Preface

The theory of stochastic processes has developed rapidly in the past decades. Martingale theory and the study of smooth diffusion processes as solutions of stochastic differential equations have been extended in several directions, such as the study of infinite dimensional diffusion processes, the study of diffusion processes with non-smooth unbounded coefficients, diffusion processes on manifolds and on singular spaces. The interplay between stochastic analysis and mathematical physics has been one of the most important and exciting research areas.

One of the best techniques to deal with the problems of these areas is Dirichlet space theory. In the original framework of this theory, the state space is a locally compact separable metric space, e.g., $\mathbb{R}^d$, or a d-dimensional manifold. This theory has given us a nice understanding about the property of diffusion processes with non-smooth unbounded coefficients. Moreover, it has been fruitfully applied to mathematical physics. This framework has been generalized to state spaces which are more general topological spaces or some infinite dimensional vector spaces or manifolds. Several key problems, such as the closability of quadratic forms and the construction of strong Markov processes associated with quasi-regular Dirichlet forms, have been solved. The study of infinite dimensional stochastic analysis as well as the study of processes on singular structures (like fractals, trees, or general metric spaces) has enriched and extended the Dirichlet space theory. In the meantime, a new framework has been introduced into Dirichlet space theory by the development of nonstandard probabilistic analysis [25, 166]. As is well-known, nonstandard analysis is an alternative setting for analysis (and, indeed, all areas of mathematics), namely by enriching the set of real numbers by infinitesimal and infinite elements. It has its origin in seminal work by Schmieden, Laugwitz [325] and most notably Robinson [310]. By now, several textbooks and surveys exist on this theory and its applications (see, e.g. [25, 63, 125, 217]). Nonstandard analysis gives a novel approach to the theory of stochastic processes. In particular, it has led to hyperfinite symmetric Dirichlet space theory. Besides being interesting by itself, it has also many applications. In the first part of the book, we extend the research to the

nonsymmetric case, and remove some restrictive conditions in the previous treatment of the subject (Chap. 5 of [25]). In addition, we shall apply the theory to present a new approach to infinite dimensional stochastic analysis.

In writing this book we have two main aims: (1) to give a presentation of research on nonsymmetric hyperfinite Dirichlet space theory and its applications in (standard) finite and infinite dimensional stochastic analysis, Chaps. 1–4; (2) to find nonstandard representations for a special class of (finite dimensional) Feller processes and their infinitesimal generators, viz. stochastically continuous processes with stationary and independent increments (i.e., *Lévy processes*), Chap. 5. Chapter 6 is a complement to illustrate the usefulness of the hyperfinite probability spaces. The first part (Chaps. 1–4) is based on Chap. 5 of Albeverio et al. [25] and the further in depth research of Sergio and Ruzong; the second part (Chaps. 5–6) is based on results obtained recently by Tom Lindstrøm and their extensions by Sergio and Frederik.

As mentioned earlier, the interplay between stochastic analysis and mathematical physics has been one of the most important and exciting themes of research in the last decades. This is already a sufficient rationale for the research of the first part of the present book. The motivation for including the second part, Chap. 5, into this book is that many of the issues discussed in the more general framework of the first part, such as existence of standard parts of hyperfinite Markov chains, become much less technical to resolve for hyperfinite Lévy processes. Furthermore, the more restrictive setting of the second part also allows one to obtain finer results on the relation between Lévy processes and their hyperfinite analogues, one example being a hyperfinite version of the Lévy–Khintchine formula.

The contents of this book are arranged as follows: In Chap. 1, we introduce the framework of hyperfinite Dirichlet forms. We develop the potential theory of hyperfinite Dirichlet forms in Chap. 2. In Chap. 3, we consider standard representations of hyperfinite Markov chains under certain conditions, and translate the conditions on hyperfinite Markov chains into the language of hyperfinite Dirichlet forms. As an interesting and important application in classical stochastic analysis, we construct tight dual strong Markov processes associated with quasi-regular Dirichlet forms by using the language of hyperfinite Dirichlet forms in Chap. 4. The results show that hyperfinite Dirichlet space theory is a powerful tool to study classical problems. In the first sections of Chap. 5, the notion of a hyperfinite Lévy process is introduced and its relation to hyperfinite random walks as well as to standard Lévy processes is investigated. These results can be used to show that the jump part of any Lévy process is essentially a hyperfinite convolution of Poisson processes. Finally, Chap. 6 is an epilogue, providing a rigorous motivation for the study of hyperfinite Loeb path spaces as generic probability spaces.

The entire book is based on nonstandard analysis. For the reader's convenience, we present some basic notions of nonstandard analysis, such as internal sets and saturation, linear spaces, Loeb measure spaces, structure of ${}^*\mathbb{R}$ and topology in the appendix. Because of its monographical character centered around the hyperfinite approach, the book has by no means the goal of including all aspects of recent developments in the theory of stochastic processes and its connections with Dirichlet forms theory or the theory of Lévy processes. For this, we rather refer to surveys and proceedings like Albeverio [2], Barndorff-Nielsen et al. [73], and Ma et al. [275], respectively.

