}{||\hat{A}||}. \tag{1.1.3}$$

Let us define new operators $Q^{\Delta t}$ and $\hat{Q}^{\Delta t}$ by

$$Q^{\Delta t} = I - \Delta t A,$$
$$\hat{Q}^{\Delta t} = I - \Delta t \hat{A}.$$

The relation (1.1.3) implies that the operators $Q^{\Delta t}$ and $\hat{Q}^{\Delta t}$ are nonnegative. Because A is nonnegative, the operator norms of $Q^{\Delta t}$ and $\hat{Q}^{\Delta t}$ are less than or equal to one. Similarly, we define the *nonnegative quadratic co-form* $\hat{\mathcal{E}}(\cdot,\cdot)$ of $\mathcal{E}(\cdot,\cdot)$ by

$$\hat{\mathcal{E}}(u,v) = \mathcal{E}(v,u) \text{ for all } u, v \in H.$$

Introduce a nonstandard time line T by

$$T = \{k\Delta t \mid k \in {}^*\mathbb{N}_0\}.$$

For each element $t = k\Delta t$ in T, define Q^t and $\hat{Q}^t$ to be the operators

$$Q^t = (Q^{\Delta t})^k,$$
$$\hat{Q}^t = (\hat{Q}^{\Delta t})^k.$$

The families $\{Q^t\}_{t\in T}$ and $\{\hat{Q}^t\}_{t\in T}$ are obviously semigroups. We shall call $\{Q^t\}_{t\in T}$ the *semigroup* and $\{\hat{Q}^t\}_{t\in T}$ the *co-semigroup* associated with $\mathcal{E}(\cdot,\cdot)$ and Δt, respectively. Whenever we refer to $\mathcal{E}(\cdot,\cdot)$, $\hat{\mathcal{E}}(\cdot,\cdot)$, $A, \hat{A}, T, Q^t$ and $\hat{Q}^t$ in the rest of this book, we shall assume that they are linked by above relations.

In applications, the primary objects will often be the semigroup $\{Q^t\}_{t\in T}$ and co-semigroup $\{\hat{Q}^t\}_{t\in T}$. We can then define A and $\hat{A}$ (and hence $\mathcal{E}(\cdot,\cdot)$) by

$$A = \frac{1}{\Delta t}\left(I - Q^{\Delta t}\right),$$
$$\hat{A} = \frac{1}{\Delta t}\left(I - \hat{Q}^{\Delta t}\right).$$

The operator A is called the *infinitesimal generator* of $\mathcal{E}(\cdot,\cdot)$, and $\hat{A}$ is called the *infinitesimal co-generator* of $\mathcal{E}(\cdot,\cdot)$. For each $t \in T$, we may define approximations $A^{(t)}$ of A and $\hat{A}^{(t)}$ of $\hat{A}$ by

$$A^{(t)} = \frac{1}{t}\left(I - Q^t\right),$$
$$\hat{A}^{(t)} = \frac{1}{t}\left(I - \hat{Q}^t\right). \tag{1.1.4}$$

From $A^{(t)}$ and $\hat{A}^{(t)}$, we get the forms

$$\begin{aligned}\mathcal{E}^{(t)}(u,v) &= \langle A^{(t)}u, v\rangle \\ &= \langle u, \hat{A}^{(t)}v\rangle,\end{aligned} \tag{1.1.5}$$

and

$$\begin{aligned}\hat{\mathcal{E}}^{(t)}(u,v) &= \mathcal{E}^{(t)}(v,u) \\ &= \langle \hat{A}^{(t)}u, v\rangle \\ &= \langle A^{(t)}v, u\rangle.\end{aligned}$$

We define the *symmetric part* $\overline{\mathcal{E}}(\cdot,\cdot)$ and *anti-symmetric part* $\mathring{\mathcal{E}}(\cdot,\cdot)$ of $\mathcal{E}(\cdot,\cdot)$ by

$$\begin{aligned}\overline{\mathcal{E}}(u,v) &= \frac{1}{2}\Big(\mathcal{E}(u,v) + \mathcal{E}(v,u)\Big), \\ \mathring{\mathcal{E}}(u,v) &= \frac{1}{2}\Big(\mathcal{E}(u,v) - \mathcal{E}(v,u)\Big).\end{aligned}$$

For $\alpha \in {}^*\mathbb{R}, \alpha \geq 0$, we set

$$\overline{\mathcal{E}}_\alpha(u,v) = \overline{\mathcal{E}}(u,v) + \alpha\langle u,v\rangle.$$

Each of these forms generates a norm (possibly a semi-norm in the case $\alpha = 0$):

$$\begin{aligned}|u|_\alpha &= \sqrt{\overline{\mathcal{E}}_\alpha(u,u)} \\ &= \sqrt{\mathcal{E}_\alpha(u,u)}.\end{aligned}$$

We recall that the original Hilbert space norm on H is denoted by $||\cdot||$. Similarly, we set for $\alpha \in {}^*\mathbb{R}, \alpha \geq 0$,

$$\begin{aligned}\mathcal{E}_\alpha(u,v) &= \mathcal{E}(u,v) + \alpha\langle u,v\rangle, \\ \hat{\mathcal{E}}_\alpha(u,v) &= \hat{\mathcal{E}}(u,v) + \alpha\langle u,v\rangle.\end{aligned}$$

Let $\overline{A}$ and $\{\overline{Q}^t\}$ be the generator and semigroup of $\overline{\mathcal{E}}(\cdot,\cdot)$, respectively. Then

$$\overline{A} = \frac{1}{2}\left(A + \hat{A}\right), \quad \overline{Q}^{\Delta t} = \frac{1}{2}\left(Q^{\Delta t} + \hat{Q}^{\Delta t}\right) \text{ and } \overline{Q}^{k\Delta t} = (\overline{Q}^{\Delta t})^k, \forall k \in {}^*\mathbb{N}.$$

Since $\overline{A}$ and $\overline{Q}^t$ are nonnegative, self-adjoint operators, they have unique nonnegative square roots, which we denote by $\overline{A}^{\frac{1}{2}}$ and $\overline{Q}^{\frac{t}{2}}$, respectively.

In the same manner as (1.1.4) and (1.1.5), we can define approximations $\overline{A}^{(t)}$ of A and $\overline{\mathcal{E}}^{(t)}(\cdot,\cdot)$ of $\overline{\mathcal{E}}(\cdot,\cdot)$ by

$$\overline{A}^{(t)} = \frac{1}{t}\left(I - \overline{Q}^t\right), \quad \overline{\mathcal{E}}^{(t)}(u,v) = \langle \overline{A}^{(t)} u, v\rangle, \quad t \in T.$$

If a nonnegative quadratic form $\mathcal{E}(\cdot,\cdot) : H \times H \longrightarrow {}^*\mathbb{R}$ satisfies

$$\mathcal{E}(u,v) = \mathcal{E}(v,u) \text{ for all } u, v \in H,$$

i.e., $\overline{\mathcal{E}}(u,v) = \mathcal{E}(u,v)$, we shall call it a *nonnegative symmetric quadratic form.* It is easy to see that a nonnegative quadratic form $\mathcal{E}(u,v)$ is symmetric if and only if $A = \hat{A}$ or $Q^t = \hat{Q}^t, \forall t \in T$.

In this book, we shall deal with nonnegative quadratic forms $\mathcal{E}(\cdot,\cdot)$ and the related theory. For the framework, potential theory and applications of nonnegative symmetric quadratic form, we refer the reader to Albeverio et al. [25], Chap. 5, Sect. 5.1 and Fan [165, 166]. We shall utilize the known results of symmetric forms in our study, and extend them to the nonsymmetric case. In particular, we need the notion of the symmetric part $\overline{\mathcal{E}}(\cdot,\cdot)$ of $\mathcal{E}(\cdot,\cdot)$, and the related notations. In Sect. 1.2, we shall define the domain $\mathcal{D}(\overline{\mathcal{E}})$ of the symmetric part $\overline{\mathcal{E}}(\cdot,\cdot)$ by using the semigroup $\{\overline{Q}^t \mid t \in T\}$. We shall introduce the resolvent $\{\overline{G}_\alpha \mid \alpha \in {}^*(-\infty, 0)\}$ of $\overline{\mathcal{E}}(\cdot,\cdot)$ in Sect. 1.3, and characterize the domain $\mathcal{D}(\overline{\mathcal{E}})$ by this resolvent. In Sect. 1.4, we shall define the domain $\mathcal{D}(\mathcal{E})$ of $\mathcal{E}(\cdot,\cdot)$ by its resolvent $\{G_\alpha \mid \alpha \in {}^*(-\infty, 0)\}$; under the hyperfinite weak sector condition, we shall show that $\mathcal{D}(\mathcal{E}) = \mathcal{D}(\overline{\mathcal{E}})$. In Sect. 1.5, we shall introduce hyperfinite Dirichlet forms and related Markov chains. For standard coercive forms, we shall construct their nonstandard representation in Sect. 1.6.

1.2 Domain of the Symmetric Part

In this section, we shall define the domain $\mathcal{D}(\overline{\mathcal{E}})$ of the symmetric part $\overline{\mathcal{E}}(\cdot,\cdot)$ for a hyperfinite nonnegative quadratic form $\mathcal{E}(\cdot,\cdot)$. Before giving a strict definition (Definition 1.2.1), we shall mention an intuitive description. At first, let $\mathrm{Fin}(H)$ be the set of all elements in H with finite norm. By defining $x \approx y$ if $||x - y|| \approx 0$, we know from Proposition A.5.2 in the Appendix that the space[5]

$$^\circ H = \mathrm{Fin}(H)/\approx$$

[5] $\approx$ stands for differing by an infinitesimal, in the sense of nonstandard analysis, refer to Albeverio et al. [25], Cutland [125], Davis [135], Hurd [216], Hurd and Loeb [217], and Lindstrøm [262].

is a Hilbert space with respect to the inner product $({}^\circ x, {}^\circ y) = \mathrm{st}(\langle x, y\rangle)$, where ${}^\circ x$ denotes the equivalence class of x and $\mathrm{st} : {}^*\mathbb{R} \longrightarrow \mathbb{R}$ is the mapping of standard part[6]. We call $({}^\circ H, (\cdot,\cdot))$ the *hull* of $(H, \langle\cdot,\cdot\rangle)$.

Consider the standard part $\overline{E}(\cdot,\cdot)$ of the nonnegative symmetric quadratic form $\overline{\mathcal{E}}(\cdot,\cdot)$. If $\overline{\mathcal{E}}(\cdot,\cdot)$ is *S-bounded*, i.e., there exists a constant $K \in \mathbb{R}_+$ such that

$$|\overline{\mathcal{E}}(u,v)| \leq K||u||||v|| \qquad \text{for all } u, v \in H,$$

we can simply define $\overline{E}(\cdot,\cdot)$ by

$$\overline{E}({}^\circ u, {}^\circ v) = {}^\circ\overline{\mathcal{E}}(u,v).$$

If $\overline{\mathcal{E}}(\cdot,\cdot)$ is not S-bounded, we shall meet two difficulties. We no longer have that $\overline{\mathcal{E}}(u,v) \approx \overline{\mathcal{E}}(\tilde{u},\tilde{v})$ whenever $u \approx \tilde{u}$ and $v \approx \tilde{v}$, and there may be elements $v \in \mathrm{Fin}(H)$ such that $\overline{\mathcal{E}}(\tilde{v},\tilde{v})$ is infinite for all $\tilde{v} \approx v$. The latter problem should not surprise us. It is an immediate consequence of the fact that unbounded forms on Hilbert spaces cannot be defined everywhere. We shall solve it by simply letting $\overline{E}({}^\circ u, {}^\circ v)$ be undefined when $\overline{\mathcal{E}}(\tilde{v},\tilde{v})$ is infinite for all $\tilde{v} \in {}^\circ v$. The most natural solution to the first problem may be to define

$$\overline{E}({}^\circ u, {}^\circ u) = \inf\{{}^\circ\overline{\mathcal{E}}(v,v) \mid v \in {}^\circ u\}, \tag{1.2.1}$$

and then extend $\overline{E}(\cdot,\cdot)$ to be a bilinear form by the usual trick

$$\overline{E}({}^\circ u, {}^\circ v) = \frac{1}{2}\left\{\overline{E}({}^\circ u + {}^\circ v, {}^\circ u + {}^\circ v) - \overline{E}({}^\circ u, {}^\circ u) - \overline{E}({}^\circ v, {}^\circ v)\right\}.$$

The disadvantage of this approach is that it gives us little understanding of how the infimum in (1.2.1) is obtained. For an easier access to the regularity properties of $\overline{\mathcal{E}}(\cdot,\cdot)$ and $\overline{E}(\cdot,\cdot)$, we prefer a more indirect way of attack. Our plan is to define a subset $\mathcal{D}(\overline{\mathcal{E}})$ of Fin(H) – we call it the *domain of* $\overline{\mathcal{E}}(\cdot,\cdot)$ – satisfying

$$\text{if} \quad {}^\circ\overline{\mathcal{E}}(u,u) < \infty, \text{there is a} \quad v \in \mathcal{D}(\overline{\mathcal{E}}) \text{ such that } v \approx u, \tag{1.2.2}$$

$$\text{if } u, v \in \mathcal{D}(\overline{\mathcal{E}}) \text{ and } u \approx v, \text{ then } {}^\circ\overline{\mathcal{E}}(u,u) = {}^\circ\overline{\mathcal{E}}(v,v) < \infty. \tag{1.2.3}$$

We then define $\overline{E}(\cdot,\cdot)$ by

$$\overline{E}({}^\circ u, {}^\circ u) = {}^\circ\overline{\mathcal{E}}(v,v), \tag{1.2.4}$$

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[6] Refer to Albeverio et al. [25].

when $v \in \mathcal{D}(\overline{\mathcal{E}}) \cap {}^\circ u$. It turns out that the two definitions (1.2.1) and (1.2.4) agree (see Proposition 1.2.4).

If we look at the standard nonsymmetric Dirichlet theory, see Albeverio et al. [9], Kim [241] and Ma and Röckner [270], the domain of a quadratic form is given from the very beginning. After that, the authors such as those of Ma and Röckner [270] introduced the symmetric and anti-symmetric parts (see page 15, [270]). This method makes the domains of the quadratic form and its symmetric part coincide. On the other hand, Albeverio et al. [25] has given us a very nice definition of domain for the symmetric hyperfinite quadratic forms by their semigroups. Therefore, we may define the domain $\mathcal{D}(\overline{\mathcal{E}})$ of $\overline{\mathcal{E}}(\cdot,\cdot)$ via the semigroup of $\{\overline{Q}^t \mid t \in T\}$. In the next section, we shall discuss the property of the resolvent $\{\overline{G}_\alpha \mid \alpha \in {}^*(-\infty, 0)\}$ of $\overline{\mathcal{E}}(\cdot,\cdot)$. We can define the domain of $\mathcal{D}(\overline{\mathcal{E}})$ through $\{\overline{G}_\alpha \mid \alpha \in {}^*(-\infty, 0)\}$.

Now it is very natural to ask: can we as well define the domain $\mathcal{D}(\mathcal{E})$ of $\mathcal{E}(\cdot,\cdot)$ directly from $\{Q^t \mid t \in T\}$? Here we would mention that it seems not easy to do the job. In Sect. 1.4, we shall define $\mathcal{D}(\mathcal{E})$ by means of the resolvent $\{G_\alpha \mid \alpha < 0\}$ of $\mathcal{E}(\cdot,\cdot)$. Under the hypothesis of weak sector condition, we shall prove $\mathcal{D}(\overline{\mathcal{E}}) = \mathcal{D}(\mathcal{E})$ by showing that the two definitions satisfy (1.2.1). This is similar to the procedure in the standard nonsymmetric Dirichlet space theory, see, e.g., Albeverio et al. [9], Albeverio et al. [47], Albeverio and Ugolini [57], Kim [241], and Ma and Röckner [270].

Notice that even when $\overline{\mathcal{E}}(\cdot,\cdot)$ is not S-bounded, $\overline{\mathcal{E}}^{(t)}(\cdot,\cdot)$ is S-bounded for all non-infinitesimal t. One of the motivations behind our definition of the domain $\mathcal{D}(\overline{\mathcal{E}})$ is that we want to single out the elements where $\overline{\mathcal{E}}(\cdot,\cdot)$ is really approximated by the bounded forms $\overline{\mathcal{E}}^{(t)}(\cdot,\cdot), t \not\approx 0$, i.e., those $u \in H$ such that

$$ {}^\circ\overline{\mathcal{E}}(u,u) = \lim_{\substack{t\downarrow 0 \\ t \not\approx 0}} {}^\circ\overline{\mathcal{E}}^{(t)}(u,u). \tag{1.2.5} $$

We could have taken this to be our definition of $\mathcal{D}(\overline{\mathcal{E}})$, but for technical and expository reasons we have chosen another one which we shall soon show to be equivalent to (1.2.5) (see Proposition 1.2.2).

Definition 1.2.1. Let $\mathcal{E}(\cdot,\cdot)$ be a nonnegative quadratic form on a hyperfinite dimensional linear space H. The domain $\mathcal{D}(\overline{\mathcal{E}})$ of the symmetric part of $\mathcal{E}(\cdot,\cdot)$ is the set of all $u \in H$ satisfying

(i) ${}^\circ\mathcal{E}_1(u,u) = {}^\circ\overline{\mathcal{E}}_1(u,u) < \infty$.
(ii) For all $t \approx 0$, $\overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u) \approx \overline{\mathcal{E}}(u,u)$.

Let us try to convey the intuition behind this definition. Thinking of $\overline{A}$ as a differential operator, the elements of $\mathcal{D}(\overline{\mathcal{E}})$ are "smooth" functions and

$\overline{Q}^t$ is a "smoothing" operator often given by an integral kernel. If an element u is already smooth, then an infinitesimal amount of smoothing $\overline{Q}^t, t \approx 0$, should not change it noticeably, and hence $\overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u) \approx \overline{\mathcal{E}}(u, u)$. We shall give a partial justification of this rather crude image later, when we show that if ${}^\circ\overline{\mathcal{E}}_1(u, u) < \infty$, then the "smoothed" elements $\overline{Q}^t u, t \not\approx 0$, are all in $\mathcal{D}(\overline{\mathcal{E}})$ (Lemma 1.2.3, see also Corollary 1.2.3).

Our first task will be to establish a list of alternative definitions of $\mathcal{D}(\overline{\mathcal{E}})$, among them (1.2.5). We begin with the following simple identity giving the relationship between $\mathcal{E}(\cdot, \cdot)$ and $\mathcal{E}^{(t)}(\cdot, \cdot)$, and also the relationship between $\overline{\mathcal{E}}(\cdot, \cdot)$ and $\overline{\mathcal{E}}^{(t)}(\cdot, \cdot)$:

Lemma 1.2.1. *For all $u \in H$ and $t \in T$, we have*

(i) $\mathcal{E}^{(t)}(u, u) \geq 0$ *and* $\overline{\mathcal{E}}^{(t)}(u, u) \geq 0$,

(ii) $$\mathcal{E}^{(t)}(u, u) = \frac{\Delta t}{t} \sum_{0 \leq s < t} \mathcal{E}(Q^s u, u) = \frac{\Delta t}{t} \sum_{0 \leq s < t} \mathcal{E}(u, \hat{Q}^s u),$$

(iii) $$\overline{\mathcal{E}}^{(t)}(u, u) = \frac{\Delta t}{t} \sum_{0 \leq s < t} \overline{\mathcal{E}}(\overline{Q}^s u, u) = \frac{\Delta t}{t} \sum_{0 \leq s < t} \overline{\mathcal{E}}(\overline{Q}^{s/2} u, \overline{Q}^{s/2} u).$$

Proof. (i) We have

$$\begin{aligned}
\mathcal{E}^{(t)}(u, u) &= \frac{1}{t} \langle (I - Q^t) u, u \rangle \\
&= \frac{1}{t} \big(\langle u, u \rangle - \langle Q^t u, u \rangle \big) \\
&\geq \frac{1}{t} \big(\langle u, u \rangle - ||Q^t|| \langle u, u \rangle \big) \\
&\geq 0,
\end{aligned}$$

since $||Q^t|| \leq ||Q^{\Delta t}||^{\frac{t}{\Delta t}} \leq 1$. It is then easy to see that $\overline{\mathcal{E}}^{(t)}(u, u) \geq 0$.

(ii) By an easy calculation, we have

$$\begin{aligned}
\mathcal{E}^{(t)}(u, u) &= \frac{1}{t} \langle (I - Q^t) u, u \rangle \\
&= \frac{1}{t} \sum_{0 \leq s < t} \langle (Q^s - Q^{s + \Delta t}) u, u \rangle \\
&= \frac{\Delta t}{t} \sum_{0 \leq s < t} \mathcal{E}(Q^s u, u) \\
&= \frac{\Delta t}{t} \sum_{0 \leq s < t} \mathcal{E}(u, \hat{Q}^s u). \qquad (1.2.6)
\end{aligned}$$

(iii) In the same way as (1.2.6), we can prove that the first equation holds. The second one is due to the symmetry and the semigroup property of $\overline{Q}^s$. □

Among other things, Lemma 1.2.1 tells us that $\mathcal{E}^{(t)}(\cdot,\cdot)$ and $\overline{\mathcal{E}}^{(t)}(\cdot,\cdot)$ are nonnegative.

Lemma 1.2.2. *Let $B, C : H \longrightarrow H$ be nonnegative, symmetric operators commuting with $\overline{A}$ and each other. Then the functions*

$$t \mapsto \langle \overline{Q}^t Bu, Cu\rangle \quad \textit{and} \quad t \mapsto \overline{\mathcal{E}}^{(s)}(\overline{Q}^t Bu, Cu)$$

are nonnegative and decreasing for all $u \in H$ and $s \in T$.

Proof. We first notice that the $\overline{\mathcal{E}}^{(s)}(\cdot,\cdot)$ part follows from the other one since

$$\overline{\mathcal{E}}^{(s)}(\overline{Q}^t Bu, Cu) = \frac{1}{s}\langle \overline{Q}^t (I - \overline{Q}^s)Bu, Cu\rangle,$$

and the operator $B' = (I - \overline{Q}^s)B$ is nonnegative and commutes with $\overline{A}$ and C. If $t > r$, then

$$\begin{aligned}\langle \overline{Q}^r Bu, Cu\rangle - \langle \overline{Q}^t Bu, Cu\rangle &= \langle (I - \overline{Q}^{t-r})\overline{Q}^r Bu, Cu\rangle \\ &= (t-r)\overline{\mathcal{E}}^{(t-r)}(\overline{Q}^{r/2}B^{1/2}C^{1/2}u, \overline{Q}^{r/2}B^{1/2}C^{1/2}u) \\ &\geq 0,\end{aligned}$$

where we used that $\overline{\mathcal{E}}^{(t-r)}(\cdot,\cdot)$ is nonnegative. Hence, $t \longrightarrow \langle \overline{Q}^t Bu, Cu\rangle$ decreases. For the positivity, we observe that

$$\begin{aligned}\langle \overline{Q}^t Bu, Cu\rangle &= \langle \overline{Q}^{t/2}B^{1/2}C^{1/2}u, \overline{Q}^{t/2}B^{1/2}C^{1/2}u\rangle \\ &\geq 0.\end{aligned}$$

□

From Lemma 1.2.2 we may now obtain our main inequalities.

