Chapter 1
"Fluctuoscopy" of Superconductors

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Abstract Study of fluctuation phenomena in superconductors (SCs) is the subject of great fundamental and practical importance. Understanding of their physics allowed to clear up the fundamental properties of SC state. Being predicted in 1968, one of the fluctuation effects, namely paraconductivity, was experimentally observed almost simultaneously. Since this time, fluctuations became a noticeable part of research in the field of superconductivity, and a variety of fluctuation effects have been discovered.

The new wave of interest to fluctuations (FL) in superconductors was generated by the discovery of cuprate oxide superconductors (high-temperature superconductors, HTS), where, due to extremely short coherence length and low effective dimensionality of the electron system, superconductive fluctuations manifest themselves in a wide range of temperatures. Moreover, anomalous properties of the normal state of HTS were attributed by many theorists to strong FL in these systems. Being studied in the framework of the phenomenological Ginzburg–Landau theory and, more extensively, in diagrammatic microscopic approach, SC FLs side by side with other quantum corrections (weak localization, etc.) became a new tool for investigation and characterization of such new systems as HTS, disordered electron systems, granular metals, Josephson structures, artificial super-lattices, etc. The characteristic feature of SC FL is their strong dependence on temperature and magnetic fields in the vicinity of phase transition. This allows one to definitely separate the fluctuation effects from other contributions and to use them as the source of information about the microscopic parameters of a material. By their origin, SC FLs are very sensitive to relaxation processes, which break phase coherence. This allows using them for versatile characterization of SC. Today, one can speak about the "fluctuoscopy" of superconductive systems.

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In review, we present the qualitative picture both of thermodynamic fluctuations close to critical temperature T_{c0} and quantum fluctuations at zero temperature and in vicinity of the second critical field $H_{c2}(0)$. Then in the frameworks of the Ginzburg–Landau theory, we discuss the characteristic crossovers in fluctuation properties of superconductive nanoparticles and layered superconductors. We present the general expression for fluctuation magneto-conductivity valid through all phase diagram of superconductor and apply it to study of the quantum phase transition close to $H_{c2}(0)$. Fluctuation analysis of this transition allows us to present the scenario of fluctuation defragmentation of the Abrikosov lattice.

1.1 Introduction

"Happy families are all alike; every unhappy family is unhappy in its own way", started Leo Tolstoy his novel "Anna Karenina". A similar statement can be made about the electronic couples in superconductors (SCs): while stable Cooper pairs forming below critical temperature T_{c0} a sort of condensate behave all in the same way, the behavior of the fluctuating Cooper pairs (FCPs) above the transition is complex and involves a lot of interesting physics. Such FCPs affect thermodynamic and transport properties of the metal both directly and through the changes which they cause in normal quasi-particle subsystem [1], and study of superconductive fluctuations (SF) presents the unique tool providing the information about the character of superconductive state formation [1]. Difficulties of such "fluctuoscopy" are caused by the quantity of these quantum corrections, necessity of their separation from unknown background, smallness of their magnitude.

The mechanisms of fluctuations in the vicinity of the superconductive critical temperature T_{c0} were deeply understood in 1970s. SFs are commonly described in terms of three principal contributions: Aslamazov–Larkin (AL) process, corresponding to the opening of the new channel of the charge transfer [2], anomalous Maki–Thompson (MT) process, which is a single-particle quantum interference on impurities in presence of SF [3–5], and the change of the single-particle density of states (DOS) due to their involvement in fluctuation pairings [6, 7]. The first two processes (AL and MT) result in appearance of positive and singular close to the superconductive critical temperature T_{c0} contributions to conductivity, while the third one (DOS) results in decrease of the Drude conductivity due to the lack of single-particle excitations at the Fermi level. The latter contribution is less singular in temperature than the first two and can compete with them only when the AL and MT processes are suppressed by some reasons (e.g., c-axis transport in layered superconductors) or far enough from T_{c0}.

The classical results obtained first in the vicinity of T_{c0} later were generalized to the temperatures far from transition [8–10] and relatively high fields [11]. More recently, quantum fluctuations (QFs), taking place in SC at low temperatures and fields close to the second critical field H_{c2} (0), entered the focus. Their manifestation strikingly differs from that one of thermal fluctuations close to T_{c0}. For instance, the direct contribution of FCPs to transport coefficients here is absent. In [12, 13] was found that in granular SC at very low temperatures and close to $H_{c2}(0)$, the positive AL contribution to magneto-conductivity (MC) decays as T^2 while the fluctuation suppression of the quasiparticle density of states (DOS) by QF results in temperature independent negative contribution to MC logarithmically growing in magnitude when $H \to H_{c2}(0)$. Effects of QF on MC and magnetization of two-dimensional (2D) SC were studied at low temperatures and fields close to $H_{c2}(0)$ in [14]. Fluctuation renormalization of the diffusion coefficient (DCR) results in appearance of a giant Nernst–Ettingshausen signal [15]. Moreover, as it was demonstrated recently [16] namely this contribution governs the behavior of fluctuation conductivity through all periphery of the phase diagram of superconductor and especially in the region of quantum phase transition in the vicinity of $H_{c2}(0)$.

1.2 Thermodynamic Superconductive Fluctuations Close to T_{c0}

1.2.1 Rather Rayleigh–Jeans Fields than Boltzmann Particles

In the BCS theory [17, 18], only the Cooper pairs forming a Bose-condensate are considered. Fluctuation theory deals with the Cooper pairs out of the condensate. In some phenomena, these FCPs behave similarly to quasiparticles but with one important difference. While for the well-defined quasiparticle, the energy has to be much larger than its inverse lifetime, for the FCPs the "binding energy" ΔE turns out to be of the same order. The FCPs lifetime τ_{GL} is determined by its decay into two free electrons. Evidently, at the transition temperature the Cooper pairs start to condense and $\tau_{\mathrm{GL}} = \infty$. Above T_{c0} τ_{GL} can be estimated using the uncertainty principle: $\tau_{\mathrm{GL}} \sim \hbar/\Delta E$, where ΔE is the difference $k_{\mathrm{B}}(T - T_{c0})$ ensuring that τ_{GL} should become infinite at the point of transition. The microscopic theory confirms this hypothesis and gives the exact coefficient:

$$\tau_{\mathrm{GL}} = \frac{\pi\hbar}{8k_{\mathrm{B}}(T - T_{\mathrm{c}})}. \tag{1.1}$$

Another important difference of the FCPs from quasiparticles lies in their large size $\xi(T)$. This size is determined by the distance by which the electrons forming the FCPs move apart during the pair lifetime τ_{GL}. In the case of an impure superconductor, the electron motion is diffusive with the diffusion coefficient $\mathcal{D} \sim v_F^2\tau$ (τ is the electron scattering time [19]), and $\xi_d(T) = \sqrt{\mathcal{D}\tau_{\mathrm{GL}}} \sim v_F\sqrt{\tau\tau_{\mathrm{GL}}}$. In the case of a clean superconductor, where $k_{\mathrm{B}}T\tau \gg \hbar$, impurity scattering no longer affects the electron correlations. In this case the time of electron ballistic motion turns out to be less than the electron–impurity scattering time τ and is determined by the uncertainty principle: $\tau_{\mathrm{bal}} \sim \hbar/k_{\mathrm{B}}T$. Then this time has to be used in this case for the determination of the effective size instead of τ: $\xi_{\mathrm{c}}(T) \sim$

$v_F\sqrt{\hbar\tau_{\mathrm{GL}}/k_{\mathrm{B}}T}$. In both cases, the coherence length grows with the approach to the critical temperature as $\epsilon^{-1/2}$, where

$$\epsilon = \ln\frac{T}{T_{\mathrm{c}}} \approx \frac{T-T_{\mathrm{c}}}{T_{\mathrm{c}}} \tag{1.2}$$

is the reduced temperature. We will write down coherence length in the unique way

$$\xi_{\mathrm{GL}}(\epsilon) = (\mathcal{D}\tau_{\mathrm{GL}})^{1/2} \sim \xi_{\mathrm{BCS}}/\sqrt{\epsilon}. \tag{1.3}$$

Here, $\xi_{\mathrm{BCS}} = \xi_{c,d}$ is the BCS coherence length. We see that the fluctuating order parameter $\Delta^{(\mathrm{fl})}(\mathbf{r},t)$ varies close to T_{c0} on the large scale $\xi_{\mathrm{GL}}(\epsilon) \gg \xi_{\mathrm{BCS}}$.

Finally, it is necessary to recognize that FCPs can really be treated as classical objects, but that these objects instead of Boltzmann particles appear as classical fields in the sense of Rayleigh–Jeans. This means that in the general Bose–Einstein distribution function only small energies $\mathcal{E}(p)$ are involved and the exponent can be expanded:

$$n(p) = \frac{1}{\exp(\mathcal{E}(p)/k_{\mathrm{B}}T)-1} = \frac{k_{\mathrm{B}}T}{\mathcal{E}(p)}. \tag{1.4}$$

That is why the more appropriate tool to study fluctuation phenomena is not the Boltzmann transport equation but the GL equation for classical fields. Nevertheless, at the qualitative level the treatment of fluctuation Cooper pairs as particles with the concentration $N_s^{(D)} = \int n(p)d^Dp/(2\pi\hbar)^D$ often turns out to be useful [20].

In the framework of both the phenomenological GL theory and the microscopic BCS theory was found that in the vicinity of the transition

$$\mathcal{E}(p) = k_{\mathrm{B}}(T-T_{\mathrm{c}}) + \frac{\mathbf{p}^2}{2m^*} = \frac{1}{2m^*}\left[\hbar^2/2\xi^2(T) + \mathbf{p}^2\right]. \tag{1.5}$$

Far from the transition temperature, the dependence $n(p)$ turns out to be more sophisticated than (1.4); nevertheless, one can always write it in the form

$$n(p) = \frac{m^*k_{\mathrm{B}}T}{\hbar^2}\xi^2(T)\, f\left(\frac{\xi(T)p}{\hbar}\right). \tag{1.6}$$

The effective GL energy of the FCPs defined by (1.5) can be understood as the sum of its kinetic energy and the binding energy $\Delta E_,$ which is nothing else as the chemical potential

$$\mu_{\mathrm{C.p.}}(T) = T_{\mathrm{c}} - T \tag{1.7}$$

of the FCPs taken with the opposite sign:

$$\mathcal{E}(p) = \frac{\mathbf{p}^2}{2m^*} - \mu(T).$$

Let us clarify the issue related to the chemical potential of fluctuating Cooper pairs, $\mu_{\mathrm{C.p.}}$. Indeed, it is known that in the thermodynamic equilibrium, the chemical potential of a system with a variable number of particles is zero, with photon and phonon gases being the textbook examples. A naïve application of this "theorem" to fluctuating Cooper pairs "gas" leads to a wrong conclusion that $\mu_{\mathrm{C.p.}} = 0$. However, a delicate issue concerning Cooper pairs is that they do not form an isolated system but are composed of the fermionic quasi-particles, which constitute another subsystem under consideration. In a multicomponent system, the chemical potential of the i'th component, μ_i, is defined as the derivative of the thermodynamic potential with respect to the number of particles of i-th sort:

$$\mu_i = (\partial\Omega/\partial N_i)_{P,V,N_j}\ , \tag{1.8}$$

provided the numbers of particles of all other species are fixed, $N_{j\neq i} = \mathrm{const}$. In deriving the condition for thermodynamic equilibrium, one should now take into account that creation of a Cooper pair must be accompanied by removing two electrons from the fermionic subsystem. This leads to $\mu_{\mathrm{C.p.}} - 2\mu_{\mathrm{q.p.}} = 0$, where $\mu_{\mathrm{q.p.}}$ is the chemical potential of quasi-particles. Therefore, the equilibrium condition does not restrict $\mu_{\mathrm{C.p.}}$to zero, even though the number of Cooper pairs is not conserved.

