Chapter 1
Invariants of CR-Hypersurfaces

Abstract In this chapter we survey the invariant theory of Levi non-degenerate CR-hypersurfaces concentrating on Chern's construction of Cartan connections.

1.1 Introduction to CR-Manifolds

We start with a brief overview of necessary definitions and facts from CR-geometry (see [2], [25], [67], [105] for more detailed expositions). Unless stated otherwise, throughout the book manifolds are assumed to be connected, and differential-geometric objects such as manifolds, distributions, fiber bundles, maps, differential forms, etc. are assumed to be C^∞-smooth. A *CR-structure* on a manifold M of dimension d is a distribution of linear subspaces of the tangent spaces $T_p^c(M) \subset T_p(M)$, $p \in M$, i.e. a subbundle of the tangent bundle $T(M)$, endowed with operators of complex structure $J_p^M : T_p^c(M) \to T_p^c(M)$, $\left(J_p^M\right)^2 = -\,\mathrm{id}$. For $p \in M$ the subspace $T_p^c(M)$ is called the *complex tangent space* at p, and a manifold equipped with a CR-structure is called a *CR-manifold*. It follows that the number $\mathrm{CRdim}\, M := \dim_{\mathbb{C}} T_p^c(M)$ does not depend on p; it is called the *CR-dimension* of M. The number $\mathrm{CRcodim}\, M := d - 2\,\mathrm{CRdim}\, M$ is called the *CR-codimension* of M. Every complex (and even almost complex) manifold is a CR-manifold of zero CR-codimension. In this book we mostly consider CR-manifolds of CR-codimension one, or *CR-hypersurfaces*. Before constraining ourselves to this case, however, we will briefly discuss general CR-manifolds.

CR-structures naturally arise on real submanifolds of complex manifolds. Indeed, if M is an immersed real submanifold of a complex manifold N, then one can consider the maximal complex subspaces of the tangent spaces to M

$$T_p'(M) := T_p(M) \cap J_p^N T_p(M), \quad p \in M, \tag{1.1}$$

where J_p^N is the operator of complex structure on $T_p(N)$. If $\dim T_p'(M)$ is constant on M, then by setting

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$$T_p^c(M) := T_p'(M), \quad J_p^M := J_p^N\Big|_{T_p^c(M)}$$

for every $p \in M$, we obtain a CR-structure on M. The CR-structure defined above is called the *CR-structure induced by* N. We note that $\dim T_p'(M)$ is constant if M is a *real hypersurface in* N (that is, an immersed real submanifold of N of codimension one), and therefore a real hypersurface in a complex manifold carries an induced CR-structure (which turns the hypersurface into a CR-hypersurface). For comparison, we remark that if the codimension of M in N is two, then $\dim T_p'(M)$ need not be constant (see [48] for a study of generic compact codimension two submanifolds of $\mathbb{C}^K$).

Let M' be an immersed submanifold of a CR-manifold M, and suppose that M' is endowed with a CR-structure. Then M' is called a *CR-submanifold* of M if for every $p \in M'$ one has $T_p^c(M') \subset T_p^c(M)$ and $J_p^{M'} = J_p^M\Big|_{T_p^c(M')}$. Clearly, if the CR-structure of a CR-manifold M is induced by a complex manifold N, then M is a CR-submanifold of N.

A map $f : M_1 \to M_2$ between two CR-manifolds is called a *CR-map* if for every $p \in M_1$ the following holds: (a) the differential $df(p)$ of f at p maps $T_p^c(M_1)$ into $T_{f(p)}^c(M_2)$, and (b) $df(p)$ is complex-linear on $T_p^c(M_1)$. Two CR-manifolds M_1, M_2 of the same dimension and the same CR-dimension are called *CR-equivalent* if there is a diffeomorphism f from M_1 onto M_2 which is a CR-map (it follows that f^{-1} is a CR-map as well). Any such diffeomorphism is called a *CR-isomorphism*, or *CR-equivalence*. A CR-isomorphism from a CR-manifold M onto itself is called a *CR-automorphism* of M. CR-automorphisms of M form a group, which we denote by $\mathrm{Aut}(M)$. A CR-isomorphism between a pair of domains in M is called a *local CR-automorphism* of M. An *infinitesimal CR-automorphism* of M is a vector field on M whose local flow near every point consists of local CR-automorphisms of M. Infinitesimal CR-automorphisms form a (possibly infinite-dimensional) Lie algebra (see Theorem 12.4.2 in [2]).

In the first instance, we are interested in the equivalence problem for CR-manifolds. This problem can be viewed as a special case of the equivalence problem for *G-structures*. Let $G \subset \mathrm{GL}(d, \mathbb{R})$ be a Lie subgroup. A G-structure on a d-dimensional manifold M is a subbundle $\mathscr{S}$ of the frame bundle $F(M)$ over M which is a principal G-bundle. Two G-structures $\mathscr{S}_1$, $\mathscr{S}_2$ on manifolds M_1, M_2, respectively, are called *equivalent* if there is a diffeomorphism f from M_1 onto M_2 such that the induced mapping $f_* : F(M_1) \to F(M_2)$ maps $\mathscr{S}_1$ onto $\mathscr{S}_2$. Any such diffeomorphism is called an *isomorphism of G-structures*. The CR-structure of a manifold M of CR-dimension n and CR-codimension k (here $d = 2n + k$) is a G-structure, where G is the group of all non-degenerate linear transformations of $\mathbb{C}^n \oplus \mathbb{R}^k$ that preserve the first component and are complex-linear on it. The notion of equivalence of such G-structures is then exactly that of CR-structures. For convenience, when speaking about G-structures below, we replace the frame bundle $F(M)$ by the coframe bundle.

É. Cartan developed a general approach to the equivalence problem for G-structures (see [18], [65], [67], [97]), which applies, for example, to Riemannian and conformal structures. In Section 1.2 we outline a solution to the CR-equivalence problem for certain classes of CR-manifolds in the spirit of Cartan's work focussing on the case of CR-hypersurfaces (for an alternative approach to the equivalence problem see, e.g. [71]). Namely, we describe some classes of CR-manifolds whose CR-structures reduce – in the sense defined below – to $\{e\}$-structures, or *absolute parallelisms*, where $\{e\}$ is the one-element group. An absolute parallelism on an ℓ-dimensional manifold $\mathscr{P}$ is a 1-form σ on $\mathscr{P}$ with values in $\mathbb{R}^\ell$ such that for every $x \in \mathscr{P}$ the linear map $\sigma(x)$ is an isomorphism from $T_x(\mathscr{P})$ onto $\mathbb{R}^\ell$. The equivalence problem for absolute parallelisms is reasonably well-understood (see [97]).

Let $\mathfrak{C}$ be a collection of manifolds equipped with G-structures. We say that the G-structures are *s-reducible to absolute parallelisms* if one can assign every $M \in \mathfrak{C}$ some principal bundles

$$\mathscr{P}^s \xrightarrow{\pi^s} \mathscr{P}^{s-1} \xrightarrow{\pi^{s-1}} \dots \xrightarrow{\pi^4} \mathscr{P}^3 \xrightarrow{\pi^3} \mathscr{P}^2 \xrightarrow{\pi^2} \mathscr{P}^1 \xrightarrow{\pi^1} M$$

and an absolute parallelism σ on $\mathscr{P}^s$ in such a way that the following holds:

(i) any isomorphism of G-structures $f : M_1 \to M_2$ for $M_1, M_2 \in \mathfrak{C}$ can be lifted to a diffeomorphism $F : \mathscr{P}_1^s \to \mathscr{P}_2^s$ satisfying $F^*\sigma_2 = \sigma_1$, and

(ii) any diffeomorphism $F : \mathscr{P}_1^s \to \mathscr{P}_2^s$ satisfying $F^*\sigma_2 = \sigma_1$ is a lift of an isomorphism of the corresponding G-structures $f : M_1 \to M_2$ for $M_1, M_2 \in \mathfrak{C}$.

In the above definition we say that F is a *lift* of f if

$$\pi_2^1 \circ \ldots \circ \pi_2^s \circ F = f \circ \pi_1^1 \circ \ldots \circ \pi_1^s.$$

Let M be a CR-manifold. For every $p \in M$ consider the complexification $T_p^c(M) \otimes_{\mathbb{R}} \mathbb{C}$ of the complex tangent space at p. Clearly, the complexification can be represented as the direct sum

$$T_p^c(M) \otimes_{\mathbb{R}} \mathbb{C} = T_p^{(1,0)}(M) \oplus T_p^{(0,1)}(M),$$

where

$$\begin{aligned} T_p^{(1,0)}(M) &:= \{\mathbf{X} - iJ_p\mathbf{X} : \mathbf{X} \in T_p^c(M)\}, \\ T_p^{(0,1)}(M) &:= \{\mathbf{X} + iJ_p\mathbf{X} : \mathbf{X} \in T_p^c(M)\}. \end{aligned} \tag{1.2}$$

The CR-structure on M is called *integrable* if for any pair of local sections $\mathfrak{z}, \mathfrak{z}'$ of the bundle $T^{(1,0)}(M)$ the commutator $[\mathfrak{z}, \mathfrak{z}']$ is also a local section of $T^{(1,0)}(M)$. It is not difficult to see that if M is a submanifold of a complex manifold N and the CR-structure on M is induced by N, then it is integrable. In this book we consider only integrable CR-structures.

A $\mathbb{C}$-valued function φ on a CR-manifold M is called a *CR-function* if for any local section $\mathfrak{z}$ of $T^{(0,1)}(M)$ we have $\mathfrak{z}\varphi \equiv 0$. If M is a real submanifold of a complex manifold N with induced CR-structure, then for any function ψ holomorphic on

N its restriction $\varphi := \psi|_M$ is a CR-function on M. Let M_1, M_2 be CR-manifolds, where M_2 is a submanifold of $\mathbb{C}^K$ with induced CR-structure. In this case any map $f : M_1 \to M_2$ is given by K component functions. It is straightforward to verify that f is a CR-map if and only if all its component functions are CR-functions on M_1.

An important characteristic of a CR-structure called the *Levi form* comes from taking commutators of local sections of $T^{(1,0)}(M)$ and $T^{(0,1)}(M)$. Let $p \in M$, $\mathbf{Z}, \mathbf{Z}' \in T_p^{(1,0)}(M)$. Choose local sections $\mathfrak{z}$, $\mathfrak{z}'$ of $T^{(1,0)}(M)$ near p such that $\mathfrak{z}(p) = \mathbf{Z}$, $\mathfrak{z}'(p) = \mathbf{Z}'$. The Levi form of M at p is the Hermitian form on $T_p^{(1,0)}(M)$ with values in $(T_p(M)/T_p^c(M)) \otimes_{\mathbb{R}} \mathbb{C}$ given by

$$\mathfrak{L}_M(p)(\mathbf{Z}, \mathbf{Z}') := i[\mathfrak{z}, \overline{\mathfrak{z}'}](p)(\operatorname{mod} T_p^c(M) \otimes_{\mathbb{R}} \mathbb{C}). \tag{1.3}$$

For fixed $\mathbf{Z}$ and $\mathbf{Z}'$ the right-hand side of the above formula is independent of the choice of $\mathfrak{z}$ and $\mathfrak{z}'$. We usually treat the Levi form as a $\mathbb{C}^k$-valued Hermitian form (i.e. a vector of k Hermitian forms) on $T_p^{(1,0)}(M)$, where k is the CR-codimension of M. As a $\mathbb{C}^k$-valued Hermitian form, the Levi form is defined uniquely up to the choice of coordinates in $T_p(M)/T_p^c(M)$. If M is a CR-hypersurface, we think of its Levi form at a given point p as a $\mathbb{C}$-valued Hermitian form on $T_p^{(1,0)}(M)$ defined up to a non-zero real multiple and speak of the *signature of the Levi form* up to sign.

Let g, $\tilde{g}$ be two $\mathbb{C}^k$-valued Hermitian forms on complex vector spaces V, $\tilde{V}$, respectively. We say that g and $\tilde{g}$ are *equivalent* if there exists a complex-linear isomorphism $A : V \to \tilde{V}$ and $B \in \mathrm{GL}(k, \mathbb{R})$ such that

$$\tilde{g}(Az, Az) = Bg(z, z)$$

for all $z \in V$. Clearly, the Levi form $\mathfrak{L}_M(p)$ defines an equivalence class of $\mathbb{C}^k$-valued Hermitian forms. When we refer to $\mathfrak{L}_M(p)$ as a $\mathbb{C}^k$-valued Hermitian form, we speak of a representative in this equivalence class.

Let $g = (g_1, \dots, g_k)$ be a $\mathbb{C}^k$-valued Hermitian form on $\mathbb{C}^n$. We say that g is *non-degenerate* if

(i) the scalar Hermitian forms $g_1, \dots, g_k$ are linearly independent over $\mathbb{R}$, and

(ii) $g(z, z') = 0$ for all $z' \in \mathbb{C}^n$ implies $z = 0$.

Observe that for a non-degenerate Hermitian form g one has $1 \le k \le n^2$. If $k = 1$ and g is non-degenerate, we write the *signature* of g as (l_1, l_2) with $l_1 + l_2 = n$, where l_1 and l_2 are the numbers of positive and negative eigenvalues of g, respectively.

A CR-manifold M is called *Levi non-degenerate* if its Levi form at any $p \in M$ is non-degenerate. Everywhere in this book, with the exception of Chapter 9, we consider only Levi non-degenerate CR-manifolds. Further, we call a CR-manifold M *strongly uniform* if $\mathfrak{L}_M(p)$ and $\mathfrak{L}_M(q)$ are equivalent for all $p, q \in M$. Every Levi non-degenerate CR-hypersurface is strongly uniform.

For any $\mathbb{C}^k$-valued Hermitian form g on $\mathbb{C}^n$ we define a CR-manifold $Q_g \subset \mathbb{C}^{n+k}$ of CR-dimension n and CR-codimension k as follows:

$$Q_g := \{(z,w) \in \mathbb{C}^{n+k} : \operatorname{Im} w = g(z,z)\}, \tag{1.4}$$

where $z = (z_1, \dots, z_n)$ is a point in $\mathbb{C}^n$ and $w \in \mathbb{C}^k$. The manifold Q_g is often called the *quadric associated to* g. The Levi form of Q_g at every point is equivalent to g.

If $k = 1$ and $g(z,z) = ||z||^2$ (where $||\cdot||$ is the Euclidean norm on $\mathbb{C}^n$), the quadric Q_g is CR-equivalent to the unit sphere in $\mathbb{C}^{n+1}$ with one point removed. Indeed, the map

$$(z,w) \mapsto \left(\frac{z}{1-w}, i\frac{w+1}{1-w}\right) \tag{1.5}$$

transforms

$$Q'_{||\cdot||^2} := \left\{(z,w) \in \mathbb{C}^{n+1} : ||z||^2 + |w|^2 = 1\right\} \setminus \{(0,1)\}$$

into $Q_{||\cdot||^2}$. More generally, for $k = 1$ and an arbitrary Hermitian form g on $\mathbb{C}^n$ set

$$S_g := \left\{(z,w) \in \mathbb{C}^{n+1} : g(z,z) + |w|^2 = 1\right\}. \tag{1.6}$$

Map (1.5) transforms

$$Q'_g := S_g \setminus \left\{(z,1) \in \mathbb{C}^{n+1} : g(z,z) = 0\right\}$$

into $Q_g \setminus \left\{(z,-i) \in \mathbb{C}^{n+1} : g(z,z) = -1\right\}$.

Assume now that g is non-degenerate. In this case every local CR-automorphism of Q_g extends to a birational map of $\mathbb{C}^{n+k}$ (see classical papers [1], [90], [99] for $k = 1$ and papers [6], [7], [44], [62], [67], [68], [70], [98], [106] for $1 < k \le n^2$). Let $\operatorname{Bir}(Q_g)$ denote the set of all such birational extensions. It turns out that $\operatorname{Bir}(Q_g)$ is a group (see [62]). For $k = 1$ every element of $\operatorname{Bir}(Q_g)$ is a linear-fractional transformation induced by an automorphism of $\mathbb{CP}^{n+1}$ (see [1], [90], [99]). For $1 < k \le n^2$ some formulas for the elements of $\operatorname{Bir}(Q_g)$ were given in [37]. It was shown in [62], [106] that the group $\operatorname{Bir}(Q_g)$ can be endowed with the structure of a Lie group with at most countably many connected components and the Lie algebra isomorphic to the Lie algebra $\mathfrak{g}_g$ of all infinitesimal CR-automorphisms of Q_g. Every infinitesimal CR-automorphism of Q_g is known to be polynomial (see [106]). We denote by $\operatorname{Bir}(Q_g)^\circ$ the connected component of $\operatorname{Bir}(Q_g)$ (with respect to the Lie group topology) that contains the identity.[1] One can show that $\operatorname{Bir}(Q_g)/\operatorname{Bir}(Q_g)^\circ$ is in fact finite.

