Chapter 2
Model

In this chapter we describe our model of a market with strategically behaving agents on both sides. First, we characterize the stage game between the firms in the model. Then, we proceed to the formulation of the countable infinite repeated game with discounting of payoffs. In Sect. 2.3 we define the solution concepts that we apply to the repeated game: SRPE and SSPE.

2.1 Stage Game

We denote the stage game by G. The stage game is a strategic form noncooperative game.

There is a nonempty finite set J of producers and a nonempty finite set I of buyers. We have $J = \{1, \ldots, \#(J)\}$ and $I = \{\#(J) + 1, \ldots, \#(J) + \#(I)\}$. Thus, $J \cup I$ is the set of players in G. (We use the terms "player(s)" and "firm(s)" interchangeably.) Buyers buy goods from producers. They either use the purchased goods to produce new good(s) (which they sell in the market(s) where customers are price takers) or sell them in the retail market to final consumers. Each producer produces one type of good, and can sell to any number of buyers in I. Therefore, we use the symbol J also for the set of goods in the model. We do not exclude the possibility that some or even all goods in J are identical.

A coalition is a nonempty subset of $J \cup I$. Thus, the set of all coalitions equals $2^{J \cup I} \setminus \{\varnothing\}$. A proper coalition is a nonempty strict subset of $J \cup I$. In subscripts and superscripts we write $-C$ instead of $(J \cup I) \setminus C$ for each coalition $C \subset J \cup I$, and $-k$ instead of $(J \cup I) \setminus \{k\}$ and k instead of $\{k\}$ for each $k \in J \cup I$.

For each $j \in J$, there exists $\chi_j > 0$, which is the upper bound on the output of good j. These upper bounds can stem, for example, from capacity constraints. We let $Y_j = [0, \chi_j]$ for each $j \in J$ and $Y = \prod_{j \in J} Y_j$. Thus, Y_j is the set of feasible outputs of producer j.

Producer $j \in J$ has cost function $c_j : Y_j \to \Re_+$.

Assumption 2.1. *For each $j \in J$, c_j is (i) continuous, (ii) strictly increasing, and (iii) $c_j(0) > 0$.*

M. Horniaček, *Cooperation and Efficiency in Markets*, Lecture Notes in Economics and Mathematical Systems 649, DOI 10.1007/978-3-642-19763-5_2,

Let

$$X = \left\{ x = (x_i)_{i \in I} = \left(\left(x_{ji} \right)_{j \in J} \right)_{i \in I} \in Y^{\#(I)} \mid \sum_{i \in I} x_{ji} \in Y_j \forall j \in J \right\}. \quad (2.1)$$

Here, x_{ji} is the quantity of good $j \in J$ purchased by buyer $i \in I$. We will call an element of X a "vector of traded quantities." Of course, in each vector of traded quantities, the sum of the purchases of all buyers from each producer $j \in J$ has to be a feasible output of j. Clearly, X is a nonempty and compact subset of $\Re_+^{\#(J)\times\#(I)}$. (It contains, for example, the zero vector in $R^{\#(J)\times\#(I)}$. It is a subset of compact set $Y^{\#(I)}$. It is closed because it is defined by linear constraints in the finite dimensional space.) We define function $\theta : X \to Y$ by

$$\theta(x) = \left(\sum_{i \in I} x_{ji} \right)_{j \in J}. \quad (2.2)$$

That is, $\theta(x)$ is the output vector, in which each producer's output equals the sum of his/her deliveries to buyers specified by x.

We set $X_i = Y$ for each $i \in I$. It is the set of feasible vectors of the purchases of buyer i from the producers in J. The cost function of buyer $i \in I$, $c_i : X_i \to \Re_+$, assigns to each vector $x_i \in X_i$ of the quantities of goods purchased from the producers in J his/her cost associated with their processing or cost associated with their sale in the retail market. These costs include, for example, transport cost, wages, depreciation allowance, expenditure on the maintenance of buildings, expenditure on the maintenance of machines (especially if the purchased goods are processed into new goods), and storage and handling cost (especially if the purchased goods are sold in the retail market). These do not include the expenditure on the purchase of goods from producers. Thus, the cost expressed by function c_i is the expenditures on the inputs used by firm $i \in I$ that are complementary to the goods purchased from the producers in J.