Proposition 1.2.1. *For all $u \in H, t \in T$:*

(i) $0 \leq \overline{\mathcal{E}}(u, u - \overline{Q}^t u) \leq \overline{\mathcal{E}}(u,u) - \overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u) \leq 2\overline{\mathcal{E}}(u, u - \overline{Q}^t u)$.

(ii) $0 \leq \overline{\mathcal{E}}(\overline{Q}^{\Delta t} u, \overline{Q}^{\Delta t} u) - \overline{\mathcal{E}}(\overline{Q}^{2\Delta t} u, \overline{Q}^{2\Delta t} u) \leq \overline{\mathcal{E}}(u,u) - \overline{\mathcal{E}}(\overline{Q}^{\Delta t} u, \overline{Q}^{\Delta t} u)$.

Proof. By trivial algebra, we have

$$\overline{\mathcal{E}}(u,u) - \overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u) = \overline{\mathcal{E}}(u, u - \overline{Q}^t u) + \overline{\mathcal{E}}(\overline{Q}^t u, u - \overline{Q}^t u).$$

Applying Lemma 1.2.2 with $B = I, C = I - \overline{Q}^t$, we see that

$$0 \le \overline{\mathcal{E}}(\overline{Q}^t u, u - \overline{Q}^t u) \le \overline{\mathcal{E}}(u, u - \overline{Q}^t u),$$

and part (i) follows.

(ii) The non-negativity is immediate from (i), and as above we have

$$\overline{\mathcal{E}}(u,u) - \overline{\mathcal{E}}(\overline{Q}^{\Delta t} u, \overline{Q}^{\Delta t} u) = \overline{\mathcal{E}}(u, u - \overline{Q}^{\Delta t} u) + \overline{\mathcal{E}}(\overline{Q}^{\Delta t} u, u - \overline{Q}^{\Delta t} u).$$

Applying Lemma 1.2.2 to each of the latter two terms, using $B = I, C = I - \overline{Q}^{\Delta t}$ in the first case, and $B = \overline{Q}^{\Delta t}, C = I - \overline{Q}^{\Delta t}$ in the second, we get

$$\begin{aligned}
\overline{\mathcal{E}}(u,u) - \overline{\mathcal{E}}(\overline{Q}^{\Delta t} u, \overline{Q}^{\Delta t} u) &\ge \overline{\mathcal{E}}(\overline{Q}^{2\Delta t} u, u - \overline{Q}^{\Delta t} u) + \overline{\mathcal{E}}(\overline{Q}^{3\Delta t} u, u - \overline{Q}^{\Delta t} u) \\
&= \overline{\mathcal{E}}(\overline{Q}^{2\Delta t} u, u) - \overline{\mathcal{E}}(\overline{Q}^{2\Delta t} u, \overline{Q}^{\Delta t} u) \\
&\quad + \overline{\mathcal{E}}(\overline{Q}^{3\Delta t} u, u) - \overline{\mathcal{E}}(\overline{Q}^{3\Delta t} u, \overline{Q}^{\Delta t} u) \\
&= \overline{\mathcal{E}}(\overline{Q}^{\Delta t} u, \overline{Q}^{\Delta t} u) - \overline{\mathcal{E}}(\overline{Q}^{2\Delta t} u, \overline{Q}^{2\Delta t} u).
\end{aligned}$$

The proposition is proved. □

The inequalities above are what we need to establish a reasonable characterization of $\mathcal{D}(\overline{\mathcal{E}})$. We first give our promised list of alternative definitions of the domain of $\overline{\mathcal{E}}(\cdot,\cdot)$.

Proposition 1.2.2. *The following statements are equivalent:*

(i) u *is in the domain* $\mathcal{D}(\overline{\mathcal{E}})$ *of* $\mathcal{E}(\cdot,\cdot)$.
(ii) $^\circ\mathcal{E}_1(u,u) = {}^\circ\overline{\mathcal{E}}_1(u,u) < \infty$, *and for all* $t \approx 0$, *we have* $\overline{\mathcal{E}}(u, u - \overline{Q}^t u) \approx 0$.
(iii) $^\circ\mathcal{E}_1(u,u) < \infty$, *and for all* $t \approx 0$, *we have* $\overline{\mathcal{E}}(u - \overline{Q}^t u, u - \overline{Q}^t u) \approx 0$.
(iv) $^\circ\mathcal{E}_1(u,u) < \infty$, *and for all* $t \approx 0$, *we have* $\overline{\mathcal{E}}^{(t)}(u,u) \approx \overline{\mathcal{E}}(u,u)$.

Proof. $(i) \iff (ii)$. Follows immediately from Proposition 1.2.1 (i).

$(ii) \implies (iii)$. We have

$$0 \le \overline{\mathcal{E}}(u - \overline{Q}^t u, u - \overline{Q}^t u) = \overline{\mathcal{E}}(u, u - \overline{Q}^t u) - \overline{\mathcal{E}}(\overline{Q}^t u, u - \overline{Q}^t u),$$

and by Lemma 1.2.2 the term $\overline{\mathcal{E}}(\overline{Q}^t u, u - \overline{Q}^t u)$ is positive.

$(iii) \implies (i)$. We recall that $|u|_0 = \sqrt{\overline{\mathcal{E}}(u,u)}$ is a semi-norm. By Lemma 1.2.2 and the triangle inequality, we have

$$0 \le |u|_0 - |\overline{Q}^t u|_0 \le |u - \overline{Q}^t u|_0.$$

Multiplying both sides by $|u|_0 + |\overline{Q}^t u|_0$, we get

$$0 \le |u|_0^2 - |\overline{Q}^t u|_0^2 \le |u - \overline{Q}^t u|_0(|u|_0 + |\overline{Q}^t u|_0) \le 2|u|_0|u - \overline{Q}^t u|_0.$$

Hence if ${}^{\circ}\mathcal{E}(u,u) < \infty$ and $\overline{\mathcal{E}}(u - \overline{Q}^t u, u - \overline{Q}^t u) \approx 0$, we have that

$$\overline{\mathcal{E}}(u,u) - \overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u) \approx 0.$$

$(ii) \implies (iv)$. Follows at once from Lemma 1.2.1.

$(iv) \implies (ii)$. Follows from Lemma 1.2.1 and the fact that $s \mapsto \overline{\mathcal{E}}(\overline{Q}^s u, u)$ is decreasing. □

The characterizations of $\mathcal{D}(\overline{\mathcal{E}})$ given in the Proposition 1.2.2 are useful for different purposes. As an illustration, we use Proposition 1.2.2 (iii) to prove that the domain has the right linear structure.

Corollary 1.2.1. *Let $u, v \in \mathcal{D}(\overline{\mathcal{E}})$, and assume that $\alpha \in {}^*\mathbb{R}$ is a nearstandard[7] number. Then αu and $u + v$ are elements of $\mathcal{D}(\overline{\mathcal{E}})$.*

Proof. The αu part is trivial. For $u+v$ we use Proposition 1.2.2 (iii) and the triangle inequality.

$$\begin{aligned} ||(u+v) - \overline{Q}^t(u+v)||_0 &= ||u - \overline{Q}^t u + v - \overline{Q}^t v||_0 \\ &\le ||u - \overline{Q}^t u||_0 + ||v - \overline{Q}^t v||_0. \end{aligned}$$

The latter two terms above are infinitesimals when $t \approx 0$. □

Corollary 1.2.2. *For any infinitesimal $\delta \in T$, we have $\mathcal{D}(\overline{\mathcal{E}}) \subset \mathcal{D}(\overline{\mathcal{E}}^{(\delta)})$.*

Proof. Let $u \in \mathcal{D}(\overline{\mathcal{E}})$. By Proposition 1.2.2 (iv), we know $\overline{\mathcal{E}}^{(k\delta)}(u,u) \approx \overline{\mathcal{E}}(u,u) \approx \overline{\mathcal{E}}^{(\delta)}(u,u)$ for all k such that $k\delta \approx 0$. By Proposition 1.2.2 (iv) again, we get $u \in \mathcal{D}(\overline{\mathcal{E}}^{(\delta)})$. □

The second part of Proposition 1.2.1 informs us that $\overline{Q}^t u$ is more likely to be in $\mathcal{D}(\overline{\mathcal{E}})$ than u is. The next lemma pins this down more precisely.

Lemma 1.2.3. *Assume ${}^{\circ}\mathcal{E}_1(u,u) < \infty$. Then for all non-infinitesimals t, we have $\overline{Q}^t u \in \mathcal{D}(\overline{\mathcal{E}})$.*

Proof. By Proposition 1.2.1, we have

$${}^{\circ}\overline{\mathcal{E}}_1(\overline{Q}^t u, \overline{Q}^t u) \le {}^{\circ}\overline{\mathcal{E}}_1(u,u) < \infty.$$

[7] See Appendix and Albeverio et al. [25] for the concept of nearstandard.

To prove that Definition 1.2.1 (ii) is satisfied, we notice that according to Proposition 1.2.1 (ii), the function

$$t \mapsto \overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u)$$

is decreasing and convex, and hence

$$\frac{1}{s}\Big(\overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u) - \overline{\mathcal{E}}(\overline{Q}^{t+s} u, \overline{Q}^{t+s} u)\Big) \leq \frac{1}{t}\Big(\overline{\mathcal{E}}(u,u) - \overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u)\Big)$$

for all $s > 0$. Multiplying through by s, we get

$$0 \leq \overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u) - \overline{\mathcal{E}}(\overline{Q}^{t+s} u, \overline{Q}^{t+s} u) \leq \frac{s}{t}\Big(\overline{\mathcal{E}}(u,u) - \overline{\mathcal{E}}(\overline{Q}^s u, \overline{Q}^s u)\Big).$$

For $s \approx 0$ and $t \not\approx 0$, the expression on the right is infinitesimal, and the lemma follows. □

We shall now strengthen the lemma above and show that if ${}^\circ\mathcal{E}_1(u,u) < \infty$, then there is an infinitesimal t such that $\overline{Q}^t u \in \mathcal{D}(\overline{\mathcal{E}})$. This is a special case of our next result. First we need to introduce a new definition. A subset F of H is called $\overline{\mathcal{E}}$*-closed* if and only if for all sequences $\{u_n\}_{n\in\mathbb{N}}$ of elements from F such that ${}^\circ|u_n - u_m|_1 \longrightarrow 0$ as $n, m \longrightarrow \infty$, there exists an element u in F such that ${}^\circ|u_n - u|_1 \longrightarrow 0$ as $n \longrightarrow \infty$.

Proposition 1.2.3. *$\mathcal{D}(\overline{\mathcal{E}})$ is $\overline{\mathcal{E}}$-closed. Moreover, if $\{u_n\}_{n\in\mathbb{N}}$ is a $|\cdot|_1$ Cauchy sequence from $\mathcal{D}(\overline{\mathcal{E}})$, and $\{u_n \mid n \in {}^*\mathbb{N}\}$ is an internal extension, then there is a $\gamma \in {}^*\mathbb{N} - \mathbb{N}$ such that $u_\eta \in \mathcal{D}(\overline{\mathcal{E}})$ for all $\eta \leq \gamma$.*

Proof. Let $\{u_n | n \in \mathbb{N}\}$ be a $|\cdot|_1$ Cauchy sequence from $\mathcal{D}(\overline{\mathcal{E}})$, and let $\{u_n | n \in {}^*\mathbb{N}\}$ be an internal extension of it. There is an element $\gamma \in {}^*\mathbb{N} - \mathbb{N}$ such that $|u_n - u_m|_1 \approx 0$ whenever n and m are infinite and less than γ. Let $\eta \in {}^*\mathbb{N} - \mathbb{N}, \eta \leq \gamma$. By the choice of γ, ${}^\circ\overline{\mathcal{E}}_1(u_\eta, u_\eta) < \infty$ and ${}^\circ|u_n - u_\eta|_1 \longrightarrow 0$ as n approaches infinity in $\mathbb{N}$. All that remains is to prove that $u_\eta \in \mathcal{D}(\overline{\mathcal{E}})$.

Assume not, then by Proposition 1.2.2 (iii) there is an $\varepsilon \in \mathbb{R}_+$ and $t \approx 0$ such that

$$|u_\eta - \overline{Q}^t u_\eta|_0 > \varepsilon.$$

Choose $m \in \mathbb{N}$ so large that

$$|u_\eta - u_m|_0 < \frac{\varepsilon}{4}.$$

Then by Proposition 1.2.1 (i), we have

$$|\overline{Q}^t u_\eta - \overline{Q}^t u_m|_0 < \frac{\varepsilon}{4}.$$

Combining the inequalities above, we have

$$\begin{aligned}\varepsilon < |u_\eta - \overline{Q}^t u_\eta|_0 &\leq |u_\eta - u_m|_0 + |u_m - \overline{Q}^t u_m|_0 + |\overline{Q}^t u_m - \overline{Q}^t u_\eta|_0 \\ &\leq \varepsilon/2 + |u_m - \overline{Q}^t u_m|_0,\end{aligned}$$

but since $u_m \in \mathcal{D}(\overline{\mathcal{E}})$, the last term is infinitesimal by Proposition 1.2.2 (iii). We have the contradiction we wanted. □

Corollary 1.2.3. *If* $^\circ\mathcal{E}_1(u,u) < \infty$, *there is a* $t_0 \approx 0$ *such that* $\overline{Q}^t u \in \mathcal{D}(\overline{\mathcal{E}})$ *for all* $t \geq t_0$.

Proof. First we notice that if $\overline{Q}^{t_0} u \in \mathcal{D}(\overline{\mathcal{E}})$, so is $\overline{Q}^t u$ for all $t > t_0$. Put $u_n = \overline{Q}^{\frac{1}{n}} u$. Then the sequence $\{|u_n|_1\}$ is increasing and bounded by $|u|_1$, and we can apply Proposition 1.2.3 to it. The corollary follows. □

Corollary 1.2.4. *If* $^\circ\mathcal{E}_1(u,u) < \infty$, *there is a* $\delta_u \approx 0$ *such that* $u \in \mathcal{D}(\overline{\mathcal{E}}^{(\delta)})$ *for all infinitesimal* $\delta \geq \delta_u$.

Proof. By Corollary 1.2.3, there is a $t_0 \approx 0$ such that $\overline{Q}^t u \in \mathcal{D}(\overline{\mathcal{E}})$ for all $t \geq t_0$. Let δ_u be an infinitesimal such that $\delta_u > t_0$ and $t_0/\delta_u \approx 0$. For all infinitesimal $\delta \geq \delta_u$, we have $v = \overline{Q}^\delta u \in \mathcal{D}(\overline{\mathcal{E}}) \subset \mathcal{D}(\overline{\mathcal{E}}^{(\delta)})$ by Corollaries 1.2.2 and 1.2.3. For all $k \in {}^*\mathbb{N}$ such that $k\delta \approx 0$, we have the following

$$\begin{aligned}\overline{\mathcal{E}}^{(\delta)}(\overline{Q}^{k\delta} u, \overline{Q}^{k\delta} u) &= \overline{\mathcal{E}}^{(\delta)}(\overline{Q}^{(k-1)\delta} v, \overline{Q}^{(k-1)\delta} v) \\ &\approx \overline{\mathcal{E}}^{(\delta)}(v,v) \\ &= \overline{\mathcal{E}}^{(\delta)}(\overline{Q}^\delta u, \overline{Q}^\delta u).\end{aligned} \tag{1.2.7}$$

By Lemma 1.2.1 (iii), we have

$$\begin{aligned}\overline{\mathcal{E}}^{(\delta)}(\overline{Q}^\delta u, \overline{Q}^\delta u) &= \frac{\Delta t}{\delta} \sum_{0 \leq s < \delta} \overline{\mathcal{E}}(\overline{Q}^{s/2} \overline{Q}^\delta u, \overline{Q}^{s/2} \overline{Q}^\delta u) \\ &= \frac{\Delta t}{\delta} \sum_{0 \leq s < \delta} \overline{\mathcal{E}}(\overline{Q}^{s/2+\delta-t_0} \overline{Q}^{t_0} u, \overline{Q}^{s/2+\delta-t_0} \overline{Q}^{t_0} u) \\ &\approx \overline{\mathcal{E}}(\overline{Q}^{t_0} u, \overline{Q}^{t_0} u),\end{aligned} \tag{1.2.8}$$

because $\overline{Q}^{t_0} u \in \mathcal{D}(\overline{\mathcal{E}})$. By Lemma 1.2.1 (iii) again, we have

$$\begin{aligned}\overline{\mathcal{E}}^{(\delta)}(u,u) &= \frac{\Delta t}{\delta} \sum_{0 \leq s < \delta} \overline{\mathcal{E}}(\overline{Q}^{s/2} u, \overline{Q}^{s/2} u) \\ &= \frac{\Delta t}{\delta} \left[\sum_{0 \leq s < 2t_0} \overline{\mathcal{E}}(\overline{Q}^{s/2} u, \overline{Q}^{s/2} u) + \sum_{2t_0 \leq s < \delta} \overline{\mathcal{E}}(\overline{Q}^{s/2} u, \overline{Q}^{s/2} u) \right]\end{aligned}$$

$$\approx \frac{\Delta t}{\delta} \sum_{2t_0 \le s < \delta} \overline{\mathcal{E}}(\overline{Q}^{s/2-t_0}\overline{Q}^{t_0}u, \overline{Q}^{s/2-t_0}\overline{Q}^{t_0}u)$$
$$\approx \overline{\mathcal{E}}(\overline{Q}^{t_0}u, \overline{Q}^{t_0}u). \tag{1.2.9}$$

By relations (1.2.7), (1.2.8), and (1.2.9), we know $u \in \mathcal{D}(\overline{\mathcal{E}}^{(\delta)})$. □

Remark 1.2.1. Proposition 1.2.3 is rather surprising since there exist standard forms which are neither closed nor closable. In fact, there are numerous applications where the main difficulty is to show that the form constructed is closed, or at least can be extended to a closed form (see, e.g., [11, 16–24, 26, 27, 36, 48, 49, 94, 98, 99, 103, 151, 175, 176, 178, 225, 232, 236, 247, 251, 259, 278, 301, 318, 345, 359]). If we know that a form comes from a hyperfinite form, this follows immediately from Proposition 1.2.3. In Albeverio et al. [25], Chap. 6, we have got various examples of how useful this observation is. For the time being, we only remark that since we shall soon show that all standard, coercive closed forms can be obtained from hyperfinite forms, the method is quite general (we refer to Sect. 1.6 of this chapter).

Notice that if we can show that whenever ${}^\circ\overline{\mathcal{E}}_1(u,u) < \infty$, then for all $t \approx 0$, $||u - \overline{Q}^t u|| \approx 0$, Corollary 1.2.3 will imply the first part of our program, i.e., (1.2.2) above.

Lemma 1.2.4. *Assume* ${}^\circ\overline{\mathcal{E}}(u,u) < \infty$*. Then for all* $t \approx 0$*, we have*

$$||u - \overline{Q}^t u|| \approx 0.$$

Proof. For $t \approx 0$, we have

$$\begin{aligned} ||u - \overline{Q}^t u||^2 &= \langle u - \overline{Q}^t u, u - \overline{Q}^t u\rangle \\ &= t\overline{\mathcal{E}}^{(t)}(u, u - \overline{Q}^t u) \\ &= t\Big[\overline{\mathcal{E}}^{(t)}(u,u) - \overline{\mathcal{E}}^{(t)}(u, \overline{Q}^t u)\Big] \\ &\le t\overline{\mathcal{E}}(u,u) \approx 0. \end{aligned}$$

□

Let us turn our attention to our second main goal (1.2.3).

Lemma 1.2.5. *If* $u, v \in \mathcal{D}(\overline{\mathcal{E}})$ *and* $u \approx v$*, then*

$$\mathcal{E}(u,u) \approx \mathcal{E}(v,v).$$

Proof. It is obviously enough to show that if $u \in \mathcal{D}(\overline{\mathcal{E}})$ and $u \approx 0$, then $\mathcal{E}(u,u) \approx 0$. But if $u \in \mathcal{D}(\overline{\mathcal{E}})$, we know from Proposition 1.2.2 (iv):

$${}^\circ\overline{\mathcal{E}}(u,u) = \lim_{\substack{t\downarrow 0 \\ t \not\approx 0}} {}^\circ\overline{\mathcal{E}}^{(t)}(u,u). \tag{1.2.10}$$

Also

$$\begin{aligned}\overline{\mathcal{E}}^{(t)}(u,u) &= \frac{1}{t}\langle (I-\overline{Q}^t)u,u\rangle \\ &= \frac{1}{t}\Big(\langle u,u\rangle - \langle \overline{Q}^t u,u\rangle\Big) \\ &\le \frac{1}{t}||u||^2,\end{aligned}$$

which is infinitesimal for $t \not\approx 0$. Combining this with (1.2.10), the lemma follows. □

We may now sum up our results on $\mathcal{D}(\overline{\mathcal{E}})$ in one statement.

Theorem 1.2.1. *Let $\mathcal{E}(\cdot,\cdot)$ be a nonnegative quadratic form on a hyperfinite dimensional space H. Then*

(i) If $u,v \in \mathcal{D}(\overline{\mathcal{E}})$ and α is a finite element of ${}^*\mathbb{R}$, then $\alpha u, u+v \in \mathcal{D}(\overline{\mathcal{E}})$.
(ii) $\mathcal{D}(\overline{\mathcal{E}})$ is $\overline{\mathcal{E}}$-closed.
(iii) If ${}^\circ\overline{\mathcal{E}}_1(u,u) < \infty$, then there exists a $v \in \mathcal{D}(\overline{\mathcal{E}})$ with $||u-v|| \approx 0$. Moreover, we have

$$\begin{aligned}{}^\circ\overline{\mathcal{E}}(v,v) &= \lim_{\substack{t\downarrow 0\\ t\not\approx 0}} {}^\circ\mathcal{E}(\overline{Q}^t u, \overline{Q}^t u) \\ &= \lim_{\substack{t\downarrow 0\\ t\not\approx 0}} {}^\circ\overline{\mathcal{E}}^{(t)}(u,u).\end{aligned}$$

(iv) If $u,v \in \mathcal{D}(\overline{\mathcal{E}})$ and $u \approx v$, then $\overline{\mathcal{E}}(u,u) \approx \overline{\mathcal{E}}(v,v)$.
(v) If $u \in \mathcal{D}(\overline{\mathcal{E}})$, then ${}^\circ\overline{\mathcal{E}}(u,u) = \inf\{{}^\circ\overline{\mathcal{E}}(v,v) \mid v \approx u\}$.
(vi) If ${}^\circ\mathcal{E}(u,u) < \infty$ and ${}^\circ\overline{\mathcal{E}}(u,u) = \inf\{{}^\circ\overline{\mathcal{E}}(v,v) \mid v \approx u\}$, then $u \in \mathcal{D}(\overline{\mathcal{E}})$.

Proof. We only need to show (v) and (vi), since we have proved the other results.

(v) Noticing (iv), we have $\mathcal{E}(u,u) \approx 0$ if $u \in \mathcal{D}(\overline{\mathcal{E}})$ and $u \approx 0$. This implies the following for general $u \in \mathcal{D}(\overline{\mathcal{E}})$:

$$\begin{aligned}\inf\{{}^\circ\overline{\mathcal{E}}(v,v) \mid v \approx u\} &\le {}^\circ\overline{\mathcal{E}}(u,u) \\ &\le \inf\left\{\left(\sqrt{{}^\circ\overline{\mathcal{E}}(v,v)} + \sqrt{{}^\circ\overline{\mathcal{E}}(u-v,u-v)}\right)^2 \,\middle|\, v \approx u\right\} \\ &= \inf\{{}^\circ\overline{\mathcal{E}}(v,v) \mid v \approx u\}.\end{aligned}$$

This is (v).