Note that Q_g is a homogeneous manifold since the subgroup $\mathfrak{H}_g \subset \operatorname{Bir}(Q_g)$ of CR-automorphisms of the form

$$(z,w) \mapsto (z+a, w+2ig(z,a)+ig(a,a)+b),$$

[1] In general, for a topological group G we denote its connected component containing the identity by G°.

with $a \in \mathbb{C}^n$, $b \in \mathbb{R}^k$, acts transitively on Q_g. Therefore, it is important to consider the subgroup of all elements of $\mathrm{Bir}(Q_g)$ that are defined and biholomorphic near a particular point in Q_g, say the origin, and preserve it. This subgroup, which we denote by $\mathrm{Bir}_0(Q_g)$, is closed in $\mathrm{Bir}(Q_g)$, and $\mathrm{Bir}(Q_g) = \mathfrak{H}_g \cdot \mathrm{Bir}_0(Q_g) \cdot \mathfrak{H}_g$ (this follows, for example, from results of [62]).

Further, let $\mathrm{Lin}(Q_g) \subset \mathrm{Bir}_0(Q_g)$ be the Lie subgroup of linear automorphisms of Q_g. Every element of $\mathrm{Lin}(Q_g)$ has the form

$$(z,w) \mapsto (Cz, \rho w),$$

with $C \in \mathrm{GL}(n,\mathbb{C})$ and $\rho \in \mathrm{GL}(k,\mathbb{R})$ satisfying $g(Cz,Cz) \equiv \rho g(z,z)$. It is shown in [37] that $\mathrm{Bir}_0(Q_g) = \mathrm{Lin}(Q_g) \cdot \mathrm{Bir}_0(Q_g)^\circ$. We call a Levi non-degenerate CR-manifold M *weakly uniform* if for any pair of points $p,q \in M$ the Lie groups $\mathrm{Lin}(Q_{\mathfrak{L}_M(p)})^\circ$, $\mathrm{Lin}(Q_{\mathfrak{L}_M(q)})^\circ$ are isomorphic by means of a map that extends to an isomorphism between $\mathrm{Bir}_0(Q_{\mathfrak{L}_M(p)})^\circ$ and $\mathrm{Bir}_0(Q_{\mathfrak{L}_M(q)})^\circ$. Clearly, for a Levi non-degenerate CR-manifold strong uniformity implies weak uniformity.

Existing results on the equivalence problem for CR-structures treat two classes of Levi non-degenerate manifolds: (i) the strongly uniform Levi non-degenerate manifolds, and (ii) the weakly uniform manifolds for which, in addition, the groups $\mathrm{Bir}_0(Q_{\mathfrak{L}_M(p)})$ are "sufficiently small", in particular $\mathrm{Bir}_0(Q_{\mathfrak{L}_M(p)}) = \mathrm{Lin}(Q_{\mathfrak{L}_M(p)})$.

In [17] (see [67] for a detailed exposition) É. Cartan solved the equivalence problem for all 3-dimensional Levi non-degenerate CR-hypersurfaces by reducing their CR-structures to absolute parallelisms (note that this reduction differs from Cartan's approach to general G-structures mentioned earlier – cf. [9]). In 1967 Tanaka obtained a solution for all Levi non-degenerate strongly uniform manifolds (see [101]), but his result became widely known only after Chern-Moser's work [24] was published in 1974 (see also [9], [10], [11], [23], [66]), where the problem was solved independently for all Levi non-degenerate CR-hypersurfaces. Although Tanaka's pioneering construction is important and applies to very general situations (which include geometric structures other than CR-structures), his treatment of CR-hypersurfaces is less detailed and clear – and is certainly less useful in calculations – than that due to Chern (see [76] for a discussion of this matter).

For example, Tanaka's construction gives 3-reducibility to absolute parallelisms, whereas Chern's construction gives 2-reducibility and in fact even 1-reducibility (see [9]). The structure group of the single bundle $\mathscr{P}^2 \to M$ that arises in Chern's construction is $\mathrm{Bir}_0(Q_g)$, where g is a Hermitian form equivalent to every $\mathfrak{L}_M(p)$, $p \in M$, and the absolute parallelism σ takes values in the Lie algebra $\mathfrak{g}_g$ (which is isomorphic to the Lie algebra of $\mathrm{Bir}(Q_g)$). The Lie algebra $\mathfrak{g}_g$ is well-understood for an arbitrary CR-codimension (see [7], [31], [34], [93]). In particular, $\mathfrak{g}_g$ is a graded Lie algebra: $\mathfrak{g}_g = \oplus_{k=-2}^{2} \mathfrak{g}_g^k$. In Tanaka's construction, however, the absolute parallelism takes values in a certain prolongation $\tilde{\mathfrak{g}}_g$ of $\oplus_{k=-2}^{0} \mathfrak{g}_g^k$. The fact that $\tilde{\mathfrak{g}}_g$ and $\mathfrak{g}_g$ coincide for an arbitrary CR-codimension is not obvious (see [31]). Further, the absolute parallelism σ from Chern's construction is in fact a *Cartan connection* (to be defined in Section 1.2). In particular, it changes in a regular way under the action of the structure group of the bundle $\mathscr{P}^2$ (see also

[9]). Namely, if for $a \in \mathrm{Bir}_0(Q_g)$ we denote by L_a the (left) action of a on $\mathscr{P}^2$, then $L_a^* \sigma = \mathrm{Ad}_{\mathrm{Bir}_0(Q_g),\mathfrak{g}_g}(a)\sigma$, where $\mathrm{Ad}_{\mathrm{Bir}_0(Q_g),\mathfrak{g}_g}$ is the adjoint representation of $\mathrm{Bir}_0(Q_g)$. It is not clear from [101] (even in the CR-hypersurface case) whether the sequence of bundles $\tilde{\mathscr{P}}^3 \to \tilde{\mathscr{P}}^2 \to \tilde{\mathscr{P}}^1 \to M$ constructed there can be reduced to a single bundle and whether the absolute parallelism defined on $\tilde{\mathscr{P}}^3$ behaves in any sense like a Cartan connection. [We note, however, that these points were clarified in Tanaka's later work [102] (see also [103]), where complete proofs of the results announced in [100] were presented (see also Tanaka's earlier work [99], where a special class of Levi non-degenerate CR-hypersurfaces was considered).]

Being more detailed, Chern's construction also allows one to investigate in detail the important *curvature form of* σ, i.e. the 2-form $\Sigma := d\sigma - 1/2[\sigma,\sigma]$ (this form is of particular importance to us throughout the book). It also can be used to introduce special invariant curves called *chains*, which have turned out to be important in the study of real hypersurfaces in complex manifolds (see, e.g. [107]). Due to these and other differences between Tanaka's and Chern's constructions, we prefer to use Chern's approach in our treatment of Levi non-degenerate CR-hypersurfaces later in the chapter. We also remark here that in a certain more general situation (namely for Levi non-degenerate *partially integrable CR-structures* of CR-codimension one) Cartan connections were constructed in [14] as part of a general approach to producing Cartan connections for parabolic geometries (see also [13]). For more details on the parabolic geometry approach we refer the reader to recent monograph [16].

We finish this introduction with a brief survey of existing results for manifolds with $\mathrm{CRcodim}\, M \geq 2$. Certain Levi non-degenerate weakly uniform CR-structures of CR-codimension two were considered in [77], [85]. Conditions imposed on the Levi form in these papers are stronger than non-degeneracy and force the groups $\mathrm{Bir}_0(Q_{\mathfrak{L}_M(p)})$ for all $p \in M$ to be minimal possible. In particular, they contain only linear transformations of a special form (in this case $\mathfrak{g}^k_{\mathfrak{L}_M(p)} = 0$ for $k = 1,2$). Further, the situation where the groups $\mathrm{Bir}_0(Q_{\mathfrak{L}_M(p)})$ are small and $\mathrm{CRcodim}\, M > 2$, $\mathrm{CRdim}\, M > (\mathrm{CRcodim}\, M)^2$ was treated in [47]. One motivation for considering manifolds with the Levi form satisfying conditions as in [85] (for $\mathrm{CRdim}\, M \geq 7$), [47], [77] is that these conditions are open, i.e. if they are satisfied at a point p, then they are also satisfied on a neighborhood of p in M. Moreover, the quadrics associated to Levi forms as in [85] (for $\mathrm{CRdim}\, M \geq 7$) and [77] are dense (in an appropriate sense) in the space of all Levi non-degenerate quadrics.

Finally, the case $\mathrm{CRdim}\, M = \mathrm{CRcodim}\, M = 2$ has been studied very extensively in recent years. This is one of only two exceptional cases among all CR-structures with $\mathrm{CRcodim}\, M > 1$ in the following sense: typically (in fact always except for the cases $\mathrm{CRdim}\, M = \mathrm{CRcodim}\, M = 2$ and $(\mathrm{CRdim}\, M)^2 = \mathrm{CRcodim}\, M$) generic Levi non-degenerate quadrics have only linear automorphisms (see [36] and also [7], [85]). However, in the case $\mathrm{CRdim}\, M = \mathrm{CRcodim}\, M = 2$ Levi non-degenerate quadrics *always* have many non-linear automorphisms. Every non-degenerate $\mathbb{C}^2$-valued Hermitian form $g = (g_1, g_2)$ on $\mathbb{C}^2$ is equivalent to one of the following:

$$\begin{aligned} g^{\mathrm{hyp}}(z,z) &:= (|z_1|^2 + |z_2|^2, z_1\bar{z}_2 + z_2\bar{z}_1), \\ g^{\mathrm{ell}}(z,z) &:= (|z_1|^2 - |z_2|^2, z_1\bar{z}_2 + z_2\bar{z}_1), \\ g^{\mathrm{par}}(z,z) &:= (|z_1|^2, z_1\bar{z}_2 + z_2\bar{z}_1). \end{aligned}$$

These forms are called *hyperbolic*, *elliptic*, and *parabolic*, respectively. The groups $\mathrm{Bir}(Q_g)^\circ$, $\mathrm{Bir}_0(Q_g)^\circ$ and the Lie algebra $\mathfrak{g}_g$, where g is one of g^{hyp}, g^{ell}, g^{par}, are quite large. They were explicitly found in [33] (see also [7], [35], [37]).

A CR-manifold whose Levi form at every point is equivalent to g^{hyp} or g^{ell} is called *hyperbolic* or *elliptic*, respectively. Clearly, the conditions of hyperbolicity and ellipticity are open. The equivalence problem for hyperbolic and elliptic CR-manifolds is of course covered by Tanaka's construction in [101]. More explicit reductions of elliptic and hyperbolic CR-structures to absolute parallelisms, and even to Cartan connections, were obtained in [32], [94], [95]. The rich geometry of hyperbolic and elliptic CR-manifolds (and their partially integrable generalizations) was also studied in [12], [15], [38], [39].

1.2 Chern's Construction

From this moment to the end of Chapter 8 we only consider Levi non-degenerate CR-hypersurfaces with integrable CR-structure. In the present section we describe Chern's construction from [24], which gives 2-reducibility of such CR-structures to absolute parallelisms. In fact, even 1-reducibility takes places for this construction (see [9]).

Let M be a Levi non-degenerate CR-hypersurface with an integrable CR-structure of CR-dimension n. Locally on M the CR-structure is given by 1-forms μ, η^α (here and below small Greek indices run from 1 to n unless specified otherwise), where μ is real-valued and vanishes exactly on the complex tangent spaces, η^α are complex-valued and complex-linear on the complex tangent spaces. The integrability condition of the CR-structure is then equivalent to the Frobenius condition, which states that $d\mu$, $d\eta^\alpha$ belong to the differential ideal generated by μ, η^α. Since μ is real-valued, this condition implies

$$d\mu \equiv i h_{\alpha\overline{\beta}} \eta^\alpha \wedge \eta^{\overline{\beta}} \pmod{\mu} \tag{1.7}$$

for some functions $h_{\alpha\overline{\beta}}$ satisfying $h_{\alpha\overline{\beta}} = h_{\overline{\beta}\alpha}$. Here and below we use the convention $\eta^{\overline{\beta}} := \overline{\eta^\beta}$, $h_{\overline{\beta}\alpha} := \overline{h_{\beta\overline{\alpha}}}$, etc. as well as the usual summation convention for subscripts and superscripts. At every point the matrix $(h_{\alpha\overline{\beta}})$ defines a Hermitian form on $\mathbb{C}^n$ equivalent to the Levi form of M, where α is the row index and β is the column index (see the footnote on the next two pages).

For $p \in M$ define E_p as the collection of all covectors $\theta \in T_p^*(M)$ such that $T_p^c(M) = \{Y \in T_p(M) : \theta(Y) = 0\}$. Clearly, all elements in E_p are real non-zero multiples of each other. Let E be the subbundle of the cotangent bundle of M whose fiber over p is E_p. Define θ^0 to be the tautological 1-form on E, that is, for $\theta \in E$ and $Y \in T_\theta(E)$ set

$$\theta^0(\theta)(Y) := \theta(d\pi_E(\theta)(Y)),$$

where $\pi_E : E \to M$ is the natural projection.

We now fix a non-degenerate Hermitian form on $\mathbb{C}^n$ with matrix $g = (g_{\alpha\overline{\beta}})$ which is equivalent to every $\mathfrak{L}_M(p)$, $p \in M$. Identity (1.7) implies that for every $\theta \in E$ there exist a real-valued covector θ^{n+1} and complex-valued covectors θ^α on $T_\theta(E)$ such that: (a) each θ^α is a lift of a complex-valued covector on $T_{\pi_E(\theta)}(M)$ which is complex-linear on $T^c_{\pi_E(\theta)}(M)$, (b) the covectors $\theta^0(\theta)$, $\operatorname{Re}\theta^\alpha$, $\operatorname{Im}\theta^\alpha$, θ^{n+1} form a basis of the cotangent space $T^*_\theta(E)$, and (c) the following identity holds:

$$d\theta^0(\theta) = \pm i g_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}} + \theta^0(\theta) \wedge \theta^{n+1}. \tag{1.8}$$

For every $p \in M$ the fiber E_p has exactly two connected components, and if the numbers of positive and negative eigenvalues of $(g_{\alpha\overline{\beta}})$ are distinct, the signs in the right-hand side of (1.8) coincide for all θ lying in the same connected component of E_p and are opposite for θ_1 and θ_2 lying in different connected components irrespectively of the choice of θ^α, θ^{n+1}. In this situation we define a bundle $\mathscr{P}^1$ over M as follows: for every $p \in M$ the fiber $\mathscr{P}^1_p$ over p is connected and consists of all elements $\theta \in E_p$ for which the plus sign occurs in the right-hand side of (1.8); we also set $\pi^1 := \pi_E\big|_{\mathscr{P}^1}$. Next, if the numbers of positive and negative eigenvalues of $(g_{\alpha\overline{\beta}})$ are equal, for every $\theta \in E$ and every choice of the sign in the right-hand side of (1.8) there are covectors θ^α, θ^{n+1} on $T_\theta(E)$ satisfying (1.8). In this case we set $\mathscr{P}^1 := E$ and $\pi^1 := \pi_E$.

For $\theta \in \mathscr{P}^1$ we now only consider covectors θ^α, θ^{n+1} on $T_\theta(\mathscr{P}^1)$ satisfying conditions (a), (b) stated above and such that

$$d\theta^0(\theta) = i g_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}} + \theta^0(\theta) \wedge \theta^{n+1}. \tag{1.9}$$

The most general linear transformation of $\theta^0(\theta)$, θ^α, $\theta^{\overline{\alpha}}$, θ^{n+1} preserving equation (1.9) and the covector $\theta^0(\theta)$ is given by the matrix (acting on the left)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ v^\alpha & u^\alpha_\beta & 0 & 0 \\ v^{\overline{\alpha}} & 0 & u^{\overline{\alpha}}_{\overline{\beta}} & 0 \\ s & i g_{\rho\overline{\sigma}} u^\rho_\beta v^{\overline{\sigma}} & -i g_{\rho\overline{\sigma}} u^{\overline{\sigma}}_{\overline{\beta}} v^\rho & 1 \end{pmatrix}, \tag{1.10}$$

where $s \in \mathbb{R}$, $u^\alpha_\beta, v^\alpha \in \mathbb{C}$ and $g_{\alpha\overline{\beta}} u^\alpha_\rho u^{\overline{\beta}}_{\overline{\sigma}} = g_{\rho\overline{\sigma}}$. In u^α_β and v^α the superscripts are used for indexing the rows and the subscript for indexing the columns.[2] Let G_1 be the

[2] We follow this convention throughout the book whenever reasonable. However, the entries of the matrices of Hermitian and bilinear forms are indexed by subscripts or superscripts alone, e.g.

group of matrices of the form (1.10). Clearly, $\mathscr{P}^1$ is equipped with a G_1-structure (upon identification of G_1 with a subgroup of $\mathrm{GL}(2n+2,\mathbb{R})$). Our immediate goal is to reduce this G_1-structure to an absolute parallelism.