1.2.2 Manifestation of SF Close to T_c

In classical field theory, the notions of the particle distribution function $n(p)$ (proportional to $\mathcal{E}^{-1}(p)$ in our case) and Cooper pair mass m^* are poorly determined. At the same time, the characteristic value of the Cooper pair center of mass momentum can be defined and it turns out to be of the order of $p_0 \sim \hbar/\xi(T)$. So for the combination $m^*\mathcal{E}(p_0)$ one can write $m^*\mathcal{E}(p_0) \sim p_0^2 \sim \hbar^2/\xi^2(T)$. The ratio of the FCPs concentration to the corresponding effective mass with the logarithmic accuracy can be expressed in terms of the coherence length:

$$\frac{N_s^{(D)}}{m^*} = \frac{k_\mathrm{B}T}{m^*\mathcal{E}(p_0)}\left(\frac{p_0}{\hbar}\right)^D \sim \frac{k_\mathrm{B}T}{\hbar^2}\xi_{\mathrm{GL}}^{2-D}(T) \tag{1.9}$$

(p_0^D here estimates the result of momentum integration).

The particles' density enters into many physical values in the combination N/m^*. For example, we can evaluate the direct FCPs contribution to conductivity (Aslamazov–Larkin paraconductivity) by using the Drude formula and noting that the role of scattering time for FCPs plays their lifetime τ_{GL}:

$$\delta\sigma_{(D)}^{\mathrm{AL}} = \frac{N_s^{(D)}\mathrm{e}^2\tau_{\mathrm{GL}}(\epsilon)}{m^*} \Rightarrow \frac{k_\mathrm{B}T}{\hbar^2}d^{D-3}\xi_{\mathrm{GL}}^{2-D}(T)(2e)^2\tau_{\mathrm{GL}}(\epsilon) \sim \epsilon^{D/2-2}. \tag{1.10}$$

This contribution to conductivity of the normal phase of superconductor corresponds to opening of the new channel of charge transfer above T_c: due to forming in it FCPs.

Analogously, a qualitative understanding of the increase in the diamagnetic susceptibility above the critical temperature may be obtained from the – known Langevin expression for the atomic susceptibility [21]:

$$\delta\chi^{\mathrm{C.p}} = -\frac{\mathrm{e}^2}{c^2}\frac{n_s^{(D)}}{m^*}\left\langle R^2\right\rangle \Rightarrow -\frac{4\mathrm{e}^2}{c^2}\frac{k_{\mathrm{B}}T}{\hbar^2}d^{D-3}\xi^{4-D}(T) \sim -\epsilon^{D/2-2}. \tag{1.11}$$

Here, we used the ratio (1.9).

Special attention has been attracted recently by the giant Nernst–Ettingshausen effect observed in the pseudogap state of the underdoped phases of HTSC [22], which motivated speculations [23] about the possibility of existence of some specific vortices and anti-vortices there or the special role of the phase fluctuations [24]. Then, very recently the giant Nernst–Ettingshausen signal (three orders of magnitude more than the value of the Nernst–Ettingshausen coefficient in typical metals) was detected also in the wide range of temperatures in a conventional disordered superconductor Nb_xSi_{1-x} [25]. All these experiments finally have been successfully explained in the frameworks of both phenomenological and microscopic fluctuation theories [15,26,27]. The proposed qualitative consideration of the FCPs allows not only to get in a simple way the correct temperature dependence of the fluctuation NEE coefficient but also to catch the reason of its giant magnitude. Indeed, as it was shown in [15, 28], the Nernst–Ettingshausen coefficient can be related to the temperature derivative of the chemical potential:

$$\delta\mathcal{N}^{\mathrm{C.p}} = \frac{\sigma}{nc\mathrm{e}^2}\left(\frac{\mathrm{d}\mu}{\mathrm{d}T}\right). \tag{1.12}$$

Applying this formula to the subsystem of FCPs close to T_{c0} with $\mu_{\mathrm{C.p.}}(T)$ defined by (1.7) and identifying its conductivity with (1.10), one finds

$$\delta\mathcal{N}^{\mathrm{C.p}} = -\frac{\sigma^{(\mathrm{C.p.})}}{N_s^{(D)}c\mathrm{e}^2} \sim \epsilon^{D/2-2}, \tag{1.13}$$

what fits well the experimental findings obtained in conventional superconductors and optimally doped phases of HTS. The reason of so strong fluctuation effect contains in the extremely strong dependence of the FCPs chemical potential on temperature: $\mathrm{d}\mu_{\mathrm{C.p.}}/\mathrm{d}T = -1$, while for the free electron gas $\mathrm{d}\mu_e/\mathrm{d}T \sim -T/E_F$.

Besides the direct FCPs effect on properties of superconductor in its normal phase, the other, indirect manifestations of SF and their effect on the quasiparticle subsystem take place. These effects, being much more sophisticated, have a purely quantum nature and, in contrast to paraconductivity, require microscopic consideration. First of them is MT contribution [3–5]. It is generated by the coherent scattering of the electrons forming a Cooper pair on the same elastic impurities and can be treated as the result of Andreev reflection of the electron by fluctuation Cooper pairs. This contribution appears only in transport coefficients and often turns out to be important. Its temperature singularity near T_c is similar to that of the paraconductivity, although being extremely sensitive to electron phase-breaking processes and to the type of orbital symmetry of pairing it can be suppressed. Let us evaluate it.

The physical origin of the MT correction consists in the fact that the Cooper interaction of electrons with nearly opposite momenta changes the mean free path (diffusion coefficient) of electrons. The amplitude of the effective BCS interaction increases drastically when $T \to T_c$:

$$g_{\mathrm{eff}} = \frac{g}{1 - \nu g \ln \frac{\omega_D}{2\pi T}} = \frac{1}{\ln \frac{T}{T_c}} \approx \frac{T}{T - T_c} = \frac{1}{\epsilon}.$$

What is the reason for this growth? One can say that the electrons scatter one at another in a resonant way with the virtual Cooper pair formation. Or, it is possible to imagine that the electrons undergo Andreev reflection by fluctuation Cooper pairs, binding in the Cooper pairs themselves. The probability of such induced pair irradiation (let us remember that Cooper pairs are Bose particles) is proportional to their number in the final state that is $n(p)$ (see (1.4)). For small momenta, $n(p) \sim 1/\epsilon$.

One can ask why such an interaction does not manifest itself considerably far from the transition point? This is due to the fact that just a small number of electrons with the total momentum $q \lesssim \xi^{-1}(T)$ interacts so intensively. In accordance with the Heisenberg principle, the minimal distance between such electrons is of the order of $\sim \xi(T)$. On the other hand, such electrons, in order to interact, have to approach one another approximately up to a distance of the Fermi length $\lambda_F \sim 1/p_F$. The probability of such event may be estimated in the spirit of the self-intersecting trajectories contribution evaluation in the weak-localization theory [29].

In the process of diffusion motion, the distance between two electrons increases with time according to the law: $R(t) \sim (\mathcal{D}t)^{1/2}$. Hence, the scattering probability

$$W \sim \int_{t_{\min}}^{t_{\max}} \frac{\lambda_F^{D-1}}{R^D(t)} v_F \mathrm{d}t.$$

The lower limit of the integral can be estimated from the condition $R(t_{\min}) \sim \xi(T)$ (only such electrons interact in the resonant way). The upper limit is determined by the phase-breaking time τ_φ since for larger time intervals the phase coherence, necessary for the pair formation, is broken. As a result, the relative correction to conductivity due to such processes is equal to the product of the scattering probability on the effective interaction constant: $\delta\sigma^{\mathrm{MT}}/\sigma = W\, g_{\mathrm{eff}}$. In the $2D$ case

$$\delta\sigma_{(2)}^{\mathrm{MT(an)}} \sim \frac{\mathrm{e}^2}{8\epsilon} \ln \frac{\mathcal{D}\tau_\varphi}{\xi^2(T)}.$$

However, positive and singular in ϵ close to T_c AL and MT contributions do not capture the complete effect of fluctuations on conductivity. The involvement of quasi-particles in the fluctuation pairing results in their lack at the Fermi level that is in the opening of the pseudo-gap in the one-electron spectrum and consequent decrease of the one-particle Drude-like conductivity. Such an indirect effect of FCPs formation is usually referred as the DOS one. Being proportional to the concentration of the FCPs $N_s^{(D)}$ the DOS contribution formally appears due to the order parameter Fourier-component $\left\langle \left|\Delta^{(\mathrm{fl})}(\mathbf{q},\omega)\right|^2\right\rangle$ integrated over all long-wave-length fluctuation modes ($q \lesssim \xi_{\mathrm{BCS}}^{-1}\sqrt{\epsilon}$):

$$\delta\sigma_{(2)}^{\mathrm{DOS}} \sim -\frac{2n_{\mathrm{c.p.}}\mathrm{e}^2\tau}{m_{\mathrm{e}}} \sim -\mathrm{e}^2\int\frac{\xi_{\mathrm{BCS}}^2\mathrm{d}^2\mathbf{q}}{\epsilon+\xi_{\mathrm{BCS}}^2q^2} \sim -\frac{\mathrm{e}^2}{\hbar}\ln\frac{1}{\epsilon}. \tag{1.14}$$

It is seen that DOS contribution has an opposite sign with respect to the AL and MT contributions, but close to T_{c0} does not compete with them since it turns to be less singular as a function of temperature [1].