Assumption 2.2. *For each $i \in I$, (i) c_i is continuous and (ii) $c_i(0) > 0$.*

For each $i \in I$, $U_i : X \to \Re_+$ is buyer i's revenue function. It assigns to each $x \in X$, buyer i's revenue from selling the output produced from the inputs given by vector x_i, or (if he/she is a retailer) his/her revenue from selling the quantities of the purchased goods given by x_i in the retail market. For example, if all buyers produce $n \in N$ goods, U_i has the form $U_i(x) = \sum_{m=1}^{n} P_{mi}\left((f_k(x_k))_{k \in I}\right) f_{mi}(x_i)$, where $f_k : X_k \to \Re_+^n$ is buyer k's (vector) production function[1] (with $f_k(x_k) = (f_{mk}(x_k))_{m \in \{1,\ldots,n\}}$) and for each $m \in \{1, \ldots, n\}$, $P_{mi} : \Re_+^{n\times\#(I)} \to \Re_+$ is the

[1] Only the quantities of the inputs purchased from the producers in J are the arguments of f_k. We assume that the other inputs are fixed unless buyer k decides to leave the analyzed market.

inverse demand function for the m-th output of buyer i. If all buyers are retailers, then U_i has the form $U_i(x) = \sum_{j \in J} \widetilde{P}_j(\theta(x))\, x_{ji}$, where for each $j \in J$, $\widetilde{P}_j : Y \to \Re_+$ is the inverse demand function for the sale of good j in the retail market. (We can view the latter situation as a special case of the former when we set $n = \#(J)$, $f_k(x_k) = x_k$ for each $k \in I$, and $P_{ji}(x) = \widetilde{P}_j(\theta(x))$ for each $j \in J$ and every $i \in I$.)

Assumption 2.3. *For each $i \in I$, U_i is (i) continuous and (ii) $x_i = 0$ implies $U_i(x) = 0$.*

The following assumption ensures that the zero vector cannot solve the maximization problem in (2.3) given below.

Assumption 2.4. *There exists $x^+ \in X$ such that*

$$\sum_{i \in I : x_i^+ > 0} \left(U_i\left(x^+\right) - c_i\left(x_i^+\right) \right) > \sum_{j \in J : \theta_j\left(x^+\right) > 0} c_j\left(\theta_j\left(x^+\right)\right).$$

Let

$$X^{\max} = \arg\max \left\{ \sum_{i \in I} U_i(x) - \sum_{j \in J : \theta_j(x) > 0} c_j\left(\theta_j(x)\right) - \sum_{i \in I : x_i > 0} c_i(x_i) \mid x \in X \right\} \tag{2.3}$$

Thus, each $x^{\max} \in X^{\max}$ is a vector of traded quantities that maximizes the surplus from the trade in the analyzed market. That is, it maximizes the difference between the sum of the buyers' revenue (from the sale of goods produced from the inputs purchased in the analyzed market or from the sale of purchased goods in the retail market) and the sum of the production costs of active producers (i.e., assuming that the producers who did not withdraw from the analyzed market have positive outputs) and the costs of active buyers (i.e., assuming that the buyers who did not withdraw from the analyzed market purchase a positive amount of at least one good). Since the sum of the producers' revenue in the analyzed market equals the sum of buyers' expenditure in it, $x^{\max}$ also maximizes the sum of the profits in the analyzed market.

$X^{\max}$ is nonempty despite the discontinuity of the objective function in the maximization program in (2.3) caused by the fact that the firms who withdraw from the analyzed market do not incur any fixed cost. We can solve the latter maximization problem in two steps as follows. In the first step, we solve for each $D \in 2^{J \cup I}$ with $D \cap J \neq \varnothing$ and $D \cap I \neq \varnothing$, the maximization problem

$$\zeta_D = \max \left\{ \begin{array}{l} \sum_{i \in I \cap D} \left[U_i(x) - c_i(x_i) \right] - \sum_{j \in J \cap D} c_j\left(\theta_j(x)\right) \mid \\ x \in X, x_{ji} = 0 \forall (j,i) \notin (D \cap J) \times (D \cap I) \end{array} \right\}. \tag{2.4}$$