We define a principal G_1-bundle $\mathscr{P}^2$ over $\mathscr{P}^1$ as follows: for $\theta \in \mathscr{P}^1$ let the fiber $\mathscr{P}^2_\theta$ over θ be the collection of all covectors $(\theta^0(\theta), \theta^\alpha, \theta^{n+1})$ on $T_\theta(\mathscr{P}^1)$ satisfying conditions (a), (b), (1.9), and let $\pi^2 : \mathscr{P}^2 \to \mathscr{P}^1$ be the natural projection. Set

$$\omega := [\pi^2]^* \, \theta^0$$

and introduce a collection of tautological 1-forms on $\mathscr{P}^2$ as follows:

$$\begin{aligned}\omega^\alpha(\Theta)(Y) &:= \theta^\alpha(d\pi^2(\Theta)(Y)),\\ \varphi(\Theta)(Y) &:= \theta^{n+1}(d\pi^2(\Theta)(Y)),\end{aligned}$$

where $\Theta = (\theta^0(\theta), \theta^\alpha, \theta^{n+1})$ is a point in $\mathscr{P}^2_\theta$ and $Y \in T_\Theta(\mathscr{P}^2)$. It is clear from (1.9) that these forms satisfy

$$d\omega = i g_{\alpha\overline{\beta}} \omega^\alpha \wedge \omega^{\overline{\beta}} + \omega \wedge \varphi. \tag{1.11}$$

Further, the integrability of the CR-structure of M yields that locally on $\mathscr{P}^2$ we have

$$d\omega^\alpha = \omega^\beta \wedge \varphi^\alpha_\beta + \omega \wedge \varphi^\alpha \tag{1.12}$$

for some 1-forms φ^α_β and φ^α. In what follows we will study consequences of identities (1.11) and (1.12). Our calculations will be entirely local, and we will impose conditions that will determine the forms φ^α_β and φ^α (as well as another 1-form ψ introduced below) uniquely. This will allow us to patch the locally defined forms φ^α_β, φ^α, ψ into globally defined 1-forms on $\mathscr{P}^2$. Together with ω, ω^α, φ these globally defined forms will be used to construct an absolute parallelism on $\mathscr{P}^2$ with required properties.

Let $(g^{\alpha\overline{\beta}})$ be the matrix inverse to $(g_{\alpha\overline{\beta}})$, that is,

$$g_{\alpha\overline{\beta}} g^{\gamma\overline{\beta}} = \delta^\gamma_\alpha, \quad g_{\alpha\overline{\beta}} g^{\alpha\overline{\gamma}} = \delta^{\overline{\gamma}}_{\overline{\beta}}.$$

As is customary in tensor analysis, we use $(g_{\alpha\overline{\beta}})$ and $(g^{\alpha\overline{\beta}})$ to lower and raise indices, respectively. For quantities that have subscripts as well as a superscript it is important to know the location where the superscript can be lowered to, and this is indicated by a dot. Thus, we write $\varphi^{\alpha}_{\beta\cdot}$ for φ^α_β and $\varphi_{\beta\overline{\gamma}}$ for $\varphi^{\alpha}_{\beta\cdot} g_{\alpha\overline{\gamma}}$, etc.

$(g_{\alpha\overline{\beta}})$ and $(g^{\alpha\overline{\beta}})$. For the matrix $(g_{\alpha\overline{\beta}})$ the first subscript is the row index and the second one is the column index, whereas for the matrix $(g^{\alpha\overline{\beta}})$ the first superscript is the column index and the second one is the row index. Further, coordinates are indexed by subscripts rather than superscripts everywhere in the book except Section 1.3. Accordingly, vectors are usually written as rows with the entries indexed by subscripts. When a matrix is applied to a row-vector on the left, it is meant that the vector needs to be transposed first.

Above we assumed the matrix g to be constant, but for all calculations below we suppose that it is a matrix-valued map on $\mathscr{P}^1$. In this case the bundle $\mathscr{P}^2$ must be replaced by a different bundle (see Section 1.3 for a precise construction). Allowing the matrix g to be variable makes our calculations more general than one needs just for the purposes of constructing an absolute parallelism on $\mathscr{P}^2$, but these more general calculations will have a further application in Section 1.3.

Differentiation of (1.11) and (1.12) yields, respectively,

$$\begin{aligned} i\left(dg_{\alpha\overline{\beta}} - \varphi_{\alpha\overline{\beta}} - \varphi_{\overline{\beta}\alpha} + g_{\alpha\overline{\beta}}\varphi\right) \wedge \omega^{\alpha} \wedge \omega^{\overline{\beta}} + \\ \left(-d\varphi + i\omega_{\overline{\beta}} \wedge \varphi^{\overline{\beta}} + i\varphi_{\overline{\beta}} \wedge \omega^{\overline{\beta}}\right) \wedge \omega = 0 \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} \left(d\varphi_{\beta\cdot}^{\alpha} - \varphi_{\beta\cdot}^{\gamma} \wedge \varphi_{\gamma\cdot}^{\alpha} - i\omega_{\beta} \wedge \varphi^{\alpha}\right) \wedge \omega^{\beta} + \\ \left(d\varphi^{\alpha} - \varphi \wedge \varphi^{\alpha} - \varphi^{\beta} \wedge \varphi_{\beta\cdot}^{\alpha}\right) \wedge \omega = 0. \end{aligned} \tag{1.14}$$

Lemma 1.1. *There exist $\varphi_{\beta\cdot}^{\alpha}$ that satisfy (1.12) and the conditions*

$$dg_{\alpha\overline{\beta}} - \varphi_{\alpha\overline{\beta}} - \varphi_{\overline{\beta}\alpha} + g_{\alpha\overline{\beta}}\varphi = 0. \tag{1.15}$$

Such $\varphi_{\beta\cdot}^{\alpha}$ are unique up to an additive term in ω.

Proof. It follows from (1.13) that

$$dg_{\alpha\overline{\beta}} - \varphi_{\alpha\overline{\beta}} - \varphi_{\overline{\beta}\alpha} + g_{\alpha\overline{\beta}}\varphi = A_{\alpha\overline{\beta}\gamma}\omega^{\gamma} + B_{\alpha\overline{\beta}\overline{\gamma}}\omega^{\overline{\gamma}} + C_{\alpha\overline{\beta}}\omega$$

for some functions $A_{\alpha\overline{\beta}\gamma}$, $B_{\alpha\overline{\beta}\overline{\gamma}}$, $C_{\alpha\overline{\beta}}$ satisfying

$$A_{\alpha\overline{\beta}\gamma} = A_{\gamma\overline{\beta}\alpha}, \quad B_{\alpha\overline{\beta}\overline{\gamma}} = B_{\alpha\overline{\gamma}\overline{\beta}}\,. \tag{1.16}$$

The Hermitian property of $g_{\alpha\overline{\beta}}$ also yields

$$A_{\overline{\alpha}\beta\overline{\gamma}} = B_{\beta\overline{\alpha}\overline{\gamma}}, \quad C_{\overline{\alpha}\beta} = C_{\beta\overline{\alpha}}\,. \tag{1.17}$$

Due to (1.16), (1.17) the forms

$$\tilde{\varphi}_{\alpha\overline{\beta}} := \varphi_{\alpha\overline{\beta}} + A_{\alpha\overline{\beta}\gamma}\omega^{\gamma} + \frac{1}{2}C_{\alpha\overline{\beta}}$$

satisfy relations (1.15) and, upon raising indices, relations (1.12). Verification of the last statement of the lemma is straightforward. □

From now on we suppose that (1.15) holds. Identity (1.13) then gives

$$d\varphi = i\omega_{\overline{\beta}} \wedge \varphi^{\overline{\beta}} + i\varphi_{\overline{\beta}} \wedge \omega^{\overline{\beta}} + \omega \wedge \psi, \tag{1.18}$$

where ψ is a real 1-form. The forms $\varphi^{\alpha}_{\beta\cdot}$, φ^{α}, ψ satisfying (1.12), (1.15), (1.18) are defined up to transformations of the form

$$\begin{aligned}\varphi^{\alpha}_{\beta\cdot} &= \tilde{\varphi}^{\alpha}_{\beta\cdot} + D^{\alpha}_{\beta\cdot}\omega,\\ \varphi^{\alpha} &= \tilde{\varphi}^{\alpha} + D^{\alpha}_{\beta\cdot}\omega^{\beta} + E^{\alpha}\omega,\\ \psi &= \tilde{\psi} + T\omega + i(E_{\alpha}\omega^{\alpha} - E_{\overline{\alpha}}\omega^{\overline{\alpha}})\end{aligned} \tag{1.19}$$

for some functions $D^{\alpha}_{\beta\cdot}$, E^{α}, T, where T is real-valued and the following holds:

$$D_{\alpha\overline{\beta}} + D_{\overline{\beta}\alpha} = 0. \tag{1.20}$$

Observe also that one can choose a subset $\mathfrak{S}$ of $\{\operatorname{Re}\varphi^{\alpha}_{\beta\cdot}, \operatorname{Im}\varphi^{\alpha}_{\beta\cdot}\}$ such that for any $\Theta \in \mathscr{P}^2$ the values at Θ of the forms in the set $\mathfrak{S}\cup\{\omega, \operatorname{Re}\omega^{\alpha}, \operatorname{Im}\omega^{\alpha}, \varphi, \operatorname{Re}\varphi^{\alpha}, \operatorname{Im}\varphi^{\alpha}, \psi\}$ constitute a basis of $T^*_{\Theta}(\mathscr{P}^2)$.

Let

$$\Pi^{\alpha}_{\beta\cdot} := d\varphi^{\alpha}_{\beta\cdot} - \varphi^{\gamma}_{\beta\cdot}\wedge\varphi^{\alpha}_{\gamma\cdot}\,. \tag{1.21}$$

Using (1.15) we obtain

$$\Pi_{\beta\overline{\alpha}} = g_{\gamma\overline{\alpha}}d\varphi^{\gamma}_{\beta\cdot} - \varphi^{\gamma}_{\beta\cdot}\wedge\varphi_{\gamma\overline{\alpha}} = d\varphi_{\beta\overline{\alpha}} - \varphi_{\beta\overline{\alpha}}\wedge\varphi - \varphi_{\overline{\alpha}\gamma}\wedge\varphi^{\gamma}_{\beta\cdot}\,.$$

Since

$$\varphi_{\beta\overline{\gamma}}\wedge\varphi^{\overline{\gamma}}_{\overline{\alpha}\cdot} = \varphi^{\gamma}_{\beta\cdot}\wedge\varphi_{\overline{\alpha}\gamma}\,,$$

it then follows that

$$\Pi_{\beta\overline{\alpha}} + \Pi_{\overline{\alpha}\beta} = d(\varphi_{\beta\overline{\alpha}} + \varphi_{\overline{\alpha}\beta}) - (\varphi_{\beta\overline{\alpha}} + \varphi_{\overline{\alpha}\beta})\wedge\varphi.$$

Differentiating (1.15) we obtain

$$\Pi_{\beta\overline{\alpha}} + \Pi_{\overline{\alpha}\beta} = g_{\beta\overline{\alpha}}d\varphi. \tag{1.22}$$

Let

$$\Gamma^{\alpha}_{\beta\cdot} := \Pi^{\alpha}_{\beta\cdot} - i\omega_{\beta}\wedge\varphi^{\alpha} + i\varphi_{\beta}\wedge\omega^{\alpha} + i\delta^{\alpha}_{\beta}(\varphi_{\sigma}\wedge\omega^{\sigma}). \tag{1.23}$$

It follows from (1.14), (1.18), (1.22), (1.23) that

$$\Gamma^{\alpha}_{\beta\cdot}\wedge\omega^{\beta} \equiv 0, \quad \Gamma_{\beta\overline{\alpha}} + \Gamma_{\overline{\alpha}\beta} \equiv 0 \pmod{\omega}. \tag{1.24}$$

Lemma 1.2. *We have*

$$\Gamma_{\beta\overline{\alpha}} \equiv S_{\beta\gamma\overline{\alpha}\overline{\sigma}}\omega^{\gamma}\wedge\omega^{\overline{\sigma}} \pmod{\omega},$$

where the functions $S_{\beta\gamma\overline{\alpha}\overline{\sigma}}$ *have the following symmetry properties:*

$$S_{\beta\gamma\overline{\alpha}\overline{\sigma}} = S_{\gamma\beta\overline{\alpha}\overline{\sigma}} = S_{\gamma\beta\overline{\sigma}\overline{\alpha}} = S_{\overline{\alpha}\overline{\sigma}\beta\gamma}. \tag{1.25}$$

Proof. From the first set of equations in (1.24) we see

$$\Gamma_{\beta\overline{\alpha}} \equiv \chi_{\beta\overline{\alpha}\gamma} \wedge \omega^\gamma \pmod{\omega},$$

where $\chi_{\beta\overline{\alpha}\gamma}$ are 1-forms. Hence, the second set of equations in (1.24) yields

$$\chi_{\beta\overline{\alpha}\gamma} \wedge \omega^\gamma + \chi_{\overline{\alpha}\beta\overline{\gamma}} \wedge \omega^{\overline{\gamma}} \equiv 0 \pmod{\omega},$$

and therefore

$$\chi_{\beta\overline{\alpha}\gamma} \wedge \omega^\gamma \equiv S_{\beta\gamma\overline{\alpha}\overline{\sigma}} \omega^\gamma \wedge \omega^{\overline{\sigma}} \pmod{\omega}$$

for some functions $S_{\beta\gamma\overline{\alpha}\overline{\sigma}}$. Symmetry properties (1.25) follow immediately from (1.24). □

We will now impose conditions on the functions $S_{\beta\gamma\overline{\alpha}\overline{\sigma}}$ from Lemma 1.2 to eliminate the remaining freedom in the choice of $\varphi_{\beta\cdot}^{\alpha}$ (see (1.19)).

Lemma 1.3. *The functions $D_{\beta\cdot}^{\alpha}$ are uniquely determined by the conditions*

$$S_{\rho\overline{\sigma}} := S_{\alpha\rho\cdot\overline{\sigma}}^{\alpha} = 0. \tag{1.26}$$

Proof. We need to understand how the functions $S_{\alpha\rho\cdot\overline{\sigma}}^{\gamma}$ change when a transformation of the form (1.19) is performed. Set

$$S := S_{\alpha\cdot}^{\alpha}, \quad D := D_{\alpha\cdot}^{\alpha}.$$

Since $g^{\alpha\overline{\beta}}$, $S_{\alpha\overline{\beta}}$ are Hermitian (see (1.25)) and $D_{\alpha\overline{\beta}}$ are skew-Hermitian (see (1.20)), it follows that S is real-valued and D is imaginary-valued. Indicating the new functions by tildas, we find

$$\tilde{S}_{\alpha\rho\cdot\overline{\sigma}}^{\gamma} = S_{\alpha\rho\cdot\overline{\sigma}}^{\gamma} - i\left(D_{\alpha\cdot}^{\gamma} g_{\rho\overline{\sigma}} + D_{\rho\cdot}^{\gamma} g_{\alpha\overline{\sigma}} - \delta_{\rho}^{\gamma} D_{\overline{\sigma}\alpha} - \delta_{\alpha}^{\gamma} D_{\overline{\sigma}\rho}\right).$$

Then we obtain

$$\tilde{S}_{\rho\overline{\sigma}} = S_{\rho\overline{\sigma}} - i\left(g_{\rho\overline{\sigma}} D + D_{\rho\overline{\sigma}} - (n+1) D_{\overline{\sigma}\rho}\right).$$

To finish the proof of the lemma, we need to show that there exist uniquely defined $D_{\beta\cdot}^{\alpha}$ satisfying (1.20) and such that

$$g_{\rho\overline{\sigma}} D + (n+2) D_{\rho\overline{\sigma}} = -i S_{\rho\overline{\sigma}}. \tag{1.27}$$

Contracting (1.27) we get

$$D = -\frac{i}{2(n+1)} S.$$

Substituting this back into (1.27) yields

$$D_{\rho\overline{\sigma}} = \frac{1}{n+2}\left(-i S_{\rho\overline{\sigma}} + \frac{i}{2(n+1)} S g_{\rho\overline{\sigma}}\right). \tag{1.28}$$

It is immediately verified that the functions $D_{\rho\overline{\sigma}}$ given by formulas (1.28) satisfy (1.20) and (1.27). $\square$

From now on we assume that conditions (1.26) are satisfied, thus $\varphi^{\alpha}_{\beta\cdot}$ are uniquely defined.