(vi) From Proposition 1.2.1, we know that $t \mapsto \overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u)$ is decreasing. This implies (vi). □

The following definition now makes sense.

Definition 1.2.2. The *symmetric standard part of* $\overline{\mathcal{E}}(\cdot,\cdot)$ is the quadratic form $\overline{E}(\cdot,\cdot)$ on ${}^\circ H$ defined by:

(i) The domain $D(\overline{E})$ of $\overline{E}(\cdot,\cdot)$ is the set of all equivalence classes ${}^\circ u \in {}^\circ H$ such that $\inf\{{}^\circ\overline{\mathcal{E}}_1(v,v) \mid v \in {}^\circ u\} < \infty$.

(ii) If $x, y \in {}^\circ H$ are in the domain of $\overline{E}(\cdot,\cdot)$, let $\overline{E}(x,y) = {}^\circ\overline{\mathcal{E}}(u,v)$, where $u \in x, v \in y$ are in $\mathcal{D}(\overline{\mathcal{E}})$.

For $\alpha \in [0,\infty)$, let us set

$$\overline{E}_\alpha(\cdot,\cdot) = \overline{E}(\cdot,\cdot) + \alpha(\cdot,\cdot).$$

We recall that $(\cdot,\cdot)$ is the inner product of ${}^\circ H$.

An $\overline{E}_1$-*Cauchy sequence* is a sequence $\{x_n\}$ of elements from $D(\overline{E})$ such that $\overline{E}_1(u_n - u_m, u_n - u_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$. We say that $\overline{E}(\cdot,\cdot)$ is *closed* if all $\overline{E}_1$-Cauchy sequences converge in $\overline{E}_1$-norm to an element in $D(\overline{E})$. The next proposition follows immediately from Theorem 1.2.1 and the definition of $\overline{E}(\cdot,\cdot)$.

Proposition 1.2.4. *Let $\overline{E}(\cdot,\cdot)$ be the standard part of $\overline{\mathcal{E}}(\cdot,\cdot)$. Then $\overline{E}(\cdot,\cdot)$ is closed, and for all $x \in {}^\circ H$*

$$\overline{E}(x,x) = \inf\{{}^\circ\overline{\mathcal{E}}(u,u) \mid u \in x\}, \tag{1.2.11}$$

where we take the value ∞ on the right to mean that the expression on the left is undefined.

We point out that (1.2.11) is just our original suggestion (1.2.1) for the standard part of $\overline{\mathcal{E}}(\cdot,\cdot)$. In Sect. 1.4, we shall study the standard part $E(\cdot,\cdot)$ of $\mathcal{E}(\cdot,\cdot)$ under the hyperfinite weak sector condition. We shall show that $D(E) = D(\overline{E})$ and that $\overline{E}(\cdot,\cdot)$ is exactly the *symmetric part of* $E(\cdot,\cdot)$, i.e., $\overline{E}(x,y) = \frac{1}{2}(E(x,y) + E(y,x))$.

1.3 Resolvent of the Symmetric Part

In Sects. 1.1 and 1.2, we have only been interested in the relationship among the form $\mathcal{E}(\cdot,\cdot)$ and the associated semigroup $\{Q^t\}$ and infinitesimal generator A, also $\overline{\mathcal{E}}(\cdot,\cdot)$ and its semigroup $\{\overline{Q}^t\}$ and generator $\overline{A}$, and so on. In this section, we turn our attention to the resolvent $\{\overline{G}_\alpha\}$ of $\overline{\mathcal{E}}(\cdot,\cdot)$. The goal is to give a description of $\overline{\mathcal{E}}(\cdot,\cdot)$ and $\mathcal{D}(\overline{\mathcal{E}})$ in terms of $\{\overline{G}_\alpha\}$, similar to the one we have given using the semigroup. The main result (Theorem 1.3.1) will allow us to reconstruct a form from its resolvent. One may want to notice that the result of Theorem 1.3.1 has played an essential part in the study of singular perturbations of operators in Albeverio et al. [25], Chap. 6.

The operator $\overline{G}_\alpha$ is defined to be $(\overline{A}-\alpha)^{-1}$ whenever this exists. A formal calculation

$$\begin{aligned}\left(\overline{A}-\alpha\right)^{-1} &= \left(I-\left(I-\Delta t(\overline{A}-\alpha)\right)\right)^{-1}\Delta t\\ &= \sum_{k=0}^{\infty}\left(I-\Delta t(\overline{A}-\alpha)\right)^{k}\Delta t\\ &= \sum_{k=0}^{\infty}\left(\overline{Q}^{\Delta t}+\alpha\Delta t\right)^{k}\Delta t \qquad (1.3.1)\end{aligned}$$

tells us that $\overline{G}_\alpha$ will exist if the series on the right hand side converges. Since $\overline{Q}^{\Delta t}$ is a nonnegative and symmetric operator with norm at most one, all its eigenvalues must be between zero and one. We get that the absolute value of all eigenvalues of $\overline{Q}^{\Delta t}+\alpha\Delta t$ must be less than $1+\alpha\Delta t$. Hence if $\alpha<0$ and $|\alpha|\Delta t<2$, the series in (1.3.1) converges, and we get the following proposition.

Proposition 1.3.1. *Let $\mathcal{E}(\cdot,\cdot)$ be a nonnegative quadratic form. Then*

(i) $(\overline{A}-\alpha)^{-1}=\overline{G}_\alpha$ *exists for all* $\alpha\in {}^*(-\infty,0)$*. Moreover, we have* $||\overline{G}_\alpha||\leq \frac{1}{|\alpha|}$ *in operator norm.*
(ii) *For* $\alpha\in {}^*\mathbb{R}, -\frac{1}{\Delta t}<\alpha<0$, *we have*

$$\overline{G}_\alpha=\sum_{k=0}^{\infty}\left(\overline{Q}^{\Delta t}+\alpha\Delta t\right)^{k}\Delta t. \qquad (1.3.2)$$

Proof. (i) We notice that for all $\alpha\in {}^*(-\infty,0)$, $u\in H$

$$\langle(\overline{A}-\alpha)u,u\rangle\geq-\alpha\langle u,u\rangle.$$

This implies $(\overline{A}-\alpha)^{-1}=\overline{G}_\alpha$ exists for all $\alpha\in {}^*(-\infty,0)$ by elementary linear algebra. Furthermore, we have

$$\langle\alpha\overline{G}_\alpha u,\alpha\overline{G}_\alpha u\rangle\leq-\alpha\overline{\mathcal{E}}_{-\alpha}(\overline{G}_\alpha u,\overline{G}_\alpha u)=-\alpha\langle u,\overline{G}_\alpha u\rangle\leq||u||||\alpha\overline{G}_\alpha u||.$$

This implies that $||\overline{G}_\alpha||\leq\frac{1}{|\alpha|}$.

(ii) We have already proved this point before stating our proposition. □

We call $\{\overline{G}_\alpha \mid \alpha\in {}^*(-\infty,0)\}$ the *resolvent of* $\overline{\mathcal{E}}(\cdot,\cdot)$.

In the standard Dirichlet space theory, the formula corresponding to (1.3.2) is

$$\overline{G}_\alpha = \int_0^\infty e^{-t(\overline{A}-\alpha)}\, dt = \int_0^\infty e^{-t\overline{A}} e^{\alpha t}\, dt,$$

giving $\overline{G}_\alpha$ as a weighted sum of the elements $e^{-t\overline{A}}$ in the semigroup. Since $-\int_0^\infty \alpha e^{\alpha t} dt = 1$, it is convenient to multiply this equation by $-\alpha$ to obtain

$$-\alpha\overline{G}_\alpha = -\int_0^\infty \alpha e^{-t\overline{A}} e^{\alpha t}\, dt. \tag{1.3.3}$$

It is not quite obvious that this result carries over to the hyperfinite setting, since the equation $(\overline{Q}^{\Delta t} + \alpha\Delta t)^k = \overline{Q}^{k\Delta t}(1+\alpha\Delta t)^k$ (corresponding to $e^{-t(\overline{A}-\alpha)} = e^{-t\overline{A}} \cdot e^{\alpha t}$) is false. But the next result shows that the two operators are close enough for our purposes.

Lemma 1.3.1. *For $\alpha \in {}^*\mathbb{R}$, $-\frac{1}{\sqrt{\Delta t}} \le \alpha < 0$, and all $u \in H$ with ${}^\circ\mathcal{E}_1(u,u) < \infty$, we have*

$$\left| \alpha\overline{G}_\alpha u - \left(\alpha \sum_{k=0}^\infty \overline{Q}^{k\Delta t} \left(1+\alpha\Delta t\right)^k \Delta t \right) u \right|_1 \approx 0. \tag{1.3.4}$$

Proof. Let $\{e_i \mid 1 \le i \le N\}$ be an orthonormal basis of eigenvectors for $\overline{A}$, and let a_i be the i-th eigenvalue. Defining $b_i = a_i + 1$, we notice that if $u = \sum_{i=1}^N u_i e_i$, then

$$\overline{\mathcal{E}}_1(u,u) = \sum_{i=1}^N b_i u_i^2. \tag{1.3.5}$$

Summing geometric series, we see that

$$\begin{aligned} \alpha\overline{G}_\alpha(e_i) &= \left(\alpha \sum_{k=0}^\infty \left(1 - \Delta t a_i + \Delta t\alpha\right)^k \Delta t \right) e_i \\ &= \frac{\alpha}{a_i - \alpha} e_i \end{aligned}$$

and similarly

$$\left(\alpha \sum_{k=0}^\infty \overline{Q}^{k\Delta t} \left(1+\alpha\Delta t\right)^k \Delta t \right)(e_i) = \frac{\alpha}{a_i - \alpha + a_i\alpha\Delta t} e_i.$$

This yields

$$\alpha\overline{G}_\alpha(u) - \left(\alpha \sum_{k=0}^{\infty} \overline{Q}^{k\Delta t}\Big(1+\alpha\Delta t\Big)^k \Delta t\right)(u)$$
$$= \sum_{i=1}^{N} \frac{a_i \Delta t u_i e_i}{\Big(1-(a_i/\alpha)\Big)\Big(1-(a_i/\alpha)-a_i\Delta t\Big)}.$$

Taking the $|\cdot|_1$-norm of this, we get from (1.3.5)

$$\left|\alpha\overline{G}_\alpha u - \left(\alpha \sum_{k=0}^{\infty} \overline{Q}^{k\Delta t}\Big(1+\alpha\Delta t\Big)^k \Delta t\right) u\right|_1^2$$
$$= \sum_{i=1}^{N} b_i u_i^2 \frac{a_i^2 \Delta t^2}{\Big(1-(a_i/\alpha)\Big)^2\Big(1-(a_i/\alpha)-a_i\Delta t\Big)^2}$$
$$\leq \eta\mathcal{E}_1(u,u),$$

where

$$\eta = \max_{1\leq i\leq N}\left(\frac{a_i^2\Delta t^2}{\Big(1-(a_i/\alpha)\Big)^2\Big(1-(a_i/\alpha)-a_i\Delta t\Big)^2}\right).$$

All that remains is to show that η is infinitesimal. First we observe that since $||\overline{A}||\Delta t \leq 1$ (recalling (1.1.3) in Sect. 1.1), we have $a_i\Delta t \leq 1$ for all i. Hence

$$\eta \leq \max_{1\leq i\leq N}\left\{\frac{a_i^2\Delta t^2}{\Big(1-(a_i/\alpha)\Big)^2 (a_i/\alpha)^2}\right\}$$
$$= \max_{1\leq i\leq N}\left\{\frac{\alpha^4\Delta t^2}{\Big(\alpha-a_i\Big)^2}\right\}.$$

Since $-1/\sqrt{\Delta t} \leq \alpha < 0$, the latter term is infinitesimal, and the proof is finished. □

Equation (1.3.4) is the nonstandard counterpart of (1.3.3). Notice that

$$\sum_{k=0}^{\infty}(-\alpha\Delta t)\Big(1+\alpha\Delta t\Big)^k = 1,$$

and that if α is infinite, then there is $t_\alpha \approx 0$ such that

$$\sum_{0 \le k\Delta t \le t_\alpha} (-\alpha\Delta t)\Big(1+\alpha\Delta t\Big)^k \approx 1. \tag{1.3.6}$$

On the other hand, if α is finite, then

$$\sum_{\substack{t \le k\Delta t \\ 0 \le k < \infty}} (-\alpha\Delta t)\Big(1+\alpha\Delta t\Big)^k \approx 1$$

for all infinitesimal t. We can now begin our description of $\overline{\mathcal{E}}(\cdot,\cdot)$ and $\mathcal{D}(\overline{\mathcal{E}})$ in terms of $\overline{G}_\alpha$.

Lemma 1.3.2. *If* $-\frac{1}{\sqrt{\Delta t}} < \alpha < 0$ *and* $^\circ\mathcal{E}(u,u) < \infty$, *then*

(i) If α *is infinite, we have* $||-\alpha\overline{G}_\alpha u - u|| \approx 0$.
(ii) If α *is finite, we have* $-\alpha\overline{G}_\alpha u \in \mathcal{D}(\overline{\mathcal{E}})$.
(iii) There is an infinite α *such that* $-\alpha\overline{G}_\alpha u \in \mathcal{D}(\overline{\mathcal{E}})$.

Proof. According to Lemma 1.3.1, it suffices to prove the statements we get after replacing $\overline{G}_\alpha$ by

$$\overline{R}_\alpha = \sum_{k=0}^{\infty} \overline{Q}^{k\Delta t}\Big(1+\alpha\Delta t\Big)^k \Delta t.$$

(i) Let $t_\alpha \approx 0$ be as in (1.3.6). Then, we have

$$\begin{aligned} \| -\alpha\overline{R}_\alpha u - u \| &= \left|\left| \sum_{k=0}^{\infty} \Big(\overline{Q}^{k\Delta t} u - u\Big)(-\alpha)\Big(1+\alpha\Delta t\Big)^k \Delta t \right|\right| \\ &\approx \left|\left| \sum_{0 < k\Delta t \le t_\alpha} \Big(\overline{Q}^{k\Delta t} u - u\Big)(-\alpha)\Big(1+\alpha\Delta t\Big)^k \Delta t \right|\right| \\ &\approx 0, \end{aligned}$$

where the last step uses Lemma 1.2.4.

(ii) Choose $t \approx 0$ such that $\overline{Q}^t u \in \mathcal{D}(\overline{\mathcal{E}})$. If $\{e_i\}_{i \le N}$ is an orthonormal basis of eigenvectors for $\overline{A}$, and a_i is the i-th eigenvalue, we have

$$\begin{aligned} \sum_{0 \le k\Delta t < t} \overline{Q}^{k\Delta t}\Big(1+\alpha\Delta t\Big)^k \Delta t e_i &= \sum_{0 \le k\Delta t < t} \Big(1 - a_i\Delta t\Big)^k \Big(1+\alpha\Delta t\Big)^k \Delta t e_i \\ &= \frac{1 - \Big(1+\alpha\Delta t\Big)^{t/\Delta t}\Big(1 - a_i\Delta t\Big)^{t/\Delta t}}{a_i - \alpha + a_i\alpha\Delta t} e_i. \end{aligned}$$

If $u = \sum_{i=1}^N u_i e_i$, we get from (1.3.5)

$$\left| \sum_{0 \leq k\Delta t < t} \overline{Q}^{k\Delta t} u(1+\alpha\Delta t)^k \Delta t \right|_1^2$$
$$= \sum_{i=1}^{N} b_i u_i^2 \left(\frac{1 - \left(1+\alpha\Delta t\right)^{t/\Delta t} \left(1 - a_i\Delta t\right)^{t/\Delta t}}{a_i - \alpha + a_i\alpha\Delta t} \right)^2,$$

where $b_i = a_i + 1$. Since α is finite, it is easy to check that

$$\frac{1 - \left(1+\alpha\Delta t\right)^{t/\Delta t} \left(1 - a_i\Delta t\right)^{t/\Delta t}}{a_i - \alpha + a_i\alpha\Delta t} \approx 0$$

for all i. Hence, we have

$$\left| \sum_{0 \leq k\Delta t < t} \overline{Q}^{k\Delta t} u\left(1+\alpha\Delta t\right)^k \Delta t \right|_1 \approx 0. \tag{1.3.7}$$

We also remark that since $\overline{Q}^{k\Delta t} u \in \mathcal{D}(\overline{\mathcal{E}})$ for all $k\Delta t \geq t$, we must have

$$\sum_{t \leq k\Delta t} \overline{Q}^{k\Delta t} u\left(1+\alpha\Delta t\right)^k \Delta t \in \mathcal{D}(\overline{\mathcal{E}}).$$

But by the relation (1.3.7), we have $|\overline{R}_\alpha u - \sum_{t \leq k\Delta t} \overline{Q}^{k\Delta t} u\left(1+\alpha\Delta t\right)^k \Delta t|_1 \approx 0$. Hence, we have $\overline{R}_\alpha u \in \mathcal{D}(\overline{\mathcal{E}})$.

(iii) We remark that

$$\begin{aligned} \mathcal{E}_1(-\alpha\overline{G}_\alpha u, -\alpha\overline{G}_\alpha u) &= \sum_{i=1}^{N} b_i u_i^2 \frac{\alpha^2}{(a_i - \alpha)^2} \\ &= \sum_{i=1}^{N} b_i u_i^2 \frac{1}{(a_i/\alpha - 1)^2} \end{aligned} \tag{1.3.8}$$

increases as $\alpha \longrightarrow -\infty$, and is bounded by $\mathcal{E}_1(u,u)$. Applying Proposition 1.2.3 to the sequence $u_n = n\overline{G}_{-n}u$, the lemma follows. □

The next proposition adds two new characterizations of $\mathcal{D}(\overline{\mathcal{E}})$ to the list in Proposition 1.2.2

Proposition 1.3.2. *The following statements are equivalent:*

(i) $u \in \mathcal{D}(\overline{\mathcal{E}})$.
(ii) ${}^\circ\overline{\mathcal{E}}_1(u,u) < \infty$ *and* $\lim_{{}^\circ\alpha \longrightarrow -\infty} {}^\circ\overline{\mathcal{E}}_1(u + \alpha\overline{G}_\alpha u, u + \alpha\overline{G}_\alpha u) = 0$.

(iii) ${}^\circ\overline{\mathcal{E}}_1(u,u) = \lim_{{}^\circ\alpha \longrightarrow -\infty} {}^\circ\overline{\mathcal{E}}_1(-\alpha\overline{G}_\alpha u, -\alpha\overline{G}_\alpha u) < \infty$.

Proof. $(i) \Longrightarrow (ii)$. Pick an infinite α such that $-\alpha\overline{G}_\alpha u \in \mathcal{D}(\overline{\mathcal{E}})$. Then $u + \alpha\overline{G}_\alpha u \in \mathcal{D}(\overline{\mathcal{E}}), u + \alpha\overline{G}_\alpha u \approx 0$, and hence $\overline{\mathcal{E}}_1(u + \alpha\overline{G}_\alpha u, u + \alpha\overline{G}_\alpha u) \approx 0$. Part (ii) follows.

$(ii) \Longrightarrow (iii)$. By (1.3.8) and the triangle inequality

$$0 \leq |u|_1 - |-\alpha\overline{G}_\alpha u|_1 \leq |u + \alpha\overline{G}_\alpha u|_1,$$

and multiplying by $|u|_1 + |-\alpha\overline{G}_\alpha u|_1 \leq 2\mathcal{E}_1(u,u)$, we get

$$0 \leq |u|_1^2 - |-\alpha\overline{G}_\alpha u|_1^2 \leq 2\mathcal{E}_1(u,u) \cdot |u + \alpha\overline{G}_\alpha u|_1,$$

which shows that $(ii) \Longrightarrow (iii)$.

$(iii) \Longrightarrow (i)$. Pick an infinite α such that $-\alpha\overline{G}_\alpha u \in \mathcal{D}(\overline{\mathcal{E}})$. Then $\| u + \alpha\overline{G}_\alpha u \| \approx 0$ and $\mathcal{E}_1(u,u) \approx \mathcal{E}_1(-\alpha\overline{G}_\alpha u, -\alpha\overline{G}_\alpha u)$, and hence $u \in \mathcal{D}(\overline{\mathcal{E}})$. □

The following results gives a way of reconstructing a form from its resolvent. In the study of singular perturbations in Albeverio et al. [25], Chap. 6, we have found it much easier to control the resolvent of the perturbed form than the form itself. Once we have a good grasp of the resolvent, Theorem 1.3.1 will give us the form.

Theorem 1.3.1. *Let $\mathcal{E}(\cdot,\cdot)$ be a nonnegative hyperfinite form on H, and let $\overline{E}(\cdot,\cdot)$ be its symmetric standard part. For all $x \in {}^\circ H$ and all $v \in x$, we have*

$$\overline{E}(x,x) = -\lim_{{}^\circ\alpha \longrightarrow -\infty} {}^\circ\Big(\alpha^2\langle\overline{G}_\alpha v, v\rangle + \alpha\langle v, v\rangle\Big). \tag{1.3.9}$$

Proof. Notice that since $\overline{G}_\alpha$ is bounded, it does not matter which $v \in x$ we use. We split the proof into two cases.

(i) *x is not in the domain of $\overline{E}(\cdot,\cdot)$:* Let $\{e_i \mid i \leq N\}$ be an orthonormal basis of eigenvectors for $\overline{A}$, and assume that the corresponding eigenvalues $\{a_i\}_{i\leq N}$ are in decreasing order. Pick $v = \sum_{i\leq N} v_i e_i$ in x. An easy calculation shows that

$$-\Big(\langle\alpha^2\overline{G}_\alpha v, v\rangle + \alpha\langle v, v\rangle\Big) = \sum_{i=1}^{N} a_i v_i^2 \frac{-\alpha}{a_i - \alpha}.$$

Assume for contradiction that the limit in (1.3.9) is finite. Then there is an infinite α such that

$${}^\circ\Big(\sum_{i=1}^{N} a_i v_i^2 \frac{-\alpha}{a_i - \alpha}\Big) < \infty.$$

If H is the largest integer such that $a_H > |\alpha|$, we have

$$\begin{aligned}\sum_{i=1}^{N} a_i v_i^2 \frac{-\alpha}{a_i - \alpha} &= \sum_{i=1}^{H} a_i v_i^2 \frac{-\alpha}{a_i - \alpha} + \sum_{i=H+1}^{N} a_i v_i^2 \frac{-\alpha}{a_i - \alpha} \\ &= -\alpha \sum_{i=1}^{H} v_i^2 \frac{1}{1 - \alpha/a_i} + \sum_{i=H+1}^{N} a_i v_i^2 \frac{1}{1 - a_i/\alpha} \\ &\geq -\frac{\alpha}{2} \sum_{i=1}^{H} v_i^2 + \frac{1}{2} \sum_{i=H+1}^{N} a_i v_i^2 .\end{aligned}$$

Hence, the last two terms $-\frac{\alpha}{2}\sum_{i=1}^{H} v_i^2$ and $\frac{1}{2}\sum_{i=H+1}^{N} a_i v_i^2$ are finite. But if $-(\alpha/2)\sum_{i=1}^{H} v_i^2$ is finite, v is infinitely close to

$$v' = \sum_{i=H+1}^{N} v_i e_i .$$

In addition, if $\frac{1}{2}\sum_{i=H+1}^{N} a_i v_i^2$ is finite, then ${}^{\circ}\mathcal{E}_1(v', v') < \infty$. This contradicts the assumption that $x \notin D(\overline{E})$.