We denote the set of its solutions by $X^{(D)\max}$. Here, ζ_D is the maximal surplus from the trade in the analyzed market among the firms in D (i.e., when all firms outside D withdraw from the analyzed market). In this auxiliary problem, we assume that no firm in D withdraws from the analyzed market (i.e., each firm in D incurs a fixed cost). Since X is a nonempty and compact set and the objective function in the maximization problem in (2.4) is continuous, $X^{(D)\max}$ is nonempty and compact. The following statement holds for each $x^{(D)\max} \in X^{(D)\max}$: if there exists a nonempty $C \subset D$ such that for each $j \in J$, $\theta_j\left(x^{(D)}\right) > 0$ if and only if $j \in C$, and for each $i \in I$, $x_i^{(D)} > 0$ if and only if $i \in C$, then

$$\zeta_C \geq \sum\nolimits_{i \in C \cap I}\left[U_i\left(x^{(D)}\right) - c_i\left(x_i^{(D)}\right)\right] - \sum\nolimits_{j \in C \cap J} c_j\left(\theta_j\left(x^{(D)}\right)\right) > \zeta_D,$$

where the second inequality follows from the fact that each firm has a positive fixed cost (see part (iii) of Assumption 2.1 and part (ii) of Assumption 2.2). Therefore, for each $F \in \arg\max\left\{\zeta_D \mid D \in 2^{J \cup I}, D \cap J \neq \varnothing, D \cap I \neq \varnothing\right\}$, we have $X^{(F)\max} \subseteq X^{\max}$. As such, in the second step, we set

$$X^{(\max)} = \bigcup\left\{\begin{array}{c} X^{(F)\max} \mid \\ F \in \arg\max\left\{\zeta_D \mid D \in 2^{J \cup I}, D \cap J \neq \varnothing, D \cap I \neq \varnothing\right\}\end{array}\right\}. \tag{2.5}$$

Since $X^{(F)\max}$ is nonempty and compact for each

$$F \in \arg\max\left\{\zeta_D \mid D \in 2^{J \cup I}, D \cap J \neq \varnothing, D \cap I \neq \varnothing\right\}$$

and the set of such coalitions F is nonempty and finite, $X^{\max}$ is nonempty and compact. We use ζ^* to denote the maximal value of the objective function in the maximization problem in (2.3). Assumption 2.4 implies that $\zeta^* > 0$. For each $x \in X$, we let $E_J(x) = \left\{j \in J \mid \theta_j(x) > 0\right\}$, $E_I(x) = \{i \in I \mid x_i > 0\}$, and $E(x) = E_J(x) \cup E_I(x)$.

It is plausible to assume that the prices in the analyzed market cannot be arbitrarily high. We denote the upper bound on a price of good $j \in J$ by $p_j^{\max}$.[2] We assume that $p_j^{\max} > 0$ for each $j \in J$ and

$$\begin{aligned} p_j^{\max} x_{ji}^{\max} \geq \max\{U_i(x) \mid x \in X\}, \forall x^{\max} \in X^{\max}, \forall j \in E_J\left(x^{\max}\right), \\ \forall i \in \left\{k \in I \mid x_{jk}^{\max} > 0\right\}. \end{aligned} \tag{2.6}$$

[2] Of course, from the modeling point of view, we need the upper bounds on the prices in the analyzed market in order to ensure that the stage game payoffs are bounded and the repeated game (described in Sect. 2.2) is continuous at infinity.

Thus, (taking into account part (ii) of Assumption 2.2) for each $x^{\max} \in X^{\max}$, every $j \in E_J(x^{\max})$, and each $i \in I$ with $x_{ji}^{\max} > 0$, no purchase, in which the buyer i buys the quantity of good j equal to $x_{ji}^{\max}$ at price $p_j^{\max}$, allows him/her to earn a nonnegative profit. We let $P = \left(\prod_{j \in J}\left[0, p_j^{\max}\right]\right)^{\#(I)}$. P is the set of feasible vectors of the prices in the analyzed market. Each of these vectors has the form $p = \left(\left(p_{ji}\right)_{j \in J}\right)_{i \in I}$.