Finally, the renormalization of the one-electron diffusion coefficient (DCR) in the presence of fluctuation pairing takes place. Close to T_{c0} this contribution is not singular in ϵ

$$\delta\sigma_{\mathrm{xx}}^{\mathrm{DCR}} \sim \frac{\mathrm{e}^2}{\hbar}\ln\ln\frac{1}{T_{c0}\tau} + O(\epsilon)$$

and was always ignored, but as was found in [15, 16] it becomes of primary importance relatively far from T_{c0}, and at very low temperatures. It is the account for $\delta\sigma_{\mathrm{xx}}^{\mathrm{DCR}}$, which changes the sign of the total contribution of fluctuations to conductivity $\delta\sigma_{(2)}^{(\mathrm{tot})}$ in the wide domain of the phase diagram and especially close to $T = 0$, in the region of quantum fluctuations [16] (see Fig. 1.1, where the regions with the dominating fluctuation contributions to magnetoconductivity are shown).

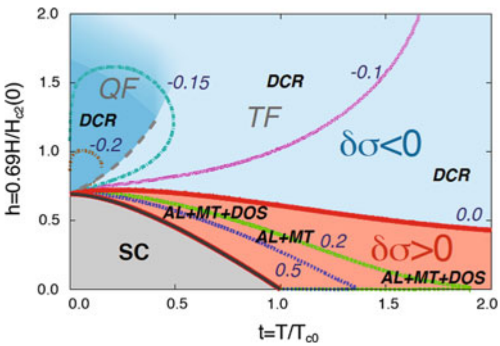

Fig. 1.1 Contours of constant fluctuation conductivity [$\delta\sigma = \delta\sigma_{\mathrm{xx}}^{(\mathrm{tot})}(t,h)$ shown in units of e^2]. The dominant FC contributions are indicated in bold-italic labels. The dashed line separates the domain of quantum fluctuations (QFs) [dark area of $\delta\sigma > 0$] and thermal fluctuations (TFs)

1.3 Ginzburg–Landau Theory

1.3.1 GL Functional

Let us consider the model of metal being close to transition to the superconductive state. The complete description of its thermodynamic properties can be done through the calculation of the partition function [30]:

$$Z = \mathrm{tr}\left\{\exp\left(-\frac{\widehat{\mathcal{H}}}{T}\right)\right\}. \tag{1.15}$$

As discussed above, in the vicinity of the superconductive transition, side by side with the fermionic electron excitations, fluctuation Cooper pairs of a bosonic nature appear in the system. They can be described by means of classical bosonic complex fields $\Psi(\mathbf{r})$, which can be treated as "Cooper pair wave functions". Therefore, the calculation of the trace in (1.15) can be separated into a summation over the "fast" electron degrees of freedom and a further functional integration carried out over all possible configurations of the "steady flow" Cooper pairs wave functions:

$$Z = \int \mathfrak{D}^2\Psi(\mathbf{r})\mathcal{Z}[\Psi(\mathbf{r})], \tag{1.16}$$

where

$$\mathcal{Z}[\Psi(\mathbf{r})] = \exp\left(-\frac{\mathcal{F}[\Psi(\mathbf{r})]}{T}\right) \tag{1.17}$$

is the system partition function in a fixed bosonic field $\Psi(\mathbf{r})$, already summed over the electronic degrees of freedom.

The "steady flow" of wave functions means that they are supposed to vary over a scale much larger than the interatomic distances. The classical part of the Hamiltonian, dependent on bosonic fields, may be chosen in the spirit of the Landau theory of phase transitions. However, in view of the space dependence of wave functions, Ginzburg and Landau included in it additionally the first nonvanishing term of the expansion over the gradient of the fluctuation field. Symmetry analysis shows that it should be quadratic. The weakness of the field coordinate dependence allows us to omit the high order terms of such an expansion. Therefore, the classical part of the Hamiltonian of a metal close to superconductive transition related to the presence of the fluctuation Cooper pairs in it (so-called GL functional) can be written as [31]:

$$\mathcal{F}[\Psi(\mathbf{r})] = F_N + \int \mathrm{d}V\left\{a|\Psi(\mathbf{r})|^2 + \frac{b}{2}|\Psi(\mathbf{r})|^4 + \frac{1}{4m}|\nabla\Psi(\mathbf{r})|^2\right\}. \tag{1.18}$$

Let us discuss the coefficients of this functional. In accordance with the Landau hypothesis, the coefficient a goes to zero at the transition point T_{c0} and depends linearly on $T - T_{c0}$. Then $a = \alpha T_c \epsilon$; all the coefficients α, b, and m are supposed to be positive and temperature independent. Concerning the magnitude of the coefficients, it is necessary to make the following comment. One of these coefficients can always be chosen arbitrarily: this option is related to the arbitrariness of the Cooper pair wave function normalization. Nevertheless, the product of two of them is fixed by dimensional analysis: $ma \sim \xi^{-2}(T)$. Another combination of the coefficients, independent of the wave function normalization and temperature, is α^2/b. One can see that it has the dimensionality of the density of states. Since these coefficients were obtained by a summation over the electronic degrees of freedom, the only reasonable candidate for this value is the one electron DOS ν (for one spin at the Fermi level). One can notice that the arbitrariness of the order parameter amplitude results in the ambiguity in the choice of the Cooper pair mass, introduced in (1.18) as $2m$. Indeed, this value enters in (1.20) as the product with the coefficient α, hence one of these parameters has to be set down.

In the phenomenological GL theory, normalization of the order parameter Ψ is usually chosen in such a way that the coefficient m corresponds to the free electron mass. At that, the coefficient α for D-dimensional clean superconductor is determined by the expression

$$\alpha_{(D)} = \frac{2D\pi^2}{7\zeta(3)} \frac{T_{c0}}{E_F}. \tag{1.19}$$

Yet, the other normalization when the order parameter, denoted as $\Delta(\mathbf{r})$, coincides with the value of the gap in spectrum of one-particle excitations of a homogeneous superconductor turns out to be more convenient. As it will be shown below, in vicinity of T_{c0} the microscopic theory allows to present the free energy of superconductor in the form of the GL expansion namely over the powers of $\Delta(\mathbf{r})$. At that turn out to be defined also the exact values of the coefficients α and b:

$$4m\alpha T_{c0} = \xi^{-2}; \alpha^2/b = \frac{8\pi^2}{7\zeta(3)}\nu, \tag{1.20}$$

where $\zeta(x)$ is the Riemann zeta function, $\zeta(3) = 1.202$.

Let us stress that at such choice of the order parameter normalization the GL parameter $C = 1/4m$ turns out to be dependent on the concentration of impurities.

1.3.2 Zero Dimensionality: The Exact Solution for the Heat Capacity Jump

In a system of finite volume, the fluctuations smear out the jump of the heat capacity. Let us demonstrate this on the example of a small superconductive sample with the characteristic size $d \ll \xi(T)$. Due to the small size of the granule with respect to the

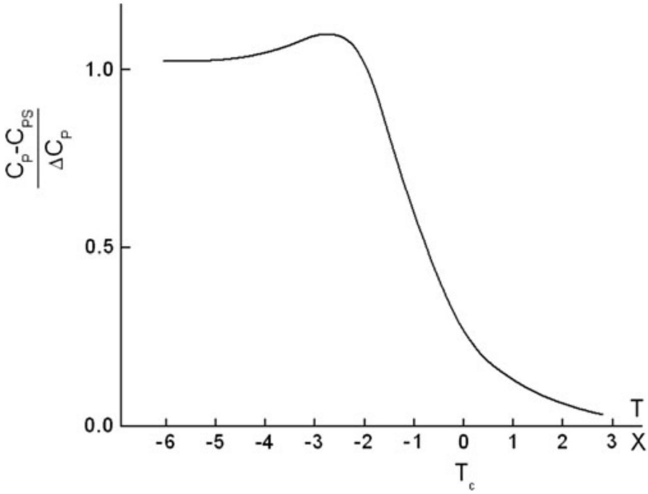

Fig. 1.2 Temperature dependence of the heat capacity of superconductive grains in the region of the critical temperature

GL coherence length, the order parameter Ψ does not depend on the space variables and the free energy can be calculated exactly for all temperatures including the critical region. It the space independent mode $\Psi_0 = \Psi\sqrt{V}$, which defines here the main contribution to the free energy:

$$Z_{(0)} = \int \mathrm{d}^2\Psi_0 \exp\left(-\frac{\mathcal{F}[\Psi_0]}{T}\right) = \pi \int \mathrm{d}|\Psi_0|^2 \exp\left(-\frac{(\,a|\Psi_0|^2 + \frac{b}{2V}|\Psi_0|^4)}{T}\right)$$
$$= \sqrt{\frac{\pi^3 VT}{2b}} \exp(x^2)(1-\mathrm{erf}(x))|_{x=a\sqrt{\frac{V}{2bT}}}. \tag{1.21}$$

By evaluating the second derivative of this exact result [32], one can find the temperature dependence of the heat capacity of the superconductive granule (see Fig. 1.2). One can see that this function is analytic in temperature, therefore fluctuations remove phase transition in the $0D$ system. The smearing of the heat capacity jump takes place in the region of temperatures in the vicinity of T_{c0} where $x \sim 1$, that is

$$\epsilon_{\mathrm{cr}} = Gi_{(0)} = \frac{\sqrt{7\zeta(3)}}{2\pi}\frac{1}{\sqrt{\nu T_{c0}V}} \approx 13.3\left(\frac{T_{\mathrm{c0}}}{E_F}\right)\sqrt{\frac{\xi^3_{\mathrm{BCS}}}{V}}.$$

Here, T_{c0} and ξ_{BCS} are the mean field critical temperature and the zero temperature coherence length of the appropriate bulk material. It is interesting that the width of this smearing does not depend on impurities concentration. From this formula, one can see that the smearing of the transition is very narrow ($\epsilon_{\mathrm{cr}} \ll 1$) when the granule volume $V \gg (\nu T_{c0})^{-1}$. This criterion means that the average spacing between the levels of the dimensional quantization:

$$\delta = (\nu V)^{-1} \tag{1.22}$$

still remains much less than the value of the mean field critical temperature, T_{c0}.

Far above the critical region, where $Gi_{(0)} \ll \epsilon \ll 1$, one can use the asymptotic expression for the erf(x) function and find

$$F_{(0)} = -T \ln Z_{(0)} = -T \ln \frac{\pi}{\alpha\epsilon}. \tag{1.23}$$

Calculation of the second derivative gives an expression for the fluctuation part of the heat capacity in this region:

$$\delta C_{(0)} = \frac{1}{V\epsilon^2}. \tag{1.24}$$

The experimental study of the heat capacity of small Sn particles in the vicinity of the transition was done in [33].