(ii) *Assume that* $x \in D(\overline{E})$: Let $v \in x$ be such that

$$^{\circ}\overline{\mathcal{E}}_1(v, v) < \infty. \tag{1.3.10}$$

Since $\overline{\mathcal{E}}_{-\alpha}(\overline{G}_\alpha u, w) = \langle u, w \rangle$, we have

$$\begin{aligned}\overline{\mathcal{E}}(-\alpha\overline{G}_\alpha v, -\alpha\overline{G}_\alpha v) &= \overline{\mathcal{E}}_{-\alpha}(\alpha\overline{G}_\alpha v, \alpha\overline{G}_\alpha v) + \alpha\langle \alpha\overline{G}_\alpha v, \alpha\overline{G}_\alpha v\rangle \\ &= \alpha^2\langle \overline{G}_\alpha v, v\rangle + \alpha^3\langle \overline{G}_\alpha v, \overline{G}_\alpha v\rangle.\end{aligned}$$

The theorem will follow from Proposition 1.3.2 (iii) if we can prove that for all v satisfying the condition (1.3.10),

$$\lim_{^{\circ}\alpha\to\infty} {}^{\circ}\Big(\alpha^2\langle \overline{G}_\alpha v, v\rangle + \alpha^3\langle \overline{G}_\alpha v, \overline{G}_\alpha v\rangle + \alpha^2\langle \overline{G}_\alpha v, v\rangle + \alpha\langle v, v\rangle\Big) = 0.$$

By simple algebra, this is the same as

$$\lim_{^{\circ}\alpha\to-\infty} {}^{\circ}\Big(\alpha||\alpha\overline{G}_\alpha v + v||^2\Big) = 0.$$

Pulling α inside the norm and reformulating the problem in nonstandard terms, we see that what we have to prove is

$$\left\| |\alpha|^{3/2}\overline{G}_\alpha v - |\alpha|^{\frac{1}{2}} v \right\|^2 \approx 0 \tag{1.3.11}$$

for all infinite, negative α of sufficiently small absolute value.

If $v = \sum_{i=1}^N v_i e_i$ is the eigenvector expansion of v, we see that

$$\begin{aligned} \left\| |\alpha|^{3/2}\overline{G}_\alpha v - |\alpha|^{1/2} v \right\|^2 &= \sum_{i=1}^N \Big(\frac{|\alpha|^{3/2}}{a_i - \alpha} - |\alpha|^{1/2} \Big)^2 v_i^2 \\ &= \sum_{i=1}^N \frac{-\alpha a_i^2}{(a_i-\alpha)^2} v_i^2 \\ &= \sum_{i=1}^N a_i v_i^2 \Big(\frac{1}{\beta_i + \beta_i^{-1} + 2} \Big), \end{aligned} \tag{1.3.12}$$

where $\beta_i = -\alpha / a_i$.

Notice that if a_i is infinitesimal compared to α or α is infinitesimal compared to a_i, then $1/(\beta_i + \beta_i^{-1} + 2)$ is infinitesimal. To get the sum on the right hand side of (1.3.12) to be infinitesimal, we only have to choose α such that the contributions from the terms satisfying neither of these requirements are infinitesimal.

Assuming that the eigenvalues $\{a_i\}$ are given in descending order, we define

$$\gamma = \sup \left\{ {}^\circ\Big(\sum_{i=1}^k a_i v_i^2 \Big) \,\Big|\, a_k \text{ is infinite} \right\}.$$

Since ${}^\circ(\sum_{i=1}^k a_i v_i^2) \leq {}^\circ\mathcal{E}(v,v)$ is finite by (1.3.10), γ is a real number.

Using saturation[8] on the sets

$$A_n = \left\{ j \in {}^*\mathbb{N} \,\Big|\, \sum_{i=1}^j a_i v_i^2 > \gamma - \frac{1}{n} \quad \text{and} \quad a_j > n \right\},$$

we find a hyperinteger K such that a_K is infinite and

$$\sum_{i=1}^K a_i v_i^2 \approx \gamma.$$

[8] Saturation is also a term of nonstandard analysis, we refer to Albeverio et al. [25].

We choose $|\alpha|$ to be infinitely large, but infinitesimal compared to a_K.

For each $\varepsilon \in \mathbb{R}_+$, let

$$M_\varepsilon = \inf\left\{ k \,\middle|\, \sum_{i=1}^{k} a_i v_i^2 \geq \gamma + \varepsilon \right\}.$$

By our choice of γ, the term a_{M_ε} must be finite. But

$$\sum_{i=1}^{n} a_i v_i^2 \frac{1}{\beta_i + \beta_i^{-1} + 2} \leq \sum_{i=1}^{K} a_i v_i^2 \frac{1}{\beta_i + \beta_i^{-1} + 2} + \sum_{i=K+1}^{M_\varepsilon - 1} a_i v_i^2 + \sum_{i=M_\varepsilon}^{N} a_i v_i^2 \frac{1}{\beta_i + \beta_i^{-1} + 2},$$

where the first term is infinitesimal since each β_i is; the second term is less than 2ε by our choice of M_ε; and the last term is infinitesimal since each β_i is infinite. Since $\varepsilon \in \mathbb{R}_+$ is arbitrary, the sum on the left must be infinitesimal. This proves the approximation (1.3.11). Hence, the theorem is also proved. □

1.4 Weak Coercive Quadratic Forms

In Sect. 1.3, we have discussed the resolvent of the symmetric part $\overline{\mathcal{E}}(\cdot,\cdot)$ of $\mathcal{E}(\cdot,\cdot)$. We have heavily depended on the eigenvectors and eigenvalues of the generator $\overline{A}$. In this section, we shall study the resolvent of $\mathcal{E}(\cdot,\cdot)$ directly. However, the generator A is not symmetric. This forces us to find an alternative way for the discussion.

Let $\mathcal{E}(\cdot,\cdot)$ be a nonnegative quadratic form on a hyperfinite dimensional linear space H. Let A and $\hat{A}$ be the infinitesimal generator and co-generator of $\mathcal{E}(\cdot,\cdot)$, respectively. For $\alpha \in {}^*(-\infty, 0), u \in H$, we have

$$\langle (A - \alpha)u, u\rangle \geq -\alpha\langle u, u\rangle \text{ and } \langle (\hat{A} - \alpha)u, u\rangle \geq -\alpha\langle u, u\rangle. \tag{1.4.1}$$

Hence, $(A - \alpha)^{-1} = G_\alpha$ and $(\hat{A} - \alpha)^{-1} = \hat{G}_\alpha$ exist for all $\alpha \in {}^*(-\infty, 0)$ by elementary linear algebra. Moreover, we have

Proposition 1.4.1. *Let $\mathcal{E}(\cdot,\cdot)$ be a nonnegative quadratic form on a hyperfinite dimensional linear space H. Then*

(i) $(A - \alpha)^{-1} = G_\alpha$ and $(\hat{A} - \alpha)^{-1} = \hat{G}_\alpha$ exist for all $\alpha \in {}^*(-\infty, 0)$. Moreover, we have $||G_\alpha|| \leq \frac{1}{|\alpha|}$ and $||\hat{G}_\alpha|| \leq \frac{1}{|\alpha|}$.

(ii) For all $\alpha, \beta \in {}^*(-\infty, 0)$, *we have*

$$\begin{aligned} G_\alpha - G_\beta &= (\alpha - \beta) G_\alpha G_\beta \\ &= (\alpha - \beta) G_\beta G_\alpha \end{aligned} \tag{1.4.2}$$

and

$$\begin{aligned} \hat{G}_\alpha - \hat{G}_\beta &= (\alpha - \beta) \hat{G}_\alpha \hat{G}_\beta \\ &= (\alpha - \beta) \hat{G}_\beta \hat{G}_\alpha . \end{aligned} \tag{1.4.3}$$

Proof. (i) From (1.4.1), we know $(A - \alpha)^{-1} = G_\alpha$ and $(\hat{A} - \alpha)^{-1} = \hat{G}_\alpha$ exist for all $\alpha \in {}^*(-\infty, 0)$. Besides, we have

$$\begin{aligned} \langle \alpha G_\alpha u, \alpha G_\alpha u \rangle &\leq -\alpha \mathcal{E}_{-\alpha}(G_\alpha u, G_\alpha u) \\ &= -\alpha \langle u, G_\alpha u \rangle \\ &\leq ||u|| ||\alpha G_\alpha u||. \end{aligned}$$

This implies that $||G_\alpha|| \leq \frac{1}{|\alpha|}$. Similarly, we have $||\hat{G}_\alpha|| \leq \frac{1}{|\alpha|}$.

(ii) We notice that

$$\alpha - \beta = (A - \beta) - (A - \alpha).$$

Hence, we have

$$(\alpha - \beta)(A - \alpha)^{-1} = (A - \alpha)^{-1}(A - \beta) - I.$$

This implies

$$(\alpha - \beta)(A - \alpha)^{-1}(A - \beta)^{-1} = (A - \alpha)^{-1} - (A - \beta)^{-1}.$$

Therefore, we have

$$(\alpha - \beta) G_\alpha G_\beta = G_\alpha - G_\beta .$$

Similarly, we can show that

$$(\alpha - \beta) G_\beta G_\alpha = G_\alpha - G_\beta .$$

This proves the relation (1.4.2). In the same manner, we can prove the relation (1.4.3). □

Thereafter, we shall call $\{G_\alpha \mid \alpha < 0\}$ the *resolvent of* $\mathcal{E}(\cdot, \cdot)$, and $\{\hat{G}_\alpha \mid \alpha < 0\}$ the *co-resolvent of* $\mathcal{E}(\cdot, \cdot)$. The relation (1.4.2) will be called the *first* *resolvent equation*, and the relation (1.4.3) will be called the *first co-resolvent equation.*

For $\alpha < 0$, we define

$$\begin{aligned} {}^{(\alpha)}\mathcal{E}(u,v) &= -\alpha\langle u + \alpha G_\alpha u, v\rangle \\ &= -\alpha\langle v + \alpha\hat{G}_\alpha v, u\rangle, u, v \in H, \end{aligned}$$

and

$$\begin{aligned} {}^{(\alpha)}\hat{\mathcal{E}}(u,v) &= -\alpha\langle v + \alpha G_\alpha v, u\rangle \\ &= -\alpha\langle u + \alpha\hat{G}_\alpha u, v\rangle, u, v \in H. \end{aligned}$$

Then

$$\begin{aligned} {}^{(\alpha)}\mathcal{E}(u, -\alpha G_\alpha u) &= -\alpha\langle u + \alpha G_\alpha u, -\alpha G_\alpha u\rangle \\ &= {}^{(\alpha)}\mathcal{E}(u,u) + \alpha\langle u + \alpha G_\alpha u, u + \alpha G_\alpha u\rangle, u \in H, \end{aligned}$$

and

$$\begin{aligned} {}^{(\alpha)}\mathcal{E}(-\alpha\hat{G}_\alpha u, u) &= -\alpha\langle u + \alpha\hat{G}_\alpha u, -\alpha\hat{G}_\alpha u\rangle \\ &= {}^{(\alpha)}\mathcal{E}(u,u) + \alpha\langle u + \alpha\hat{G}_\alpha u, u + \alpha\hat{G}_\alpha u\rangle, u \in H. \end{aligned}$$

Therefore, we get

$$ {}^{(\alpha)}\mathcal{E}(u, -\alpha G_\alpha u) \le {}^{(\alpha)}\mathcal{E}(u,u), u \in H, \tag{1.4.4}$$

and

$$ {}^{(\alpha)}\mathcal{E}(-\alpha\hat{G}_\alpha u, u) \le {}^{(\alpha)}\mathcal{E}(u,u), u \in H.$$

Actually, we have

Lemma 1.4.1. *Let $\mathcal{E}(\cdot,\cdot)$ be a nonnegative quadratic form on a hyperfinite dimensional linear space H. Then*

(i) ${}^{(\alpha)}\mathcal{E}(u,v) = \mathcal{E}(-\alpha G_\alpha u, v)$ *and* ${}^{(\alpha)}\hat{\mathcal{E}}(u,v) = \mathcal{E}(-\alpha\hat{G}_\alpha u, v)$ *for all* $u, v \in H$.
(ii) $\mathcal{E}(-\alpha G_\alpha u, -\alpha G_\alpha u) \le {}^{(\alpha)}\mathcal{E}(u,u)$ *and* $\mathcal{E}(-\alpha\hat{G}_\alpha u, -\alpha\hat{G}_\alpha u) \le {}^{(\alpha)}\mathcal{E}(u,u)$ *for all* $u \in H$.

Proof. (i) We have

$$\begin{aligned} {}^{(\alpha)}\mathcal{E}(u,v) &= -\alpha\langle u, v\rangle - \alpha^2\langle G_\alpha u, v\rangle \\ &= -\alpha\mathcal{E}_{-\alpha}(G_\alpha u, v) - \alpha^2\langle G_\alpha u, v\rangle \\ &= \mathcal{E}(-\alpha G_\alpha u, v). \end{aligned}$$

(ii) It follows from (i) and (1.4.4) that

$$\begin{aligned}\mathcal{E}(-\alpha G_\alpha u, -\alpha G_\alpha u) &= {}^{(\alpha)}\mathcal{E}(u, -\alpha G_\alpha u)\\ &\leq {}^{(\alpha)}\mathcal{E}(u, u).\end{aligned}$$

Notice that

$$\mathcal{E}(-\alpha G_\alpha u, -\alpha G_\alpha u) = \mathcal{E}(-\alpha \hat{G}_\alpha u, -\alpha \hat{G}_\alpha u).$$

This implies (ii). □

Definition 1.4.1. Let $\mathcal{E}(\cdot,\cdot)$ be a nonnegative quadratic form on a hyperfinite dimensional linear space H. We call $\mathcal{E}(\cdot,\cdot)$ a *hyperfinite weak coercive quadratic form* if and only if there exists a constant $C \in {}^*\mathbb{R}_+$ with $0 \leq {}^\circ C < \infty$ such that

$$|\mathcal{E}_1(u, v)| \leq C\sqrt{\mathcal{E}_1(u, u)}\sqrt{\mathcal{E}_1(v, v)}, \forall u, v \in H. \tag{1.4.5}$$

We call (1.4.5) the *hyperfinite weak sector condition* and C a *continuity constant.*

Lemma 1.4.2. *Let $\mathcal{E}(\cdot,\cdot)$ be a nonnegative quadratic form on a hyperfinite dimensional linear space H. Then the following statements are equivalent:*

(i) $\mathcal{E}(\cdot,\cdot)$ *satisfies the hyperfinite weak sector condition.*
(ii) For $\beta \in {}^*(0, \infty), \infty > {}^\circ\beta > 0$, *there exists* $C_\beta \in {}^*\mathbb{R}_+$ *with* $0 \leq {}^\circ C_\beta < \infty$ *such that*

$$|\mathcal{E}_\beta(u, v)| \leq C_\beta\sqrt{\mathcal{E}_\beta(u, u)}\sqrt{\mathcal{E}_\beta(v, v)}, \forall u, v \in H.$$

(iii) For $\beta \in {}^*(0, \infty), \infty > {}^\circ\beta > 0$, *there exists* $C'_\beta \in {}^*\mathbb{R}_+$ *with* $0 \leq {}^\circ C'{}_\beta < \infty$ *such that*

$$|\mathcal{E}(u, v)| \leq C'_\beta\sqrt{\mathcal{E}_\beta(u, u)}\sqrt{\mathcal{E}_\beta(v, v)}, \forall u, v \in H.$$

Proof. The proof is easy and is left as an exercise. □

Lemma 1.4.3. *Let $\mathcal{E}(\cdot,\cdot)$ be a hyperfinite weak coercive quadratic form on a hyperfinite dimensional linear space H. Then for all $\alpha \in {}^*(-\infty, 0)$*

(i) $|{}^{(\alpha)}\mathcal{E}_1(u, v)| \leq (C'_1 + 1)\sqrt{\mathcal{E}_1(u, u)}\sqrt{{}^{(\alpha)}\mathcal{E}_1(v, v)}$ *for all* $u, v \in H$.
(ii) $\mathcal{E}_1(-\alpha G_\alpha u, -\alpha G_\alpha u) \leq (C'_1 + 1)^2\mathcal{E}_1(u, u)$ *for all* $u \in H$.

Proof. (i) By using Lemma 1.4.1 and Lemma 1.4.2, we have

$$\begin{aligned}\left|{}^{(\alpha)}\mathcal{E}(u, v)\right| &= \left|\mathcal{E}(u, -\alpha\hat{G}_\alpha v)\right|\\ &\leq C'_1\sqrt{\mathcal{E}_1(u, u)}\sqrt{\mathcal{E}_1(-\alpha\hat{G}_\alpha v, -\alpha\hat{G}_\alpha v)}\\ &\leq C'_1\sqrt{\mathcal{E}_1(u, u)}\sqrt{{}^{(\alpha)}\mathcal{E}(v, v)}.\end{aligned}$$

(ii) From (i) and Lemma 1.4.1 (ii), we have

$$\begin{aligned}\mathcal{E}_1(-\alpha G_\alpha u, -\alpha G_\alpha u) &\le {}^{(\alpha)}\mathcal{E}(u,u) \\ &\le (C_1'+1)^2\mathcal{E}_1(u,u).\end{aligned}$$

□

Lemma 1.4.4. *Let $\mathcal{E}(\cdot,\cdot)$ be a hyperfinite weak coercive quadratic form on a hyperfinite dimensional linear space H. If ${}^\circ\mathcal{E}_1(u,u) < \infty$, then for all infinite $\alpha < 0$, we have $||u+\alpha G_\alpha u|| \approx 0$.*

Proof. From Lemma 1.4.1 (i) and Lemma 1.4.3 (i), we have

$$\begin{aligned}||u+\alpha G_\alpha u||^2 &= \langle u+\alpha G_\alpha u, u+\alpha G_\alpha u\rangle \\ &= -\frac{{}^{(\alpha)}\mathcal{E}(u, u+\alpha G_\alpha u)}{\alpha} \\ &= -\frac{1}{\alpha}\left[{}^{(\alpha)}\mathcal{E}(u,u) + {}^{(\alpha)}\mathcal{E}(u,\alpha G_\alpha u)\right] \\ &= -\frac{1}{\alpha}\left[{}^{(\alpha)}\mathcal{E}(u,u) - \mathcal{E}(\alpha G_\alpha u, \alpha G_\alpha u)\right] \\ &\le -\frac{{}^{(\alpha)}\mathcal{E}(u,u)}{\alpha} \\ &\le -\frac{(C_1'+1)^2}{\alpha}\mathcal{E}_1(u,u) \\ &\approx 0.\end{aligned}$$

□

It is the time to introduce the definition of the domain $\mathcal{D}(\mathcal{E})$ of $\mathcal{E}(\cdot,\cdot)$.

Definition 1.4.2. Let $\mathcal{E}(\cdot,\cdot)$ be a nonnegative quadratic form on a hyperfinite dimensional linear space H. The domain $\mathcal{D}(\mathcal{E})$ of $\mathcal{E}(\cdot,\cdot)$ is the set of all $u \in H$ satisfying

(i) ${}^\circ\mathcal{E}_1(u,u) < \infty$.
(ii) For all infinite $\alpha < 0$, $\mathcal{E}(u+\alpha G_\alpha u, u+\alpha G_\alpha u) \approx 0$ and $\mathcal{E}(u+\alpha\hat{G}_\alpha u, u+\alpha\hat{G}_\alpha u) \approx 0$.

Proposition 1.4.2. *Let $\mathcal{E}(\cdot,\cdot)$ be a hyperfinite weak coercive quadratic form on a hyperfinite dimensional linear space H. Then the following statements are equivalent:*

(i) $u \in \mathcal{D}(\mathcal{E})$.
(ii) ${}^\circ\mathcal{E}_1(u,u) < \infty$, *and for all infinite* $\alpha < 0$, $\mathcal{E}(u+\alpha G_\alpha u, u) \approx 0$ *and* $\mathcal{E}(u+\alpha\hat{G}_\alpha u, u) \approx 0$.
(iii) ${}^\circ\mathcal{E}_1(u,u) < \infty$, *and for all infinite* $\alpha < 0$, ${}^{(\alpha)}\mathcal{E}(u,u) \approx \mathcal{E}(u,u)$ *and* ${}^{(\alpha)}\hat{\mathcal{E}}(u,u) \approx \mathcal{E}(u,u)$.

Proof. $(i) \implies (ii)$. This is easily seen from Lemma 1.4.2 (iii) and Lemma 1.4.4.

$(ii) \implies (i)$. From Lemma 1.4.1, we have

$$\begin{aligned}
0 &\le \mathcal{E}(u+\alpha G_\alpha u, u+\alpha G_\alpha u) \\
&= \mathcal{E}(u, u+\alpha G_\alpha u) + \mathcal{E}(\alpha G_\alpha u, u+\alpha G_\alpha u) \\
&= \mathcal{E}(u, u+\alpha G_\alpha u) + \mathcal{E}(-\alpha G_\alpha u, -\alpha G_\alpha u) - {}^{(\alpha)}\mathcal{E}(u,u) \\
&\le \mathcal{E}(u, u+\alpha G_\alpha u) \\
&= \mathcal{E}(u+\alpha \hat{G}_\alpha u, u) \\
&\approx 0.
\end{aligned}$$

Similarly, we can show

$$\mathcal{E}(u+\alpha \hat{G}_\alpha u, u+\alpha \hat{G}_\alpha u) \approx 0.$$

$(ii) \iff (iii)$. This is easily seen from Lemma 1.4.1. □

Lemma 1.4.5. *Let $\mathcal{E}(\cdot,\cdot)$ be a hyperfinite weak coercive quadratic form on a hyperfinite dimensional linear space H. If ${}^\circ\mathcal{E}(u,u) < \infty$, then for all finite $\beta < 0, {}^\circ\beta \neq 0$, $G_\beta u \in \mathcal{D}(\mathcal{E})$.*

Proof. We have from Lemma 1.4.3 (ii) that

$$\mathcal{E}_1(G_\beta u, G_\beta u) \le {}^\circ\left[\frac{(C_1'+1)^2}{\beta^2}\mathcal{E}_1(u,u)\right] < \infty.$$

Hence, it suffices to show that for all infinite $\alpha < 0$ by Proposition 1.4.2 (ii)

$$\mathcal{E}(G_\beta u + \alpha G_\alpha G_\beta u, G_\beta u) \approx 0 \tag{1.4.6}$$

and

$$\mathcal{E}(G_\beta u + \alpha \hat{G}_\alpha G_\beta u, G_\beta u) \approx 0. \tag{1.4.7}$$

Actually, we have

$$\begin{aligned}
&\mathcal{E}(G_\beta u + \alpha G_\alpha G_\beta u, G_\beta u) \\
&\quad = \langle u+\alpha G_\alpha u, G_\beta u\rangle + \beta\langle G_\beta u + \alpha G_\alpha G_\beta u, G_\beta u\rangle.
\end{aligned} \tag{1.4.8}$$

From Lemma 1.4.4, we see that

$$\langle u+\alpha G_\alpha u, G_\beta u\rangle \approx 0. \tag{1.4.9}$$

From the relation (1.4.2), we have

$$\begin{aligned}\beta\langle G_\beta u+\alpha G_\alpha G_\beta u, G_\beta u\rangle &= \beta\langle G_\beta u+\frac{\alpha}{\alpha-\beta}(G_\alpha-G_\beta)u, G_\beta u\rangle\\ &= \beta\langle(1+\frac{-\alpha}{\alpha-\beta})G_\beta u+\frac{\alpha}{\alpha-\beta}G_\alpha u, G_\beta u\rangle\\ &= \frac{-\beta^2}{\alpha-\beta}\langle G_\beta u, G_\beta u\rangle+\frac{\alpha\beta}{\alpha-\beta}\langle G_\alpha u, G_\beta u\rangle\\ &\approx 0. \end{aligned}\tag{1.4.10}$$

By the relations (1.4.8), (1.4.9), and (1.4.10), we have proved the relation (1.4.6). Similarly, we can prove the relation (1.4.7). □

We recall that the norm $|\cdot|_1$ is defined by $|u|_1=\sqrt{\mathcal{E}_1(u,u)}$. A subset F of H is called *$\mathcal{E}$-closed* if and only if for all sequences $\{u_n\}_{n\in\mathbb{N}}$ of elements from F such that $^\circ|u_n-u_m|_1\longrightarrow 0$ as $n,m\longrightarrow\infty$, there exists an element u in F such that $^\circ|u_n-u|_1\longrightarrow 0$ as $n\longrightarrow\infty$. One may want to notice that a subset F of H is $\mathcal{E}$-closed if and only if it is $\overline{\mathcal{E}}$-closed.