Further, Lemma 1.2 yields

$$\Gamma^{\alpha}_{\beta\cdot} = S^{\alpha}_{\beta\rho\cdot\overline{\sigma}}\omega^{\rho}\wedge\omega^{\overline{\sigma}} + \lambda^{\alpha}_{\beta\cdot}\wedge\omega, \tag{1.29}$$

where $\lambda^{\alpha}_{\beta\cdot}$ are 1-forms. It follows from (1.14), (1.21), (1.23), (1.29) that

$$d\varphi^{\alpha} - \varphi\wedge\varphi^{\alpha} - \varphi^{\beta}\wedge\varphi^{\alpha}_{\beta\cdot} - \lambda^{\alpha}_{\beta\cdot}\wedge\omega^{\beta} = \kappa^{\alpha}\wedge\omega, \tag{1.30}$$

where κ^{α} are also 1-forms. From (1.24), (1.25), (1.29) we get

$$\lambda_{\beta\overline{\alpha}} + \lambda_{\overline{\alpha}\beta} + g_{\beta\overline{\alpha}}\psi \equiv 0 \pmod{\omega}. \tag{1.31}$$

We now differentiate (1.29) retaining only the terms that involve $\omega^{\rho}\wedge\omega^{\overline{\sigma}}$. In doing so we use the following formulas, which are immediately obtained from (1.12), (1.15), (1.30):

$$\begin{aligned} d\omega_{\alpha} &= -\omega^{\overline{\beta}}\wedge\varphi_{\alpha\overline{\beta}} + \omega_{\alpha}\wedge\varphi + \omega\wedge\varphi_{\alpha},\\ d\varphi_{\alpha} &= \varphi_{\alpha\overline{\beta}}\wedge\varphi^{\overline{\beta}} + \lambda_{\overline{\beta}\alpha}\wedge\omega^{\overline{\beta}} + \kappa_{\alpha}\wedge\omega. \end{aligned} \tag{1.32}$$

Identities (1.11) (1.12), (1.21), (1.23), (1.29), (1.30), (1.32) then yield

$$\begin{aligned} &dS^{\alpha}_{\beta\rho\cdot\overline{\sigma}} - S^{\alpha}_{\gamma\rho\cdot\overline{\sigma}}\varphi^{\gamma}_{\beta\cdot} - S^{\alpha}_{\beta\gamma\cdot\overline{\sigma}}\varphi^{\gamma}_{\rho\cdot} + S^{\gamma}_{\beta\rho\cdot\overline{\sigma}}\varphi^{\alpha}_{\gamma\cdot} - S^{\alpha}_{\beta\rho\cdot\overline{\gamma}}\varphi^{\overline{\gamma}}_{\overline{\sigma}} \equiv \\ &\qquad i(\lambda^{\alpha}_{\beta\cdot}g_{\rho\overline{\sigma}} + \lambda^{\alpha}_{\rho\cdot}g_{\beta\overline{\sigma}} - \delta^{\alpha}_{\beta}\lambda_{\overline{\sigma}\rho} - \delta^{\alpha}_{\rho}\lambda_{\overline{\sigma}\beta}) \pmod{\omega,\omega^{\gamma},\omega^{\overline{\gamma}}}, \end{aligned}$$

and by contraction we get

$$dS_{\rho\overline{\sigma}} - S_{\gamma\overline{\sigma}}\varphi^{\gamma}_{\rho\cdot} - S_{\rho\overline{\gamma}}\varphi^{\overline{\gamma}}_{\overline{\sigma}\cdot} \equiv i(\lambda^{\beta}_{\beta\cdot}g_{\rho\overline{\sigma}} + \lambda_{\rho\overline{\sigma}} - (n+1)\lambda_{\overline{\sigma}\rho}) \pmod{\omega,\omega^{\gamma},\omega^{\overline{\gamma}}}.$$

Now (1.26) and (1.31) imply

$$\lambda_{\rho\overline{\sigma}} \equiv -\frac{1}{2}g_{\rho\overline{\sigma}}\psi \pmod{\omega,\omega^{\gamma},\omega^{\overline{\gamma}}}.$$

Hence,

$$\lambda_{\rho\overline{\sigma}} \equiv -\frac{1}{2}g_{\rho\overline{\sigma}}\psi + V_{\rho\overline{\sigma}\beta}\omega^{\beta} + W_{\rho\overline{\sigma}\overline{\beta}}\omega^{\overline{\beta}} \pmod{\omega} \tag{1.33}$$

for some functions $V_{\rho\overline{\sigma}\beta}$, $W_{\rho\overline{\sigma}\overline{\beta}}$. Substituting this expression into (1.31) we obtain

$$V_{\rho\overline{\sigma}\beta} + W_{\overline{\sigma}\rho\beta} = 0. \tag{1.34}$$

It now follows from (1.29) that

$$\Phi^{\alpha}_{\beta\cdot} := \Gamma^{\alpha}_{\beta\cdot} + \frac{1}{2}\delta^{\alpha}_{\beta}\psi \wedge \omega = S^{\alpha}_{\beta\rho\cdot\overline{\sigma}}\omega^{\rho} \wedge \omega^{\overline{\sigma}} + V^{\alpha}_{\beta\cdot\rho}\omega^{\rho} \wedge \omega - V^{\alpha}_{\cdot\beta\overline{\sigma}}\omega^{\overline{\sigma}} \wedge \omega. \quad (1.35)$$

Therefore, substitution of $\Gamma^{\alpha}_{\beta\cdot} - i\varphi_{\beta} \wedge \omega^{\alpha} - i\delta^{\alpha}_{\beta}(\varphi_{\sigma} \wedge \omega^{\sigma})$ into (1.14) implies

$$\begin{aligned}\Phi^{\alpha} := d\varphi^{\alpha} - \varphi \wedge \varphi^{\alpha} - \varphi^{\beta} \wedge \varphi^{\alpha}_{\beta\cdot} + \frac{1}{2}\psi \wedge \omega^{\alpha} = \\ -V^{\alpha}_{\beta\cdot\gamma}\omega^{\beta} \wedge \omega^{\gamma} + V^{\alpha}_{\cdot\beta\overline{\sigma}}\omega^{\beta} \wedge \omega^{\overline{\sigma}} + \nu^{\alpha} \wedge \omega,\end{aligned} \quad (1.36)$$

where ν^{α} are 1-forms.

Formulas (1.35) yield that under transformation (1.19) with $D^{\alpha}_{\beta\cdot} = 0$ the functions $V^{\alpha}_{\beta\cdot\rho}$ change as follows:

$$\tilde{V}^{\alpha}_{\beta\cdot\rho} = V^{\alpha}_{\beta\cdot\rho} + i\left(\delta^{\alpha}_{\rho}E_{\beta} + \frac{1}{2}\delta^{\alpha}_{\beta}E_{\rho}\right).$$

Contracting we obtain

$$\tilde{V}^{\rho}_{\beta\cdot\rho} = V^{\rho}_{\beta\cdot\rho} + i\left(n + \frac{1}{2}\right)E_{\beta}.$$

This calculation leads to the following lemma.

Lemma 1.4. *The functions E_{β} are uniquely determined by the conditions*

$$V^{\rho}_{\beta\cdot\rho} = 0. \quad (1.37)$$

From now on we assume that conditions (1.37) are satisfied, thus φ^{α} are uniquely defined.

Next, we differentiate identity (1.18). Using (1.11), (1.12), (1.31), (1.32), (1.36), we get

$$\omega \wedge (-d\psi + \varphi \wedge \psi + 2i\varphi^{\beta} \wedge \varphi_{\beta} - i\omega^{\beta} \wedge \nu_{\beta} - i\nu^{\beta} \wedge \omega_{\beta}) = 0.$$

Therefore, we have

$$\Psi := d\psi - \varphi \wedge \psi - 2i\varphi^{\beta} \wedge \varphi_{\beta} = -i\omega^{\beta} \wedge \nu_{\beta} - i\nu^{\beta} \wedge \omega_{\beta} + \xi \wedge \omega, \quad (1.38)$$

where ξ is a 1-form.

We now differentiate (1.36) retaining only the terms that involve $\omega^{\rho} \wedge \omega^{\overline{\sigma}}$. Using identities (1.11), (1.12), (1.18), (1.21), (1.23), (1.35), (1.36), (1.38), we obtain

$$\begin{aligned}dV^{\alpha}_{\cdot\rho\overline{\sigma}} - V^{\alpha}_{\cdot\beta\overline{\sigma}}\varphi^{\beta}_{\rho\cdot} + V^{\beta}_{\cdot\rho\overline{\sigma}}\varphi^{\alpha}_{\beta\cdot} - V^{\alpha}_{\cdot\rho\overline{\gamma}}\varphi^{\overline{\gamma}}_{\overline{\sigma}\cdot} - V^{\alpha}_{\cdot\rho\overline{\sigma}}\varphi \equiv \\ S^{\alpha}_{\beta\rho\cdot\overline{\sigma}}\varphi^{\beta} + ig_{\rho\overline{\sigma}}\nu^{\alpha} + \frac{i}{2}\delta^{\alpha}_{\rho}\nu_{\overline{\sigma}} \quad (\operatorname{mod}\omega, \omega^{\gamma}, \omega^{\overline{\gamma}}).\end{aligned} \quad (1.39)$$

Conditions (1.37) are equivalent to

$$V^{\alpha}_{\cdot\rho\overline{\sigma}}g^{\rho\overline{\sigma}}=0.$$

Differentiating these identities and using (1.15), (1.25), (1.26), (1.37), (1.39), we obtain

$$\nu^{\alpha}\equiv 0 \quad (\operatorname{mod}\omega,\omega^{\gamma},\omega^{\overline{\gamma}}).$$

Hence, we have

$$\nu^{\alpha}\equiv P^{\alpha}_{\beta\cdot}\omega^{\beta}+Q^{\alpha}_{\overline{\beta}\cdot}\omega^{\overline{\beta}} \quad (\operatorname{mod}\omega) \tag{1.40}$$

for some functions $P^{\alpha}_{\beta\cdot}$, $Q^{\alpha}_{\overline{\beta}\cdot}$. Substitution of (1.40) into (1.36) now yields

$$\Phi^{\alpha}=-V^{\alpha}_{\beta\cdot\gamma}\omega^{\beta}\wedge\omega^{\gamma}+V^{\alpha}_{\cdot\beta\overline{\sigma}}\omega^{\beta}\wedge\omega^{\overline{\sigma}}+P^{\alpha}_{\beta\cdot}\omega^{\beta}\wedge\omega+Q^{\alpha}_{\overline{\beta}\cdot}\omega^{\overline{\beta}}\wedge\omega. \tag{1.41}$$

Further, substituting (1.40) into (1.38) and absorbing into ξ the indeterminacy of ν^{α} in ω, we obtain

$$\Psi=iQ_{\alpha\beta}\omega^{\alpha}\wedge\omega^{\beta}-iQ_{\overline{\alpha}\overline{\beta}}\omega^{\overline{\alpha}}\wedge\omega^{\overline{\beta}}-i\widehat{P}_{\alpha\overline{\beta}}\omega^{\alpha}\wedge\omega^{\overline{\beta}}+\xi\wedge\omega, \tag{1.42}$$

where

$$\widehat{P}_{\alpha\overline{\beta}}:=P_{\alpha\overline{\beta}}+P_{\overline{\beta}\alpha}. \tag{1.43}$$

Formulas (1.36), (1.41) imply that under transformation (1.19) with $D^{\alpha}_{\beta\cdot}=0$ and $E^{\alpha}=0$ the functions $P^{\alpha}_{\beta\cdot}$ change as follows:

$$\tilde{P}^{\alpha}_{\beta\cdot}=P^{\alpha}_{\beta\cdot}+\frac{1}{2}\delta^{\alpha}_{\beta}T,$$

which gives

$$\tilde{P}^{\alpha}_{\alpha\cdot}=P^{\alpha}_{\alpha\cdot}+\frac{n}{2}T. \tag{1.44}$$

On the other hand, from (1.43) we see

$$\widehat{P}^{\alpha}_{\alpha\cdot}=2\operatorname{Re}P^{\alpha}_{\alpha\cdot},$$

and therefore (1.44) yields

$$\widehat{\tilde{P}}{}^{\alpha}_{\alpha\cdot}=\widehat{P}^{\alpha}_{\alpha\cdot}+nT.$$

This leads us to the following lemma.

Lemma 1.5. *The function T is uniquely determined by the condition*

$$\widehat{P}^{\alpha}_{\alpha\cdot}=0. \tag{1.45}$$

With condition (1.45) satisfied, the form ψ is uniquely defined. Thus, the locally defined forms $\varphi^{\alpha}_{\beta\cdot}$, φ^{α}, ψ give rise to 1-forms (which we denote by the same respective symbols) defined on all of $\mathscr{P}^2$.

We will now finalize our formula for Ψ. We differentiate (1.42) retaining only the terms that involve $\omega^\rho \wedge \omega^{\overline{\sigma}}$. Using identities (1.11), (1.12), (1.18), (1.32), (1.33), (1.34), (1.36), (1.38), (1.42), we obtain

$$\begin{aligned} d\widehat{P}_{\rho\overline{\sigma}} - \widehat{P}_{\beta\overline{\sigma}}\varphi^{\beta}_{\rho\cdot} - \widehat{P}_{\rho\overline{\gamma}}\varphi^{\overline{\gamma}}_{\overline{\sigma}\cdot} - \widehat{P}_{\rho\overline{\sigma}}\varphi \equiv & \\ 2V^{\beta}_{\cdot\rho\overline{\sigma}}\varphi_\beta + 2V_{\beta\overline{\sigma}\rho}\varphi^\beta - g_{\rho\overline{\sigma}}\xi \quad & (\operatorname{mod}\omega, \omega^\gamma, \omega^{\overline{\gamma}}). \end{aligned} \tag{1.46}$$

Clearly, condition (1.45) can be written as follows:

$$\widehat{P}_{\alpha\overline{\beta}}g^{\alpha\overline{\beta}} = 0.$$

Differentiating this identity and using (1.15), (1.37), (1.45), (1.46), we get

$$\xi \equiv 0 \quad (\operatorname{mod}\omega, \omega^\gamma, \omega^{\overline{\gamma}}).$$

Since Ψ is real-valued, we can write (1.42) in the form

$$\begin{aligned} \Psi = iQ_{\alpha\beta}\,\omega^\alpha \wedge \omega^\beta - iQ_{\overline{\alpha}\overline{\beta}}\omega^{\overline{\alpha}} \wedge \omega^{\overline{\beta}} - i\widehat{P}_{\alpha\overline{\beta}}\omega^\alpha \wedge \omega^{\overline{\beta}} + & \\ R_\alpha\omega^\alpha \wedge \omega + R_{\overline{\alpha}}\omega^{\overline{\alpha}} \wedge \omega & \end{aligned} \tag{1.47}$$

for some functions R_α.

Remark 1.1. For $n = 1$ all formulas derived above reduce to those given by É. Cartan in [17].

We now assume that the matrix $g = (g_{\alpha\overline{\beta}})$ is constant and define a Hermitian form $\mathscr{H}^g$ on $\mathbb{C}^{n+2}$ with matrix $\left(\mathscr{H}^g_{l\overline{m}}\right)_{l,m=0,\dots,n+1}$ by setting

$$\mathscr{H}^g_{\alpha\overline{\beta}} := g_{\alpha\overline{\beta}}, \quad \mathscr{H}^g_{0\,\overline{n+1}} := -\frac{i}{2}, \quad \mathscr{H}^g_{n+1\,\overline{0}} := \frac{i}{2}, \tag{1.48}$$

and letting the remaining matrix entries to be zero. Let $\mathrm{SU}^{\pm}_{\mathscr{H}^g}$ be the group of matrices $A \in \mathrm{SL}(n+2,\mathbb{C})$ such that $A\mathscr{H}^gA^* = \pm\mathscr{H}^g$. The choice $A\mathscr{H}^gA^* = -\mathscr{H}^g$ is only possible if the numbers of positive and negative eigenvalues of the form g coincide, in which case the group $\mathrm{SU}^{\pm}_{\mathscr{H}^g}$ has exactly two connected components. If the numbers of positive and negative eigenvalues of g are distinct, $\mathrm{SU}^{\pm}_{\mathscr{H}^g}$ is connected. Let $\mathrm{PSU}^{\pm}_{\mathscr{H}^g} := \mathrm{SU}^{\pm}_{\mathscr{H}^g}/\mathscr{Z}$, where $\mathscr{Z}$ is the center of $\mathrm{SU}^{\pm}_{\mathscr{H}^g}$. We denote by H_1 the subgroup of $\left(\mathrm{SU}^{\pm}_{\mathscr{H}^g}\right)^\circ$ that consists of all matrices

$$\begin{pmatrix} t & 0 & 0 \\ t^\alpha & t^\alpha_{\beta\cdot} & 0 \\ \tau & \tau_\beta & t \end{pmatrix}, \tag{1.49}$$

where $|t| = 1$ and the following holds:

$$
\begin{aligned}
&\text{(i)}\ \ t^{\alpha}=-2it\sum_{\beta\overline{\gamma}}t^{\alpha}_{\beta\cdot}g_{\beta\overline{\gamma}}\tau_{\overline{\gamma}},\\
&\text{(ii)}\ \ t^{2}\det(t^{\alpha}_{\beta\cdot})=1,\\
&\text{(iii)}\ \sum_{\rho,\overline{\sigma}}t^{\alpha}_{\rho\cdot}t^{\overline{\beta}}_{\overline{\sigma}\cdot}g_{\rho\overline{\sigma}}=g_{\alpha\overline{\beta}},\\
&\text{(iv)}\ \sum_{\rho,\overline{\sigma}}g_{\rho\overline{\sigma}}\tau_{\rho}\tau_{\overline{\sigma}}+\frac{i}{2}(\overline{\tau}t-\tau t^{-1})=0.
\end{aligned}
\tag{1.50}
$$

Let $\chi : H_1 \to G_1$ be the homomorphism that assigns matrix (1.10), with

$$
\begin{aligned}
&v^{\alpha}=it\sum_{\overline{\beta}}t^{\overline{\beta}}g^{\alpha\overline{\beta}},\\
&(u^{\alpha}_{\beta\cdot})=t\left[(t^{\alpha}_{\beta\cdot})^{T}\right]^{-1},\\
&s=4\,\mathrm{Re}(\tau t^{-1}),
\end{aligned}
$$

to matrix (1.49). The homomorphism χ is onto and its kernel coincides with $\mathscr{Z}$. Hence, G_1 is isomorphic to $H_1/\mathscr{Z}\subset \mathrm{PSU}^{\pm}_{\mathscr{H}^g}$, and we denote by χ_1 the isomorphism between $H_1/\mathscr{Z}$ and G_1 induced by χ.