Now we describe the sets of pure strategies and payoff functions in stage game $G = \langle J \cup I, (A_k)_{k \in J \cup I}, (g_k)_{k \in J \cup I}\rangle$. We have

$$A_j = \left\{ \left(p_{ji}, q_{ji}\right)_{i \in I} \in \left(\left[0, p_j^{\max}\right] \times Y_j\right)^{\#(I)} \mid \sum_{i \in I} q_{ji} \in Y_j \right\}, \forall j \in J \tag{2.7}$$

and

$$A_i = \left(\left[0, p_j^{\max}\right] \times Y_j\right)^{\#(J)}, \forall i \in I. \tag{2.8}$$

A pure strategy of producer $j \in J$ in G is a collection of contracts (one for each buyer) that he/she proposes to the buyers. Each contract proposal specifies a proposed price and a proposed quantity.[3] Of course, the sum of the proposed quantities cannot exceed a producer's capacity. There is no need to treat a producer's decision not to propose a contract to some buyer as a special case. We identify such decision with the contract proposal $(0, 0)$. For each $j \in J$, we identify the producers' pure strategy $(0, 0)^{\#(I)}$ with his/her decision to withdraw from the analyzed market. For such a decision, the producer does not incur a fixed cost and his/her payoff in G equals zero.[4]

A pure strategy of buyer $i \in I$ in G is a collection of contracts (one for each producer) that he/she proposes to the producers. Thus, each $a_i \in A_i$ has the form $a_i = \left(p_{ij}, q_{ij}\right)_{j \in J}$. Again, we identify the contract proposal $(0, 0)$ made to $j \in J$ with the decision not to trade with producer j. Further, for each $i \in I$, we identify the buyer's pure strategy $(0, 0)^{\#(J)}$ with his/her decision to withdraw from the analyzed market. For such a decision, the buyer does not incur a fixed cost and his/her payoff in G equals zero.

We let $A = \prod_{k \in J \cup I} A_k$. For each $a \in A$ and every $(j, i) \in J \times I$ trade between producer j and buyer i takes place if and only if j's contract proposal to i differs from $(0, 0)$ and coincides with i's contract proposal to j, i.e., if and only if $\left(p_{ji}, q_{ji}\right) = \left(p_{ij}, q_{ij}\right) \neq (0, 0)$. For every $a \in A$, $x(a)$ is the vector of

---

[3] The producers can sell their product to different buyers at different prices. This enables us to construct a punishment for a deviation by a proper coalition of buyers, which does not harm the buyers who did not deviate, in the repeated game.

[4] We assume here that a firm can leave the analyzed market in one period. In the repeated game (described in the following section), we assume that a firm can enter the analyzed market in one period and the entry requires only paying the fixed cost. Our qualitative results hold if the exit from and entry into the analyzed market took more than one period and entry required the incurring of a sunk cost (exceeding the single period fixed cost).

traded quantities generated by strategy profile a. We define it by $x_{ji}(a) = q_{ji}$ if $(p_{ji}, q_{ji}) = (p_{ij}, q_{ij})$ and by $x_{ji}(a) = 0$ if $(p_{ji}, q_{ji}) \neq (p_{ij}, q_{ij})$. Further, for each $a \in A$, $y(a) = \theta(x(a))$ is the output vector generated by strategy profile a. Similarly, for each $a \in A$, $\widetilde{p}(a) \in P$ is the vector of prices at which the quantities given by $x(a)$ are traded. We use the convention that a zero price is assigned to each zero traded quantity. Thus, $\widetilde{p}_{ji}(a) = p_{ji} = p_{ij}$ if $x_{ji}(a) > 0$ and $\widetilde{p}_{ji}(a) = 0$ if $x_{ji}(a) = 0$.

For $k \in J \cup I$, player k's payoff function in G, $g_k : A \to \Re$, is defined by

$$\forall j \in J : g_j(a) = \sum_{i \in I} \widetilde{p}_{ji}(a) x_{ji}(a) - c_j(y_j(a)),$$
$$\textit{if}\ \exists i \in I\ \textit{with}\ (p_{ji}, q_{ji}) \neq (0,0), \tag{2.9}$$

$$\forall j \in J : g_j(a) = 0,\ \textit{if}\ (p_{ji}, q_{ji}) = (0,0)\ \forall i \in I, \tag{2.10}$$

$$\forall i \in I : g_i(a) = U_i(x(a)) - \sum_{j \in J} \widetilde{p}_{ji}(a) x_{ji}(a) - c_i(x_i(a)),$$
$$\textit{if}\ \exists j \in J\ \textit{with}\ (p_{ij}, q_{ij}) \neq (0,0), \tag{2.11}$$

$$\forall i \in I : g_i(a) = 0,\ \textit{if}\ (p_{ij}, q_{ij}) = (0,0)\ \forall j \in J. \tag{2.12}$$

We define function $g : A \to \Re^{\#(J \cup I)}$ by $g(a) = (g_k(a))_{k \in J \cup I}$ and let

$$V = \left\{ v \in \Re^{\#(J \cup I)} \mid \exists a \in A\ \textit{such that}\ g(a) = v \right\}. \tag{2.13}$$