One can estimate the fluctuation contribution to heat capacity for a specimen of an arbitrary effective dimensionality on the basis of the following observation. The volume of the specimen may be divided into regions of size $\xi(T)$, which are weakly correlated with each other. Then the whole free energy can be estimated as the free energy of one such $0D$ specimen (1.23), multiplied by their number $N_{(D)} = V\xi^{-D}(T)$:

$$F_{(D)} = -TV\xi^{-D}(T) \ln \frac{\pi}{\alpha\epsilon}. \tag{1.25}$$

This formula gives the correct temperature dependence of the free energy not too close to T_{c} for the specimens of the even dimensionalities. As we will demonstrate below, a more accurate treatment removes the $\ln\epsilon$ dependence from it in the case of the odd dimensions.

In the Ginzburg–Landau region, one can omit the fourth-order term in $\Psi(\mathbf{r})$ with respect to the quadratic one and write down the GL functional, expanding the order parameter in a Fourier series:

$$F[\Psi_{\mathbf{k}}] = F_N + \sum_{\mathbf{k}} \left[a + \frac{\mathbf{k}^2}{4m} \right] |\Psi_{\mathbf{k}}|^2 = F_N + \alpha T_{\mathrm{c}} \sum_{\mathbf{k}} \left(\epsilon + \xi^2 \mathbf{k}^2 \right) |\Psi_{\mathbf{k}}|^2. \tag{1.26}$$

Here, $\Psi_{\mathbf{k}} = \frac{1}{\sqrt{V}} \int \Psi(\mathbf{r}) \mathrm{e}^{-\mathrm{i}\mathbf{kr}} \mathrm{d}V$ and the summation is carried out over the wave vectors $\mathbf{k}$ (fluctuation modes). For the specimen of dimensions L_x, L_y, L_z $k_i L_i = 2\pi n_i$. The functional integral for the partition function (1.17) can be factored out to a product of Gaussian-type integrals over these modes:

$$Z = \prod_{\mathbf{k}} \int \mathrm{d}^2\Psi_{\mathbf{k}} \exp\left\{-\alpha\left(\epsilon + \frac{\mathbf{k}^2}{4m\alpha T_{\mathrm{c}}}\right)|\Psi_{\mathbf{k}}|^2\right\}. \tag{1.27}$$

Carrying out these integrals, one gets the fluctuation contribution to the free energy:

$$F(\epsilon > 0) = -T \ln Z = -T \sum_{\mathbf{k}} \ln \frac{\pi}{\alpha\left(\epsilon + \frac{\mathbf{k}^2}{4m\alpha T_{\mathrm{c}}}\right)}. \tag{1.28}$$

1.3.3 Zero Dimensionality: The Exact Solution for the Fluctuation Magnetization

For quantitative analysis of the fluctuation diamagnetism, we start from the GL functional for the free energy written down in the presence of the magnetic field. The generalization of the functional (1.18) in the presence of magnetic field requires first of all the gauge invariance; therefore, the momentum operator $-\mathrm{i}\nabla$ must be substituted by its gauge invariant form $-\mathrm{i}\nabla - 2e\mathbf{A}(\mathbf{r})$ [34]. Moreover, the presence of a magnetic field results in the accumulation of some residual energy of the magnetic field in the volume of superconductor. Finally, the superconductor itself interacts with the external magnetic field $\mathbf{H}$. Taking into account these three observations one can write the generalization of the functional (1.18) in the form

$$\mathcal{F}[\Psi(\mathbf{r})] = F_n + \int \mathrm{d}V \left\{ a|\Psi(\mathbf{r})|^2 + \frac{b}{2}|\Psi(\mathbf{r})|^4 + \frac{1}{4m}|\left(-\mathrm{i}\nabla - 2e\mathbf{A}(\mathbf{r})\right)\Psi(\mathbf{r})|^2 \right.$$
$$\left. + \frac{[\nabla \times \mathbf{A}(\mathbf{r})]^2}{8\pi} - \frac{\nabla \times \mathbf{A}(\mathbf{r}) \cdot \mathbf{H}}{4\pi} \right\}. \tag{1.29}$$

The fluctuation contribution to the diamagnetic susceptibility in the simplest case of a "zero-dimensional" superconductor (spherical superconductive granule of diameter $d \ll \xi(\epsilon)$) was considered by Shmidt [32]. As above, the smallness $d \ll \xi(T)$ allows us to omit in (1.29) the term $-\mathrm{i}\nabla$. Then, due to the smallness of the granule size with respect to the magnetic field penetration depth in superconductor λ, one can assume the equivalence of the average magnetic field in metal $\mathbf{B}$ with the external field $\mathbf{H}$. This allows us to omit also the last two terms in (1.29) since in the assumed approximation they do not depend on fluctuations. It is why formally the effect of a magnetic field in this case is reduced to the renormalization of the coefficient a, or, in other words, to the suppression of the critical temperature:

$$T_{\mathrm{c}}(H) = T_{\mathrm{c0}}\left(1 - \frac{4\pi^2\xi^2}{\Phi_0^2}\langle\mathbf{A}^2\rangle\right). \tag{1.30}$$

Here, $\Phi_0 = \pi/e$ is the magnetic flux quantum and $\langle\cdots\rangle$ means the averaging over the sample volume. That is why for the granule in a magnetic field one can use the partition function in the same form (1.21) as in the absence of the field but with the renormalized GL parameter $a(H) = a + \frac{\mathrm{e}^2}{m}\langle\mathbf{A}^2\rangle$:

$$Z_{(0)}(H) = \pi\int \mathrm{d}|\Psi_0|^2 \exp\left(-\frac{\left[a + \frac{\mathrm{e}^2}{m}\langle\mathbf{A}^2\rangle\right]|\Psi_0|^2 + \frac{b}{2V}|\Psi_0|^4}{T}\right)$$
$$= \sqrt{\frac{\pi^3 VT}{2b}}\exp\left[\frac{a^2(H)V}{2bT}\right]\left\{1 - \mathrm{erf}\left[a(H)\sqrt{\frac{V}{2bT}}\right]\right\}. \quad (1.31)$$

Such a trivial dependence of the properties of $0D$ samples on the magnetic field immediately allows us to understand its effect on the heat capacity of a granular sample. Indeed, with the growth of the field the temperature dependence of the heat capacity presented in Fig. 1.2 just moves in the direction of lower temperatures. Equation (1.31) allows to calculate exactly the fluctuation part of the free energy and corresponding magnetization as the function of temperature and magnetic field, which can be used for the quantitative analysis of the experiments on nanoparticles (see below).

In the GL region $Gi_{(0)} \lesssim \epsilon$, one can easily write the asymptotic expression (1.23) for the free energy:

$$F_{(0)}(\epsilon, H) = -T\ln\frac{\pi}{\alpha\left(\epsilon + \frac{4\pi^2\xi^2}{\Phi_0^2}\langle\mathbf{A}^2\rangle\right)}.$$

In the case of a spherical particle, one has to choose the gauge of the vector-potential $\mathbf{A} = \frac{1}{2}\mathbf{H}\times\mathbf{r}$ yielding $\langle\mathbf{A}^2\rangle = \frac{1}{40}H^2d^2$ (calculation of this average value is completely analogous to the calculation of the moment of inertia of a solid sphere). In this way, an expression for the $0D$ fluctuation magnetization valid for all fields $H \ll H_{c2}(0)$ can be found:

$$M_{(0)}(\epsilon, H) = -\frac{1}{V}\frac{\partial F_{(0)}(\epsilon, H)}{\partial H} = -\frac{6\pi T\xi^2}{5\Phi_0^2 d}\frac{H}{\left(\epsilon + \frac{\pi^2\xi^2}{10\Phi_0^2}H^2d^2\right)}. \quad (1.32)$$

One can see that the fluctuation magnetization turns out to be negative and linear up to some crossover field $H_{\mathrm{up}}(\epsilon) \sim \frac{\Phi_0}{d\xi(\epsilon)} \sim \frac{\xi}{d}H_{c2}(0)\sqrt{\epsilon}$ [35] at which it reaches a minimum (this field can be called the temperature dependent upper critical field of the granule). At higher fields, $H_{\mathrm{up}}(\epsilon) \lesssim H \ll H_{c2}(0)$ the fluctuation magnetization of the $0D$ granule decreases as $1/H$. In the weak field region $H \ll H_{c2(0)}(\epsilon)$ the diamagnetic susceptibility is:

$$\chi_{(0)}(\epsilon, H) = -\frac{6\pi T\xi_0^2}{5\Phi_0^2 d}\frac{1}{\epsilon} \approx -10^2\chi_P\left(\frac{\xi}{d}\right)\frac{1}{\epsilon}.$$

Let us underline that the temperature dependence of the $0D$ fluctuation diamagnetic susceptibility turns out to be less singular than the $0D$ heat capacity correction: ϵ^{-1} instead of ϵ^{-2}.

The expression for the fluctuation part of free energy (1.28) is also applicable to the cases of a wire or a film placed in a parallel field: as was already mentioned above all its dependence on the magnetic field is manifested by the shift of the critical temperature (1.30). In the case of a wire in a parallel field the gauge of the vector-potential can be chosen as above what yields $\left\langle \mathbf{A}^2 \right\rangle_{(\text{wire},\|)} = H^2 d^2/32$. For a wire in a perpendicular field, or a film in a parallel field, the gauge has to be chosen in the form $\mathbf{A} = (0, Hx, 0)$. One can find $\left\langle \mathbf{A}^2 \right\rangle_{(\text{wire},\perp)} = H^2 d^2/16$ for a wire and $\left\langle \mathbf{A}^2 \right\rangle_{(\text{film},\|)} = H^2 d^2/12$ for a film.

Calculating the second derivative of (1.28) with the appropriate magnetic field dependencies of the critical temperature, one can find the following expressions for the diamagnetic susceptibility:

$$\chi_{(D)}(\epsilon) = -2\pi \frac{\xi T}{v_F} \chi_P \begin{cases} \frac{1}{\sqrt{\epsilon}}, \text{ wire in parallel field,} \\ \frac{2}{\sqrt{\epsilon}}, \text{ wire in perpendicular field,} \\ \frac{d}{3\xi} \ln \frac{1}{\epsilon}, \text{ film in parallel field.} \end{cases} \tag{1.33}$$

1.3.4 Fluctuation Diamagnetism in Lead Nanoparticles

Recently, in [36] the $0D$ fluctuating diamagnetism was carefully studied in lead nanoparticles with size $d \ll \xi$ by means of high-resolution superconductive quantum interference device magnetization measurements. In result, the diamagnetic magnetization $M_{\text{dia}}(H, T = \text{const})$ was reported as a function of the applied magnetic field H at constant temperatures in a wide range of temperatures around T_{c0} including the critical region. The magnetization curves were analyzed in the framework of the presented above exact fluctuation theory based on the Ginzburg–Landau functional.