As we just mention above, the reader may already note that the definitions of $\overline{\mathcal{E}}$-closedness and $\mathcal{E}$-closedness are formally identical. However, in the definition of $\overline{\mathcal{E}}$-closedness, the norm $|\cdot|_1$ is defined via $|u|_1=\sqrt{\overline{\mathcal{E}}(u,u)}$, whereas in the definition of $\mathcal{E}$-closedness, it is defined via $|u|_1=\sqrt{\mathcal{E}(u,u)}$. It will always be clear from the context which norm we actually mean by $|\cdot|_1$.

Proposition 1.4.3. *Let $\mathcal{E}(\cdot,\cdot)$ be a hyperfinite weak coercive quadratic form on a hyperfinite dimensional linear space H. Then $\mathcal{D}(\mathcal{E})$ is $\mathcal{E}$-closed. Moreover, if $\{u_n\}_{n\in\mathbb{N}}$ is a $|\cdot|_1$ Cauchy sequence from $\mathcal{D}(\mathcal{E})$, and $\{u_n \mid n\in{}^*\mathbb{N}\}$ is an internal extension of $\{u_n\}_{n\in\mathbb{N}}$, then there is a $\gamma\in{}^*\mathbb{N}-\mathbb{N}$ such that $u_\eta\in\mathcal{D}(\mathcal{E})$ for all $\eta\le\gamma$.*

Proof. Let $\{u_n \mid n\in\mathbb{N}\}$ be a $|\cdot|_1$ Cauchy sequence from $\mathcal{D}(\mathcal{E})$, and let $\{u_n \mid n\in{}^*\mathbb{N}\}$ be an internal extension of it. There is an element $\gamma\in{}^*\mathbb{N}-\mathbb{N}$ such that $|u_n-u_m|_1\approx 0$ whenever n and m are infinite and less than γ. Let $\eta\in{}^*\mathbb{N}-\mathbb{N}, \eta\le\gamma$. By the choice of γ, $^\circ\mathcal{E}_1(u_\eta,u_\eta)<\infty$ and $^\circ|u_n-u_\eta|_1\longrightarrow 0$ as n approaches infinity in $\mathbb{N}$. All that remains is to prove that $u_\eta\in\mathcal{D}(\mathcal{E})$.

Assuming not, there is an $\varepsilon\in\mathbb{R}_+$ and infinite $\beta<0$ such that

$$|u_\eta+\beta G_\beta u_\eta|_1>\varepsilon \tag{1.4.11}$$

or

$$|u_\eta-\beta\hat{G}_\beta u_\eta|_1>\varepsilon. \tag{1.4.12}$$

(1) Assume that the relation (1.4.11) holds. Choose $m\in\mathbb{N}$ so large that

$$|u_\eta-u_m|_1<\frac{\varepsilon}{4(C_1'+1)^2}.$$

Then by Lemma 1.4.3 (ii), we have

$$|\beta G_\beta u_\eta - \beta G_\beta u_m|_1 < \frac{\varepsilon}{4}.$$

Combining the inequalities above, we get

$$\begin{aligned}\varepsilon &< |u_\eta + \beta G_\beta u_\eta|_1 \\ &\le |u_\eta - u_m|_1 + |u_m + \beta G_\beta u_m|_1 + |\beta G_\beta u_m - \beta G_\beta u_\eta|_1 \\ &\le \frac{\varepsilon}{2} + |u_m + \beta G_\beta u_m|_1.\end{aligned}$$

However, the last term $|u_m + \beta G_\beta u_m|_1$ is infinitesimal by Lemma 1.4.4 since $u_m \in \mathcal{D}(\mathcal{E})$. We have the contradiction we wanted.

(2) If the relation (1.4.12) holds, then we can get a corresponding contradiction. □

As in Sect. 1.2, we may now sum up our results on $\mathcal{D}(\mathcal{E})$ in one statement.

Theorem 1.4.1. *Let $\mathcal{E}(\cdot,\cdot)$ be a hyperfinite weak coercive quadratic form on a hyperfinite dimensional space H. Then*

(i) If $u, v \in \mathcal{D}(\mathcal{E})$ and α is a finite element of ${}^*\mathbb{R}$, then $\alpha u, u+v \in \mathcal{D}(\mathcal{E})$.
(ii) $\mathcal{D}(\mathcal{E})$ is $\mathcal{E}$-closed.
(iii) If ${}^\circ\mathcal{E}_1(u,u) < \infty$, then there exists a $v \in \mathcal{D}(\mathcal{E})$ with $||u-v|| \approx 0$. Moreover, we have

$$\begin{aligned}{}^\circ\mathcal{E}(v,v) &= \lim_{\substack{{}^\circ\alpha\downarrow-\infty \\ {}^\circ\alpha\neq-\infty}} {}^\circ\mathcal{E}(\alpha G_\alpha u, \alpha G_\alpha u) \\ &= \lim_{\substack{{}^\circ\alpha\downarrow-\infty \\ {}^\circ\alpha\neq-\infty}} {}^\circ[{}^{(\alpha)}\mathcal{E}(u,u)].\end{aligned}$$

(iv) If $u, v \in \mathcal{D}(\mathcal{E})$ and $u \approx v$, then $\mathcal{E}(u,u) \approx \mathcal{E}(v,v)$.
(v) If $u \in \mathcal{D}(\mathcal{E})$, then ${}^\circ\mathcal{E}(u,u) = \inf\{{}^\circ\mathcal{E}(v,v) \mid v \approx u\}$.
(vi) If ${}^\circ\mathcal{E}(u,u) = \inf\{{}^\circ\mathcal{E}(v,v) \mid v \approx u\} < \infty$, then $u \in \mathcal{D}(\mathcal{E})$.

Proof. (i) We can prove this in the same way as the proof of Corollary 1.2.1.

(ii) It is proved in Proposition 1.4.3.

(iii) Since ${}^\circ\mathcal{E}_1(u,u) < \infty$, we have from Lemma 1.4.3 (ii) that for all $\alpha \in {}^*(-\infty, 0)$

$${}^\circ[\mathcal{E}_1(-\alpha G_\alpha u, -\alpha G_\alpha u)] \le {}^\circ\left[(C_1' + 1)^2 \mathcal{E}_1(u,u)\right] < \infty.$$

Let $\{-\alpha_n G_{\alpha_n}\}$ be a $|\cdot|_1$ Cauchy sequence, ${}^\circ\alpha \downarrow -\infty$. Let $\{-\alpha_n G_{\alpha_n} \mid n \in {}^*\mathbb{N}\}$ be an internal extension of it. There exists an infinite $\eta \in {}^*\mathbb{N}$ such that $-\alpha_n G_{\alpha_n} u \in \mathcal{D}(\mathcal{E}), n \leq \eta$. By Lemma 1.4.4, we have $u \approx -\alpha_\eta G_{\alpha_\eta} u$. Hence by letting $v = -\alpha_\eta G_{\alpha_\eta} u$, we have gotten (iii).

(iv) It is obviously enough to show that if $u \in \mathcal{D}(\mathcal{E})$ and $u \approx 0$, then $\mathcal{E}(u,u) \approx 0$. But if $u \in \mathcal{D}(\mathcal{E})$, we know from (iii):

$$^\circ\mathcal{E}(u,u) = \lim_{\substack{^\circ\alpha \downarrow -\infty \\ ^\circ\alpha \neq -\infty}} {}^{(\alpha)}[^\circ\mathcal{E}(u,u)]. \tag{1.4.13}$$

Also

$$\begin{aligned} {}^{(\alpha)}\mathcal{E}(u,u) &= -\alpha\langle u + \alpha G_\alpha u, u\rangle \\ &= -\alpha\langle u,u\rangle - \alpha^2\langle G_\alpha u, u\rangle \\ &\leq -\alpha||u||^2, \end{aligned}$$

which is infinitesimal for $\alpha \not\approx -\infty$. Combining this with (1.4.13), (iv) follows.

(v) This follows in the same way as for the proof of Theorem 1.2.1 (v).

(vi) For simplicity, we assume that $u \approx 0$. Then $\mathcal{E}(u,u) \approx 0$. From Lemma 1.4.3 (i), we have for all $\alpha \in {}^*(-\infty, 0)$

$$\begin{aligned} 0 &\leq |{}^{(\alpha)}\mathcal{E}_1(u,u)| \\ &\leq (C_1' + 1)^2 \mathcal{E}_1(u,u) \\ &\approx 0. \end{aligned}$$

Hence, ${}^{(\alpha)}\mathcal{E}_1(u,u) \approx 0, \forall \alpha \in {}^*(-\infty,0)$. Similarly, we can show that ${}^{(\alpha)}\hat{\mathcal{E}}_1(u,u) \approx 0$. By Proposition 1.4.2, we have $u \in \mathcal{D}(\mathcal{E})$. □

The following definition now makes sense.

Definition 1.4.3. Let $\mathcal{E}(\cdot,\cdot)$ be a hyperfinite weak coercive quadratic form on a hyperfinite dimensional space H. The *standard part of* $\mathcal{E}(\cdot,\cdot)$ is the quadratic form $E(\cdot,\cdot)$ on $^\circ H$ defined by:

(i) The domain $D(E)$ of $E(\cdot,\cdot)$ is the set of all equivalence classes $^\circ u \in {}^\circ H$ such that $\inf\{^\circ\mathcal{E}_1(v,v) \mid v \in {}^\circ u\} < \infty$.
(ii) If $x, y \in {}^\circ H$ are in the domain of $E(\cdot,\cdot)$, let $E(x,y) = {}^\circ\mathcal{E}(u,v)$, where $u \in x, v \in y$ are in $\mathcal{D}(\mathcal{E})$.

For $\alpha \in [0,\infty)$, let us set

$$E_\alpha(\cdot,\cdot) = E(\cdot,\cdot) + \alpha(\cdot,\cdot).$$

We recall that $(\cdot,\cdot)$ is the inner product of $^\circ H$.

An E_1-*Cauchy sequence* is a sequence $\{x_n\}$ of elements from $D(E)$ such that $E_1(u_n - u_m, u_n - u_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$. We say that $E(\cdot,\cdot)$ is *closed* if all E_1-Cauchy sequences converge in E_1-norm to an element in $D(E)$. The next proposition follows immediately from Theorem 1.4.1 and the definition of $E(\cdot,\cdot)$.

Proposition 1.4.4. *Let $\mathcal{E}(\cdot,\cdot)$ be a hyperfinite weak coercive quadratic form on a hyperfinite dimensional space H. Let $E(\cdot,\cdot)$ be the standard part of $\mathcal{E}(\cdot,\cdot)$. Then $E(\cdot,\cdot)$ is closed, and for all $x \in {}^\circ H$*

$$E(x,x) = \inf\{{}^\circ\mathcal{E}(u,u) \mid u \in x\}, \tag{1.4.14}$$

where we take the value ∞ on the right to mean that the expression on the left is undefined.

Proof. If ${}^\circ\mathcal{E}_1(u,u) = \infty$ for all $u \in x$, it is easy to see (1.4.14) holds. Assume that $\inf\{{}^\circ\mathcal{E}(u,u) \mid u \in x\} < \infty$, then $E(x,x) = {}^\circ\mathcal{E}(v,v)$ for some $v \in \mathcal{D}(\mathcal{E}), v \in x$. Hence, we only need to show ${}^\circ\mathcal{E}(v,v) = \inf\{{}^\circ\mathcal{E}(u,u) \mid u \in x\}$. But this is implied by Theorem 1.4.1 (iv). □

In Sect. 1.2, we have gotten the symmetric standard part $\overline{E}(\cdot,\cdot)$ of $\overline{\mathcal{E}}(\cdot,\cdot)$. We have studied the domain $\mathcal{D}(\overline{\mathcal{E}})$ of $\overline{\mathcal{E}}(\cdot,\cdot)$ also. Now it is very natural to discuss the relation between the results of Sect. 1.2 and those of this section. Actually, we have

Theorem 1.4.2. *Let $\mathcal{E}(\cdot,\cdot)$ be a hyperfinite weak coercive quadratic form on a hyperfinite dimensional space H. Then $\mathcal{D}(\mathcal{E}) = \mathcal{D}(\overline{\mathcal{E}})$ and $D(E) = D(\overline{E})$. Moreover, we have*

$$\overline{E}(x,y) = \frac{1}{2}\Big(E(x,y) + E(y,x)\Big). \tag{1.4.15}$$

Proof. From Theorem 1.4.1 (v) and (vi), we know that $v \in \mathcal{D}(\mathcal{E})$ if and only if ${}^\circ\mathcal{E}(v,v) = \inf\{{}^\circ\mathcal{E}(u,u) \mid v \approx u\}$ and ${}^\circ\mathcal{E}(v,v) < \infty$. This statement is also true for the elements in $\mathcal{D}(\overline{\mathcal{E}})$ from Theorem 1.2.1 (v) and (vi). Hence, $\mathcal{D}(\mathcal{E}) = \mathcal{D}(\overline{\mathcal{E}})$. The relation (1.4.15) follows immediately. □

Remark 1.4.1. Let $\mathcal{E}(\cdot,\cdot)$ be a hyperfinite weak coercive quadratic form on a hyperfinite dimensional space H. It is easy to see that $(E(\cdot,\cdot), D(E))$ satisfies the following *weak sector condition*

$$|E_1(x,y)| \le C\sqrt{E_1(x,x)}\sqrt{E_1(y,y)} \text{ for all } x, y \in D(E),$$

where $C \in \mathbb{R}_+$ is a positive real number. Hence, $(E(\cdot,\cdot), D(E))$ is a coercive closed form on ${}^\circ H$ (the definition of coercive closed form will be given in Sect. 1.6).

1.5 Hyperfinite Dirichlet Forms

It is time to discuss the hyperfinite forms associated with Markov processes, i.e., the Dirichlet forms. The aim is to give a reasonably detailed account of the relationship between the properties of these forms and the behavior of the associated processes.

Consider a particle which can be in $N+1$ different states $s_0, s_1, s_2, \cdots, s_N$. Assume that if the particle is in state s_i at some instant t, then – independently of what its past history may be – the probability that it will be in state s_j at the next instant $t + \Delta t$ is given by a fixed number q_{ij}. This is the familiar setting for the theory of stationary Markov chains with finite state space, see, e.g., Chung [119], Dynkin [147], and Dynkin and Yushkevich [148]. We shall be interested in the case where $S = \{s_0, s_1, \cdots, s_N\}$ is a hyperfinite set, and $T = \{k\Delta t \mid k \in {}^*\mathbb{N}_0\}$ is a hyperfinite time line with $\Delta t \approx 0$ (for technical reasons it is convenient to have an *-infinite time line to work with). The idea is to use the hyperfinite setup to reduce the highly sophisticated theory of continuous parameter Markov processes taking values in topological spaces (see [96, 175, 193, 270, 330, 333, 334]) to the much simpler theory of finite Markov chains.

Let Y be a *Hausdorff space*, i.e., Y is a topological space such that for each pair x, y of distinct points in Y, there are open sets U, V satisfying $x \in U, y \in V$, and $U \cap V = \emptyset$ [121]. Here $\emptyset$ denotes the empty set. Let $^*Y$ be the nonstandard extension of Y. Let $S = \{s_0, s_1, \cdots, s_N\}$ be an S-dense subset of $^*Y$ for some $N \in {}^*\mathbb{N} - \mathbb{N}$ and m be a hyperfinite measure on S. Denote by $\mathcal{S}$ the internal algebra of subsets of S. Assume that $Q = \{q_{ij}\}$ is an $(N+1) \times (N+1)$ matrix with nonnegative entries, and assume that

$$\sum_{j=0}^{N} q_{ij} = 1 \quad \text{for all} \quad i = 0, 1, \cdots, N, \tag{1.5.1}$$

and the state s_0 is a trap, i.e.,

$$q_{0i} = 0 \quad \text{for all} \quad i \neq 0. \tag{1.5.2}$$

In the sequel, we shall write m_i for $m(\{s_i\})$ and q_{ij} for $q_{s_i s_j}$ respectively, whenever it is convenient.

If (Ω, P) is an internal measure space, and $X : \Omega \times T \longrightarrow S$ is an internal process, let

$$[\omega]_t = \{\omega' \in \Omega \mid X(\omega', s) = X(\omega, s) \text{ for all } s \leq t\}. \tag{1.5.3}$$

For each $t \in T$, let $\mathcal{F}_t$ be the internal algebra on Ω generated by the sets $[\omega]_t$.

If for all $\omega \in \Omega$

$$P([\omega]_0) = m\{X(\omega, 0)\}, \tag{1.5.4}$$

and whenever $X(\omega, t) = s_i$,

$$P\{\omega' \in [\omega]_t \mid X(t + \Delta t, \omega') = s_j\} = q_{ij} P([\omega]_t), \tag{1.5.5}$$

then we call X a *hyperfinite Markov chain* with initial distribution m and transition matrix Q. Notice that we do not assume that m and P are probability measures. $P(\Omega)$ could be an infinite, hyperfinite number.

In fact, given m and Q, it is easy to construct an associated Markov chain $X(\omega, t)$. Let Ω be the set of all internal functions $\omega : T \longrightarrow S$. Denote by X the coordinate function $X(\omega, t) = \omega(t)$. Let P be the measure defined by

$$P\Big([\omega]_{k\Delta t}\Big) = m(\{\omega(0)\}) \prod_{n=0}^{k-1} q_{\omega(n\Delta t), \omega((n+1)\Delta t)}. \tag{1.5.6}$$

In particular, we define a family $(\Omega, \mathcal{F}_t, P_i, i \in S)$ of internal probability spaces by

$$P_i\Big([\omega]_{k\Delta t}\Big) = \delta_{i\omega(0)} \prod_{n=0}^{k-1} q_{\omega(n\Delta t), \omega((n+1)\Delta t)} \tag{1.5.7}$$

for each $i \in S$, where δ_{ij} is the Kronecker symbol.

Similarly, let $\hat{Q} = \{\hat{q}_{ij}\}$ be an $(N + 1) \times (N + 1)$ matrix with nonnegative entries, and assume that

$$\sum_{j=0}^{N} \hat{q}_{ij} = 1 \quad \text{for all} \quad i = 0, 1, \cdots, N, \tag{1.5.8}$$

and the state s_0 is a trap, i.e.,

$$\hat{q}_{0i} = 0 \quad \text{for all} \quad i \neq 0. \tag{1.5.9}$$

In the same manner as above, we shall write $\hat{q}_{ij}$ for $\hat{q}_{s_i s_j}$, respectively, whenever it is convenient.

If $(\hat{\Omega}, \hat{P})$ is an internal measure space, and $\hat{X} : \hat{\Omega} \times T \longrightarrow S$ is an internal process, let

$$[\hat{\omega}]_t = \{\hat{\omega}' \in \hat{\Omega} \mid \hat{X}(\hat{\omega}', s) = \hat{X}(\hat{\omega}, s) \text{ for all } s \leq t\}.$$

For each $t \in T$, let $\hat{\mathcal{F}}_t$ be the internal algebra on $\hat{\Omega}$ generated by the sets $[\hat{\omega}]_t$.

If for all $\hat{\omega} \in \hat{\Omega}$

$$\hat{P}([\hat{\omega}]_0) = m\{\hat{X}(\hat{\omega}, 0)\}, \tag{1.5.10}$$

and whenever $\hat{X}(\hat{\omega}, t) = s_i$,

$$\hat{P}\{\hat{\omega}' \in [\hat{\omega}]_t \mid \hat{X}(t + \Delta t, \hat{\omega}') = s_j\} = \hat{q}_{ij} \hat{P}([\hat{\omega}]_t), \tag{1.5.11}$$

then we call $\hat{X}$ a *hyperfinite Markov chain* with initial distribution m and transition matrix $\hat{Q}$.

Given m and $\hat{Q}$, we can construct an associated Markov chain $\hat{X}(\hat{\omega}, t)$ as that of (1.5.6). Moreover, we define a family $(\hat{\Omega}, \hat{\mathcal{F}}_t, \hat{P}_i, i \in S)$ of internal probability spaces by

$$\hat{P}_i\Big([\hat{\omega}]_{k\Delta t}\Big) = \delta_{i\hat{\omega}(0)} \prod_{n=0}^{k-1} \hat{q}_{\hat{\omega}(n\Delta t), \hat{\omega}((n+1)\Delta t)} \tag{1.5.12}$$

for each $i \in S$. It is easy to see that we can take

$$\Omega = \hat{\Omega}, \quad X = \hat{X}, \quad [\omega]_t = [\hat{\omega}]_t, \quad \mathcal{F}_t = \hat{\mathcal{F}}_t.$$

However, $\hat{Q}$, $\hat{P}$, and $\hat{P}_i$ are different from Q, P, and P_i.

Now let us introduce some regularity conditions. We assume that the measure m and the transition matrices Q and $\hat{Q}$ satisfy the dual conditions

$$m_i q_{ij} = m_j \hat{q}_{ji} \quad \text{for all} \quad i \neq 0, j \neq 0. \tag{1.5.13}$$

Besides, we assume that

$$m_i \neq 0 \quad \text{for at least one} \quad i \neq 0. \tag{1.5.14}$$

It is easy to find examples of transition matrices Q and $\hat{Q}$ such that no m satisfies the conditions (1.5.13) and (1.5.14). Thus, these assumptions may be regarded as conditions on Q and $\hat{Q}$.

Notice that for most i, the transition probabilities q_{i0} and $\hat{q}_{i0}$ should be of order of magnitude Δt, since if not the process will die in infinitesimal time.

Given m, Q and $\hat{Q}$ which satisfy conditions (1.5.1), (1.5.2), (1.5.4), (1.5.5), (1.5.8), (1.5.9), (1.5.10), (1.5.11), (1.5.13), and (1.5.14), the processes X and $\hat{X}$ as above are called *dual hyperfinite Markov chains*. These are the processes we will study in detail. We shall first associate to a given process a hyperfinite quadratic form.