Clearly, taking into account (2.9)–(2.12), we have

$$V = \bigcup_{D \in 2^{J \cup I}} \left\{ \begin{array}{c} v \in \Re^{\#(J \cup I)} \mid v_j = \sum_{i \in I} p_{ji} x_{ji} - c_j(\theta_j(x))\ \forall j \in J \backslash D, \\ v_i = U_i(x) - c_i(x_i) - \sum_{j \in J} p_{ji} x_{ji} \forall i \in I \backslash D, v_k = 0 \forall k \in D, \\ x \in X, x_{ji} = 0 \forall (j,i) \in [(J \cap D) \times I] \cup [J \times (I \cap D)], \\ p_{ji} \in \left[0, p_j^{\max}\right] \forall (j,i) \in J \times I \end{array} \right\}. \tag{2.14}$$

That is, V is the set of payoff vectors in G that can result from the pure strategy profiles. It follows from (2.13) that V is nonempty. Clearly, V is a compact subset of $\Re^{\#(J \cup I)}$. (For each $D \in 2^{J \cup I}$ the corresponding set in the union in (2.14) is compact because X is a compact set, function c_j, $j \in J$ is continuous, and functions U_i and c_i, $i \in I$ are continuous.) Therefore, V has a strict Pareto efficient frontier that we denote by $\wp(V)$.

Since V contains payoff vectors generated by the pure strategy profiles in G, it need not be convex. Thus, a vector in $\wp(V)$ can be weakly (or even strictly) Pareto dominated by a convex combination of other vectors in $\wp(V)$. We let $V^+ = conV$.

In order to enable the firms to achieve payoff vectors in $V^+ \backslash V$, we also allow objectively correlated strategies (see [Aumann (1974)]; henceforth, we use only the term "correlated strategies") in G. We assume that all firms can observe the signals of a public randomizing device generating uniformly distributed signals from interval $[0,1]$. A correlated strategy of firm $k \in J \cup I$ is the mapping $\xi_k : [0,1] \to A_k$. It assigns to each signal $\omega \in [0,1]$, firm k's pure strategy in G. We denote the set of correlated strategies of firm $k \in J \cup I$ by Ξ_k and let $\Xi = \prod_{k \in J \cup I} \Xi_k$ and $\Xi_C = \prod_{k \in C} \Xi_k$ for each $C \in 2^{J \cup I} \backslash \{\varnothing\}$. Of course, a pure strategy is a special case of a correlated strategy. (That is, $a_k \in A_k$ is – from the point of view of the outcome of G – identical to $\xi_k \in \Xi_k$ as defined by $\xi_k(\omega) = a_k$ for each $\omega \in [0,1]$.) Taking into account the Carathéodory theorem (see [Hildenbrand (1974), p. 37]), in order to obtain any vector in V^+, it is enough to use a correlated strategy profile that uses at most $\#(J \cup I) + 1$ pure strategy profiles with a positive probability.

With a slight abuse of the notation, we will use the symbol g_k, $k \in J \cup I$ also for the payoff functions in G defined on Ξ and the symbol g for the function that assigns to each $\xi \in \Xi$, the vector of expected payoffs in G.

Since V is nonempty and compact, V^+ is also nonempty and compact. We use $\wp\left(V^+\right)$ to denote its strict Pareto efficient frontier and let $\wp^*\left(V^+\right) = \wp\left(V^+\right) \cap \Re_+^{\#(J \cup I)}$. Thus, $\wp^*\left(V^+\right)$ is the set of individually rational strictly Pareto efficient payoff vectors in G. (Recall that each firm can guarantee itself a zero payoff by withdrawing from the analyzed market. The firms on the other side of the analyzed market can – as a group – prevent it from achieving a positive payoff when they refuse to trade with it, i.e., when each of them proposes the contract $(0,0)$ to it. Thus, each firm's minimax payoff in G equals zero.) We use $\wp^{**}\left(V^+\right)$ to denote the subset of $\wp^*\left(V^+\right)$ with the following property: if $v \in \wp^{**}\left(V^+\right)$ and $v_k = 0$, then there exists $\xi \in \Xi$ such that $v = g(\xi)$ and ξ_k assigns to each $\omega \in [0,1]$, firm k's pure strategy in G leading to its withdrawal from the analyzed market (i.e., $\xi_k(\omega) = (0,0)^{\#(I)}$ for each $\omega \in [0,1]$ if $k \in J$, and $\xi_k(\omega) = (0,0)^{\#(J)}$ for each $\omega \in [0,1]$ if $k \in I$). Thus, the payoff vectors in $\wp^{**}\left(V^+\right)$, besides being individually rational and strictly Pareto efficient, are also strictly individually rational for firms that do not withdraw from the analyzed market.