The representative isothermal magnetization curves in the temperature range around T_{c0} are reported in Fig. 1.3. The extraction of the diamagnetic contribution from the magnetization requires a detailed subtraction procedure when the magnetic field is increased to relatively strong values. In fact, in the range $H > H_{\text{up}}$ $|M_{\text{dia}}|$ decreases on increasing the field (see Fig. 1.3), while the paramagnetic contributions due to the Pauli paramagnetism and to a small amount of paramagnetic impurities continue to increase on increasing H. Thus, from the computer-stored raw magnetization data around T_{c0}, the magnetization values measured at a higher temperature (around 8 K) where the SFs are negligible have been subtracted. The slight variation of the paramagnetic contribution with temperature did not prevent reliable estimates of M_{dia} for magnetic field up to about 600 Oe, as indicated by the error bars in Fig. 1.5b.

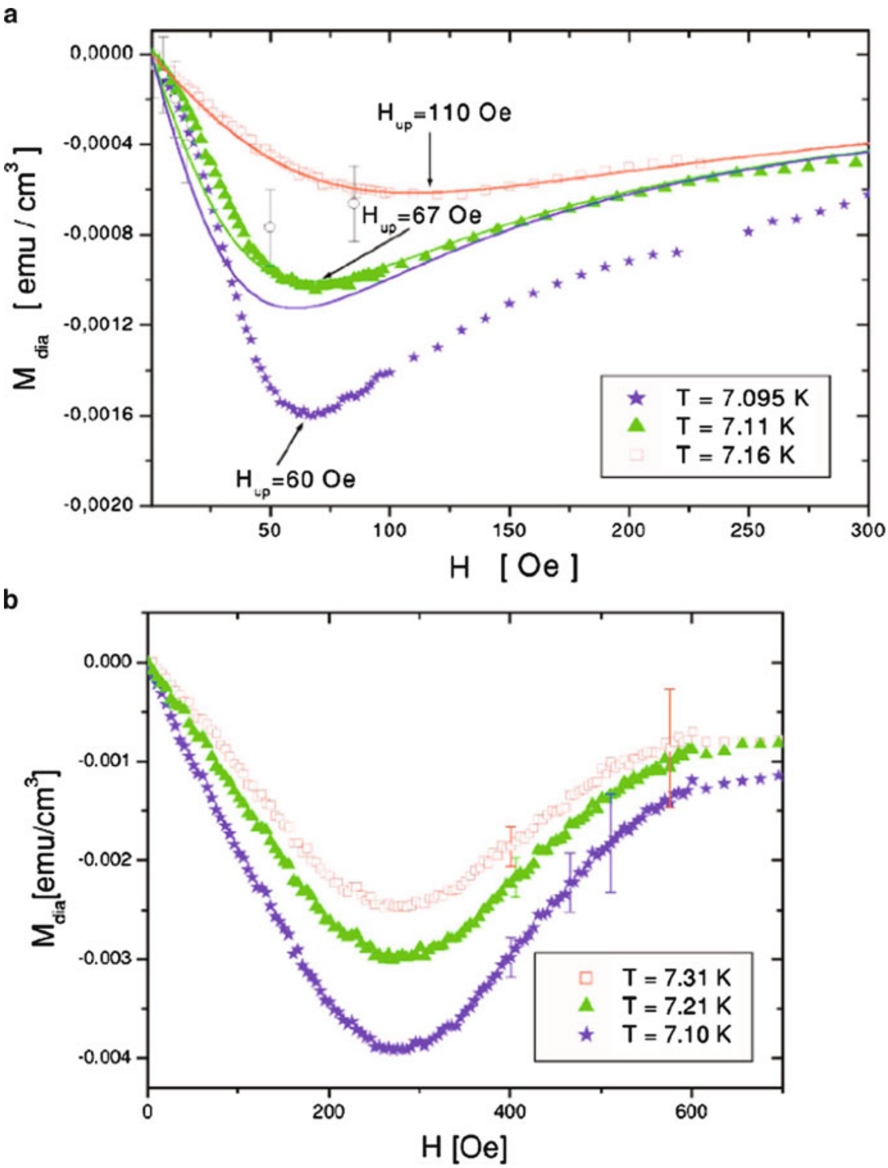

Fig. 1.3 (**a**) Magnetization M_{dia} vs H for the sample with the characteristic diameter of grains $d_3 \simeq 75\,\text{nm}$ at representative temperatures above T_{c0}. The solid lines correspond to (1.32) in the text for critical field of the grain 1,150 Oe . For $\epsilon \lesssim \epsilon_c$, the curves depart from the behavior described by (1.32). (**b**) Magnetization curves for sample with the characteristic size of grains $d_1 \simeq 16\,\text{nm}$, all corresponding to temperature range where $\epsilon \lesssim \epsilon_c$, namely, within the critical region. The open circles in part (a) correspond to the data obtained from the iso-field measurements as a function of temperature, with large experimental errors

The first-order fluctuation correction is found to be valid only outside the critical region $\epsilon \gtrsim \epsilon_c$, where it accurately describes the behavior $M_{\rm dia}$ for magnetic fields $H \lesssim H_{\rm up}$. Also, the scaling properties of $\mathrm{d}T_c(H)/\mathrm{d}H$ for small fields and of the upturn field $H_{\rm up}$ in the magnetization curves are well described within that approximation.

In the critical region, however, the role of the field and the limits of validity of the first-order fluctuation correction have been analyzed by comparing the experimental findings to the derivation of $M_{\rm dia}$ as a function of the magnetic field starting from the complete form of the GL functional and with the exact expression of the zero-dimensional partition function. The authors found that the role of the $|\Psi(\mathbf{r})|^4$ term in the GL functional is crucial in describing the data in the critical region. For the sample with average grain diameter of 75 nm, the fluctuating diamagnetism can be well described by our extended model even in the critical region, without introducing any adjustable parameters. For the sample with the smallest average diameter of 16 nm, the agreement of the numerically derived $M_{\rm dia}$ with the experimental findings is again good for fields of the order of $H_{\rm up}$. Poor agreement between the theoretically predicted $M_{\rm dia}$ vs H and the authors data is observed for fields above $H_{\rm up}$, when the fluctuating diamagnetic contribution is approaching zero and the subtraction procedure of the paramagnetic term introduces large errors.

The temperature dependence of the upturn field and the scaling properties with the grain size are also well described by the exact theory both outside and inside the critical region, with the product $(H_{\rm up}d)$ vs reduced temperature being approximately size independent and following the predicted temperature dependence, even though the mean field result $H_{\rm up} \sim \epsilon^{1/2}/d$ evidently breaks down. The relevance of the magnetization curves vs H and of the upturn field $H_{\rm up}$ for the study of the fluctuating diamagnetism above the superconductive transition temperature has been emphasized.

1.4 Fluctuation Thermodynamics of Layered Superconductor in Magnetic Field

1.4.1 Lawrence–Doniach Model

Let us pass now to the quantitative analysis of the temperature and field dependencies of the fluctuation magnetization of a layered superconductor. This system has a great practical importance because of its direct applicability to HTS, where the fluctuation effects are very noticeable. Moreover, the general results obtained will allow us to analyze as limiting cases $3D$ and already familiar $2D$ situations. The effects of a magnetic field are more pronounced for a perpendicular orientation, so let us first consider this case.

The generalization of the GL functional for a layered superconductor (Lawrence–Doniach (LD) functional [37]) in a perpendicular magnetic field can be written as

$$\mathcal{F}_{\mathrm{LD}}\left[\Psi\right] = \sum_{l} \int \mathrm{d}^2 r \left(a \left|\Psi_l\right|^2 + \frac{b}{2} \left|\Psi_l\right|^4 + \frac{1}{4m} \left| \left(\nabla_{\parallel} - 2\mathrm{i}e\mathbf{A}_{\parallel}\right) \left|\Psi_l\right|^2 \right. \right.$$
$$\left. + \mathcal{J} \left|\Psi_{l+1} - \Psi_l\right|^2 \right), \tag{1.34}$$

where Ψ_l is the order parameter of the l-th superconductive layer and the phenomenological constant $\mathcal{J}$ is proportional to the energy of the Josephson coupling between adjacent planes. The gauge with $A_z = 0$ is chosen in (1.34). In the immediate vicinity of T_{c}, the LD functional is reduced to the GL one with the effective mass $M = (4\mathcal{J}s^2)^{-1}$ along c-direction, where s is the inter-layer spacing. One can relate the value of $\mathcal{J}$ to the coherence length along the z-direction: $\mathcal{J} = 2\alpha T_{\mathrm{c}}\xi_z^2/s^2$. Since we are dealing with the GL region, the fourth order term in (1.34) can be omitted.

The Landau representation is the most appropriate for solution of the problems related to the motion of a charged particle in a uniform magnetic field. The fluctuation Cooper pair wave function can be written as the product of a plane wave propagating along the magnetic field direction and a Landau state wave function $\phi_n(\mathbf{r})$. Let us expand the order parameter $\Psi_l(\mathbf{r})$ on the basis of these eigenfunctions:

$$\Psi_l(\mathbf{r}) = \sum_{\mathbf{n},k_z} \Psi_{n,k_z}\phi_n(\mathbf{r}) \exp(\mathrm{i}k_z l), \tag{1.35}$$

where $\mathbf{n}$ is the quantum number related to the degenerate Landau state and k_z is the momentum component along the direction of the magnetic field. Substituting this expansion into (1.34), one can find the LD free energy as a functional of the Ψ_{n,k_z} coefficients:

$$\mathcal{F}_{\mathrm{LD}}\left[\Psi_{\{n,k_z\}}\right] = \sum_{n,k_z} \left\{ \alpha T_{\mathrm{c}}\epsilon + \omega_c \left(n + \frac{1}{2}\right) + \mathcal{J}\left[1 - \cos(k_z s)\right] \right\} |\Psi_{n,k_z}|^2. \tag{1.36}$$

In complete analogy with the case of an isotropic spectrum, the functional integral over the order parameter configurations Ψ_{n,k_z} in the partition function can be reduced to a product of ordinary Gaussian integrals, and the fluctuation part of the free energy of a layered superconductor in magnetic field takes the form:

$$F(\epsilon, H) = -\frac{SH}{\Phi_0} T \sum_{n,k_z} \ln \frac{\pi T}{\alpha T_{\mathrm{c}}\epsilon + \ \omega_{\mathrm{c}}\left(n + \frac{1}{2}\right) + \mathcal{J}\left[1 - \cos(k_z s)\right]} \tag{1.37}$$

(compare this expression with the (1.28)).