If

$$S_0 = \{s_1, s_2, \cdots, s_N\}$$

is the state space S without the trap s_o, let us set $\mathcal{S}_0 = \mathcal{S} \cap S_0$. Let H be the linear space of all internal functions $u : S_0 \longrightarrow {}^*\mathbb{R}$ with the inner product

$$\begin{aligned}\int_{S_0} uv\ dm &= \langle u, v\rangle \\ &= \sum_{i=1}^{N} u(s_i)v(s_i)m(s_i).\end{aligned} \tag{1.5.15}$$

Just as we usually write m_i for $m(s_i)$, we shall write $u(i)$ or u_i for $u(s_i)$. And we shall identify H with the set of all internal functions $u : S \longrightarrow {}^*\mathbb{R}$ such that $u(s_0) = 0$.

Our convention of letting the trap s_0 be the zeroth element is notationally convenient, but we call attention of the reader to the fact that she/he should distinguish between sums of the forms $\sum_{i=0}^{N}$ and $\sum_{i=1}^{N}$, e.g.

For $t \in T$ and $u \in H$, we define new functions $Q^t u, \hat{Q}^t u \in H$ by

$$\begin{aligned}Q^t u(i) &= E_i u(X(t)), \\ \hat{Q}^t u(i) &= \hat{E}_i u(X(t)),\end{aligned} \tag{1.5.16}$$

where E_i and $\hat{E}_i$ are the expectations with respect to the measures P_i and $\hat{P}_i$ defined in (1.5.7) and (1.5.12), respectively. Intuitively, $Q^t u(i)$ and $\hat{Q}^t u(i)$ are the expected values of $u(X(t))$ for a particle starting in state s_i. Notice that

$$\begin{aligned}Q^{\Delta t} u(i) &= (Q \cdot u)(i) \\ &= \sum_{j=1}^{N} u(j) q_{ij}, \\ \hat{Q}^{\Delta t} u(i) &= (\hat{Q} \cdot u)(i) \\ &= \sum_{j=1}^{N} u(j) \hat{q}_{ij},\end{aligned}$$

where $\cdot$ are the matrix multiplications in the middle terms. Since

$$q_{ij}^{(t+s)} = \sum_{k=1}^{N} q_{ik}^{(t)} q_{kj}^{(s)},$$
$$\hat{q}_{ij}^{(t+s)} = \sum_{k=1}^{N} \hat{q}_{ik}^{(t)} \hat{q}_{kj}^{(s)},$$

we must have

$$Q^{t+s} = Q^t \cdot Q^s,$$
$$\hat{Q}^{t+s} = \hat{Q}^t \cdot \hat{Q}^s,$$

where $q_{ij}^{(t)}$ and $\hat{q}_{ij}^{(t)}$ are the transition probabilities given by operator Q^t and $\hat{Q}^t$, respectively. Hence, the families $\{Q^t \mid t \in T\}$ and $\{\hat{Q}^t \mid t \in T\}$ are semigroups of operators on H. Actually, $\{\hat{Q}^t \mid t \in T\}$ is the co-semigroup of $\{Q^t \mid t \in T\}$.

The *infinitesimal generator* A of the semigroup $\{Q^t \mid t \in T\}$ is given by

$$Au(i) = \frac{1}{\Delta t}\Big(u(i) - \sum_{j=1}^{N} u(j) q_{ij}\Big). \tag{1.5.17}$$

The *hyperfinite quadratic form* associated with Q and m is defined to be

$$\begin{aligned} \mathcal{E}(u,v) &= \langle Au, v\rangle \\ &= \sum_{i=1}^{N} Au(i)v(i)m(i). \end{aligned} \tag{1.5.18}$$

Combining (1.5.17) and (1.5.18), we get

$$\mathcal{E}(u,v) = \frac{1}{\Delta t}\sum_{i=1}^{N}\left[u(i)v(i)m(i) - \sum_{j=1}^{N} u(j)v(i)q_{ij}m(i)\right]. \tag{1.5.19}$$

The *infinitesimal co-generator* $\hat{A}$ of the co-semigroup $\{\hat{Q}^t \mid t \in T\}$ is given by

$$\hat{A}u(i) = \frac{1}{\Delta t}\Big(u(i) - \sum_{j=1}^{N} u(j) \hat{q}_{ij}\Big). \tag{1.5.20}$$

The *hyperfinite quadratic co-form* associated with $\hat{Q}$ and m is defined to be

$$\begin{aligned}\hat{\mathcal{E}}(u,v) &= \mathcal{E}(v,u)\\ &= \langle Av, u\rangle\\ &= \sum_{i=1}^{N} Av(i)u(i)m(i).\end{aligned} \tag{1.5.21}$$

Combining (1.5.13), (1.5.17), (1.5.20), and (1.5.21), we get

$$\begin{aligned}\hat{\mathcal{E}}(u,v) &= \frac{1}{\Delta t}\sum_{i=1}^{N}\left[u(i)v(i)m(i) - \sum_{j=1}^{N} u(i)v(j)q_{ij}m(i)\right]\\ &= \frac{1}{\Delta t}\sum_{i=1}^{N}\left[u(i)v(i)m(i) - \sum_{j=1}^{N} u(i)v(j)\hat{q}_{ji}m(j)\right]\\ &= \langle \hat{A}u, v\rangle.\end{aligned} \tag{1.5.22}$$

Now let us look at the symmetric part $\overline{\mathcal{E}}(\cdot,\cdot)$ of $\mathcal{E}(\cdot,\cdot)$. Let $\{\overline{Q}^t \mid t\in T\}$ be the semigroup and $\overline{A}$ be the generator of $\overline{\mathcal{E}}(\cdot,\cdot)$. Then, we have

$$\overline{q}_{ij} = \frac{1}{2}(q_{ij} + \hat{q}_{ij}),$$

where $(\overline{q}_{ij}) = \overline{Q} = \overline{Q}^{\Delta t}$. It is easy to see that

$$m_i\overline{q}_{ij} = m_j\overline{q}_{ji}.$$

Moreover, the generator $\overline{A}$ of $\overline{\mathcal{E}}(\cdot,\cdot)$ is given by

$$\begin{aligned}\overline{A}u(i) &= \frac{1}{2}\Big(Au(i) + \hat{A}u(i)\Big)\\ &= \frac{1}{\Delta t}\Big(u(i) - \sum_{j=1}^{N} u(j)\frac{1}{2}(q_{ij}+\hat{q}_{ij})\Big)\\ &= \frac{1}{\Delta t}\Big(u(i) - \sum_{j=1}^{N} u(j)\overline{q}_{ij}\Big).\end{aligned}$$

The *symmetric hyperfinite quadratic form* $\overline{\mathcal{E}}(\cdot,\cdot)$ is given by

$$\begin{aligned}\overline{\mathcal{E}}(u,v) &= \sum_{i=1}^{N} \overline{A}u(i)v(i)m(i) \\ &= \frac{1}{\Delta t}\sum_{i=1}^{N}\left[u(i)v(i)m(i) - \sum_{j=1}^{N} u(j)v(i)\overline{q}_{ij}m(i)\right]. \qquad (1.5.23)\end{aligned}$$

Our first result in the following lemma gives alternative ways of expressing hyperfinite quadratic forms in term of m_i, q_{ij}, $\hat{q}_{ij}$ and $\overline{q}_{ij}$. They are nonstandard versions of the Beurling–Deny formulae [87, 88], and are often more useful than the expressions (1.5.19), (1.5.22), and (1.5.23).

Lemma 1.5.1. *Let $\mathcal{E}(\cdot,\cdot)$ be the hyperfinite quadratic form as above. Then, we have*

$$(i) \quad \mathcal{E}(u,v) = \frac{1}{\Delta t}\Big[\sum_{1\le i,j\le N}\Big(u(i)-u(j)\Big)v(i)q_{ij}m_i + \sum_{i=1}^{N} u(i)v(i)q_{i0}m_i\Big]. \qquad (1.5.24)$$

$$(ii) \quad \mathcal{E}(u,v) = \frac{1}{\Delta t}\Big[\sum_{1\le i,j\le N}\Big(v(i)-v(j)\Big)u(i)\hat{q}_{ij}m_i + \sum_{i=1}^{N} u(i)v(i)\hat{q}_{i0}m_i\Big]. \qquad (1.5.25)$$

$$\begin{aligned}(iii) \quad \overline{\mathcal{E}}(u,v) = \frac{1}{\Delta t}\Big[&\sum_{1\le i<j\le N}\Big(u(i)-u(j)\Big)\Big(v(i)-v(j)\Big)\overline{q}_{ij}m(i) \\ &+\sum_{i=1}^{N} u(i)v(i)\overline{q}_{i0}m(i)\Big]. \qquad (1.5.26)\end{aligned}$$

Proof. We only prove (iii) since the other proofs are similar. Notice that

$$\begin{aligned}\overline{\mathcal{E}}(u,v) &= \frac{1}{\Delta t}\left[\sum_{i=1}^{N} u(i)v(i)m_i - \sum_{i=1}^{N}\sum_{j=1}^{N} u(j)v(i)\overline{q}_{ij}m_i\right] \\ &= \frac{1}{\Delta t}\left[\sum_{i=1}^{N}\sum_{j=0}^{N} u(i)v(i)\overline{q}_{ij}m_i - \sum_{i=1}^{N}\sum_{j=1}^{N} u(j)v(i)\overline{q}_{ij}m_i\right] \\ &= \frac{1}{\Delta t}\left[\sum_{1\le i,j\le N}\Big(u(i)-u(j)\Big)v(i)\overline{q}_{ij}m_i + \sum_{i=1}^{N} u(i)v(i)\overline{q}_{i0}m_i\right],\end{aligned}$$

where the first line is a trivial modification of the expression (1.5.23), the second line follows from the first since $\sum_{j=0}^{N}\overline{q}_{ij} = 1$, and the last line is just a rearrangement of the second line.

Fix a pair (i,j) and consider the terms in the last expression above involving both i and j. If $i=j$, there is only one such term, and that term is zero. If $i\neq j$, there are two terms to consider, i.e.,

$$(u(i)-u(j))v(i)\overline{q}_{ij}m_i \quad \text{and} \quad (u(j)-u(i))v(j)\overline{q}_{ji}m_j.$$

Since $\overline{q}_{ij}m_i=\overline{q}_{ji}m_j$, the sum of the two terms equals

$$(u(i)-u(j))(v(i)-v(j))\overline{q}_{ij}m_i.$$

Summing over all pairs (i,j), the relation (1.5.26) follows. □

As an immediate consequence, we have:

Corollary 1.5.1. *The hyperfinite quadratic form $\mathcal{E}(\cdot,\cdot)$ associated with Q and m as above is nonnegative.*

In order to get some further properties of the hyperfinite quadratic form $\mathcal{E}(\cdot,\cdot)$, let us introduce the following definitions.

For $u\in H, v\in H$, we define

$$(u\vee v)(s)=\max\{u(s),v(s)\} \text{ and } (u\wedge v)(s)=\min\{u(s),v(s)\}, s\in S_0.$$

Moreover, let $u^+=u\vee 0$ and $u^-=-u\wedge 0$.

If $u\in H$, the function $\tilde{u}=(0\vee u)\wedge 1$ is called the *unit contraction* of u. A quadratic form $\mathcal{E}(\cdot,\cdot)$ is said to have the *Markov property* if for all u

$$\mathcal{E}(\tilde{u},\tilde{u})\leq\mathcal{E}(u,u).$$

Corollary 1.5.2. *Let $\mathcal{E}(\cdot,\cdot)$ be the hyperfinite quadratic form as above. Then*

(1) For all $u\in H, \alpha\in {}^*\mathbb{R}, \alpha\geq 0$, we have

$$\begin{aligned}\mathcal{E}(u\wedge\alpha, u-u\wedge\alpha)&\geq 0,\\ \mathcal{E}(u-u\wedge\alpha, u\wedge\alpha)&\geq 0.\end{aligned}$$

(2) $\mathcal{E}(\tilde{u},\tilde{u})\leq\mathcal{E}(u,u)$.

Proof. (1) By the expression (1.5.24), we have

$$\begin{aligned}\mathcal{E}(u\wedge\alpha, u-u\wedge\alpha)=\frac{1}{\Delta t}\Big[&\sum_{1\leq i,j\leq N}\Big(u(i)\wedge\alpha-u(j)\wedge\alpha\Big)\Big(u-u\wedge\alpha\Big)(i)q_{ij}m_i\\ &+\sum_{i=1}^{N}(u\wedge\alpha)(i)\Big(u(i)-u(i)\wedge\alpha\Big)q_{i0}m_i\Big].\end{aligned} \tag{1.5.27}$$

If $u(i) \geq \alpha$, then

$$\Big(u(i) \wedge \alpha - u(j) \wedge \alpha\Big)\Big(u - u \wedge \alpha\Big)(i) = \Big(\alpha - u(j) \wedge \alpha\Big)\Big(u(i) - \alpha\Big) \\ \geq 0 \tag{1.5.28}$$

and

$$\Big(u \wedge \alpha\Big)(i)\Big(u(i) - u(i) \wedge \alpha\Big) = \alpha\Big(u(i) - \alpha\Big) \\ \geq 0. \tag{1.5.29}$$

If $u(i) < \alpha$, then

$$\Big(u(i) \wedge \alpha - u(j) \wedge \alpha\Big)\Big(u - u \wedge \alpha\Big)(i) = (u \wedge \alpha)(i)\Big(u - u \wedge \alpha\Big)(i) \\ = 0. \tag{1.5.30}$$

It follows from the relations (1.5.27), (1.5.28), (1.5.29), and (1.5.30) that $\mathcal{E}(u \wedge \alpha, u - u \wedge \alpha) \geq 0$. Similarly, we can show that $\mathcal{E}(u - u \wedge \alpha, u \wedge \alpha) \geq 0$ by using the relation (1.5.25).

(2) From the relation (1.5.26), we have $\overline{\mathcal{E}}(\tilde{u}, \tilde{u}) \leq \overline{\mathcal{E}}(u, u)$. This implies the conclusion (2), since $\mathcal{E}(u, u) = \overline{\mathcal{E}}(u, u)$ for all $u \in H$. □

We call a map $T : H \longrightarrow H$ a *Markov operator*, if it maps nonnegative functions into nonnegative functions, and never increases the supremum norm, i.e.,

$$||Tu||_\infty \leq ||u||_\infty$$

for all $u \in H$, where $||u||_\infty = \max_{1 \leq i \leq N} |u(i)|$.

The next result gives us four ways of deciding whether a given quadratic form is a hyperfinite quadratic form associated with some Q and m without actually constructing an associated Markov process.

Proposition 1.5.1. *Let*

$$\mathcal{E}(u, v) = \sum_{i,j=1}^{N} b_{ij} u(i) v(j)$$

be a nonnegative quadratic form which is non-zero. The following statements are equivalent:

(i) *$\mathcal{E}(\cdot,\cdot)$ is the hyperfinite quadratic form of some Q, $\hat{Q}$, and m which satisfy the conditions (1.5.1), (1.5.2), (1.5.4), (1.5.5), (1.5.8), (1.5.9), (1.5.10), (1.5.11), (1.5.13), and (1.5.14).*

(ii) *There exist a hyperfinite measure m on S_0 and two Markov operators $Q^{\Delta t}, \hat{Q}^{\Delta t} : H \longrightarrow H$ such that*

$$\langle Q^{\Delta t} u, v\rangle = \langle u, \hat{Q}^{\Delta t} v\rangle$$

and

$$\begin{aligned}\mathcal{E}(u,v) &= \frac{1}{\Delta t}\langle (I - Q^{\Delta t})u, v\rangle \\ &= \frac{1}{\Delta t}\langle u, (I - \hat{Q}^{\Delta t})v\rangle,\end{aligned}$$

where $(H, \langle\cdot,\cdot\rangle)$ is defined through (1.5.15) by m, and Δt is an infinitesimal.

(iii) *For all $u \in H, \alpha \in {}^*\mathbb{R}, \alpha \geq 0$, we have*

$$\begin{aligned}&\mathcal{E}(u \wedge \alpha, u - u \wedge \alpha) \geq 0, \\ &\mathcal{E}(u - u \wedge \alpha, u \wedge \alpha) \geq 0.\end{aligned}$$

(iv) *$\mathcal{E}(\cdot,\cdot)$ satisfies $\mathcal{E}(\tilde{u}, u - \tilde{u}) \geq 0$ and $\mathcal{E}(u - \tilde{u}, \tilde{u}) \geq 0, \forall u \in H$, where $\tilde{u}$ is the unit contraction of u.*

(v) *Whenever $i \neq j, b_{ij} \leq 0$; but $b_{ii} \geq -\sum_{j\neq i} b_{ij}$ and $b_{ii} \geq -\sum_{j \neq i} b_{ji}$ for all i.*

Proof. We shall prove $(i) \Longrightarrow (iii) \Longrightarrow (iv) \Longrightarrow (v) \Longrightarrow (i) \Longrightarrow (ii) \Longrightarrow (v)$.

$(i) \Longrightarrow (iii)$. This follows immediately from Corollary 1.5.2.

$(iii) \Longrightarrow (iv)$. We have

$$\begin{aligned}\mathcal{E}(\tilde{u}, u - \tilde{u}) &= \mathcal{E}(\tilde{u}, (u \vee 0) - \tilde{u}) + \mathcal{E}(\tilde{u}, u - (u \vee 0)) \\ &= \mathcal{E}((u \vee 0) \wedge 1, (u \vee 0) - (u \vee 0) \wedge 1) + \mathcal{E}(\tilde{u}, u - u \vee 0) \\ &\geq \mathcal{E}(\tilde{u}, u - u \vee 0) \\ &= \mathcal{E}((u \wedge 1)^+, -(u \wedge 1)^-).\end{aligned}$$

Let $v = u \wedge 1$, then

$$\begin{aligned}\mathcal{E}(\tilde{u}, u - \tilde{u}) &\geq \mathcal{E}((u \wedge 1)^+, -(u \wedge 1)^-) \\ &= -\mathcal{E}(v^+, v^-) \\ &= -\mathcal{E}(v^+, v^+ - v) \\ &= \mathcal{E}((-v) \wedge 0, (-v) - (-v) \wedge 0) \\ &\geq 0.\end{aligned}$$

Similarly, we can show

$$\mathcal{E}(u - \tilde{u}, \tilde{u}) \geq 0.$$

$(iv) \implies (v)$. Fix $k, l \in S_0, k \neq l$, define u by

$$u(i) = \begin{cases} -1 & \text{if } i = k, \\ 1 & \text{if } i = l, \\ 0 & \text{otherwise.} \end{cases}$$

If $\tilde{u}$ is the unit contraction of u, we have

$$\begin{aligned} 0 &\leq \mathcal{E}(\tilde{u}, u) - \mathcal{E}(\tilde{u}, \tilde{u}) \\ &= b_{ll} - b_{lk} - b_{ll} \\ &= -b_{lk}. \end{aligned}$$

Therefore, $b_{lk} \leq 0$.

To get the second half of (v), we fix a $k \in S_0$ and consider the function

$$v(i) = \begin{cases} 2 & \text{if } i = k, \\ 1 & \text{otherwise} \end{cases}$$

If $\tilde{v}$ is the unit contraction of v, then

$$\begin{aligned} 0 &\leq \mathcal{E}(\tilde{v}, v - \tilde{v}) \\ &= \sum_{i=1}^{N} b_{ik} \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \mathcal{E}(v - \tilde{v}, \tilde{v}) \\ &= \sum_{j=1}^{N} b_{kj}. \end{aligned}$$

$(v) \implies (i)$. We first observe

$$\begin{aligned} \sum_{1 \leq i,j \leq N} (v(i) - v(j))u(i)b_{ij} &= \sum_{1 \leq i,j \leq N} u(i)v(i)b_{ij} - \sum_{1 \leq i,j \leq N} u(i)v(j)b_{ij} \\ &= \sum_{1 \leq i,j \leq N} u(i)v(i)b_{ij} - \mathcal{E}(u, v). \end{aligned}$$

Thus, we have

$$\mathcal{E}(u,v) = -\sum_{1\le i,j\le N} (v(i)-v(j))u(i)b_{ij} + \sum_{1\le i,j\le N} u(i)v(i)b_{ij}. \quad (1.5.31)$$

Similarly, we have

$$\begin{aligned}\sum_{1\le i,j\le N} (u(i)-u(j))v(i)b_{ji} &= \sum_{1\le i,j\le N} u(i)v(i)b_{ji} - \sum_{1\le i,j\le N} u(j)v(i)b_{ji} \\ &= \sum_{1\le i,j\le N} u(i)v(i)b_{ji} - \mathcal{E}(u,v).\end{aligned}$$

Hence, we have

$$\mathcal{E}(u,v) = -\sum_{1\le i,j\le N} (u(i)-u(j))v(i)b_{ji} + \sum_{1\le i,j\le N} u(i)v(i)b_{ji}. \quad (1.5.32)$$

In the following, we shall define Q and m by marching the two expressions (1.5.25) and (1.5.31), (1.5.24) and (1.5.32) term by term. It turns out that we have one degree of freedom for each i. We may choose

$$0 \le q_{ii} < 1, \quad q_{ii} = \hat{q}_{ii}.$$

Comparing (1.5.25) and (1.5.31), (1.5.24) and (1.5.32), we notice that we must have

$$\begin{aligned}\frac{1}{\Delta t} m_i \hat{q}_{ij} &= -b_{ij} \\ &= \frac{1}{\Delta t} q_{ji} m_j \quad \text{for} \quad j \ne 0, i \end{aligned} \quad (1.5.33)$$

and

$$\begin{aligned}\frac{1}{\Delta t} m_i \hat{q}_{i0} &= \sum_{j=1}^{N} b_{ij}, \\ \frac{1}{\Delta t} m_i q_{i0} &= \sum_{j=1}^{N} b_{ji}. \end{aligned} \quad (1.5.34)$$

In order to have $\sum_{j=0}^{N} \hat{q}_{ij} = 1$ satisfied, we must have

$$
\begin{aligned}
m_i &= \sum_{j=0}^{N} m_i \hat{q}_{ij} \\
&= m_i \hat{q}_{ii} - \Delta t \sum_{j \neq i,0} b_{ij} + \Delta t \sum_{j=1}^{N} b_{ij} \\
&= m_i \hat{q}_{ii} + b_{ii} \Delta t.
\end{aligned}
$$

Hence, this gives us

$$
\begin{aligned}
m_i &= \frac{b_{ii} \Delta t}{1 - q_{ii}} \\
&= \frac{b_{ii} \Delta t}{1 - \hat{q}_{ii}}.
\end{aligned}
$$

Combining this with (1.5.33), we get

$$
\begin{aligned}
q_{ij} &= -\frac{b_{ji}(1 - q_{ii})}{b_{ii}}, \\
\hat{q}_{ij} &= -\frac{b_{ij}(1 - q_{ii})}{b_{ii}} \quad \text{for} \quad j \neq 0, i,
\end{aligned}
$$

and from (1.5.34), we have

$$
\begin{aligned}
\hat{q}_{i0} &= \frac{\sum_{j=1}^{N} b_{ij}(1 - \hat{q}_{ii})}{b_{ii}}, \\
q_{i0} &= \frac{\sum_{j=1}^{N} b_{ji}(1 - q_{ii})}{b_{ii}}.
\end{aligned}
$$

This construction breaks down when $b_{ii} = 0$, but in that case $b_{ij} = 0$ for all j, and we get $m_i = 0$ and can choose the q_{ij}'s arbitrarily.