We assume that when a firm contemplates (either unilaterally or as a member of a coalition of deviating firms), a deviation from its correlated strategy in G, it does so before the signal from the public randomizing device is observed. Thus, the deviation strategy of a firm has to assign its pure strategy to each $\omega \in [0,1]$.

2.2 Repeated Game

The repeated game is a countable infinite repetition of G with discounting of future profits (without discounting of current profit). That is, G is played in periods numbered by positive integers. All firms use the common discount factor $\delta \in (0,1)$. We denote the repeated game with the discount factor δ by $\Gamma(\delta)$ and its game form by Γ. The actions in Γ are observable. That is, at the end of each period $t \in N$, every firm $k \in J \cup I$ observes – for every firm $r \in (J \cup I) \setminus \{k\}$ – r's pure strategy in G in period t. We denote the set of histories in Γ by H and the set of histories leading to period $t \in N$ by $H^{(t)}$. Each element of $H^{(t)}$ contains decisions made by the players and signals of the public randomizing device prior to period t. $H^{(1)}$ contains only the empty history. We set $H_f = \bigcup_{t \in N} H^{(t)}$. Thus, H_f is the set of nonterminal (i.e., finite) histories. We use H_∞ to denote the set of terminal (i.e., infinite) histories. Hence, $H = H_f \cup H_\infty$. Each $h \in H_\infty$ has the form $h = \left\{\left(a^{(t)}, \omega^{(t)}\right)\right\}_{t \in N}$, where $a^{(t)} \in A$ and $\omega^{(t)} \in [0, 1]$ for each $t \in N$. For each $h \in H_\infty$, $x(h) = \left\{x\left(a^{(t)}\right)\right\}_{t \in N}$ is the sequence of vectors of traded quantities generated by terminal history h, and $y(h) = \left\{y\left(a^{(t)}\right)\right\}_{t \in N}$ is the sequence of output vectors generated by h.

The behavioral strategy of firm $k \in J \cup I$ in Γ is the function $s_k : H_f \to \Xi_k$. It assigns to each nonterminal history $h \in H_f$, one of k's correlated strategies in G. We denote the set of the behavioral strategies of firm $k \in J \cup I$ in Γ by S_k and let $S = \prod_{k \in J \cup I} S_k$ and $S_C = \prod_{k \in C} S_k$ for each $C \in 2^{J \cup I} \setminus \{\varnothing\}$. For $s \in S$ and $C \in 2^{J \cup I} \setminus \{\varnothing\}$ we let $s_C = (s_k)_{k \in C}$. We use the term "strategy profile" for $s^{(C)} \in S_C$.[5] For every $k \in J \cup I$, the terms "strategy of k" and "strategy profile of $\{k\}$" are equivalent. For $C \in 2^{J \cup I} \setminus \{\varnothing\}$ with $\#(C) \geq 2$, $s^{(C)} = \left(s_k^{(C)}\right)_{k \in C} \in S_C$, and $D \in 2^C \setminus \{\varnothing\}$, we let $s_D^{(C)} = \left(s_k^{(C)}\right)_{k \in D}$.

We express the firms' payoffs in $\Gamma(\delta)$ (computed before they implement their behavioral strategies and a signal of the public randomizing device in the first period is observed) as their expected average discounted profits (i.e., their expected average discounted stage game payoffs). For each $k \in J \cup I$, function $\pi_k : S \to \Re$ is firm k's payoff function in $\Gamma(\delta)$.[6] That is, when $s \in S$ generates a sequence $\left\{v_k^{(t)}\right\}_{t \in N}$ of expected stage game payoffs of firm $k \in J \cup I$, we have

$$\pi_k(s) = (1 - \delta) \sum_{t \in N} \delta^{t-1} v_k^{(t)}. \tag{2.15}$$

We define function $\pi : S \to \Re^{\#(J \cup I)}$ by $\pi(s) = (\pi_k(s))_{k \in J \cup I}$. With this formulation of the payoff functions, the set of the vectors of the firms' payoffs in $\Gamma(\delta)$ equals V^+.