In the limit of weak fields, one can carry out the summation over the Landau states by means of the Euler–Maclaurin's transformation and obtain

$$F(\epsilon, H) = F(\epsilon, 0) + \frac{\pi S T H^2}{24 m \Phi_0^2} \int_{-\pi/s}^{\pi/s} \frac{\mathcal{N} s \mathrm{d}k_z}{2\pi} \left\{ \frac{1}{\alpha T_\mathrm{c}\epsilon + \mathcal{J}\left(1 - \cos(k_z s)\right)} \right\}. \tag{1.38}$$

Here $\mathcal{N}$, is the total number of layers. Carrying out the final integration over the transversal momentum, one gets:

$$F(\epsilon, H) = F(\epsilon, 0) + \frac{TV}{24\pi s \xi_{\mathrm{xy}}^2} \frac{h^2}{\sqrt{\epsilon(\epsilon + r)}}$$

with the anisotropy parameter defined as

$$r = \frac{2\mathcal{J}}{\alpha T} = \frac{4\xi_z^2(0)}{s^2} \tag{1.39}$$

and $h = 2\pi\xi^2 H/\Phi_0$ as reduced magnetic field. The diamagnetic susceptibility in a weak field turns out [38, 39] to be

$$\chi_{(\mathrm{layer},\perp)} = -\frac{\mathrm{e}^2 T}{3\pi s} \frac{\xi_{\mathrm{xy}}^2}{\sqrt{\epsilon(\epsilon + r)}}. \tag{1.40}$$

In the $2D$ and 3D limits, this formula reproduces (1.33). Note that (1.40) predicts a nontrivial increase of diamagnetic susceptibility for clean metals [39]. The usual statement that fluctuations are most important in dirty superconductors with a short electronic mean free path does not hold in the particular case of susceptibility because here ξ turns out to be in the numerator of the fluctuation correction.

1.4.2 General Formula for the Fluctuation Free Energy in Magnetic Field

Now we will demonstrate that, besides the crossovers in its temperature dependence, the fluctuation-induced magnetization and heat capacity are also nonlinear functions of magnetic field. These nonlinearities, different for various dimensionalities, take place at relatively weak fields. This, strong in comparison with the expected scale of $H_{\mathrm{c}2}(0)$, manifestation of the nonlinear regime in fluctuation magnetization and hence, field-dependent fluctuation susceptibility was the subject of the intensive debates in early 1970s [40–49] (see also the old but excellent review of Scokpol and Tinkham [50]) and after the discovery of HTS [51–54]. We will mainly follow here the recent essay of Mishonov and Penev [55] and the paper of Buzdin et al. [56], dealing with the fluctuation magnetization of a layered superconductor, which allows observing in a unique way all variety of the crossover phenomena in temperature and magnetic field.

One can evaluate the general expression (1.37) without taking the magnetic field to be small and get

$$F_{\mathrm{LD}}(\epsilon,h) = -\frac{TV}{2\pi s\xi_{\mathrm{xy}}^2}\Big[h\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\ln\frac{\Gamma(1/2+\widetilde{\epsilon}(\theta)/2h)}{\sqrt{2\pi}} + \frac{1}{2}\Big(\epsilon+\frac{r}{2}\Big)\ln h + \mathrm{const}\Big]. \tag{1.41}$$

This formula is valid for any anisotropy parameter.

1.4.3 Fluctuation Magnetization of Layered Superconductor and its Crossovers

Direct derivation of (1.41) over magnetic field gives for fluctuation part of magnetization:

$$M_{\mathrm{LD}}(\epsilon,h;r) = -\frac{T}{\Phi_0 s}\int_0^{\pi/2}\frac{\mathrm{d}\phi}{\pi/2}\left\{\frac{\epsilon+r\sin^2\phi}{2h}\left[\psi\left(\frac{\epsilon+r\sin^2\phi}{2h}+\frac{1}{2}\right)-1\right] - \ln\Gamma\left(\frac{\epsilon+r\sin^2\phi}{2h}+\frac{1}{2}\right)+\frac{1}{2}\ln(2\pi)\right\}.$$

Handling with the Hurvitz zeta functions the general formula for an arbitrary magnetic field in $3D$ case ($\epsilon < r$) can be carried out [44, 55]:

$$M_{(3)}(\epsilon \ll r,h) = 3\frac{T}{\Phi_0 s}\left(\frac{2}{r}\right)^{1/2}\sqrt{h} \times\left[\zeta\left(-\frac{1}{2},\frac{1}{2}+\frac{\epsilon}{2h}\right)-\zeta\left(\frac{1}{2},\frac{1}{2}+\frac{\epsilon}{2h}\right)\frac{\epsilon}{6h}\right], \tag{1.42}$$

while in the opposite case of extremely high anisotropy $r < |\epsilon|, h \ll 1$ one obtains the $2D$ result.

Let us comment on the different crossovers in the $M(\epsilon, H)$ field dependence. Let us fix the temperature $\epsilon \ll r$. In this case, the c-axis coherence length exceeds the interlayer distance ($\xi_z \gg s$) and in the absence of a magnetic field the fluctuation Cooper pairs motion has a $3D$ character. For weak fields ($h \ll \epsilon$), the magnetization grows linearly with magnetic field, justifying our preliminary qualitative results:

$$M_{(3)}(\epsilon \ll r, h\to 0) = -\frac{\mathrm{e}^2 TH}{6\pi}\xi_{\mathrm{xy}}(\epsilon). \tag{1.43}$$

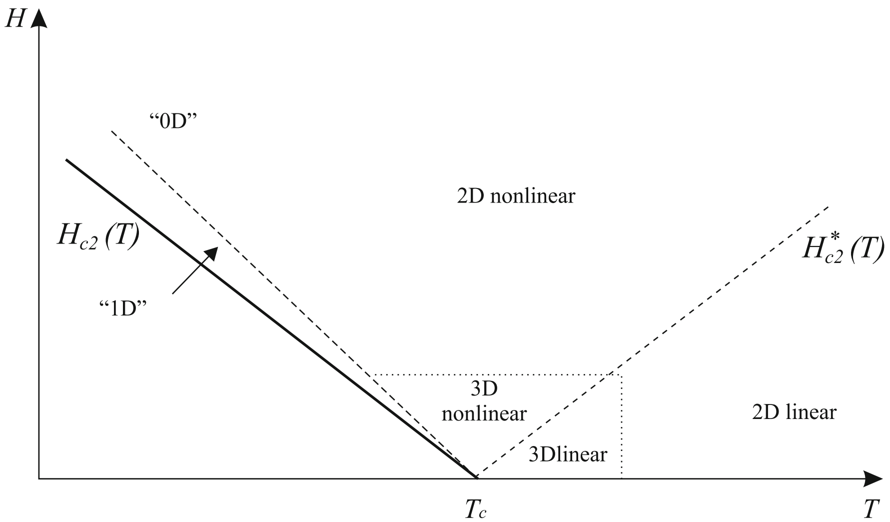

Fig. 1.4 Schematic representation of the different regimes for fluctuation magnetization in the (H, T) diagram. The line $H^*_{c2}(T)$ is mirror-symmetric to the $H_{c2}(T)$ line with respect to a y-axis passing through $T = T_c$. This line defines the crossover between linear and nonlinear behavior of the fluctuation magnetization above T_c [56]

Nevertheless, this linear growth is changed to the nonlinear $3D$ high field regime $M \sim \sqrt{H}$ already in the region of a relatively small fields $H_{c2}(\epsilon) \lesssim H$ $(\epsilon \lesssim h)$ (see Fig. 1.4). The further increase of magnetic field at $h \sim r$ leads to the next $3D \to 2D$ crossover in the magnetization field dependence. In the limit $\epsilon \ll h$, magnetization saturates at the value M_∞.

The substitution of $\epsilon = 0$ gives the result typical of $2D$ superconductors. Therefore, at $h \sim r$ we have a $3D \to 2D$ crossover in $M(H)$ behavior in spite of the fact that all sizes of fluctuation Cooper pair exceed considerably the lattice parameters. Let us stress that this crossover occurs in the region of already strongly nonlinear dependence of $M(H)$ and therefore for a rather strong magnetic field from the experimental point of view in HTS.

Let us mention the particular case of strong magnetic fields $\epsilon \ll h$ (1.42) reproduces the result by Prange [42] with an anisotropy correction multiplier [55] $\xi_{xy}(0)/\xi_z(0)$:

$$M_{(3)}(0, h) = -\frac{0.32T}{\Phi_0^{3/2}} \frac{\xi_{xy}(0)}{\xi_z(0)} \sqrt{H}. \tag{1.44}$$

Near the line of the upper critical field ($h_{c2}(\epsilon) = -\epsilon$), the contribution of the term with $n = 0$ in the sum (1.37) becomes the most important and for the magnetization the expression

$$M(h) = -0.346 \left(\frac{T}{\Phi_0 s}\right) \frac{h}{\sqrt{(h - h_{c2}(\epsilon))(h - h_{c2}(\epsilon) + r)}} \tag{1.45}$$

can be obtained [56]. It contains the already familiar for us "$0D$" regime ($r \ll h - h_{c2} \ll 1$), where the magnetization decreases as $-M(h) \sim \frac{1}{h-h_{c2}}$ (compare with (1.32)), while for $h - h_{c2} << r$ the regime becomes "$1D$" and the magnetization decreases slower, as $-M(h) \sim \frac{1}{\sqrt{h-h_{c2}}}$.

Such an analogy is observed in the next orders in Gi too. In the [57], the analogy was demonstrated for the example of the first eleven terms for the $2D$ case and nine for the $3D$ case. Summation of the series of high-order fluctuation contributions to the heat capacity by the Pade–Borel method resulted in its temperature dependence similar to the $0D$ and $1D$ cases without a magnetic field. Nevertheless, a considerable difference is not to be forgotten: in the $0D$ and $1D$ cases, no phase transition takes place while in the $2D$ and $3D$ cases in a magnetic field a phase transition of first order to the Abrikosov vortex lattice state occurs.

In conclusion, the fluctuation magnetization of a layered superconductor in the vicinity of the transition temperature turns out to be a complicated function of temperature and magnetic field, and it evidently cannot be factorized in these variables. The fit of the experimental data is very sensitive to the anisotropy parameter r and allows determination of the latter with a rather high precision [58, 59]. In Fig. 1.5, the successful application of the described approach to fit the experimental data on $YBa_2Cu_3O_7$ is shown [60].