We finally observe that (1.5.33) implies (1.5.13), and that the non-triviality of $\mathcal{E}(\cdot, \cdot)$ implies (1.5.14).

$(i) \implies (ii)$. This follows immediately from (1.5.16).

$(ii) \implies (v)$. Notice that

$$
Q^{\Delta t} u(i) = u(i) - \Delta t \sum_{j=1}^{N} b_{ji} u(j) \frac{1}{m(i)},
$$

for $m(i) \neq 0$. For $m(i) = 0$, we simplify here $Q^{\Delta t} u(i) = u(i)$. For each $j \in S_0$, let u_j be given by $u_j(i) = \delta_{ij}$. Since $Q^{\Delta t}$ is Markov, we have for all $i \neq j$

$$\begin{aligned}0 &\le Q^{\Delta t} u_j(i) \\ &= -\Delta t b_{ji} \frac{1}{m(i)},\end{aligned}$$

and thus $b_{ji} \le 0$.

On the other hand, applying $Q^{\Delta t}$ to the function which is constant one and using that $Q^{\Delta t}$ cannot increase the supremum norm, we get

$$\begin{aligned}1 &\ge Q^{\Delta t} 1(i) \\ &= 1 - \Delta t \sum_{j=1}^{N} b_{ji} \frac{1}{m(i)}, \text{ if } m(i) \ne 0,\end{aligned}$$

and $Q^{\Delta t} 1(i) = 1$ if $m(i) = 0$. It follows that $b_{ii} \ge -\sum_{j \ne i} b_{ji}$.

Besides, we have

$$\hat{Q}^{\Delta t} u(i) = u(i) - \Delta t \sum_{j=1}^{N} b_{ij} u(j) \frac{1}{m(i)}.$$

Hence, we have

$$\begin{aligned}1 &\ge \hat{Q}^{\Delta t} 1(i) \\ &= 1 - \Delta t \sum_{j=1}^{N} b_{ij} \frac{1}{m(i)}.\end{aligned}$$

From this, we have $b_{ii} \ge -\sum_{j \ne i} b_{ij}$. □

Now it is the time to introduce the following:

Definition 1.5.1. The hyperfinite quadratic form $\mathcal{E}(\cdot, \cdot)$ associated with Q and m is called a *hyperfinite Dirichlet form* if it satisfies the hyperfinite weak sector condition in the sense of Definition 1.4.1.

If $\delta \in T$, let T_δ be the sub-line

$$T_\delta = \{k\delta \mid k \in {}^*\mathbb{N}_0\}.$$

We write $X^{(\delta)}$ for the restriction $X|T_\delta$. For each $t \in T_\delta$, let $\mathcal{F}_t^{(\delta)}$ be the internal algebra on Ω generated by the sets

$$[\omega]_t^{(\delta)} = \left\{\omega' \in \Omega \,\middle|\, X^{(\delta)}(\omega', s) = X^{(\delta)}(\omega, s) \text{ for all } s \in T_\delta, s \le t\right\}. \tag{1.5.35}$$

It is easy to verify the following for all $k \in {}^*\mathbb{N}$,

$$P\Big([\omega]_{k\delta}^{(\delta)}\Big) = m(\omega(0)) \prod_{n=0}^{k-1} q_{\omega(n\delta),\omega((n+1)\delta)}^{(\delta)}$$

and

$$P_i\Big([\omega]_{k\delta}^{(\delta)}\Big) = \delta_{i\omega(0)} \prod_{n=0}^{k-1} q_{\omega(n\delta),\omega((n+1)\delta)}^{(\delta)}.$$

Therefore, for any $t \in T_\delta$ and $u \in H$, we have

$$\begin{aligned} E_i u(X^{(\delta)}(t)) &= Q^t u(i) \\ &= E_i u(X(t)). \end{aligned}$$

In particular, we get

$$\begin{aligned} E_i u(X^{(\delta)}(\delta)) &= Q^\delta u(i) \\ &= \sum_{j=1}^{N} q_{ij}^{(\delta)} u(j). \end{aligned}$$

This implies that the semigroup of $X^{(\delta)}$ is $\{Q^t \mid t \in T_\delta\}$. The infinitesimal generator $A^{(\delta)}$ of $X^{(\delta)}$ is given by

$$A^{(\delta)} u(i) = \frac{1}{\delta}\Big(u(i) - \sum_{j=1}^{N} u(j) q_{ij}^{(\delta)}\Big).$$

Similarly, we can get $\hat{X}^{(\delta)}, \hat{\mathcal{F}}_t^{(\delta)}, [\hat{\omega}]_t^{(\delta)}, \{\hat{Q}^t \mid t \in T_\delta\}, \hat{A}^{(\delta)}$. It is easy to see that $A^{(\delta)}$ and $\hat{A}^{(\delta)}$ are nonnegative, i.e.,

$$\langle A^{(\delta)} u, u\rangle \geq 0 \text{ and } \langle \hat{A}^{(\delta)} u, u\rangle \geq 0 \text{ for all } u \in H.$$

Moreover, we can easily show that

$$\sum_{j=0}^{N} q_{ij}^{(\delta)} = 1 \text{ and } \sum_{j=0}^{N} \hat{q}_{ij}^{(\delta)} = 1 \text{ for all } i = 0, 1, 2 \cdots, N,$$

$$q_{0i}^{(\delta)} = 0 \text{ and } \hat{q}_{0i}^{(\delta)} = 0 \qquad \text{for all} \quad i \neq 0$$

and

$$m_i q_{ij}^{(\delta)} = m_j \hat{q}_{ji}^{(\delta)} \quad \text{for all} \quad i \neq 0, j \neq 0.$$

The hyperfinite quadratic form associated with $Q^{(\delta)}$ and m is defined to be

$$\begin{aligned}\mathcal{E}^{(\delta)}(u,v) &= \langle A^{(\delta)}u, v\rangle \\ &= \sum_{i=1}^{N} A^{(\delta)}u(i)v(i)m(i).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathcal{E}^{(\delta)}(u,v) &= \frac{1}{\delta}\sum_{i=1}^{N}\left[u(i)v(i)m(i) - \sum_{j=1}^{N} u(j)v(i)q_{ij}^{(\delta)}m(i)\right] \\ &= \frac{1}{\delta}\left[\sum_{1\leq i,j\leq N}\Big(u(i)-u(j)\Big)v(i)q_{ij}^{(\delta)}m(i) + \sum_{i=1}^{N} u(i)v(i)q_{i0}^{(\delta)}m(i)\right].\end{aligned}$$

As in Sect. 1.4, let $G_\alpha^{(\delta)} = (A^{(\delta)} - \alpha)^{-1}$ and $\hat{G}_\alpha^{(\delta)} = (\hat{A}^{(\delta)} - \alpha)^{-1}$ be the resolvents of $A^{(\delta)}$ and $\hat{A}^{(\delta)}$, $\alpha \in {}^*(-\infty, 0)$, respectively. Similarly, the domain $\mathcal{D}(\mathcal{E}^{(\delta)})$ of $\mathcal{E}^{(\delta)}(\cdot,\cdot)$ is the set of all $u \in H$ such that

(i) ${}^\circ\mathcal{E}_1^{(\delta)}(u,u) = {}^\circ\Big[\mathcal{E}^{(\delta)}(u,u) + \langle u,u\rangle\Big] < \infty$.

(ii) For all infinitesimal $\alpha < 0$, we have ${}^{(\alpha)}\mathcal{E}(u+\alpha G_\alpha^{(\delta)}u, u+\alpha G_\alpha^{(\delta)}u) \approx 0$ and ${}^{(\alpha)}\mathcal{E}(u+\alpha \hat{G}_\alpha^{(\delta)}u, u+\alpha \hat{G}_\alpha^{(\delta)}u) \approx 0$.

Moreover, we can define $\hat{\mathcal{E}}^{(\delta)}(\cdot,\cdot)$ and $\overline{\mathcal{E}}^{(\delta)}(\cdot,\cdot)$, $\overline{A}^{(\delta)}$, $\{\overline{Q}^t \mid t \in T_\delta\}$ and $\mathcal{D}(\overline{\mathcal{E}}^{(\alpha)})$ as in Sects. 1.1 and 1.2. We can get similar results of the Beurling–Deny formulae for these hyperfinite quadratic forms; in addition, we may get similar results as Proposition 1.5.1 for $\mathcal{E}^{(\delta)}(\cdot,\cdot)$.

Let us try to illustrate the theory by a simple example.

Example 1.5.1. (Brownian motion on a circle). Let $N = (\Delta t)^{-1/2}$ be an even hyperfinite integer, and let $S_0 = \{s_1, \cdots, s_N\}$ be uniformly distributed on a circle of circumference one. If $i, j \in \{1, 2, \cdots, N\}$, let the transition probability q_{ij} be $\frac{1}{2}$ if s_i and s_j are neighbors, and 0 otherwise. The semigroup $\{Q^t\}$ is given by

$$Q^{\Delta t}u(i) = \frac{1}{2}u(i+1) + \frac{1}{2}u(i-1),$$

where the addition is modulo N inside the u's. The infinitesimal generator is

$$Au(i) = -\frac{u(i+1) - 2u(i) + u(i-1)}{2\Delta t}. \tag{1.5.36}$$

If $m_i = \frac{1}{N}$ for all i, the associated hyperfinite quadratic form $\mathcal{E}(\cdot,\cdot)$ is given by

$$\mathcal{E}(u,v) = -\frac{N}{2}\sum_{i=1}^{N}\Big(u(i+1) - 2u(i) + u(i-1)\Big)v(i),$$

or – in the Beurling–Deny formulation –

$$\mathcal{E}(u,v) = \frac{N}{2}\sum_{i=0}^{N}\Big(u(i+1) - u(i)\Big)\Big(v(i+1) - v(i)\Big). \tag{1.5.37}$$

Notice that by (1.5.36), the infinitesimal generator A is a nonstandard version of the operator

$$\tilde{A}f = -\frac{1}{2}f'',$$

and by (1.5.37), the form $\mathcal{E}(\cdot,\cdot)$ is a representation of

$$E(f,g) = \frac{1}{2}\int_C f'g'\,dm,$$

where m is the Lebesgue measure on the circle C, and all derivatives are taken along the circle. Passing from the original expression for $\mathcal{E}(\cdot,\cdot)$ to the Beurling–Deny version amounts to an integration by parts.

1.6 Hyperfinite Representations

In this section, we will relate the results we have obtained so far to the standard Dirichlet space theory. We have proved that a hyperfinite weak coercive quadratic form on a hyperfinite dimensional space H induces a closed form on the hull ${}^\circ H$ of H. For most applications, the space ${}^\circ H$ is too large. What we really want is a form defined on a Hilbert space K given in advance of the nonstandard construction. If K can be identified with a subspace of ${}^\circ H$, we get the desired form by restricting the form on ${}^\circ H$ to K. Noticing that since K is a closed subspace of ${}^\circ H$, the restricted form is also closed. The result we shall prove in this section states that any coercive closed form on any Hilbert space K can be obtained from a nonnegative quadratic form in this way. There are two reasons for proving such a representation theorem. The first and most important is to obtain the “correct” relationship between the already developed standard theory on the one hand, and the new hyperfinite theory on the other. In fact, we shall utilize this relation to solve the problem in classical theory, referring to Theorem 4.1.1 in Chap. 4. The second, closely related reason is to show that no generality is lost by working within the nonstandard framework.

Let K be a standard Hilbert space with an inner product $(\cdot,\cdot)$. A hyperfinite dimensional subspace H of ${}^*K$ is called *S-dense* in ${}^*K$ if for all $x \in K$, there is a $y \in H$ such that $||x-y|| \approx 0$. We recall that $\approx$ is the equivalent relation on H given by $u \approx v$ if and only if $||u-v|| \approx 0$, and that ${}^\circ u$ denotes the equivalence class of u under $\approx$. If H is S-dense in ${}^*K$, we can identify K with a subspace of ${}^\circ H$ by identifying x and ${}^\circ u$ whenever $||x-u|| \approx 0$.

If $\mathcal{E}(\cdot,\cdot)$ is a hyperfinite weak coercive quadratic form on H, we let $E(\cdot,\cdot)$ denote the standard part of $\mathcal{E}(\cdot,\cdot)$ as defined in Definition 1.4.3, and we let $E_K(\cdot,\cdot)$ be the restriction of $E(\cdot,\cdot)$ to K. As mentioned above, our goal is to show that all coercive closed forms can be obtained in this way. Before we prove this, we shall recall a few results from the standard theory of quadratic forms ([270] is a convenient reference).

Let $D(F)$ be a linear subspace of K, and let $F : D(F) \times D(F) \longrightarrow \mathbb{R}$ be a bilinear map which is nonnegative, i.e., $F(x,x) \geq 0$ for all $x \in D(F)$. For $\alpha \in \mathbb{R}_+$, we set

$$F_\alpha(\cdot,\cdot) = F(\cdot,\cdot) + \alpha(\cdot,\cdot).$$

$(F(\cdot,\cdot), D(F))$ is said to satisfy the *weak sector condition* if

$$|F_1(x,y)| \leq C\sqrt{F_1(x,x)}\sqrt{F_1(y,y)} \text{ for all } x,y \in D(F),$$

where C is positive real number which is called a *continuity constant.*

$(F(\cdot,\cdot), D(F))$ is said to be *closed* if $D(F)$ is complete with respect to the norm $\sqrt{F_1(\cdot,\cdot)}$.

We say that $(F(\cdot,\cdot), D(F))$ is a *coercive closed form* on K if and only if

(1) $D(F)$ is a dense subspace of K.
(2) $(F(\cdot,\cdot), D(F))$ satisfies the weak sector condition.
(3) $(F(\cdot,\cdot), D(F))$ is a closed form.

Proposition 1.6.1. *Let $(F(\cdot,\cdot), D(F))$ be a coercive closed form on K with continuity constant C. Then there exist unique strongly continuous contraction resolvent $\{R_\alpha \mid \alpha \in (-\infty,0)\}$ and co-resolvent $\{\hat{R}_\alpha \mid \alpha \in (-\infty,0)\}$ on K such that*

$$R_\alpha(K), \hat{R}_\alpha(K) \subset D(F) \text{ and } F_{-\alpha}(R_\alpha f, u) = (f,u) = F_{-\alpha}(u, \hat{R}_\alpha f) \tag{1.6.1}$$

for all $f \in K, u \in D(F), \alpha \in (-\infty, 0)$. *In particular, we have*

$$(R_\alpha f, g) = (f, \hat{R}_\alpha g) \text{ for all } f, g \in K,$$

i.e., $\hat{R}_\alpha$ *is the adjoint of* R_α *for all* $\alpha \in (-\infty, 0)$.

Proof. We refer to Ma and Röckner [270], Chap. I, Theorem 2.8 and use negative value of α for the resolvents. □

We assume that $(F(\cdot,\cdot), D(F))$ is a coercive closed form on K with continuity constant C. Let $(L, D(L))$ and $(\hat{L}, D(\hat{L}))$ be the generators of $\{R_\alpha \mid \alpha \in (-\infty, 0)\}$ and $\{\hat{R}_\alpha \mid \alpha \in (-\infty, 0)\}$, respectively. Then we have from Ma and Röckner [270], Chap. I, Corollary 2.10,

$$R_\alpha(K) = D(L) \subset D(F) \text{ and } F(u, v) = (Lu, v), \forall u \in D(L), v \in D(F)$$

and

$$\hat{R}_\alpha(K) = D(\hat{L}) \subset D(F) \text{ and } F(v, u) = (\hat{L}u, v), \forall u \in D(\hat{L}), v \in D(F).$$

Define for $\alpha \in (-\infty, 0), u, v \in K$,

$$\begin{aligned} {}^{(\alpha)}F(u, v) &= -\alpha(u + \alpha R_\alpha u, v) \\ &= -\alpha(v + \alpha \hat{R}_\alpha v, u) \end{aligned}$$

and

$$\begin{aligned} {}^{(\alpha)}\hat{F}(u, v) &= -\alpha(v + \alpha R_\alpha v, u) \\ &= -\alpha(u + \alpha \hat{R}_\alpha u, v). \end{aligned}$$

Then for all $\alpha \in (-\infty, 0), u \in K$, we have

$${}^{(\alpha)}F(u, -\alpha R_\alpha u) = {}^{(\alpha)}F(u, u) + \alpha(u + \alpha R_\alpha u, u + \alpha R_\alpha u)$$

and

$${}^{(\alpha)}F(-\alpha \hat{R}_\alpha u, u) = {}^{(\alpha)}F(u, u) + \alpha(u + \alpha \hat{R}_\alpha u, u + \alpha \hat{R}_\alpha u).$$

Therefore, we have for all $\alpha \in (-\infty, 0), u \in K$,

$$\begin{aligned} {}^{(\alpha)}F(u, -\alpha R_\alpha u) &\leq {}^{(\alpha)}F(u, u), \\ {}^{(\alpha)}F(-\alpha \hat{R}_\alpha u, u) &\leq {}^{(\alpha)}F(u, u). \end{aligned}$$

Proposition 1.6.2. *Let* $(F(\cdot,\cdot), D(F))$ *be a coercive closed form on* K *with continuity constant* $C > 0$. *Then, we have*

(i) ${}^{(\alpha)}F(u,v) = F(-\alpha R_\alpha u, v)$ *and* ${}^{(\alpha)}\hat{F}(u,v) = F(-\alpha \hat{R}_\alpha u, v)$ *for all* $u \in K, v \in D(F)$.
(ii) $F(\alpha R_\alpha u, \alpha R_\alpha u) \leq {}^{(\alpha)}F(u,u)$ *and* $F(\alpha \hat{R}_\alpha u, \alpha \hat{R}_\alpha u) \leq {}^{(\alpha)}F(u,u)$ *for all* $u \in K$ *and* $\alpha \in (-\infty, 0)$.
(iii) $|{}^{(\alpha)}F_1(u,v)| \leq (C+1)\sqrt{F_1(u,u)}\sqrt{{}^{(\alpha)}F_1(v,v)}$ *for all* $u \in D(F), v \in K$.
(iv) $F_1(\alpha R_\alpha u, \alpha R_\alpha u) \leq (C+1)^2 F_1(u,u)$ *for all* $u \in D(F)$.

Proof. We refer to Ma and Röckner [270], Chap. I, Lemma 2.11. □

Proposition 1.6.3. *Let* $(F(\cdot,\cdot), D(F))$ *be a coercive closed form on* K *with continuity constant* $C > 0$. *Then*

(i) *Let* $u \in K$. *Then* $u \in D(F)$ *if and only if* $\sup_{\alpha \in (-\infty,0)} {}^{(\alpha)}F(u,u) < \infty$.
(ii) $D(L) = R_\alpha(K)$ *is dense in* $D(F)$ *with respect to* $\sqrt{F_1(\cdot,\cdot)}$. *Moreover, we have for all* $u \in D(F)$

$$\lim_{\alpha \longrightarrow -\infty} F_1(\alpha R_\alpha u + u, \alpha R_\alpha u + u) = 0.$$

In particular, $(F(\cdot,\cdot), D(F))$ *is determined by* $\{R_\alpha \mid \alpha \in (-\infty, 0)\}$ *via (1.6.1) uniquely.*
(iii) $\lim_{\alpha \longrightarrow -\infty} {}^{(\alpha)}F(u,v) = F(u,v)$ *for all* $u, v \in D(F)$.

Proof. We refer to Ma and Röckner [270], Chap. I, Theorem 2.13. □

Let $(F(\cdot,\cdot), D(F))$ be a coercive closed form on K with continuity constant $C > 0$. Denote by $\{T_t \mid t \in [0,\infty)\}$ and $\{\hat{T}_t \mid t \in [0,\infty)\}$ the strongly continuous contraction semigroups corresponding to $\{R_\alpha \mid \alpha \in (-\infty, 0)\}$ and $\{\hat{R}_\alpha \mid \alpha \in (-\infty, 0)\}$, respectively. Then, we have from Ma and Röckner [270], Chap. I, Theorem 2.8 that

$$(T_t f, g) = (f, \hat{T}_t g) \text{ for all } f, g \in K, t > 0.$$

Thereafter, $\{R_\alpha \mid \alpha \in (-\infty, 0)\}$ and $\{\hat{R}_\alpha \mid \alpha \in (-\infty, 0)\}$ shall be called *resolvent* and *co-resolvent* of $(F(\cdot,\cdot), D(F))$, respectively. Similarly, $\{T_t \mid t \in [0,\infty)\}$ and $\{\hat{T}_t \mid t \in [0,\infty)\}$ shall be called *semigroup* and *co-semigroup of* $(F(\cdot,\cdot), D(F))$, respectively. For $t \in (0,\infty)$, we define

$$F^{(t)}(u,v) = \frac{1}{t}(u - T_t u, v), u, v \in K.$$

As in Proposition 1.6.3, we have

Proposition 1.6.4. *Let* $(F(\cdot,\cdot), D(F))$ *be a coercive closed form on* K *with continuity constant* $C > 0$. *Then*

(i) *Let* $u \in K$*. Then* $u \in D(F)$ *if and only if* $\sup_{t>0} F^{(t)}(u,u) < \infty$*.*
(ii) *For all* $u, v \in D(F)$*, we have* $\lim_{t \downarrow 0} F^{(t)}(u,v) = F(u,v)$*.*
(iii) $\lim_{t \downarrow 0} F_1(u - T_t u, u - T_t u) = 0$ *for all* $u \in D(F)$*.*

Proof. We refer to Albeverio et al. [9], Theorem 3.4. □

Proposition 1.6.5. *Let* $(F(\cdot,\cdot), D(F))$ *be a coercive closed form on* K *with continuity constant* $C > 0$*. Then, we have for* $n \in \mathbb{N}$ *and* $\forall f \in K$

$$T_t f = \lim_{n \longrightarrow \infty} \left(I + \frac{t}{n} L \right)^{-n} f$$
$$= \lim_{n \longrightarrow \infty} \left[\frac{n}{t} R_{-n/t} \right]^n f$$

and

$$\hat{T}_t f = \lim_{n \longrightarrow \infty} \left(I + \frac{t}{n} \hat{L} \right)^{-n} f$$
$$= \lim_{n \longrightarrow \infty} \left[\frac{n}{t} \hat{R}_{-n/t} \right]^n f.$$

Proof. We refer to Pazy [299], Theorem 8.3, Chap. I, page 33. □

Let st_K be the standard part map from ${}^*K$ to K.