For each $h \in H^{(t)}$ with $t \in N \setminus \{1\}$, we use h^- to denote the subhistory of h leading to period $t-1$. The subhistory contains all information contained in h except for the firms' decisions and the signal of the public randomizing device in period

[5] We use the symbol s_C for a profile of the behavioral strategies of the members of a coalition C determined by a previously mentioned profile of the behavioral strategies (of all firms) $s \in S$ and the symbol $s^{(C)}$ for a profile of the behavioral strategies of the members of a coalition C that is not determined by any previously mentioned $s \in S$.

[6] Of course, the functional values of π_k depend on δ. Nevertheless, in order to avoid unnecessary notational complication, we use the symbol π_k instead of $\pi_{k,\delta}$.

$t-1$. (That is, if $t>2$ and $h=\left\{\left(a^{(n)},\omega^{(n)}\right)\right\}_{n=1}^{t-1}$, then $h^{-}=\left\{\left(a^{(n)},\omega^{(n)}\right)\right\}_{n=1}^{t-2}$. If $h\in H^{(2)}$, then $h^{-}=\varnothing$.)

For each nonterminal history $h\in H_f$, $\Gamma_{(h)}(\delta)$ is the subgame of $\Gamma(\delta)$ following h. $\Gamma_{(h)}$ is its game form. Since $\Gamma(\delta)$ is a game with observable actions, each of its subgames is a proper subgame.

For any set B defined for $\Gamma(\delta)$ and any $h\in H_f$, the symbol $B_{(h)}$ stands for the restriction of B to subgame $\Gamma_{(h)}(\delta)$ (e.g., $H_{f(h)}$ is the set of nonterminal histories in $\Gamma_{(h)}(\delta)$). Similarly, for any function f defined for $\Gamma(\delta)$ and any $h\in H_f$, the symbol $f_{(h)}$ stands for the restriction of f to subgame $\Gamma_{(h)}(\delta)$ (e.g., $\pi_{k(h)}$ is firm k's payoff function in $\Gamma_{(h)}(\delta)$). For $h\in H_f$ and $h'\in H_{f(h)}$, the subgame of (the subgame) $\Gamma_{(h)}(\delta)$ following history h' in $\Gamma_{(h)}(\delta)$ is identical to the subgame $\Gamma_{(h,h')}(\delta)$ of $\Gamma(\delta)$. We use the latter symbol to denote this. For $h\in H_f$, $C\in 2^{J\cup I}\setminus\{\varnothing\}$, $s^{(C)}\in S_{C(h)}$, and $h'\in H_{f(h)}$, $s^{(C)}_{(h')}$ is the restriction of strategy profile $s^{(C)}$ of coalition C in subgame $\Gamma_{(h)}(\delta)$ to subgame $\Gamma_{(h,h')}(\delta)$.

We assume that when a firm contemplates (either unilaterally or as a member of a coalition of deviating firms) a deviation from its behavioral strategy in a subgame $\Gamma_{(h)}(\delta)$ with $h\in H^{(t)}$, it does so before the signal of the public randomizing device for period t is observed. This assumption corresponds to the excluding of a signal of the public randomizing device in period t from $h\in H^{(t)}$ and defining a firm's behavioral strategy as a mapping from the set of nonterminal histories to the set of its stage game correlated strategies (instead of as a mapping from the Cartesian product of the set of nonterminal histories with [0, 1] to the set of its stage game pure strategies). This allows the grand coalition to use any profile of stage game correlated strategies – and hence, (if V is not convex) to achieve a stage game payoff vector belonging to $V^{+}\setminus V$ – in the first period of a subgame. (If the firms would have contemplated a deviation by the grand coalition only after observing the signal of the public randomizing device in the first period of a subgame, they would not have been able to use a profile of stage game correlated strategies with non-singleton support in the first period of a subgame.) The ability of the grand coalition to weakly Pareto improve the vector of current period stage game payoffs by a deviation contemplated after the current period signal of the public randomizing device is observed implies its ability to do so also by a deviation contemplated before the current period signal of the public randomizing device is observed, but not vice versa. An analogous comment holds for the deviations by any non-singleton coalition.

2.3 Solution Concepts

As already stated in the Introduction, the SRPE and the SSPE are the solutions concepts that we apply to $\Gamma(\delta)$.