The Lie algebra $\mathfrak{su}_{\mathscr{H}^g}$ of $\mathrm{SU}^{\pm}_{\mathscr{H}^g}$ consists of all matrices $\mathfrak{A}\in\mathfrak{sl}(n+2,\mathbb{C})$ such that $\mathfrak{A}\mathscr{H}^g+\mathscr{H}^g\mathfrak{A}^*=0$. We now define an $\mathfrak{su}_{\mathscr{H}^g}$-valued absolute parallelism $\sigma=(\sigma^m_l)_{l,m=0,\dots,n+1}$ on $\mathscr{P}^2$ by the formulas

$$
\begin{array}{lll}
\sigma^0_0 := -\dfrac{1}{n+2}\left(\varphi^{\alpha}_{\alpha\cdot}+\varphi\right), & \sigma^0_{\alpha} := \omega^{\alpha}, & \sigma^0_{n+1} := 2\omega,\\[2ex]
\sigma^{\alpha}_0 := -i\varphi_{\alpha}, & \sigma^{\alpha}_{\beta} := \varphi^{\beta}_{\alpha\cdot}+\delta^{\beta}_{\alpha}\sigma^0_0, & \sigma^{\alpha}_{n+1} := 2i\omega_{\alpha},\\[2ex]
\sigma^{n+1}_0 := -\dfrac{1}{4}\psi, & \sigma^{n+1}_{\alpha} := \dfrac{1}{2}\varphi^{\alpha}, & \sigma^{n+1}_{n+1} := -\overline{\sigma^0_0}.
\end{array}
\tag{1.51}
$$

It is easy to observe that σ defines an isomorphism between $T_{\Theta}(\mathscr{P}^2)$ and $\mathfrak{su}_{\mathscr{H}^g}$ for every $\Theta\in\mathscr{P}^2$ (see (1.15)).

Consider the following form called the *curvature form of* σ:

$$
\Sigma := d\sigma-\frac{1}{2}[\sigma,\sigma]=d\sigma-\sigma\wedge\sigma. \tag{1.52}
$$

This is an $\mathfrak{su}_{\mathscr{H}^g}$-valued 2-form with

$$
\Sigma=(\Sigma^m_l)_{l,m=0,\dots,n+1},\quad \Sigma^m_l := d\sigma^m_l-\sigma^m_k\wedge\sigma^k_l.
$$

It is often referred to as the *CR-curvature form* of M. The components Σ^0_{α}, Σ^0_{n+1}, Σ^{α}_{n+1} are called the *torsion* of σ. Conditions (1.11), (1.12), (1.18) yield that the torsion of σ in fact vanishes. Further, a straightforward calculation shows

$$\begin{aligned}
&\Sigma_0^0 = -\frac{1}{n+2}\Phi_{\alpha\cdot}^{\alpha}, \\
&\Sigma_0^{\alpha} = -i\Phi_{\alpha}, \qquad \Sigma_{\beta}^{\alpha} = \Phi_{\alpha\cdot}^{\beta} - \frac{1}{n+2}\delta_{\alpha}^{\beta}\Phi_{\gamma\cdot}^{\gamma}, \\
&\Sigma_0^{n+1} = -\frac{1}{4}\Psi, \qquad \Sigma_{\alpha}^{n+1} = \frac{1}{2}\Phi^{\alpha}, \qquad \Sigma_{n+1}^{n+1} = -\overline{\Sigma_0^0}.
\end{aligned} \tag{1.53}$$

For any 2-form Ω on $\mathscr{P}^2$ in ω^{α}, $\omega^{\overline{\alpha}}$, ω such that

$$\Omega \equiv a_{\alpha\overline{\beta}}\omega^{\alpha} \wedge \omega^{\overline{\beta}} + \text{terms quadratic in } \omega^{\gamma}, \omega^{\overline{\gamma}} \pmod{\omega}$$

set

$$\mathrm{Tr}\,\Omega := a_{\alpha\cdot}^{\alpha}.$$

Then conditions (1.26), (1.37), (1.45) can be restated, respectively, as follows:

$$\begin{aligned}
&\text{(i)}\ \ \mathrm{Tr}\,\Sigma_{\beta}^{\alpha} = 0, \quad \mathrm{Tr}\,\Sigma_0^0 = 0, \\
&\text{(ii)}\ \ \mathrm{Tr}\,\Sigma_0^{\alpha} = 0, \quad \mathrm{Tr}\,\Sigma_{\alpha}^{n+1} = 0, \\
&\text{(iii)}\ \ \mathrm{Tr}\,\Sigma_0^{n+1} = 0,
\end{aligned}$$

and their totality can be summarized by the equation

$$\mathrm{Tr}\,\Sigma = 0. \tag{1.54}$$

It follows from Chern's construction described above that the absolute parallelism σ defined in (1.51) is uniquely determined by the vanishing of its torsion and by condition (1.54).

To describe further properties of σ, we need a general definition. Let R be a Lie group with Lie algebra $\mathfrak{r}$ and S a closed subgroup of R with Lie algebra $\mathfrak{s} \subset \mathfrak{r}$ acting by diffeomorphisms on a manifold $\mathscr{P}$ such that $\dim \mathscr{P} = \dim R$. For every element $s \in \mathfrak{s}$ denote by X_s the fundamental vector field arising from the one-parameter subgroup $\{\exp(\mathtt{t}s), \mathtt{t} \in \mathbb{R}\}$ of S, i.e.

$$X_s(x) := \frac{d}{d\mathtt{t}}\Big(\exp(-\mathtt{t}s)\,x\Big)\Big|_{\mathtt{t}=0}, \quad x \in \mathscr{P}.$$

A *Cartan connection of type R/S* on the manifold $\mathscr{P}$ is an $\mathfrak{r}$-valued absolute parallelism ρ on $\mathscr{P}$ such that

(i) $\rho(x)(X_s(x)) = s$ for all $s \in \mathfrak{s}$ and $x \in \mathscr{P}$, and
(ii) $L_a^*\rho = \mathrm{Ad}_{S,\mathfrak{r}}(a)\rho$ for all $a \in S$,

where L_a denotes the action of a on $\mathscr{P}$ and $\mathrm{Ad}_{S,\mathfrak{r}}$ is the adjoint representation of S.

A straightforward calculation shows that, upon identification of the group G_1 with the group $H_1/\mathscr{Z} \subset \mathrm{PSU}^{\pm}_{\mathscr{H}^g}$ by means of the isomorphism χ_1, the absolute parallelism σ is in fact a Cartan connection of type $\mathrm{PSU}^{\pm}_{\mathscr{H}^g}/G_1$ on the bundle $\mathscr{P}^2 \to \mathscr{P}^1$. Thus, we have proved the following theorem.

Theorem 1.1. [24] *If g is a non-degenerate Hermitian form on $\mathbb{C}^n$ and $\mathfrak{C}^g$ the collection of CR-hypersurfaces of CR-dimension n whose Levi form at every point is equivalent to g, then the CR-structures of the manifolds in $\mathfrak{C}^g$ are 2-reducible to absolute parallelisms. For $M \in \mathfrak{C}^g$ the absolute parallelism σ on $\mathscr{P}^2 \to \mathscr{P}^1 \to M$ establishes an isomorphisms between $T_\Theta(\mathscr{P}^2)$ and the Lie algebra $\mathfrak{su}_{\mathscr{H}^g}$ at every point $\Theta \in \mathscr{P}^2$. Furthermore, σ is a Cartan connection on $\mathscr{P}^2 \to \mathscr{P}^1$ and is determined by the vanishing of the torsion and curvature condition* (1.54).

As was noted by S. Webster (see the Appendix to [24]), there are further symmetries for the functions occurring in formulas (1.35), (1.41), (1.47), which give expansions of the components of the CR-curvature form Σ with respect to ω^α, $\omega^{\overline{\alpha}}$, ω. The additional symmetries follow from the *Bianchi identities*, which one obtains by differentiating equation (1.52). Namely, differentiation of (1.52) yields

$$d\Sigma = \sigma \wedge \Sigma - \Sigma \wedge \sigma,$$

which in terms of components is written as follows:

$$d\Sigma_l^m = \sigma_k^m \wedge \Sigma_l^k - \Sigma_k^m \wedge \sigma_l^k. \tag{1.55}$$

Webster shows that the Bianchi identities imply

$$V_{\alpha\cdot\beta}^{\alpha} = 0, \quad V_{\alpha\overline{\beta}\gamma} = V_{\gamma\overline{\beta}\alpha}, \quad Q_{\alpha\beta} = Q_{\beta\alpha}, \quad P_{\alpha\overline{\beta}} = P_{\overline{\beta}\alpha}. \tag{1.56}$$

Hence, (1.41) and (1.47) become, respectively,

$$\begin{aligned} \Phi^\alpha &= V_{\cdot\beta\overline{\sigma}}^{\alpha}\omega^\beta \wedge \omega^{\overline{\sigma}} + P_{\beta\cdot}^{\alpha}\omega^\beta \wedge \omega + Q_{\overline{\beta}\cdot}^{\alpha}\omega^{\overline{\beta}} \wedge \omega, \\ \Psi &= -2iP_{\alpha\overline{\beta}}\omega^\alpha \wedge \omega^{\overline{\beta}} + R_\alpha\omega^\alpha \wedge \omega + R_{\overline{\alpha}}\omega^{\overline{\alpha}} \wedge \omega. \end{aligned} \tag{1.57}$$

Identities (1.26), (1.35), (1.37), (1.56) yield $\Phi_{\alpha\cdot}^{\alpha} = 0$. Thus, (1.53) implies

$$\Sigma_0^0 = 0, \quad \Sigma_{n+1}^{n+1} = 0, \quad \Sigma_\beta^\alpha = \Phi_{\alpha\cdot\,\cdot}^{\beta}. \tag{1.58}$$

In addition, from (1.56) we see

$$\widehat{P}_{\alpha\cdot}^{\alpha} = 2\,\mathrm{Re}\,P_{\alpha\cdot}^{\alpha} = 2P_{\alpha\cdot}^{\alpha},$$

and therefore condition (1.45) is equivalent to

$$P_{\alpha\cdot}^{\alpha} = 0. \tag{1.59}$$

For a non-degenerate $\mathbb{C}$-valued Hermitian form g on $\mathbb{C}^n$ consider the quadric Q_g associated to g (see (1.4)). We will now give an explicit description of the group $\mathrm{Bir}(Q_g)$. Consider $\mathbb{CP}^{n+1}$ with homogeneous coordinates $\mathfrak{z} = (\zeta_0 : \zeta_1 : \ldots : \zeta_{n+1})$ and realize $\mathbb{C}^{n+1}$ in $\mathbb{CP}^{n+1}$ as the set of points $(1 : z_1 : \ldots : z_n : w)$. Let $\overline{Q}_g$ be the closure of Q_g in $\mathbb{CP}^{n+1}$. Clearly, we have

$$\overline{Q}_g = \left\{\mathfrak{z} \in \mathbb{CP}^{n+1} : \mathscr{H}^g(\mathtt{Z},\mathtt{Z}) = 0\right\}, \tag{1.60}$$

where $\mathtt{Z} := (\zeta_0, \zeta_1, \ldots, \zeta_{n+1})$ and $\mathscr{H}^g$ is the Hermitian form defined in (1.48), that is, $\mathscr{H}^g(\mathtt{Z},\mathtt{Z}) = g(\zeta,\zeta) + i/2(\zeta_{n+1}\overline{\zeta}_0 - \zeta_0\overline{\zeta}_{n+1})$ with $\zeta := (\zeta_1, \ldots, \zeta_n)$. We consider $\overline{Q}_g$ with the CR-structure induced by $\mathbb{CP}^{n+1}$. If g is sign-definite, $\overline{Q}_g$ is CR-equivalent to the unit sphere in $\mathbb{C}^{n+1}$. In general, $\overline{Q}_g$ is CR-equivalent to the closure $\overline{S}_g$ in $\mathbb{CP}^{n+1}$ of the hypersurface S_g defined in (1.6). Indeed, we have

$$\overline{S}_g = \left\{\mathfrak{z} \in \mathbb{CP}^{n+1} : g(\zeta,\zeta) + |\zeta_{n+1}|^2 - |\zeta_0|^2 = 0\right\},$$

and the map

$$\mathfrak{z} \mapsto (\zeta_0 - \zeta_{n+1} : \zeta_1 : \ldots : \zeta_n : i(\zeta_0 + \zeta_{n+1})) \tag{1.61}$$

transforms $\overline{S}_g$ into $\overline{Q}_g$ (observe that the restriction of map (1.61) to $\mathbb{C}^{n+1} \setminus \{w = 1\}$ coincides with map (1.5)).

We define an action of the group $\mathrm{SU}^{\pm}_{\mathscr{H}^g}$ on $\mathbb{CP}^{n+1}$ by assigning a matrix $A \in \mathrm{SU}^{\pm}_{\mathscr{H}^g}$ the holomorphic automorphism of $\mathbb{CP}^{n+1}$ given by $\mathfrak{z} \mapsto \left[A^T\right]^{-1}\mathfrak{z}$. Clearly, every such automorphism preserves $\overline{Q}_g$, thus its restriction to $\overline{Q}_g$ is a CR-automorphism of $\overline{Q}_g$. The kernel of this action is the center $\mathscr{Z}$ of $\mathrm{SU}^{\pm}_{\mathscr{H}^g}$, hence the group $\mathrm{PSU}^{\pm}_{\mathscr{H}^g}$ acts on $\overline{Q}_g$ effectively and transitively by CR-automorphisms. One can show that every local automorphism of Q_g extends to a CR-automorphism of $\overline{Q}_g$ induced by this action. This continuation result goes back to Poincaré for the case $n = 1$ (see [90]). It was obtained by Tanaka in [99] for arbitrary $n \geq 1$ and g for all local CR-automorphisms of Q_g that can be holomorphically continued to a neighborhood in $\mathbb{C}^{n+1}$ of a domain in Q_g (see also [1]). In fact, every local CR-automorphism of Q_g admits a local holomorphic continuation required by Tanaka's result. Indeed, let $f : V \to V'$ be a CR-isomorphism between domains V and V' in Q_g. If the form g is indefinite, the existence of a holomorphic continuation of f to a neighborhood of V in $\mathbb{C}^{n+1}$ follows from a well-known fact that appears as Theorem 3.3.2 in [22] (see references therein for details). If the form g is sign-definite, a continuation of f to a neighborhood of V is provided by [88]. [Note that the existence of a local holomorphic continuation also follows from Theorem 3.1 of [3].] Thus, the group $\mathrm{Bir}(Q_g)$ endowed with the compact-open topology arising from its action on $\overline{Q}_g$ admits the structure of a Lie group isomorphic to $\mathrm{PSU}^{\pm}_{\mathscr{H}^g}$. It can be shown that the Lie algebra of $\mathrm{Bir}(Q_g)$ with respect to this structure is isomorphic to the Lie algebra of infinitesimal CR-automorphisms of Q_g. As a Lie group, $\mathrm{Bir}(Q_g)$ acts on $\overline{Q}_g$ transitively by CR-automorphisms. Clearly, $\mathrm{Bir}(Q_g)$ is connected if the numbers of positive and negative eigenvalues of g are distinct and has exactly two connected components otherwise. From now on we identify the group $\mathrm{Bir}(Q_g)$ with $\mathrm{PSU}^{\pm}_{\mathscr{H}^g}$ and its Lie algebra with $\mathfrak{su}_{\mathscr{H}^g}$. [We note in passing that the effect of continuation of local CR-automorphisms and, more generally, locally defined CR-isomorphisms to globally defined maps for manifolds other than Q_g has been observed by many authors (see, e.g. [62], [69], [78], [86], [89], [107]). A related continuation result for *global* CR-automorphisms in the case where the Hermitian form g is degenerate was obtained in [63].]

Let $H := \mathrm{Bir}_0(Q_g)$ and H_0 be the subgroup of $\mathrm{SU}^{\pm}_{\mathscr{H}^g}$ that consists of all matrices of the form

$$\begin{pmatrix} t & 0 & 0 \\ t^{\alpha} & t^{\alpha}_{\beta\cdot} & 0 \\ \tau & \tau_{\beta} & \pm\bar{t}^{-1} \end{pmatrix},$$

where conditions (1.50) are replaced by the conditions

$$\begin{aligned}
&\text{(i)}\ \ t^{\alpha} = \mp 2it \sum_{\beta\overline{\gamma}} t^{\alpha}_{\beta\cdot} g_{\beta\overline{\gamma}} \tau_{\overline{\gamma}},\\
&\text{(ii)}\ \ \pm t\bar{t}^{-1} \det(t^{\alpha}_{\beta\cdot}) = 1,\\
&\text{(iii)}\ \ \sum_{\rho,\overline{\sigma}} t^{\alpha}_{\rho\cdot} t^{\overline{\beta}}_{\overline{\sigma}\cdot} g_{\rho\overline{\sigma}} = \pm g_{\alpha\overline{\beta}},\\
&\text{(iv)}\ \ \sum_{\rho,\overline{\sigma}} g_{\rho\overline{\sigma}} \tau_{\rho} \tau_{\overline{\sigma}} \pm \frac{i}{2}(\overline{\tau}\bar{t}^{-1} - \tau t^{-1}) = 0,
\end{aligned}$$

with the bottom choice of the sign only possible if the numbers of positive and negative eigenvalues of the form g are equal. Clearly, H_1 is the codimension one subgroup of H_0 given by the top choice of the sign and the condition $|t| = 1$ (see (1.49), (1.50)). It is straightforward to check that the isomorphism $\mathrm{PSU}^{\pm}_{\mathscr{H}^g} \to \mathrm{Bir}(Q_g)$ identifies the subgroup $H_0/\mathscr{Z}$ with H, thus the group $G_1 \simeq H_1/\mathscr{Z}$ can be viewed as a codimension one subgroup of H.