The germ of this book goes back to the year 1989 when the second author, Ruzong Fan, worked on the construction of symmetric Markov processes associated with Dirichlet forms at Peking University, Beijing ([165] and Chap. 4). At that time, Ruzong was unaware that Sergio's group was working on the same project using standard methods [41]. The second author, Dr. Zhiming Ma, of [41] did privately inquire Ruzong about the progress of Ruzong's research in 1989 at the Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing. In response to Dr. Ma's request of a private meeting, Ruzong presented his work to Dr. Ma in a classroom with Dr. Ma as the only audience. Dr. Ma, however, did not mention his ongoing work with Sergio in any way. Thus, Ruzong was totally unaware of Sergio's research. In the spring of 1990, Ruzong first realized this when he saw a manuscript of Albeverio and Ma [41] in Beijing with a surprise. These events notwithstanding, Ruzong continued to work on a "symmetric version" of Chaps. 1–4 using non-standard language when he was at Peking University till 1991 and when he visited the Humboldt-University, Berlin, between 1991 and 1992. Under Sergio's supervision and encouragement, Ruzong extended the project to the current "nonsymmetric version" from 1992 to 1994 at Ruhr-University, Bochum. In 2006, Frederik kindly joined the project with a contribution on hyperfinite Lévy processes (Chap. 5) and the Epilogue (Chap. 6). In the summer of 2006, the three authors gathered at the University of Bonn to finalize this monograph. We gratefully acknowledge the manifold support of various institutions in the long process of work on this project.

In the run-up to its completion, Sergio and Frederik were supported partially by the collaborative research center SFB 611 of the German Research Foundation (DFG), Germany; in addition, Ruzong's visit to Bonn was partially funded through a research fellowship from the Alexander von Humboldt Foundation, Germany.

Over the course of his career, Ruzong has received a lot of generous support from Sergio. As a Ph.D candidate in Beijing around 1987–1988, Ruzong was greatly fascinated by Sergio and Raphael Høegh-Krohn's novel work on infinite dimensional stochastic analysis, in which Ruzong finished his Ph.D thesis. Unfortunately, Ruzong got no chance to meet Raphael Høegh-Krohn; right before Ruzong went to Europe, he was shocked to learn that Raphael

Høegh-Krohn died of a heart attack. In a relatively isolated environment, Ruzong mostly worked on himself by reading numerous papers and books of Sergio and Raphael Høegh-Krohn; and many times, Ruzong had to spend a few days on a single equation or lemma to guess and to understand it. Whilst it seemed like a helpless or hopeless situation for Ruzong at that time, Ruzong eventually came to the forefront of research in areas of infinite dimensional stochastic analysis: he studied the hard and central questions regarding Beurling–Deny formulae, representation of martingale additive functionals and absolute continuity of symmetric diffusion processes on Banach spaces, potential theory of symmetric hyperfinite Dirichlet forms, and construction of the symmetric strong Markov processes associated with quasi-regular Dirichlet forms by using the non-standard analysis language. This direction of research was initiated by Sergio, although Ruzong was unaware that Sergio's group already worked on the construction of Markov processes using the language of standard stochastic analysis.

In early 1989, Ruzong applied for a fellowship from the Alexander von Humboldt Foundation from Peking University, Beijing; soon after a rejection from the Foundation in the fall 1989, Ruzong received a warm letter from Sergio with encouragement and a kind offer to nominate, as an academic host, Ruzong for the fellowship and by writing a strong letter of recommendation. This is just one anecdote to illustrate how Ruzong has constantly been able to count on Sergio's help via communications by either mail or face-to-face conversations starting from 1989. Between 1992 and 1994, Sergio generously supported Ruzong at Ruhr-University Bochum to complete the main part of Chaps. 1–4 of this monograph, and helped Ruzong to pass the hard period of time in his career.

The story of Ruzong is an example how Sergio has helped many young mathematicians to grow and to mature. Quite probably, Ruzong would have disappeared from academia a long time ago without the support of Sergio. In a true sense, Sergio has been an academic father figure for Ruzong when he desperately needed one. In recent years, after his departure from Sergio's research group, Ruzong has been mainly working on statistical genetics guided by his beloved American mentor, Dr. Kenneth Lange, at the University of Michigan and UCLA. Nevertheless, Ruzong has fond memories and deep appreciation of numerous communications with his European academic father Sergio; and both Ruzong and Frederik are deeply grateful for Sergio's mentoring.

Thus, especially right after Sergio's 70th birthday in 2009 – which also marks the 50th anniversary of his remarkable scientific career –, Ruzong and Frederik are sure that they will be joined by many other young mathematicians in thanking Sergio for his wonderful role in our professional and personal development and in wishing him all the best for the rest of his life: Not just continued productivity, but most of all good health, happiness, joy, and peace.

We owe a huge debt of gratitude to our families: In the summer of 2006, Dr. Li Zhu (Ruzong's wife) kindly took care of two young children when her husband was visiting Bonn. Their adorable daughter, Olivia Wenlu Fan, was with the second author in Germany for the "hot and interesting" summer of Bonn, where she liked everything except German milk. Frederik thanks his wife, Angélique Herzberg, for her love and manifold support with the words of Proverbs 31,10–12: "A wife of noble character [...] is worth far more than rubies. Her husband [...] lacks nothing of value. She brings him good [...] all the days of her life." We are all very grateful to our families for their love and understanding during the entire process of writing this book.

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