Definition 2.1. A strict renegotiation-proof equilibrium of $\Gamma(\delta)$ is a strategy profile $s^{*}\in S$ with the following properties.

(i) There do not exist a nonterminal history $h \in H_f$, a firm $k \in J \cup I$, and its strategy $s_k \in S_{k(h)}$ such that

$$\pi_{k(h)}\left(\left(s_k, s^*_{-k(h)}\right)\right) > \pi_{k(h)}\left(s^*_{(h)}\right). \tag{2.16}$$

(ii) There do not exist a nonterminal history $h \in H_f$ and a strategy profile $s \in S_{(h)}$ such that

$$\pi_{k(h)}(s) \geq \pi_{k(h)}\left(s^*_{(h)}\right), \forall k \in J \cup I \tag{2.17}$$

and

$$\exists\, k \in J \cup I \ with\ \pi_{k(h)}(s) > \pi_{k(h)}\left(s^*_{(h)}\right). \tag{2.18}$$

It follows from part (i) of Definition 2.1 that an SRPE is a subgame perfect equilibrium. Part (ii) implies that all continuation equilibrium payoff vectors[7] are strictly Pareto efficient.

Definition 2.2. A strict strong perfect equilibrium of $\Gamma(\delta)$ is a strategy profile $s^* \in S$ with the following properties.

(i) There do not exist a nonterminal history $h \in H_f$, a coalition $C \in 2^{J\cup I} \setminus \{\varnothing, J \cup I\}$, and its strategy profile $s^{(C)} \in S_{C(h)}$ such that

$$\pi_{k(h)}\left(\left(s^{(C)}, s^*_{-C(h)}\right)\right) \geq \pi_{k(h)}\left(s^*_{(h)}\right), \forall k \in C \tag{2.19}$$

and

$$\exists\, k \in C \ with\ \pi_{k(h)}\left(\left(s^{(C)}, s^*_{-C(h)}\right)\right) > \pi_{k(h)}\left(s^*_{(h)}\right). \tag{2.20}$$

(ii) There do not exist a nonterminal history $h \in H_f$ and a strategy profile $s \in S_{(h)}$ such that

$$\pi_{k(h)}(s) \geq \pi_{k(h)}\left(s^*_{(h)}\right), \forall k \in J \cup I \tag{2.21}$$

and

$$\exists\, k \in J \cup I \ with\ \pi_{k(h)}(s) > \pi_{k(h)}\left(s^*_{(h)}\right). \tag{2.22}$$

Thus, a strategy profile s^* is an SSPE if no coalition in no subgame can increase the expected average discounted profit of at least one of its members without decreasing the expected average discounted profit of any other member by a deviation from the prescriptions of s^*. That is, no coalition in no subgame has a deviation leading to a vector of expected average discounted profits of its members that weakly Pareto dominates the vector of their expected average discounted profits generated by s^*. Part (ii) of Definition 2.2 implies that all continuation equilibrium payoff

[7] Let $s^* \in S$ be an equilibrium strategy profile and let $h \in H_f$. Then, the continuation equilibrium in subgame $\Gamma_{(h)}(\delta)$ is $s^*_{(h)}$, and the continuation equilibrium payoff vector is $\pi_{(h)}\left(s^*_{(h)}\right)$.

vectors in an SSPE are strictly Pareto efficient. Part (i) also holds for singleton coalitions. Therefore, it implies that each SSPE is a subgame perfect equilibrium. Thus, each SSPE is an SRPE.

Definition 2.2 does not exclude the possibility that a deviation by a coalition from its continuation equilibrium strategy profile in some subgame leaves the payoffs of all its members unchanged. Thus, taking into account the meaning of the term "strict Nash equilibrium" (which is a Nash equilibrium where each unilateral deviation by any player decreases his/her payoff), we should use the term "semi-strict strong perfect equilibrium" instead of SSPE. Nevertheless, in order to avoid terminological complexities, we use the latter term.

References

Aumann, R.J.: "Subjectivity and Correlation in Randomized Strategies," *Journal of Mathematical Economics* 1 (1974), 67-96.

Hildenbrand, W.: *Core and Equilibria of a Large Economy.* Princeton, N.J.: Princeton University Press, 1974.

Springer

http://www.springer.com/978-3-642-19762-8

Cooperation and Efficiency in Markets
Horniaček, M.
2011, VII, 92 p., Softcover
ISBN: 978-3-642-19762-8