It was shown in [9] that the manifold $\mathscr{P}^2$ constructed above is in fact a principal H-bundle over M with the projection $\pi := \pi^1 \circ \pi^2$ and, upon identification of H and $H_0/\mathscr{Z}$, the parallelism σ is a Cartan connection of type $\mathrm{PSU}^{\pm}_{\mathscr{H}^g}/H$ on the bundle $\mathscr{P}^2 \to M$. Thus, the following variant of Theorem 1.1 holds.

Theorem 1.2. [9], [24] *The CR-structures of the manifolds from $\mathfrak{C}^g$ are 1-reducible to absolute parallelisms. For $M \in \mathfrak{C}^g$ the absolute parallelism σ on $\mathscr{P}^2 \to M$ establishes an isomorphisms between $T_{\Theta}(\mathscr{P}^2)$ and the Lie algebra $\mathfrak{su}_{\mathscr{H}^g}$ at every point $\Theta \in \mathscr{P}^2$. Furthermore, σ is a Cartan connection on $\mathscr{P}^2 \to M$ and is determined by the vanishing of the torsion and curvature condition* (1.54).

Inspection of Chern's construction yields that for the manifold $\overline{Q}_g$ the bundle $\mathscr{P}^2 \to \overline{Q}_g$ is the bundle $\mathrm{Bir}(Q_g) \xrightarrow{\pi_g} \mathrm{Bir}(Q_g)/H$, where the quotient $\mathrm{Bir}(Q_g)/H$ is identified with the $\mathrm{Bir}(Q_g)$-homogeneous manifold $\overline{Q}_g$ in the usual way and π_g is the quotient map. In this case the Cartan connection σ is the *Maurer-Cartan form* $\sigma_{\mathrm{Bir}(Q_g)}$ on the group $\mathrm{Bir}(Q_g)$.

Recall that the Maurer-Cartan form σ_R on a Lie group R is the right-invariant 1-form with values in the Lie algebra $\mathfrak{r}$ of R such that $\sigma_R(e) : \mathfrak{r} \to \mathfrak{r}$ is the identity map. The Maurer-Cartan form satisfies the *Maurer-Cartan equation*

$$d\sigma_R - \frac{1}{2}[\sigma_R, \sigma_R] = 0$$

and under the left multiplication L_a by $a \in R$ transforms as follows:

$$L_a^* \sigma_R = \mathrm{Ad}_{R,\mathfrak{r}}(a)\sigma_R.$$

The Maurer-Cartan equation implies that the CR-curvature form of $\overline{Q}_g$ vanishes. Conversely, suppose that the CR-curvature form of a manifold $M \in \mathfrak{C}^g$ is zero. Then for every point $\Theta \in \mathscr{P}^2$ there is a neighborhood U of Θ, a neighborhood V of the identity in $\mathrm{Bir}(Q_g)$, and a diffeomorphism $F : U \to V$ such that $F^*\left(\sigma_{\mathrm{Bir}(Q_g)}\big|_V\right) = \sigma\big|_U$. By Theorem 1.2 the diffeomorphism F is a lift of a CR-isomorphism $f : \pi(U) \to \pi_{Q_g}(V)$. Therefore, every point of M has a neighborhood CR-equivalent to an open subset of Q_g.

A CR-hypersurface $M \in \mathfrak{C}^g$ is called *spherical* if it is locally CR-equivalent to Q_g, i.e. if every point in M has a neighborhood CR-equivalent to an open subset of Q_g. If the signature of the non-degenerate Hermitian form g is $(k, n-k)$ for some $0 \le k \le n$, and M is locally CR-equivalent to Q_g, we also say that M is $(k, n-k)$-*spherical*. It is usually assumed, without loss of generality, that $n \le 2k$. [We will generalize the above definition of sphericity to the Levi degenerate case in Section 9.1. Until then we only consider Levi non-degenerate CR-hypersurfaces.] Further, a CR-hypersurface with vanishing CR-curvature form is called *CR-flat*. We summarize the content of the preceding paragraph as follows.

Corollary 1.1. *A CR-hypersurface is spherical if and only if it is CR-flat.*

In this book we study spherical CR-hypersurfaces. Corollary 1.1 and formulas (1.53), (1.58) yield that such CR-hypersurfaces are characterized by the conditions

$$\Phi_{\alpha\cdot}^{\beta} = 0, \quad \Phi^{\alpha} = 0, \quad \Psi = 0,$$

or, equivalently, by the conditions

$$S_{\alpha\rho\cdot\overline{\sigma}}^{\beta} = 0, \quad V_{\alpha\cdot\rho}^{\beta} = 0, \quad P_{\beta\cdot}^{\alpha} = 0, \quad Q_{\beta\cdot}^{\alpha} = 0, \quad R_{\beta} = 0. \tag{1.62}$$

Due to the transformation law

$$L_a^* \sigma = \mathrm{Ad}_{H,\mathfrak{su}_{\mathscr{H}g}}(a)\sigma, \quad a \in H,$$

where L_a is the (left) action of a on the bundle $\mathscr{P}^2 \to M$, the CR-curvature form Σ transforms in a similar way

$$L_a^* \Sigma = \mathrm{Ad}_{H,\mathfrak{su}_{\mathscr{H}g}}(a)\Sigma. \tag{1.63}$$

Transformation law (1.63) implies that conditions (1.62) hold everywhere on $\mathscr{P}^2$ if for every $p \in M$ there is a local section Γ_W of $\mathscr{P}^2$ over a neighborhood W of p in M such that these conditions hold on the submanifold $\Gamma_W(W)$ of $\mathscr{P}^2$.

Throughout the book we only consider real hypersurfaces in complex manifolds with induced CR-structure, and our next step is to write sphericity conditions (1.62) on a certain local section of $\mathscr{P}^2$ defined in terms of a local defining function of the hypersurface (cf. [76], Section 5).

1.3 Chern's Invariants on Section of Bundle $\mathscr{P}^2 \to M$

Let M be a Levi non-degenerate CR-hypersurface with an integrable CR-structure of CR-dimension n. Fix a Hermitian form on $\mathbb{C}^n$ with matrix g which is equivalent to every $\mathfrak{L}_M(p)$, $p \in M$, and consider the fiber bundle $\mathscr{P}^1$ over M and the tautological 1-form θ^0 on $\mathscr{P}^1$ as constructed in Section 1.2. Let W be an open subset of M and $U := [\pi^1]^{-1}(W)$. Further, let $\mathscr{G} = (\mathscr{G}_{\alpha\overline{\beta}})$ be a matrix-valued map on U such that for every $\theta \in U$ the value $\mathscr{G}(\theta)$ is the matrix of a Hermitian form whose signature coincides with that of the Hermitian form defined by g. Then for every $\theta \in U$ there exist a real-valued covector θ^{n+1} and complex-valued covectors θ^α on $T_\theta(\mathscr{P}^1)$ such that: (a) each θ^α is a lift of a complex-valued covector on $T_{\pi^1(\theta)}(M)$ which is complex-linear on $T^c_{\pi^1(\theta)}(M)$, (b) the covectors $\theta^0(\theta)$, $\operatorname{Re}\theta^\alpha$, $\operatorname{Im}\theta^\alpha$, θ^{n+1} form a basis of the cotangent space $T^*_\theta(\mathscr{P}^1)$, and (c) the following identity holds:

$$d\theta^0(\theta) = i\mathscr{G}_{\alpha\overline{\beta}}(\theta)\theta^\alpha \wedge \theta^{\overline{\beta}} + \theta^0(\theta) \wedge \theta^{n+1}. \tag{1.64}$$

The most general linear transformation of $\theta^0(\theta)$, θ^α, $\theta^{\overline{\alpha}}$, θ^{n+1} preserving equation (1.64) and the covector $\theta^0(\theta)$ is given by the matrix (acting on the left)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ v^\alpha & u^\alpha_\beta & 0 & 0 \\ v^{\overline{\alpha}} & 0 & u^{\overline{\alpha}}_{\overline{\beta}} & 0 \\ s & i\mathscr{G}_{\rho\overline{\sigma}}(\theta)u^\rho_\beta v^{\overline{\sigma}} & -i\mathscr{G}_{\rho\overline{\sigma}}(\theta)u^{\overline{\sigma}}_{\overline{\beta}} v^\rho & 1 \end{pmatrix},$$

where $s \in \mathbb{R}$, $u^\alpha_\beta, v^\alpha \in \mathbb{C}$ and $\mathscr{G}_{\alpha\overline{\beta}}(\theta)u^\alpha_\rho u^{\overline{\beta}}_{\overline{\sigma}} = \mathscr{G}_{\rho\overline{\sigma}}(\theta)$.

For $\theta \in U$ let $\mathscr{P}^2_{\theta,\mathscr{G}}$ be the collection of all covectors $(\theta^0(\theta), \theta^\alpha, \theta^{n+1})$ on $T_\theta(\mathscr{P}^1)$ satisfying conditions (a), (b), (c) above. The sets $\mathscr{P}^2_{\theta,\mathscr{G}}$, $\theta \in U$, form a fiber bundle over U, which we denote by $\mathscr{P}^2_{\mathscr{G}}$. Let $\pi^2_{\mathscr{G}} : \mathscr{P}^2_{\mathscr{G}} \to U$ be the projection $(\theta^0(\theta), \theta^\alpha, \theta^{n+1}) \mapsto \theta$. For every point $\theta_0 \in U$ there is a neighborhood U_0 of θ_0 in U such that the open sets $[\pi^2_{\mathscr{G}}]^{-1}(U_0)$ and $[\pi^2]^{-1}(U_0)$ are diffeomorphic, with the fiber $\mathscr{P}^2_{\theta,\mathscr{G}}$ mapped onto the fiber $\mathscr{P}^2_\theta$ for every $\theta \in U_0$ as follows:

$$\mathscr{F} : (\theta^0(\theta), \theta^\alpha, \theta^{n+1}) \mapsto (\theta^0(\theta), \mathscr{C}^\alpha_\beta(\theta)\theta^\beta, \theta^{n+1}),$$

where $\mathscr{C}^\alpha_\beta$ are complex-valued functions on U_0 and the matrix $(\mathscr{C}^\alpha_\beta)$ is everywhere non-degenerate. Next, set

$$\omega_{\mathscr{G}} := [\pi^2_{\mathscr{G}}]^*\,\theta^0$$

and introduce a collection of tautological 1-forms on $\mathscr{P}^2_{\mathscr{G}}$ as follows:

$$\begin{aligned} \omega^\alpha_{\mathscr{G}}(\Theta)(Y) &:= \theta^\alpha(d\pi^2_{\mathscr{G}}(\Theta)(Y)), \\ \varphi_{\mathscr{G}}(\Theta)(Y) &:= \theta^{n+1}(d\pi^2_{\mathscr{G}}(\Theta)(Y)), \end{aligned}$$

where $\Theta = (\theta^0(\theta), \theta^\alpha, \theta^{n+1})$ is a point in $\mathscr{P}^2_{\theta,\mathscr{G}}$ and $Y \in T_\Theta(\mathscr{P}^2_{\mathscr{G}})$. Identity (1.64) implies

$$d\omega_{\mathscr{G}} = i\left([\pi^2_{\mathscr{G}}]^* \mathscr{G}_{\alpha\overline{\beta}}\right) \omega^\alpha_{\mathscr{G}} \wedge \omega^{\overline{\beta}}_{\mathscr{G}} + \omega_{\mathscr{G}} \wedge \varphi_{\mathscr{G}}.$$

As in Section 1.2, starting with the forms $\omega_{\mathscr{G}}$, $\omega^\alpha_{\mathscr{G}}$, $\varphi_{\mathscr{G}}$ we can construct 1-forms $\varphi^\alpha_{\beta\cdot,\mathscr{G}}$, $\varphi^\alpha_{\mathscr{G}}$, $\psi_{\mathscr{G}}$ and 2-forms $\Phi^\alpha_{\beta\cdot,\mathscr{G}}$, $\Phi^\alpha_{\mathscr{G}}$, $\Psi_{\mathscr{G}}$ on $\mathscr{P}^2_{\mathscr{G}}$ (recall that in our calculations in Section 1.2 we allowed $(g_{\alpha\overline{\beta}})$ to be a matrix-valued map). A straightforward calculation yields that on $[\pi^2_{\mathscr{G}}]^{-1}(U_0)$ we have

$$\begin{aligned}
\omega_{\mathscr{G}} &= \mathscr{F}^*\omega,\\
\omega^\alpha_{\mathscr{G}} &= \mathscr{D}^\alpha_\beta\ \mathscr{F}^*\omega^\beta,\\
\varphi_{\mathscr{G}} &= \mathscr{F}^*\varphi,\\
\varphi^\alpha_{\beta\cdot,\mathscr{G}} &= -d\mathscr{D}^\alpha_\gamma \cdot \mathscr{C}^\gamma_\beta + \mathscr{D}^\alpha_\gamma \mathscr{C}^\nu_\beta\ \mathscr{F}^*\varphi^\gamma_{\nu\cdot},\\
\varphi^\alpha_{\mathscr{G}} &= \mathscr{D}^\alpha_\beta\ \mathscr{F}^*\varphi^\beta,\\
\psi_{\mathscr{G}} &= \mathscr{F}^*\psi
\end{aligned}$$

and

$$\begin{aligned}
S^\alpha_{\beta\rho\cdot\overline{\sigma},\mathscr{G}} &= \mathscr{D}^\alpha_\gamma \mathscr{C}^\nu_\beta \mathscr{C}^\mu_\rho \mathscr{C}^{\overline{\eta}}_{\overline{\sigma}}\ \mathscr{F}^* S^\gamma_{\nu\mu\cdot\overline{\eta}},\\
V^\alpha_{\beta\cdot\rho,\mathscr{G}} &= \mathscr{D}^\alpha_\gamma \mathscr{C}^\nu_\beta \mathscr{C}^\mu_\rho\ \mathscr{F}^* V^\gamma_{\nu\cdot\mu},\\
P^\alpha_{\beta\cdot,\mathscr{G}} &= \mathscr{D}^\alpha_\gamma \mathscr{C}^\nu_\beta\ \mathscr{F}^* P^\gamma_{\nu\cdot},\\
Q^\alpha_{\overline{\beta}\cdot,\mathscr{G}} &= \mathscr{D}^\alpha_\gamma \mathscr{C}^{\overline{\eta}}_{\overline{\beta}}\ \mathscr{F}^* Q^\gamma_{\overline{\eta}\cdot},\\
R_{\alpha,\mathscr{G}} &= \mathscr{C}^\gamma_\alpha\ \mathscr{F}^* R_\gamma,
\end{aligned} \tag{1.65}$$

where $(\mathscr{D}^\alpha_\beta)$ is the matrix inverse to $(\mathscr{C}^\alpha_\beta)$ and $S^\alpha_{\beta\rho\cdot\overline{\sigma},\mathscr{G}}$, $V^\alpha_{\beta\cdot\rho,\mathscr{G}}$, $P^\alpha_{\beta\cdot,\mathscr{G}}$, $Q^\alpha_{\overline{\beta}\cdot,\mathscr{G}}$, $R_{\alpha,\mathscr{G}}$ are the corresponding functions in the expansions of the forms $\Phi^\alpha_{\beta\cdot,\mathscr{G}}$, $\Phi^\alpha_{\mathscr{G}}$, $\Psi_{\mathscr{G}}$ with respect to the forms $\omega_{\mathscr{G}}$, $\omega^\rho_{\mathscr{G}}$, $\omega^{\overline{\rho}}_{\mathscr{G}}$.

Let $\gamma_{U,\mathscr{G}} : U \to \mathscr{P}^2_{\mathscr{G}}$ be a section of $\mathscr{P}^2_{\mathscr{G}}$ and γ_W a local section of $\mathscr{P}^1$ over W. Formulas (1.65) imply that if the functions $S^\alpha_{\beta\rho\cdot\overline{\sigma},\mathscr{G}}$, $V^\alpha_{\beta\cdot\rho,\mathscr{G}}$, $P^\alpha_{\beta\cdot,\mathscr{G}}$, $Q^\alpha_{\overline{\beta}\cdot,\mathscr{G}}$, $R_{\alpha,\mathscr{G}}$ vanish on the submanifold $(\gamma_{U,\mathscr{G}} \circ \gamma_W)(W)$ of $\mathscr{P}^2_{\mathscr{G}}$, then conditions (1.62) hold on the submanifold $\Gamma_W(W)$ of $\mathscr{P}^2$, where $\Gamma_W := \mathscr{F} \circ \gamma_{U,\mathscr{G}} \circ \gamma_W$ is a section of the bundle $\mathscr{P}^2 \to M$ over the set W. [Here we assume for simplicity that $\mathscr{F}$ is defined on all of $[\pi^2_{\mathscr{G}}]^{-1}(U)$. To be absolutely precise, one must consider for every $\theta_0 \in U$ a neighborhood U_0 as above.]