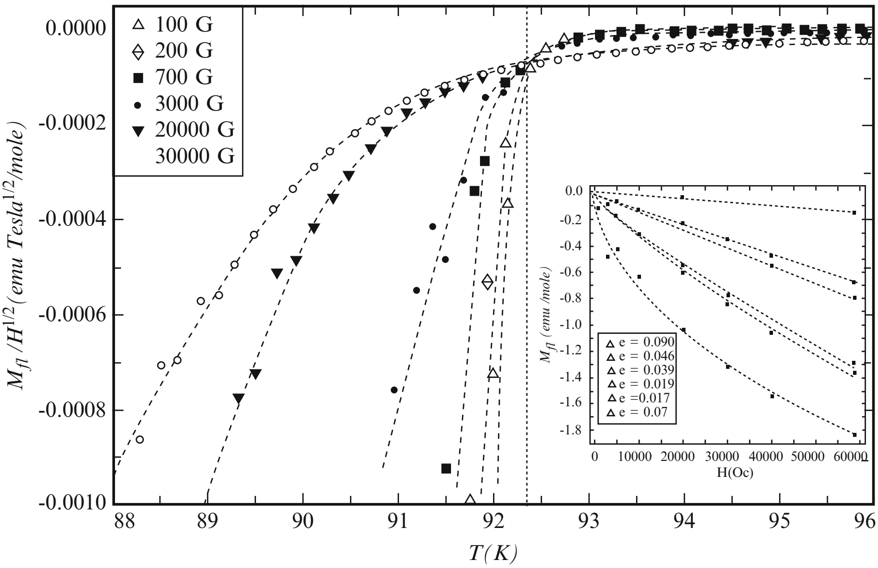

Fig. 1.5 Fluctuation magnetization of a YBaCO123 normalized on $\sqrt{H}$ as the function of temperature in accordance with the described theory shows the crossing of the iso-field curves at $T = T_c(0) = 92.3$ K. The best fit obtained for anisotropy parameter $r = 0.09$. In the inset, the magnetization curves as the function of magnetic field are reported

1.5 Fluctuation Conductivity of Layered Superconductor

The appearance of fluctuating Cooper pairs above T_c leads to the opening of a "new channel" for charge transfer. The fluctuation Cooper pairs were treated above as carriers with charge $2e$, while their lifetime τ_{GL} was chosen to play the role of the scattering time in the Drude formula. Such a qualitative consideration results in the Aslamazov–Larkin (AL) pair contribution to conductivity (1.10) (the so-called paraconductivity [61]). Below we will present the generalization of the phenomenological GL functional approach to transport phenomena. Dealing with the fluctuation order parameter, it is possible to describe correctly the paraconductivity-type fluctuation contributions to the normal resistance and magnetoconductivity, Hall effect, thermoelectric power, and thermal conductivity at the edge of the transition. Unfortunately, the indirect fluctuation contributions are beyond the possibilities of the description by time-dependent GL (TDGL) approach, and they can be calculated only in the framework of the microscopic theory (see below).

1.5.1 Time-Dependent GL Equation

In previous sections, we have demonstrated how the GL functional formalism allows one to accounting for fluctuation corrections to thermodynamic quantities. Let us discuss the effect of fluctuations on the transport properties of a superconductor above the critical temperature.

To find the value of paraconductivity, some time-dependent generalization of the GL equations is required. Indeed, the conductivity characterizes the response of the system to the applied electric field. It can be defined as $\mathbf{E} = -\partial\mathbf{A}/\partial t$ but, in contrast to the previous section, $\mathbf{A}$ has to be regarded as being time dependent. The general nonstationary BCS equations are very complicated, even in the limit of slow time and space variations of the field and the order parameter. For our purposes, it will be sufficient, following [62–70], to write a model equation in the vicinity of T_c, which in general correctly reflects the qualitative aspects of the order parameter dynamics and in some cases is exact.

Let us revise the GL functional formalism introduced above. One can see that the derived above stationary GL equations do not describe correctly the superconductive properties when a deviation from equilibrium is assumed. Indeed, in the absence of equilibrium, the order parameter Ψ becomes time dependent and this in no way was included in the scheme. Nevertheless, the scheme can be improved. For small deviations from the equilibrium, it is natural to assume that in the process of order parameter relaxation its time derivative $\partial\Psi/\partial t$ is proportional to the variational derivative of the free energy $\delta\mathcal{F}/\delta\Psi^*$, which is equal to zero at the equilibrium. But this is not all: side by side with the normal relaxation of the order parameter the effect of thermodynamic fluctuations on it has to be taken into account. This can be done by the introduction, the Langevin forces $\zeta(\mathbf{r}, t)$ in the right-hand side of the equation describing the order parameter dynamics. Finally, gauge invariance requires that $\partial\Psi/\partial t$ should be included in the equation in the combination $\partial\Psi/\partial t + 2ie\varphi\Psi$, where φ is the scalar potential of the electric field. By including all these considerations, one can write the model time-dependent GL equation in the form

$$-\gamma_{\mathrm{GL}}\left(\frac{\partial}{\partial t} + 2ie\varphi\right)\Psi = \frac{\delta\mathcal{F}}{\delta\Psi^*} + \zeta(\mathbf{r},t) \tag{1.46}$$

with the GL functional $\mathcal{F}$ determined by (1.18), (1.29), (1.34) [71]. The dimensionless coefficient γ_{GL} in the left-hand-side of the equation can be related to pair lifetime τ_{GL} (1.1): $\gamma_{\mathrm{GL}} = \alpha T_{\mathrm{c}}\epsilon\tau_{\mathrm{GL}} = \pi\alpha/8$ by the substitution in (1.46) of the first term of (1.18) only [72].

Neglecting the fourth-order term in the GL functional, (1.46) can be rewritten in operator form as

$$[\widehat{L}^{-1} - 2ie\gamma_{\mathrm{GL}}\varphi(r,t)]\Psi(\mathbf{r},t) = \zeta(\mathbf{r},t) \tag{1.47}$$

with the TDGL operator $\widehat{L}$ and Hamiltonian $\widehat{\mathcal{H}}$ defined as

$$\widehat{L} = \left[\gamma_{\mathrm{GL}}\frac{\partial}{\partial t} + \widehat{\mathcal{H}}\right]^{-1}, \widehat{\mathcal{H}} = \alpha T_{\mathrm{c}}\left[\epsilon - \widehat{\xi}^2(\widehat{\nabla} - 2ie\mathbf{A})^2\right]. \tag{1.48}$$

We have introduced here the formal operator of the coherence length $\widehat{\xi}$ to have the possibility to deal with an arbitrary type of spectrum. For example, in the most interesting case for our applications to layered superconductors, the action of this operator is defined by (1.34).

In the absence of an electric field, one can write the formal solution of (1.47) as

$$\Psi^{(0)}(\mathbf{r},t) = \widehat{L}\zeta(\mathbf{r},t). \tag{1.49}$$

The correlator of the Langevin forces introduced above must satisfy the fluctuation–dissipation theorem. This requirement is fulfilled if the Langevin forces $\zeta(\mathbf{r},t)$ and $\zeta^*(\mathbf{r},t)$ are correlated by the Gaussian white-noise law

$$\langle\zeta^*(\mathbf{r},t)\zeta(\mathbf{r}',t')\rangle = 2T\,\mathrm{Re}\,\gamma_{\mathrm{GL}}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{1.50}$$

The fundamental solution $L(\mathbf{p},\Omega)$ can be found by making a Fourier transform of (1.48), what gives:

$$L(\mathbf{p},\Omega) = (-i\gamma_{\mathrm{GL}}\Omega + \varepsilon_{\mathbf{p}})^{-1}. \tag{1.51}$$

with

$$\varepsilon_{\mathbf{p}} = \alpha T_{\mathrm{c}}(\epsilon + \widehat{\xi}^2\mathbf{p}^2) \tag{1.52}$$

as the fluctuation Cooper pair energy spectrum.

1.5.2 General Expression for Paraconductivity

By means of the qualitative consideration based on the Drude formula, we obtained in the Introduction the expression for paraconductivity, which correctly reflects its temperature singularity in any dimension. Following this way, one could write down some kind of master equation for fluctuation Cooper pairs and obtain indeed the precise expression for paraconductivity (see [1]). Unfortunately, the applicability of the derived master equation is restricted to relatively weak electric and magnetic fields. For stronger fields $H_{c2}(\epsilon) \lesssim H \ll H_{c2}(0)$, the density matrix has to be introduced and the master equation loses its attractive simplicity. At the same time, as we already know, these fields, quantizing the fluctuation Cooper pair motion, present special interest. That is why to include in the scheme the magnetic field and frequency dependencies of the paraconductivity, we return to the analysis of the general TDGL equation (1.46) without the objective to reduce it to a Boltzmann-type transport equation.

Let us solve it in the case, when the applied electric field can be considered as a perturbation. The method will much resemble an exercise from a course on quantum mechanics. To impose the necessary generality side by side with a formal simplicity of expressions, we will introduce a subscript of the kind $\{i\}$, which includes the complete set of quantum numbers and time. By a repeated subscript, a summation over a discrete and integration over continuous variables (time in particular) is implied.