Theorem 1.6.1. *Let* $(F(\cdot,\cdot), D(F))$ *be a coercive closed form on a Hilbert space* K*, and let* H *be an* S*-dense, hyperfinite dimensional subspace of* ${}^*K$*. Then, there exists a nonnegative quadratic form* $\mathcal{E}(\cdot,\cdot)$ *on* H *– associated with an internal time line* $T = \{0, \Delta t, 2\Delta t, \cdots, k\Delta t, \cdots\} = \{k\Delta t \mid k \in {}^*\mathbb{N}_0\}$ *with an infinitesimal* $\Delta t > 0$ *– such that*

$$F(\cdot,\cdot) = E_K(\cdot,\cdot). \tag{1.6.2}$$

Moreover, if $\{G_\alpha \mid \alpha \in {}^*(-\infty, 0)\}$ *and* $\{\hat{G}_\alpha \mid \alpha \in {}^*(-\infty, 0)\}$ *are the resolvent and co-resolvent generated by* $\mathcal{E}(\cdot,\cdot)$*, then for all* $\beta \in (-\infty, 0), \alpha \in {}^*(-\infty, 0), u \in K, v \in H$ *such that* $\beta = {}^\circ\alpha, u = st_K(v)$*, we have*

$$st_K G_\alpha v = R_\beta u \text{ and } st_K \hat{G}_\alpha v = \hat{R}_\beta u. \tag{1.6.3}$$

On the other hand, if $\{Q^s \mid s \in T\}$ *and* $\{\hat{Q}^s \mid s \in T\}$ *are the semigroup and co-semigroup generated by* $\mathcal{E}(\cdot,\cdot)$*, then for all* $t \in [0, \infty), s \in T, u \in K, v \in H$ *such that* $t = {}^\circ s, u = st_K(v)$*, we have*

$$st_K Q^s v = T_t u \text{ and } st_K \hat{Q}^s v = \hat{T}_t u. \tag{1.6.4}$$

Proof. Let P be the projection of ${}^*K$ on H. Let us write $\{{}^*R_\alpha\}_{\alpha\in{}^*(-\infty,0)}$ for ${}^*(\{R_\alpha\}_{\alpha\in(-\infty,0)})$, and $\{{}^*\hat{R}_\alpha\}_{\alpha\in{}^*(-\infty,0)}$ for ${}^*(\{\hat{R}_\alpha\}_{\alpha\in(-\infty,0)})$. Our plan is first to define an internal semigroup by putting $Q^{\Delta t} = -P^*(\gamma R_\gamma)$ for a carefully chosen infinitesimal $\Delta t = -\frac{1}{\gamma}$, and then let

$$\begin{aligned}\mathcal{E}(u,v) &= \frac{1}{\Delta t}\langle (I - Q^{\Delta t})u, v\rangle \\ &= -\gamma\langle (I + P^*(\gamma R_\gamma))u, v\rangle,\end{aligned} \tag{1.6.5}$$

by using Proposition 1.6.3. Notice that if $u \in H$, then

$$\begin{aligned}\langle Q^{\Delta t}u, u\rangle &= -\langle P^*(\gamma R_\gamma)u, u\rangle \\ &= -\langle {}^*(\gamma R_\gamma)u, u\rangle \\ &\geq 0.\end{aligned}$$

This shows that $Q^{\Delta t}$ is positive on H. Also, since the operator norm of ${}^*(\gamma R_\gamma)$ is less than or equal to one, so is the norm of $Q^{\Delta t}$. Hence, the conditions in Sect. 1.1 are satisfied.

We shall now choose Δt such that the relations (1.6.2), (1.6.3), and (1.6.4) hold. If $u \in H$ is nearstandard and $-{}^\circ\alpha < \infty$, then

$$||P^*(\alpha R_\alpha)u - {}^*(\alpha R_\alpha)u|| \approx 0 \text{ and } ||P^*(\alpha \hat{R}_\alpha)u - {}^*(\alpha \hat{R}_\alpha)u|| \approx 0$$

because ${}^*(\alpha R_\alpha)$ and ${}^*(\alpha\hat{R}_\alpha)$ take the nearstandard elements to nearstandard elements, and because H is S-dense in ${}^*K$. By induction, we get

$$||[P^*(\alpha R_\alpha)]^n u - {}^*(\alpha R_\alpha)^n u|| \approx 0 \text{ and } ||[P^*(\alpha \hat{R}_\alpha)]^n u - {}^*(\alpha \hat{R}_\alpha)^n u|| \approx 0$$

for all $n \in \mathbb{N}$. For each $u \in K$, let $v_u = Pu$. Then $v_u \in H$ and $||u - v_u|| \approx 0$. We consider the set

$$\begin{aligned}A_u = \Bigg\{ n \in {}^*\mathbb{N} \,\Bigg|\, \forall k \leq 2^{2n} \Big(&||[P^*(2^n R_{-2^n})]^k v_u - [{}^*(2^n R_{-2^n})]^k u|| \leq \frac{1}{n} \text{ and} \\ &||[P^*(2^n \hat{R}_{-2^n})]^k v_u - [{}^*(2^n \hat{R}_{-2^n})]^k u|| \leq \frac{1}{n}\Big)\Bigg\}.\end{aligned} \tag{1.6.6}$$

For each $u \in K$, this set contains $\mathbb{N}$, and hence an internal segment $\{n \in {}^*\mathbb{N} \mid n \leq n_u\}$. By saturation, there is an infinite n smaller than all the n_u's, $u \in K$.

Next, we consider

$$B_u = \left\{ m \in {}^*\mathbb{N} \middle| \forall k \leq 2^{2m} \Big(|k\langle (I + kR_{-k})v_u, v_u \rangle - k\langle (I + P^*(kR_{-k}))v_u, v_u \rangle| \leq \frac{1}{m} \Big) \right\}.$$

For each $u \in K$, this set contains $\mathbb{N}$, and hence an internal segment $\{m \in {}^*\mathbb{N} \mid m \leq m_u\}$. By saturation, there is an infinite m smaller than all the m_u's, $u \in K$.

We now take Δt to be the infinitesimal $2^{-m} \vee 2^{-n}$. That is, $\Delta t = 2^{-m} \vee 2^{-n}$, or $\gamma = -2^m \wedge 2^n$. Since $R_\alpha = (L - \alpha)^{-1}$, we have $L = \alpha + R_\alpha^{-1}$ for all $\alpha \in (-\infty, 0)$. Hence, we have for all $u \in D(L) = R_1(K)$,

$$\begin{aligned} -\alpha(I + \alpha R_\alpha)u &= -\alpha R_\alpha (R_\alpha^{-1} + \alpha)u \\ &= -\alpha R_\alpha(Lu) \\ &\longrightarrow Lu, \alpha \longrightarrow -\infty, \alpha \in (-\infty, 0). \end{aligned}$$

Therefore, we get for all infinite $\alpha \in {}^*(-\infty, 0)$

$$\begin{aligned} 0 &\approx ||Lu + \alpha R_\alpha(Lu)|| \\ &= ||Lu + \alpha(I + \alpha R_\alpha)u||, \forall u \in D(L). \end{aligned} \tag{1.6.7}$$

From the relations (1.6.5) and (1.6.7), we have for all $u \in D(L) = R_1(K)$ that

$$^\circ\mathcal{E}(Pu, Pu) = F(u, u).$$

Because of the closedness of $F(\cdot, \cdot)$, we have got the following equation for all $u \in D(F)$

$$\begin{aligned} ^\circ\mathcal{E}(Pu, Pu) &= E_K(u, u) \\ &= F(u, u). \end{aligned}$$

This is (1.6.2). From (1.6.6) and Proposition 1.6.5, we have the results (1.6.4).

In the following, we shall show that $\mathcal{E}(\cdot, \cdot)$ satisfies (1.6.3). For all $\beta \in (-\infty, 0)$, $\alpha \in {}^*(-\infty, 0), u \in K, v \in H$ such that $\beta = {}^\circ\alpha, u = st_K(v)$, we have

$$\mathcal{E}_{-\alpha}(G_\alpha v, G_\alpha v - R_\alpha v_u) = \langle v, G_\alpha v - R_\alpha v_u \rangle$$

and

$$\begin{aligned}
&\mathcal{E}_{-\alpha}(R_\alpha v_u, G_\alpha v - R_\alpha v_u)\\
&\quad = -\gamma\langle (I + P^*(\gamma R_\gamma))R_\alpha v_u, G_\alpha v - R_\alpha v_u\rangle - \alpha\langle R_\alpha v_u, G_\alpha v - R_\alpha v_u\rangle\\
&\quad \approx -\gamma\langle (I + \gamma R_\gamma))R_\alpha v_u, G_\alpha v - R_\alpha v_u\rangle - \alpha\langle R_\alpha v_u, G_\alpha v - R_\alpha v_u\rangle\\
&\quad \approx \langle L R_\alpha v_u, G_\alpha v - R_\alpha v_u\rangle - \alpha\langle R_\alpha v_u, G_\alpha v - R_\alpha v_u\rangle\\
&\quad \approx \langle v, G_\alpha v - R_\alpha v_u\rangle.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
|\alpha| \cdot ||G_\alpha v - R_\alpha v_u|| &\leq \mathcal{E}_{-\alpha}(G_\alpha v - R_\alpha v_u, G_\alpha v - R_\alpha v_u)\\
&\approx 0.
\end{aligned}$$

This implies the relations (1.6.3). □

Now let us make a few comments on Theorem 1.6.1.

Remark 1.6.1. In the proof of Theorem 1.6.1, we apply Proposition 1.6.3 by using the resolvent $\{R_\alpha \mid \alpha \in (-\infty, 0)\}$ to construct $Q^{\Delta t}$. An alternative way is to apply Proposition 1.6.4 by using the semigroup $\{T_t \mid t \in (0, \infty)\}$ to construct $Q^{\Delta t}$, and the readers may refer to the proof of 5.2.1. Proposition, Albeverio et al. [25] for the details.

Remark 1.6.2. The assumption that $(F(\cdot,\cdot), D(F))$ is densely defined is for convenience only. If it is not satisfied, we just apply the proposition to the closure of $D(F)$. If $F(\cdot,\cdot)$ is not closed, we obviously cannot obtain $F(\cdot,\cdot)$ as $E_K(\cdot,\cdot)$ for any hyperfinite form $\mathcal{E}(\cdot,\cdot)$ since we need the closedness of $F(\cdot,\cdot)$ in the proof of Theorem 1.6.1. However, if $F(\cdot,\cdot)$ is closable (i.e., there exists a closed form extending $F(\cdot,\cdot)$), all closed extensions of $F(\cdot,\cdot)$ can be represented as standard parts of hyperfinite forms. A natural representation for a closed form $F(\cdot,\cdot)$ would be a representation of its smallest closed extension – the Friedrichs extension. If $F(\cdot,\cdot)$ is not closable, no hyperfinite representation (in our sense) is possible. Any representation we try will change some $F(\cdot,\cdot)$ values, and restrict and extend $D(F)$ in different directions in order to turn $F(\cdot,\cdot)$ into a closed form. As we commented in Sect. 1.2, the fact that non-closable forms do not have hyperfinite representation is more than a curse. In the standard theory, a lot of effort goes into showing that the forms one constructs are closable. In the hyperfinite theory, this is an immediate consequence of the construction.

Remark 1.6.3. In Theorem 1.6.1, the space H was just any S-dense, hyperfinite dimensional subspace of ${}^*K$. In applications, we often want to choose special kind of subspaces which are appropriate for the problems we have in mind. In Chap. 4, we shall study the case $K = L^2(Y, \nu)$ for some Hausdorff space Y.

Remark 1.6.4. The lifting $\mathcal{E}(\cdot,\cdot)$ for a coercive closed form $F(\cdot,\cdot)$ in Theorem 1.6.1 needs not satisfy the hyperfinite weak sector condition. Hence, $\mathcal{E}(\cdot,\cdot)$ is not necessarily a hyperfinite weak coercive quadratic form. This makes it impossible for us to use the results in Sect. 1.4. However, we do have some good property about $\mathcal{E}(\cdot,\cdot)$ which will be very useful in Chap. 4. Actually, we have the following from Proposition 1.6.2:

Corollary 1.6.1. *Let $(F(\cdot,\cdot),D(F))$ be a coercive closed form on K with continuity constant $C>0$. For $\alpha\in{}^*(-\infty,0)$, we define for $u,v\in{}^*K$*

$$\begin{aligned}{}^{(\alpha)}F(u,v)&=-\alpha(u+\alpha R_\alpha u,v)\\&=-\alpha(v+\alpha\hat{R}_\alpha v,u)\end{aligned}$$

and

$$\begin{aligned}{}^{(\alpha)}\hat{F}(u,v)&=-\alpha(v+\alpha R_\alpha v,u)\\&=-\alpha(u+\alpha\hat{R}_\alpha u,v).\end{aligned}$$

Then

(i) ${}^{(\alpha)}F(u,v)=F(-\alpha R_\alpha u,v)$ *and* ${}^{(\alpha)}\hat{F}(u,v)=F(-\alpha\hat{R}_\alpha u,v)$ *for all* $u\in{}^*K,v\in{}^*(D(F))$ *and* $\alpha\in{}^*(-\infty,0)$.
(ii) $F(\alpha R_\alpha u,\alpha R_\alpha u)\le{}^{(\alpha)}F(u,u)$ *and* $F(\alpha\hat{R}_\alpha u,\alpha\hat{R}_\alpha u)\le{}^{(\alpha)}F(u,u)$ *for all* $u\in{}^*K$ *and* $\alpha\in{}^*(-\infty,0)$.
(iii) $|{}^{(\alpha)}F_1(u,v)|\le(C+1)\sqrt{F_1(u,u)}\sqrt{{}^{(\alpha)}F_1(v,v)}$ *for all* $u\in{}^*(D(F))$, $v\in{}^*K$ *and* $\alpha\in{}^*(-\infty,0)$.
(iv) $F_1(\alpha R_\alpha u,\alpha R_\alpha u)\le(C+1)^2F_1(u,u)$ *for all* $u\in{}^*(D(F))$ *and* $\alpha\in{}^*(-\infty,0)$.

Proof. The proof follows from Proposition 1.6.2. □

When are the standard forms generated by two hyperfinite forms different? The last result in this section we shall prove shows that to answer this question, it is enough to check whether the forms have the same resolvents. We recall that in Theorem 1.4.1 we found a way to construct a form from its resolvent. This representation will be helpful to solve our problem.

Theorem 1.6.2. *Let K be a Hilbert space and H be an S-dense, hyperfinite dimensional subspace of ${}^*K$. Let $\mathcal{E}(\cdot,\cdot)$ and $\breve{\mathcal{E}}(\cdot,\cdot)$ be two hyperfinite weak coercive quadratic forms on H inducing $E_K(\cdot,\cdot)$ and $\breve{E}_K(\cdot,\cdot)$ on K, respectively. Let $\{G_\alpha\}$ and $\{\breve{G}_\alpha\}$ be the resolvents of $\mathcal{E}(\cdot,\cdot)$ and $\breve{\mathcal{E}}(\cdot,\cdot)$. Assume that for some finite, non-infinitesimal $\alpha\in{}^*(-\infty,0)$, there is a $u\in H$ with ${}^\circ\mathcal{E}_1(u,u)<\infty$ such that $v=G_\alpha u$, $w=\breve{G}_\alpha u$ are both nearstandard, but ${}^\circ||v-w||\neq 0$. Then $E_K(\cdot,\cdot)\neq\breve{E}_K(\cdot,\cdot)$.*

Proof. Assume for contradiction that $E_K(\cdot,\cdot)=\breve{E}_K(\cdot,\cdot)$. Pick $\tilde{v}\approx v$, $\tilde{w}\approx w$ such that $\tilde{v}\in\mathcal{D}(\breve{\mathcal{E}}),\tilde{w}\in\mathcal{D}(\mathcal{E})$. Notice that by Lemma 1.4.5, $v\in\mathcal{D}(\mathcal{E}),w\in\mathcal{D}(\breve{\mathcal{E}})$. We have

$$\begin{aligned}\langle u, v - w\rangle &\approx \langle u, v - \tilde{w}\rangle \\ &= \mathcal{E}_{-\alpha}(v, v - \tilde{w}).\end{aligned} \tag{1.6.8}$$

Since v, w are nearstandard and $E_K(\cdot,\cdot) = \breve{E}_K(\cdot,\cdot)$, we have

$$\begin{aligned}{}^\circ\mathcal{E}_{-\alpha}(v, v - \tilde{w}) &= E_K({}^\circ v, {}^\circ v - {}^\circ w) - ({}^\circ\alpha)({}^\circ v, {}^\circ v - {}^\circ w) \\ &= \breve{E}_K({}^\circ v, {}^\circ v - {}^\circ w) - ({}^\circ\alpha)({}^\circ v, {}^\circ v - {}^\circ w) \\ &= {}^\circ\breve{\mathcal{E}}_{-\alpha}(\tilde{v}, \tilde{v} - w).\end{aligned} \tag{1.6.9}$$

On the other hand, we have

$$\begin{aligned}\langle u, v - w\rangle &\approx \langle u, \tilde{v} - w\rangle \\ &= \breve{\mathcal{E}}_{-\alpha}(w, \tilde{v} - w).\end{aligned} \tag{1.6.10}$$

Combining the relations (1.6.8), (1.6.9), and (1.6.10), we see that

$$\begin{aligned}0 &= {}^\circ\breve{\mathcal{E}}_{-\alpha}(\tilde{v} - w, \tilde{v} - w) \\ &\geq {}^\circ|\alpha|^\circ||v - w||^2 \\ &> 0.\end{aligned}$$

The theorem is proved. □

1.7 Weak Coercive Quadratic Forms, Revisited

Let $\mathcal{E}(\cdot,\cdot)$ be a hyperfinite weak coercive quadratic form on a hyperfinite dimensional space H. Let $\{G_\alpha \mid \alpha \in {}^*(-\infty, 0)\}$ be the resolvent of $\mathcal{E}(\cdot,\cdot)$, and let $\{\hat{G}_\alpha \mid \alpha \in {}^*(-\infty, 0)\}$ be the co-resolvent of $\mathcal{E}(\cdot,\cdot)$, respectively. Let us still denote by A the infinitesimal generator of $\mathcal{E}(\cdot,\cdot)$, and by $\hat{A}$ the infinitesimal co-generator of $\mathcal{E}(\cdot,\cdot)$. Such as in Sect. 1.1, we fix an infinitesimal Δt, and we define new operators $Q^{\Delta t}$ and $\hat{Q}^{\Delta t}$ by

$$\begin{aligned}Q^{\Delta t} &= I - \Delta t A, \\ \hat{Q}^{\Delta t} &= I - \Delta t \hat{A}.\end{aligned}$$

Introduce a nonstandard time line T by $T = \{k\Delta t \mid k \in {}^*\mathbb{N}_0\}$. For each element $t = k\Delta t \in T$, define the semigroup Q^t and the co-semigroup $\hat{Q}^t$ to be the families of operators

$$\begin{aligned}Q^t &= (Q^{\Delta t})^k, \\ \hat{Q}^t &= (\hat{Q}^{\Delta t})^k, t \in T.\end{aligned}$$

For each $t \in T$, we may define approximations $A^{(t)}$ of A and $\hat{A}^{(t)}$ of $\hat{A}$ by

$$A^{(t)} = \frac{1}{t}\left(I - Q^t\right),$$
$$\hat{A}^{(t)} = \frac{1}{t}\left(I - \hat{Q}^t\right).$$

From $A^{(t)}$ and $\hat{A}^{(t)}$, we get the forms $\mathcal{E}^{(t)}(u,v) = \langle A^{(t)}u, v\rangle = \langle u, \hat{A}^{(t)}v\rangle$.

Let $E(\cdot,\cdot)$ be the standard part of $\mathcal{E}(\cdot,\cdot)$. Then $E(\cdot,\cdot)$ is closed. In addition, $(E(\cdot,\cdot), D(E))$ satisfies the weak sector condition by Remark 1.4.1. Hence, $(E(\cdot,\cdot), D(E))$ is a coercive closed form on ${}^\circ H$. Let $\{R_\beta \mid \beta \in (-\infty, 0)\}$ and $\{\hat{R}_\beta \mid \beta \in (-\infty, 0)\}$ be the resolvent and co-resolvent of $(E(\cdot,\cdot), D(E))$, respectively. Similarly, let $\{T_t \mid t \in [0,\infty)\}$ and $\{\hat{T}_t \mid t \in [0,\infty)\}$ be the semigroup and co-semigroup of $(E(\cdot,\cdot), D(E))$, respectively. For $t \in (0,\infty)$, we define

$$E^{(t)}(x,y) = \frac{1}{t}(x - T_t x, y), x, y \in {}^\circ H.$$

By Albeverio et al. [9], Theorem 3.4 (or referring to Proposition 1.6.4), we have

Lemma 1.7.1. *(i) Let $x \in {}^\circ H$. Then $x \in D(E)$ if and only if* $\sup_{t>0} E^{(t)}$ $(x,x) < \infty$.

(ii) For all $x, y \in D(E)$, we have

$$\lim_{t\downarrow 0} E^{(t)}(x,y) = E(x,y).$$

(iii) For all $x \in D(E)$, we have

$$\lim_{t\downarrow 0} E_1(x - T_t x, x - T_t x) = 0.$$

By applying Lemma 1.7.1, we can see from the proof of Theorem 1.6.1 that there exists an infinitesimal $\delta \in T$ such that $(E(\cdot,\cdot), D(E))$ is the standard part of $\mathcal{E}^{(\delta)}(u,v)$ (referring to Remark 1.6.1, and replacing $-\gamma R_\gamma$ by Q^δ in the proof of Theorem 1.6.1). In addition, for any $x \in {}^\circ H, u \in x$, and for all $t \in [0,\infty), s \in \{k\delta \mid k \in {}^*\mathbb{N}_0\}$, $t = {}^\circ s$, we have that $Q^s u \in T_t x$. Since $Q^{s_1+s_2} \approx Q^{s_1}$ if $s_2 \approx 0$, we have that $Q^s u \in T_t x$ for all $t \in [0,\infty), s \in T = \{k\varDelta t \mid k \in {}^*\mathbb{N}_0\}$, $t = {}^\circ s$. Notice that $(E(\cdot,\cdot), D(E))$ is the standard part of both $\mathcal{E}(u,v)$ and $\mathcal{E}^{(\delta)}(u,v)$, and so the resolvent of $\mathcal{E}(u,v)$ is almost the same as that of $\mathcal{E}^{(\delta)}(u,v)$ by Theorem 1.6.2.

Summarizing above results, we have

Theorem 1.7.1. *Let $\mathcal{E}(\cdot,\cdot)$ be a hyperfinite weak coercive quadratic form on a hyperfinite dimensional space H. Then, we have* *(i)* *Let* $u \in H$*. Then* $u \in \mathcal{D}(\mathcal{E})$ *if and only if* $\sup_s {}^\circ\mathcal{E}^{(s)}(u,u) < \infty$*.*

(ii) *For all* $u, v \in \mathcal{D}(\mathcal{E})$*, we have*

$$\lim_{^\circ s \downarrow 0} \mathcal{E}^{(s)}(u,v) = \mathcal{E}(u,v).$$

(iii) *For all* $u \in \mathcal{D}(\mathcal{E})$*, we have*

$$\lim_{^\circ s \downarrow 0} \mathcal{E}_1(u - Q^s u, u - Q^s u) \approx 0.$$

Proof. Let $x = {}^\circ u$. Then for all $t \in [0,\infty), s \in T = \{k\Delta t \mid k \in {}^*\mathbb{N}_0\}$, $t = {}^\circ s$, we have that $Q^s u \in T_t x$. By Lemma 1.7.1, the theorem follows easily. □

Proposition 1.7.1. *Let* $\mathcal{E}(\cdot,\cdot)$ *be a hyperfinite weak coercive quadratic form on a hyperfinite dimensional linear space* H*. If* ${}^\circ\mathcal{E}(u,u) < \infty$*, then for all finite* $s > 0, s \in T$*, and* ${}^\circ s \neq 0$*, we have* ${}^\circ[Q^s u] \in D(E)$*.*

Proof. Let $x = {}^\circ u$. For $t = {}^\circ s$, we have that $Q^s u \in T_t x$. Therefore, we have ${}^\circ[Q^s u] = T_t x \in D(E)$. □

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