Suppose now that M is an immersed Levi non-degenerate real hypersurface in a complex manifold N of dimension $n+1$ with $n \geq 1$. Fix $p \in M$ and consider a neighborhood M' of p in M which is locally closed in N.[3] Then there exist a neighborhood $\mathscr{W}$ of p in N, holomorphic coordinates z^0, $z = (z^1, \dots, z^n)$ in $\mathscr{W}$, and

---

[3] We say that an immersed submanifold S of a manifold R is locally closed if the immersion $\iota : S \to R$, $\iota(x) := x$, is a locally proper map, or, equivalently, if S is a closed submanifold of an open submanifold of R. We say that S is closed in R if ι is a proper map.

a real-valued function $r(z^0,\bar{z}^0,z,\bar{z})$ on $\mathscr{W}$ such that the set $W := M' \cap \mathscr{W}$ coincides with the set $\{r=0\}$ and $r_0 := \partial r/\partial z^0 \neq 0$ on W (note that $r_{\bar{0}} := \partial r/\partial \bar{z}^0 \neq 0$ on W as well since $r_0 = \overline{r_{\bar{0}}}$).[4]

The CR-structure of M, being induced by N, is given on W by setting

$$\begin{aligned} \mu &= i\partial r\Big|_W := i\left(\frac{\partial r}{\partial z^\beta}dz^\beta + r_0 dz^0\right)\Big|_W, \\ \eta^\alpha &= dz^\alpha\Big|_W \end{aligned} \tag{1.66}$$

(cf. the beginning of Section 1.2). Then on W we have

$$d\mu = ih_{\alpha\bar{\beta}}dz^\alpha \wedge dz^{\bar{\beta}} + \mu \wedge \phi, \tag{1.67}$$

with

$$\begin{aligned} h_{\alpha\bar{\beta}} &= -r_{\alpha\bar{\beta}} + r_0^{-1} r_\alpha r_{0\bar{\beta}} + r_{\bar{0}}^{-1} r_{\bar{\beta}} r_{\bar{0}\alpha} - |r_0|^{-2} r_{0\bar{0}} r_\alpha r_{\bar{\beta}}, \\ \phi &:= -r_{\bar{0}}^{-1} r_{\bar{0}\gamma} dz^\gamma - r_0^{-1} r_{0\bar{\gamma}} dz^{\bar{\gamma}} + |r_0|^{-2} r_{0\bar{0}}\left(r_\gamma dz^\gamma + r_{\bar{\gamma}} dz^{\bar{\gamma}}\right) \end{aligned} \tag{1.68}$$

(cf. (1.7)), where we use the following notation:

$$r_\alpha := \frac{\partial r}{\partial z^\alpha}, \quad r_{\bar{\beta}} := \frac{\partial r}{\partial z^{\bar{\beta}}}, \quad r_{\alpha\bar{\beta}} := \frac{\partial^2 r}{\partial z^\alpha \partial z^{\bar{\beta}}}, \quad \text{etc.}$$

Clearly, for every $q \in W$ the Levi form of M at q is equivalent to the Hermitian form with the matrix $h(q) := (h_{\alpha\bar{\beta}}(q))$.

We now choose a matrix g such that the Hermitian form defined by g has the same signature as the Hermitian form defined by $h(q)$ for every $q \in W$. Then the fiber of the bundle $\mathscr{P}^1$ over q is $\{u\mu(q) : u > 0\}$ in the case where the numbers of positive and negative eigenvalues of g are distinct and $\{u\mu(q) : u \in \mathbb{R}^*\}$ otherwise. For the form θ^0 on $U = [\pi^1]^{-1}(W)$ we have

$$d\theta^0 = iu\left([\pi^1]^* h_{\alpha\bar{\beta}}\right)[\pi^1]^* dz^\alpha \wedge [\pi^1]^* dz^{\bar{\beta}} + \theta^0 \wedge \left(-\frac{du}{u} + [\pi^1]^* \phi\right).$$

We now let $\mathscr{G}_{\alpha\bar{\beta}} = u[\pi^1]^* h_{\alpha\bar{\beta}}$ on U and choose the section $\gamma_{U,\mathscr{G}}$ as follows:

$$\gamma_{U,\mathscr{G}}(u\mu(q)) = \left(\theta^0(u\mu(q)), \left([\pi^1]^* dz^\alpha\right)(u\mu(q)), -\frac{du}{u} + ([\pi^1]^* \phi)(u\mu(q))\right).$$

Next, choose the section γ_W by setting $u=1$, i.e. $\gamma_W(q) = \mu(q)$. Our goal is to compute the forms $\omega_{\mathscr{G}}$, $\omega^\alpha_{\mathscr{G}}$, $\varphi_{\mathscr{G}}$, $\varphi^\alpha_{\beta\cdot,\mathscr{G}}$, $\varphi^\alpha_{\mathscr{G}}$, $\psi_{\mathscr{G}}$ and the functions $S^\alpha_{\beta\rho\cdot\bar{\sigma},\mathscr{G}}$, $V^\alpha_{\beta\cdot\rho,\mathscr{G}}$, $P^\alpha_{\beta\cdot,\mathscr{G}}$, $Q^\alpha_{\bar{\beta}\cdot,\mathscr{G}}$, $R_{\alpha,\mathscr{G}}$ on the submanifold $\mathfrak{W} := (\gamma_{U,\mathscr{G}} \circ \gamma_W)(W)$ of $\mathscr{P}^2_{\mathscr{G}}$. In fact, we

[4] For notational convenience, in this section we index coordinates by superscripts rather than subscripts. We will return to indexing coordinates by subscripts in Chapter 2.

compute the push-forwards of these quantities to W under the diffeomorphism $\pi^1 \circ \pi^2_{\mathscr{G}}\big|_{\mathfrak{W}} : \mathfrak{W} \to W$.

Clearly, on $\mathfrak{W}$ we have

$$\begin{aligned}
\omega_{\mathscr{G}} &= [\pi^2_{\mathscr{G}}]^* \left([\pi^1]^* \mu\right), \\
\omega^{\alpha}_{\mathscr{G}} &= [\pi^2_{\mathscr{G}}]^* \left([\pi^1]^* dz^{\alpha}\right), \\
\varphi_{\mathscr{G}} &= [\pi^2_{\mathscr{G}}]^* \left([\pi^1]^* \phi\right),
\end{aligned}$$

thus the push-forwards of $\omega_{\mathscr{G}}\big|_{\mathfrak{W}}$, $\omega^{\alpha}_{\mathscr{G}}\big|_{\mathfrak{W}}$, $\varphi_{\mathscr{G}}\big|_{\mathfrak{W}}$ from $\mathfrak{W}$ to W are μ, dz^{α}, ϕ, respectively.

Differentiating (1.67) we obtain

$$i(dh_{\alpha\overline{\beta}} + h_{\alpha\overline{\beta}}\phi) \wedge dz^{\alpha} \wedge dz^{\overline{\beta}} - \mu \wedge d\phi = 0.$$

Hence,

$$\begin{aligned}
& dh_{\alpha\overline{\beta}} + h_{\alpha\overline{\beta}}\phi = a_{\alpha\overline{\beta}\gamma} dz^{\gamma} + a_{\overline{\beta}\alpha\overline{\gamma}} dz^{\overline{\gamma}} + c_{\alpha\overline{\beta}}\mu, \\
& d\phi = ic_{\alpha\overline{\beta}} dz^{\alpha} \wedge dz^{\overline{\beta}} + \mu \wedge \zeta^{(1)}
\end{aligned} \tag{1.69}$$

for some 1-form $\zeta^{(1)}$ and functions $a_{\alpha\overline{\beta}\gamma}$, $c_{\alpha\overline{\beta}}$ satisfying

$$a_{\alpha\overline{\beta}\gamma} = a_{\gamma\overline{\beta}\alpha}, \quad c_{\alpha\overline{\beta}} = c_{\overline{\beta}\alpha}.$$

With μ given in (1.66) and $h_{\alpha\overline{\beta}}$, ϕ given in (1.68), the functions $a_{\alpha\overline{\beta}\gamma}$, $c_{\alpha\overline{\beta}}$ and the form $\zeta^{(1)}$ are completely determined by formulas (1.69) if we assume that $\zeta^{(1)}$ is a linear combination of dz^{α} and $dz^{\overline{\alpha}}$. These quantities involve partial derivatives of the function r up to order 3.

Everywhere below indices are lowered by means of the matrix $h = (h_{\alpha\overline{\beta}})$ and raised by means of its inverse $(h^{\alpha\overline{\beta}})$, where $h_{\alpha\overline{\beta}} h^{\gamma\overline{\beta}} = \delta^{\gamma}_{\alpha}$, $h_{\alpha\overline{\beta}} h^{\alpha\overline{\gamma}} = \delta^{\overline{\gamma}}_{\overline{\beta}}$. Set

$$\phi^{\alpha\,(1)}_{\beta\cdot} := a^{\alpha}_{\beta\cdot\gamma} dz^{\gamma} + \frac{1}{2} c^{\alpha}_{\beta\cdot}\mu, \quad \phi^{\alpha\,(1)} := \frac{1}{2} c^{\alpha}_{\beta\cdot} dz^{\beta}. \tag{1.70}$$

Identities (1.69) imply

$$\begin{aligned}
& dz^{\beta} \wedge \phi^{\alpha\,(1)}_{\beta\cdot} + \mu \wedge \phi^{\alpha\,(1)} = 0, \\
& dh_{\alpha\overline{\beta}} + h_{\alpha\overline{\beta}}\phi - \phi^{(1)}_{\alpha\overline{\beta}} - \phi^{(1)}_{\overline{\beta}\alpha} = 0, \\
& d\phi = idz_{\overline{\beta}} \wedge \phi^{\overline{\beta}\,(1)} + i\phi^{(1)}_{\overline{\beta}} \wedge dz^{\overline{\beta}} + \mu \wedge \zeta^{(1)}.
\end{aligned} \tag{1.71}$$

On the other hand, let 1-forms $\phi^{\alpha}_{\beta\cdot}$, ϕ^{α}, ζ be the push-forwards of $\varphi^{\alpha}_{\beta\cdot,\mathscr{G}}\big|_{\mathfrak{W}}$, $\varphi^{\alpha}_{\mathscr{G}}\big|_{\mathfrak{W}}$, $\psi_{\mathscr{G}}\big|_{\mathfrak{W}}$ from $\mathfrak{W}$ to W, respectively. It follows from identities (1.12), (1.15), (1.18) applied to the forms $\omega_{\mathscr{G}}$, $\omega^{\alpha}_{\mathscr{G}}$, $\varphi_{\mathscr{G}}$, $\varphi^{\alpha}_{\beta\cdot,\mathscr{G}}$, $\varphi^{\alpha}_{\mathscr{G}}$, $\psi_{\mathscr{G}}$ that $\phi^{\alpha}_{\beta\cdot}$, ϕ^{α}, ζ satisfy

$$
\begin{aligned}
&dz^{\beta} \wedge \phi_{\beta\cdot}^{\alpha} + \mu \wedge \phi^{\alpha} = 0,\\
&dh_{\alpha\overline{\beta}} + h_{\alpha\overline{\beta}}\phi - \phi_{\alpha\overline{\beta}} - \phi_{\overline{\beta}\alpha} = 0, \qquad (1.72)\\
&d\phi = i dz_{\overline{\beta}} \wedge \phi^{\overline{\beta}} + i\phi_{\overline{\beta}} \wedge dz^{\overline{\beta}} + \mu \wedge \zeta.
\end{aligned}
$$

It is straightforward to see from (1.71), (1.72) that $\phi_{\beta\cdot}^{\alpha}$, ϕ^{α}, ζ are related to $\phi_{\beta\cdot}^{\alpha\,(1)}$, $\phi^{\alpha\,(1)}$, $\zeta^{(1)}$ as follows:

$$
\begin{aligned}
&\phi_{\beta\cdot}^{\alpha\,(1)} = \phi_{\beta\cdot}^{\alpha} + d_{\beta\cdot}^{\alpha}\mu,\\
&\phi^{\alpha\,(1)} = \phi^{\alpha} + d_{\beta\cdot}^{\alpha} dz^{\beta} + e^{\alpha}\mu, \qquad (1.73)\\
&\zeta^{(1)} = \zeta + t\mu + i(e_{\alpha}dz^{\alpha} - e_{\overline{\alpha}}dz^{\overline{\alpha}}),
\end{aligned}
$$

where $d_{\beta\cdot}^{\alpha}$, e^{α}, t are functions on W, t is real-valued and the following holds:

$$
d_{\alpha\overline{\beta}} + d_{\overline{\beta}\alpha} = 0. \qquad (1.74)
$$

We will now find $d_{\beta\cdot}^{\alpha}$, e^{α}, t from conditions (1.26), (1.37), (1.59).

Identities (1.35) imply

$$
\begin{aligned}
&d\phi_{\beta\cdot}^{\alpha} - \phi_{\beta\cdot}^{\gamma} \wedge \phi_{\gamma\cdot}^{\alpha} - i dz_{\beta} \wedge \phi^{\alpha} +\\
&\qquad i\phi_{\beta} \wedge dz^{\alpha} + i\delta_{\beta}^{\alpha}(\phi_{\sigma} \wedge dz^{\sigma}) \equiv \mathscr{S}_{\beta\gamma\cdot\overline{\sigma}}^{\alpha} dz^{\gamma} \wedge dz^{\overline{\sigma}} \pmod{\mu},
\end{aligned} \qquad (1.75)
$$

where $\mathscr{S}_{\beta\gamma\cdot\overline{\sigma}}^{\alpha}$ are the push-forwards of the functions $S_{\beta\gamma\cdot\overline{\sigma},\mathscr{G}}^{\alpha}\big|_{\mathfrak{W}}$ from $\mathfrak{W}$ to W. It follows from (1.67), (1.73), (1.75) that

$$
\begin{aligned}
&d\phi_{\beta\cdot}^{\alpha\,(1)} - \phi_{\beta\cdot}^{\gamma(1)} \wedge \phi_{\gamma\cdot}^{\alpha\,(1)} - i dz_{\beta} \wedge \phi^{\alpha\,(1)} +\\
&\qquad i\phi_{\beta}^{(1)} \wedge dz^{\alpha} + i\delta_{\beta}^{\alpha}\left(\phi_{\sigma}^{(1)} \wedge dz^{\sigma}\right) \equiv \mathscr{S}_{\beta\gamma\cdot\overline{\sigma}}^{\alpha\,(1)} dz^{\gamma} \wedge dz^{\overline{\sigma}} \pmod{\mu},
\end{aligned} \qquad (1.76)
$$

where

$$
\mathscr{S}_{\beta\gamma\cdot\overline{\sigma}}^{\alpha\,(1)} := \mathscr{S}_{\beta\gamma\cdot\overline{\sigma}}^{\alpha} + i(d_{\beta\cdot}^{\alpha}h_{\gamma\overline{\sigma}} + d_{\gamma}^{\alpha}h_{\beta\overline{\sigma}} - \delta_{\gamma}^{\alpha}d_{\overline{\sigma}\beta} - \delta_{\beta}^{\alpha}d_{\overline{\sigma}\gamma}). \qquad (1.77)
$$

Note that with $\phi_{\beta\cdot}^{\alpha\,(1)}$, $\phi^{\alpha\,(1)}$ given by (1.70), where $a_{\alpha\overline{\beta}\gamma}$, $c_{\alpha\overline{\beta}}$ are found from (1.69), the functions $\mathscr{S}_{\beta\gamma\cdot\overline{\sigma}}^{\alpha\,(1)}$ are completely determined by formulas (1.76) and involve partial derivatives of r up to order 4.

Define

$$
\mathscr{S}_{\gamma\overline{\sigma}}^{(1)} := \mathscr{S}_{\alpha\gamma\cdot\overline{\sigma}}^{\alpha\,(1)},\ \mathscr{S}^{(1)} := \mathscr{S}_{\gamma\cdot}^{\gamma(1)}. \qquad (1.78)
$$

Contracting (1.77) and using conditions (1.26) we obtain

$$
h_{\gamma\overline{\sigma}}d + d_{\gamma\overline{\sigma}} - (n+1)d_{\overline{\sigma}\gamma} = -i\mathscr{S}_{\gamma\overline{\sigma}}^{(1)}, \qquad (1.79)
$$

where $d := d_{\alpha\cdot}^{\alpha}$. Identities (1.74) and (1.79) imply

$$h_{\gamma\overline{\sigma}} d + (n+2) d_{\gamma\overline{\sigma}} = -i \mathscr{S}^{(1)}_{\gamma\overline{\sigma}}. \tag{1.80}$$

Contracting (1.80) we get

$$d = -\frac{i}{2(n+1)} \mathscr{S}^{(1)}.$$

Substituting this back into (1.80) yields

$$d_{\gamma\overline{\sigma}} = \frac{i}{n+2} \left(-\mathscr{S}^{(1)}_{\gamma\overline{\sigma}} + \frac{1}{2(n+1)} \mathscr{S}^{(1)} h_{\gamma\overline{\sigma}} \right). \tag{1.81}$$

Formulas (1.81) determine the functions $d^{\alpha}_{\beta\cdot}$ in terms of partial derivatives of r up to order 4, and we set

$$\phi^{\alpha\,(2)} := \phi^{\alpha\,(1)} - d^{\alpha}_{\beta\cdot} dz^{\beta}. \tag{1.82}$$

Next, identities (1.35) imply

$$\begin{aligned} d\phi^{\alpha}_{\beta\cdot} - \phi^{\gamma}_{\beta\cdot} \wedge \phi^{\alpha}_{\gamma\cdot} - i dz_{\beta} \wedge \phi^{\alpha} + i \phi_{\beta} \wedge dz^{\alpha} + i \delta^{\alpha}_{\beta} (\phi_{\sigma} \wedge dz^{\sigma}) + \\ \frac{1}{2} \delta^{\alpha}_{\beta} \zeta \wedge \mu = \mathscr{S}^{\alpha}_{\beta\gamma\cdot\overline{\sigma}} dz^{\gamma} \wedge dz^{\overline{\sigma}} + \mathscr{V}^{\alpha}_{\beta\cdot\gamma} dz^{\gamma} \wedge \mu - \mathscr{V}^{\alpha}_{\cdot\beta\overline{\sigma}} dz^{\overline{\sigma}} \wedge \mu, \end{aligned} \tag{1.83}$$

where $\mathscr{V}^{\alpha}_{\beta\cdot\gamma}$ are the push-forwards of the functions $V^{\alpha}_{\beta\cdot\gamma,\mathscr{G}}\big|_{\mathfrak{W}}$ from $\mathfrak{W}$ to W. It follows from (1.73), (1.82), (1.83) that

$$\begin{aligned} d\phi^{\alpha}_{\beta\cdot} - \phi^{\gamma}_{\beta\cdot} \wedge \phi^{\alpha}_{\gamma\cdot} - i dz_{\beta} \wedge \phi^{\alpha\,(2)} + i \phi^{(2)}_{\beta} \wedge dz^{\alpha} + i \delta^{\alpha}_{\beta} \left(\phi^{(2)}_{\sigma} \wedge dz^{\sigma} \right) + \\ \frac{1}{2} \delta^{\alpha}_{\beta} \zeta^{(1)} \wedge \mu = \mathscr{S}^{\alpha}_{\beta\gamma\cdot\overline{\sigma}} dz^{\gamma} \wedge dz^{\overline{\sigma}} + \mathscr{V}^{\alpha\,(1)}_{\beta\cdot\gamma} dz^{\gamma} \wedge \mu - \mathscr{V}^{\alpha\,(1)}_{\cdot\beta\overline{\sigma}} dz^{\overline{\sigma}} \wedge \mu, \end{aligned} \tag{1.84}$$

where

$$\mathscr{V}^{\alpha\,(1)}_{\beta\cdot\gamma} := \mathscr{V}^{\alpha}_{\beta\cdot\gamma} - i \left(\delta^{\alpha}_{\gamma} e_{\beta} + \frac{1}{2} \delta^{\alpha}_{\beta} e_{\gamma} \right). \tag{1.85}$$

Note that with $\zeta^{(1)}$ found from (1.69), $\phi^{\alpha}_{\beta\cdot}$ given by

$$\phi^{\alpha}_{\beta\cdot} = a^{\alpha}_{\beta\cdot\gamma} dz^{\gamma} + \left(\frac{1}{2} c^{\alpha}_{\beta\cdot} - d^{\alpha}_{\beta\cdot} \right) \mu, \tag{1.86}$$

$\phi^{\alpha\,(2)}$ given by (1.82), $\phi^{\alpha\,(1)}$ given by (1.70), where $a_{\alpha\overline{\beta}\gamma}$, $c_{\alpha\overline{\beta}}$ are found from (1.69) and $d^{\alpha}_{\beta\cdot}$ are found from (1.81), the functions $\mathscr{V}^{\alpha\,(1)}_{\beta\cdot\gamma}$ are completely determined by formulas (1.84) and involve partial derivatives of r up to order 5. Contracting (1.85) and using conditions (1.37) we obtain

$$e_{\beta} = \frac{2i}{2n+1} \mathscr{V}^{\alpha\,(1)}_{\beta\cdot\alpha}. \tag{1.87}$$

Formulas (1.87) determine the functions e^{α} in terms of partial derivatives of r up to order 5, and we set

$$\zeta^{(2)} := \zeta^{(1)} - i\left(e_\alpha dz^\alpha - e_{\overline{\alpha}} dz^{\overline{\alpha}}\right). \tag{1.88}$$

Further, identities (1.36), (1.57) imply

$$\begin{aligned} d\phi^\alpha - \phi\wedge\phi^\alpha - \phi^\beta\wedge\phi^\alpha_{\beta\cdot} + \frac{1}{2}\zeta\wedge dz^\alpha = \\ \mathscr{V}^\alpha_{\cdot\beta\overline{\sigma}} dz^\beta\wedge dz^{\overline{\sigma}} + \mathscr{P}^\alpha_{\beta\cdot} dz^\beta\wedge\mu + \mathscr{Q}^\alpha_{\overline{\beta}\cdot} dz^{\overline{\beta}}\wedge\mu, \end{aligned} \tag{1.89}$$

where $\mathscr{P}^\alpha_{\beta\cdot}$ and $\mathscr{Q}^\alpha_{\overline{\beta}\cdot}$ are the push-forwards of the functions $P^\alpha_{\beta\cdot,\mathscr{G}}\big|_{\mathfrak{W}}$ and $Q^\alpha_{\overline{\beta}\cdot,\mathscr{G}}\big|_{\mathfrak{W}}$ from $\mathfrak{W}$ to W, respectively. It follows from (1.73), (1.88), (1.89) that

$$\begin{aligned} d\phi^\alpha - \phi\wedge\phi^\alpha - \phi^\beta\wedge\phi^\alpha_{\beta\cdot} + \frac{1}{2}\zeta^{(2)}\wedge dz^\alpha = \\ \mathscr{V}^\alpha_{\cdot\beta\overline{\sigma}} dz^\beta\wedge dz^{\overline{\sigma}} + \mathscr{P}^{\alpha\,(1)}_{\beta\cdot} dz^\beta\wedge\mu + \mathscr{Q}^\alpha_{\overline{\beta}\cdot} dz^{\overline{\beta}}\wedge\mu, \end{aligned} \tag{1.90}$$

where

$$\mathscr{P}^{\alpha\,(1)}_{\beta\cdot} := \mathscr{P}^\alpha_{\beta\cdot} - \frac{1}{2}\delta^\alpha_\beta t. \tag{1.91}$$

Note that with ϕ given in (1.68), $\phi^\alpha_{\beta\cdot}$ given by (1.86), ϕ^α given by

$$\phi^\alpha = \left(\frac{1}{2}c^\alpha_{\beta\cdot} - d^\alpha_{\beta\cdot}\right)dz^\beta - e^\alpha\mu, \tag{1.92}$$

$\zeta^{(2)}$ given by (1.88), where $a_{\alpha\overline{\beta}\gamma}$, $c_{\alpha\overline{\beta}}$, $\zeta^{(1)}$ are found from (1.69), $d^\alpha_{\beta\cdot}$ are found from (1.81), and e^α are found from (1.87), the functions $\mathscr{P}^{\alpha\,(1)}_{\beta\cdot}$ and $\mathscr{Q}^\alpha_{\overline{\beta}\cdot}$ are completely determined by formulas (1.90) and involve partial derivatives of r up to order 6. Contracting (1.91) and using condition (1.59) we obtain

$$t = -\frac{2}{n}\mathscr{P}^{\alpha\,(1)}_{\alpha\cdot}. \tag{1.93}$$

Formula (1.93) determines the function t in terms of partial derivatives of r up to order 6, and ζ is given by

$$\zeta = \zeta^{(1)} - i\left(e_\alpha dz^\alpha - e_{\overline{\alpha}} dz^{\overline{\alpha}}\right) - t\mu. \tag{1.94}$$

Finally, identities (1.38), (1.57) imply

$$d\zeta - \phi\wedge\zeta - 2i\phi^\beta\wedge\phi_\beta = -2i\mathscr{P}_{\alpha\overline{\beta}} dz^\alpha\wedge dz^{\overline{\beta}} + \mathscr{R}_\alpha dz^\alpha\wedge\mu + \mathscr{R}_{\overline{\alpha}} dz^{\overline{\alpha}}\wedge\mu, \tag{1.95}$$

where $\mathscr{R}_\alpha$ are the push-forwards of the functions $R_{\alpha,\mathscr{G}}\big|_{\mathfrak{W}}$ from $\mathfrak{W}$ to W. Since the forms $\phi^\alpha_{\beta\cdot}$, ϕ^α, ζ have now been determined, identities (1.83), (1.89), (1.95) can be used to find the functions $\mathscr{S}^\alpha_{\beta\gamma\cdot\overline{\sigma}}$, $\mathscr{V}^\alpha_{\beta\cdot\gamma}$, $\mathscr{P}^\alpha_{\beta\cdot}$, $\mathscr{Q}^\alpha_{\overline{\beta}\cdot}$, $\mathscr{R}_\alpha$ in terms of partial derivatives of r up to order 7. More precisely, $\mathscr{S}^{\alpha}_{\beta\gamma\cdot\overline{\sigma}}$ are determined by the partial derivatives of order 4, $\mathscr{V}^{\alpha}_{\beta\cdot\gamma}$ by the partial derivatives of order 5, $\mathscr{P}^{\alpha}_{\beta\cdot}$ and $\mathscr{Q}^{\alpha}_{\overline{\beta}\cdot}$ by the partial derivatives of order 6, and $\mathscr{R}_{\alpha}$ by the partial derivatives of order 7.

The discussion at the end of Section 1.2 and transformation law (1.65) now yield that the system of equations

$$\mathscr{S}^{\alpha}_{\beta\gamma\cdot\overline{\sigma}} = 0, \quad \mathscr{V}^{\alpha}_{\beta\cdot\gamma} = 0, \quad \mathscr{P}^{\alpha}_{\beta\cdot} = 0, \quad \mathscr{Q}^{\alpha}_{\overline{\beta}\cdot} = 0, \quad \mathscr{R}_{\alpha} = 0 \tag{1.96}$$

is equivalent to the sphericity of the locally closed portion W of the real hypersurface M. System (1.96) involves partial derivatives of r up to order 7 and is hard to deal with in general. However, for special classes of hypersurfaces it can be simplified and becomes a rather useful tool for identifying spherical hypersurfaces. In this book we consider hypersurfaces of such a kind.

1.4 Umbilicity

Before we turn to special classes of hypersurfaces, we will show that system (1.96) can be simplified to some extent in general. To describe this simplification, we introduce the notion of *umbilic point* in a Levi non-degenerate CR-hypersurface M of CR-dimension n. For $n \geq 2$ a point $p \in M$ is called umbilic if all functions $S^{\alpha}_{\beta\gamma\cdot\overline{\sigma}}$ vanish on the fiber $\pi^{-1}(p)$ of the bundle $\mathscr{P}^2 \to M$. For $n = 1$ conditions (1.26), (1.37), (1.59) become

$$S^{1}_{11\cdot\overline{1}} = 0, \quad V^{1}_{1\cdot 1} = 0, \quad P^{1}_{1\cdot} = 0, \tag{1.97}$$

respectively, and for $n = 1$ we call a point $p \in M$ umbilic if $Q^{1}_{\overline{1}\cdot}$ vanishes on the fiber $\pi^{-1}(p)$. Due to transformation law (1.63), it is sufficient to require in the definition of umbilicity that $S^{\alpha}_{\beta\gamma\cdot\overline{\sigma}}$ and $Q^{1}_{\overline{1}\cdot}$ vanish only at some point of the fiber $\pi^{-1}(p)$ for $n \geq 2$ and $n = 1$, respectively.

We will now prove the following useful proposition.

Proposition 1.1. [9] *A Levi non-degenerate CR-hypersurface M is spherical if and only if every point of M is umbilic.*

Proof. If M is spherical, then its every point is umbilic due to conditions (1.62). Conversely, assume that every point of M is umbilic. To show that conditions (1.62) hold on $\mathscr{P}^2$, we use the Bianchi identities (see (1.55)).

First, suppose $n = 1$. Due to (1.35), (1.53), (1.57), (1.58), (1.97), all components of the curvature form Σ are equal to zero except possibly for

$$\Sigma^2_0 = -\frac{1}{4}\Psi = -\frac{1}{4}\left(R_1\omega^1 \wedge \omega + R_{\overline{1}}\omega^{\overline{1}} \wedge \omega\right).$$

From identities (1.55) for $m = 1$, $l = 0$ and (1.51) we see

$$\omega_1 \wedge \Psi = 0,$$

which implies

$$R_1 \omega_1 \wedge \omega^1 \wedge \omega = 0.$$

Hence $R_1 = 0$, and therefore $\Sigma = 0$ as required.

Now, suppose $n \geq 2$. In this case due to (1.35) we have

$$\Phi_{\alpha\cdot}^{\beta} = V_{\alpha\cdot\rho}^{\beta} \omega^{\rho} \wedge \omega - V_{\cdot\alpha\overline{\sigma}}^{\beta} \omega^{\overline{\sigma}} \wedge \omega. \tag{1.98}$$

Further, from identities (1.55) for $m = \alpha$, $l = \beta$ and (1.51), (1.53), (1.57), (1.58) we obtain

$$\begin{aligned} d\Phi_{\alpha}^{\beta} = \sum_{\gamma} \sigma_{\gamma}^{\alpha} \wedge \Phi_{\gamma}^{\beta} + i\omega_{\alpha} \wedge \left(V_{\cdot\gamma\overline{\sigma}}^{\beta} \omega^{\gamma} \wedge \omega^{\overline{\sigma}} + P_{\gamma\cdot}^{\beta} \omega^{\gamma} \wedge \omega + Q_{\overline{\gamma}\cdot}^{\beta} \omega^{\overline{\gamma}} \wedge \omega \right) + \\ i \left(V_{\alpha\overline{\gamma}\rho} \omega^{\overline{\gamma}} \wedge \omega^{\rho} + P_{\overline{\gamma}\alpha} \omega^{\overline{\gamma}} \wedge \omega + Q_{\gamma\alpha} \omega^{\gamma} \wedge \omega \right) \wedge \omega^{\beta} - \sum_{\gamma} \Phi_{\alpha}^{\gamma} \wedge \sigma_{\beta}^{\gamma}. \end{aligned} \tag{1.99}$$

Considering in identities (1.99) the terms not involving ω and using (1.11), (1.98), we get $V_{\alpha\cdot\rho}^{\beta} = 0$. Hence, (1.99) yields

$$\omega_{\alpha} \wedge \left(P_{\gamma\cdot}^{\beta} \omega^{\gamma} \wedge \omega + Q_{\overline{\gamma}\cdot}^{\beta} \omega^{\overline{\gamma}} \wedge \omega \right) + \left(P_{\overline{\gamma}\alpha} \omega^{\overline{\gamma}} \wedge \omega + Q_{\gamma\alpha} \omega^{\gamma} \wedge \omega \right) \wedge \omega^{\beta} = 0,$$

which implies $P_{\gamma\cdot}^{\beta} = 0$ and $Q_{\overline{\gamma}\cdot}^{\beta} = 0$. Thus, all components of the curvature form Σ are equal to zero except possibly for

$$\Sigma_0^{n+1} = -\frac{1}{4}\Psi = -\frac{1}{4}\left(R_{\alpha}\omega^{\alpha} \wedge \omega + R_{\overline{\alpha}}\omega^{\overline{\alpha}} \wedge \omega \right).$$

From identities (1.55) for $m = \beta$, $l = 0$ and (1.51) we see

$$\omega_{\beta} \wedge \Psi = 0.$$

Hence $R_{\alpha} = 0$, and therefore $\Sigma = 0$ as required. $\square$

Due to Proposition 1.1 and transformation laws (1.63), (1.65), system of equations (1.96), which characterizes the sphericity of a locally closed portion of an immersed real hypersurface in a complex $(n+1)$-dimensional manifold, can be replaced by the system of equations

$$\mathscr{S}_{\beta\gamma\cdot\overline{\sigma}}^{\alpha} = 0 \tag{1.100}$$

for $n \geq 2$ and by the single equation

$$\mathscr{Q}_{1\cdot}^{1} = 0 \tag{1.101}$$

for $n = 1$. System (1.100) involves partial derivatives of r up to order 4, whereas equation (1.101) involves partial derivatives of r up to order 6.

We also remark that in the real-analytic case the sphericity condition can be expressed in terms of a so-called *complex defining function* (see [83], [84]). In this case, analogously to (1.100), (1.101), sphericity is equivalent to a system of equations involving partial derivatives up to order 4 for $n \geq 2$ and to a single equation involving partial derivatives up to order 6 for $n = 1$.

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