We will look for the response of the order parameter to a weak electric field applied in the form

$$\Psi_{k_z}(\mathbf{r},t) = \Psi^{(0)}_{\{i\}} + \Psi^{(1)}_{\{i\}}, \tag{1.53}$$

where $\Psi^{(0)}_{\{i\}}$ is determined by (1.49). Substituting this expression into (1.47) and restricting our consideration to linear terms in the electric field, we can write

$$(\widehat{L}^{-1})_{\{ik\}}\Psi^{(1)}_{\{k\}} = 2\mathrm{i}e\gamma_{\mathrm{GL}}\varphi_{\{il\}}\Psi^{(0)}_{\{l\}} \tag{1.54}$$

with the solution in the form

$$\Psi^{(1)}_{\{i\}} = 2\mathrm{i}e\gamma_{\mathrm{GL}}\widehat{L}_{\{ik\}}\varphi_{\{kl\}}\widehat{L}_{\{lm\}}\zeta_{\{m\}}. \tag{1.55}$$

Let us substitute the order parameter (1.53) in the quantum mechanical expression for current:

$$\mathbf{j} = 2e\,\mathrm{Re}\left[\Psi^{(0)*}_{\{i\}}\widehat{\mathbf{v}}_{\{ik\}}\Psi^{(1)}_{\{k\}} + \Psi^{(1)*}_{\{i\}}\widehat{\mathbf{v}}_{\{ik\}}\Psi^{(0)}_{\{k\}}\right], \tag{1.56}$$

where $\widehat{\mathbf{v}}_{\{ik\}}$ is the velocity operator, which can be expressed by means of the commutator of $\mathbf{r}$ with Hamiltonian (1.48):

$$\widehat{\mathbf{v}}_{\{ik\}} = \mathrm{i}\{\widehat{\mathcal{H}}, \mathbf{r}\}_{\{ik\}} \tag{1.57}$$

and average now (1.56) over the Langevin forces. Moving the operator $\widehat{L}^{*}_{\{ki\}}$ from the beginning to the end of the trace, one finds

$$\mathbf{j} = -16T\mathrm{e}^2\,\mathrm{Re}(\gamma_{\mathrm{GL}})\,\mathrm{Im}\{\gamma_{\mathrm{GL}}\widehat{\mathbf{v}}_{\{il\}}\widehat{L}_{\{lm\}}\varphi_{\{mn\}}\widehat{L}_{\{np\}}\widehat{L}^{*}_{\{pi\}}\}. \tag{1.58}$$

Now we choose the representation where the $\widehat{L}_{\{lm\}}$ operator is diagonal (it is evidently given by the eigenfunctions of the Hamiltonian (1.48)):

$$L_{\{m\}}(\Omega) = \frac{1}{-\mathrm{i}\Omega\gamma_{\mathrm{GL}} + \varepsilon_{\{m\}}}, \tag{1.59}$$

where $\varepsilon_{\{m\}}$ are the appropriate energy eigenvalues. Then we assume that the electric field is coordinate independent but is a monochromatic periodic function of time:

$$\varphi(r,t) = -E^{\beta}r^{\beta}\exp(-\mathrm{i}\omega t). \tag{1.60}$$

In doing the Fourier transform in (1.58), one has to remember that the time dependence of the matrix elements $\varphi_{\{mn\}}$ results in a shift of the frequency variable of integration $\Omega \to \Omega - \omega$ in both L-operators placed after $\varphi_{\{mn\}}$ or, what is the same, to a shift of the argument of the previous $\widehat{L}_{\{lm\}}$ for ω:

$$\mathbf{j}^{\alpha}_{\omega} = 16T\mathrm{e}^2\,\mathrm{Re}(\gamma_{\mathrm{GL}})\int\frac{\mathrm{d}\Omega}{2\pi}\Re\{\gamma_{\mathrm{GL}}\widehat{\mathbf{v}}^{\alpha}_{\{il\}}\widehat{L}_{\{l\}}(\Omega+\omega)[-\mathrm{i}r^{\beta}_{\{li\}}]\widehat{L}_{\{i\}}(\Omega)\widehat{L}^{*}_{\{i\}}(\Omega)\}\mathbf{E}^{\beta}, \tag{1.61}$$

where $\Re f(\omega) \equiv [f(\omega) + f^{*}(-\omega)]/2$.

Let us express the matrix element $\mathbf{r}_{\{li\}}$ by means of $\widehat{\mathbf{v}}_{\{li\}}$ using the commutation relation (1.57). One can see that in the representation chosen

$$\widehat{\mathbf{r}}^{\beta}_{\{li\}} = \mathrm{i}\frac{\widehat{\mathbf{v}}^{\beta}_{\{li\}}}{\varepsilon_{\{i\}} - \varepsilon_{\{l\}}} \tag{1.62}$$

and, carrying out the frequency integration in (1.61), finally write for the fluctuation conductivity tensor ($\mathbf{j}^{\alpha}_{\omega} = \sigma^{\alpha\beta}(\omega)\mathbf{E}^{\beta}$):

$$\begin{aligned}&\sigma^{\alpha\beta}(\epsilon, H, \omega)\\ &= 8\mathrm{e}^2T\,\mathrm{Re}(\gamma_{\mathrm{GL}})\sum_{\{i,l\}=0}^{\infty}\Re\left[\gamma_{\mathrm{GL}}\frac{\widehat{\mathbf{v}}^{\alpha}_{\{il\}}\widehat{\mathbf{v}}^{\beta}_{\{li\}}}{\varepsilon_{\{i\}}(\gamma_{\mathrm{GL}}\varepsilon_{\{i\}} + \gamma^{*}_{\mathrm{GL}}\varepsilon_{\{l\}} - \mathrm{i}|\gamma_{\mathrm{GL}}|^2\omega)(\varepsilon_{\{l\}} - \varepsilon_{\{i\}})}\right].\end{aligned} \tag{1.63}$$

This is the most general expression which describes the d.c., galvanomagnetic and high frequency paraconductivity contributions.

The microscopic analysis of the coefficient γ_{GL} demonstrates that its imaginary part $\mathrm{Im}\,\gamma_{\mathrm{GL}}$ usually is much smaller than $\mathrm{Re}\,\gamma_{\mathrm{GL}}$. Its origin can be related to the electron–hole asymmetry or other peculiarities of the electron spectrum. In the case when one is interested in the diagonal effects only it is enough to accept γ_{GL} as real: ($\gamma_{\mathrm{GL}} = \mathrm{Re}\,\gamma_{\mathrm{GL}} = \pi\alpha/8$). In this way, (1.64) can be simplified and after symmetrization of the summation variables the d.c. contribution of fluctuation Cooper pairs to magnetoconductivity takes the form:

$$\sigma^{\alpha\alpha}(\epsilon, H) = \frac{\pi}{2}\alpha \mathrm{e}^2 T \sum_{\{i,l\}=0}^{\infty} \Re\left[\frac{\widehat{\mathbf{v}}^{\alpha}_{\{il\}}\widehat{\mathbf{v}}^{\alpha}_{\{li\}}}{\varepsilon_{\{i\}}\varepsilon_{\{l\}}(\varepsilon_{\{i\}} + \varepsilon_{\{l\}})}\right]. \tag{1.64}$$

Let us demonstrate the calculation of the d.c. paraconductivity in the simplest case of a metal with an isotropic spectrum. In this case, we choose a plane wave representation. By using $\varepsilon_{\mathbf{p}}$ defined by (1.52), one has

$$\widehat{\mathbf{v}}_{\{\mathbf{p}\mathbf{p}'\}} = \mathbf{v}_{\mathbf{p}}\delta_{\mathbf{p}\mathbf{p}'},\ \mathbf{v}_{\mathbf{p}} = \frac{\partial \varepsilon_{\mathbf{p}}}{\partial \mathbf{p}} = 2\alpha T_{\mathrm{c}}\xi^2\mathbf{p}. \tag{1.65}$$

We do not need to keep here the imaginary part of γ_{GL}, which is necessary to calculate particle-hole asymmetric effects only. As a result, one reproduces the AL formula:

$$\sigma^{\alpha\beta}_{(D)} = 2\mathrm{e}^2 T\,\mathrm{Re}\,\gamma_{\mathrm{GL}} \sum_{\mathbf{p}} \frac{\mathbf{v}^{\alpha}_{\mathbf{p}}\mathbf{v}^{\beta}_{\mathbf{p}}}{\varepsilon^3_{\mathbf{p}}} = \delta^{\alpha\beta}\begin{cases} \dfrac{\mathrm{e}^2}{32\xi}\dfrac{1}{\sqrt{\epsilon}}, & \text{3D case},\\ \dfrac{\mathrm{e}^2}{16d}\dfrac{1}{\epsilon}, & \text{2D film, thickness}: d \ll \xi,\\ \dfrac{\pi \mathrm{e}^2\xi}{16S}\dfrac{1}{\epsilon^{3/2}}, & \text{1D wire, cross} - \text{section}: S \ll \xi^2. \end{cases}$$

1.5.3 Paraconductivity of a Layered Superconductor

Let us return to the discussion of our general formula (1.64) for the fluctuation conductivity tensor. A magnetic field directed along the c-axis still allows separation of variables even in the case of a layered superconductor. The Hamiltonian in this case can be written as in (1.36), (1.48):

$$\widehat{\mathcal{H}} = \alpha T_{\mathrm{c}}\left(\epsilon - \xi^2_{\mathrm{xy}}(\nabla_{\mathrm{xy}} - 2\mathrm{i}e\mathbf{A}_{\mathrm{xy}})^2 - \frac{r}{2}(1 - \cos(k_z s)\right). \tag{1.66}$$

It is convenient to work in the Landau representation, where the summation over $\{i\}$ is reduced to one over the ladder of Landau levels $i = 0, 1, 2 \ldots$. (each is degenerate with a density H/Φ_0 per unit square) and integration over the c-axis momentum in the limits of the Brillouin zone. The eigenvalues of the Hamiltonian (1.66) can be written in the form

$$\varepsilon_{\{n\}} = \alpha T_{\rm c}\left[\epsilon + \frac{r}{2}(1-\cos(k_z s)) + h(2n+1)\right] = \varepsilon_{k_z} + \alpha T_{\rm c} h(2n+1), \quad (1.67)$$

where $h = eH/2m\alpha T_{\rm c}$. For the velocity operators one can write

$$\widehat{\mathbf{v}}^{x,y} = \frac{1}{2m}(-{\rm i}\nabla - 2{\rm i}e\mathbf{A})^{x,y};\ \widehat{\mathbf{v}}^{z} = -\frac{\alpha r s}{2}T_{\rm c}\sin(k_z s). \quad (1.68)$$

1.5.4 In-Plane Conductivity

Let us start from the calculation of the in-plane components. The calculation of the velocity operator matrix elements requires some special consideration. First of all, let us stress that the required matrix elements have to be calculated for the eigenstates of a quantum oscillator whose motion is equivalent to the motion of a charged particle in a magnetic field. The commutation relation for the velocity components follows from (1.68) (see [73]):

$$[\widehat{\mathbf{v}}^{x}, \widehat{\mathbf{v}}^{y}] = {\rm i}\frac{eH_z}{2m^2} = \frac{{\rm i}\alpha T_{\rm c}}{m}h. \quad (1.69)$$

To calculate the necessary matrix elements, let us present the velocity operator components in the form of boson-type creation and annihilation operators $\widehat{a}^{+}, \widehat{a}$:

$$\langle l|\widehat{a}|n\rangle = \langle n|\widehat{a}^{+}|l\rangle = \sqrt{n}\delta_{n,l+1},$$

which satisfy the commutation relation $[\widehat{a}, \widehat{a}^{+}] = 1$. We obtain

$$\widehat{\mathbf{v}}^{x,y} = \sqrt{\frac{\alpha T_{\rm c} h}{2m}}\begin{pmatrix} \widehat{a}^{+} + \widehat{a} \\ {\rm i}\widehat{a}^{+} - {\rm i}\widehat{a} \end{pmatrix}.$$

One can check that the correct commutation relation (1.69) is fulfilled and see that the only nonzero matrix elements of the velocity operator are

$$\langle l|\widehat{\mathbf{v}}^{x,y}|n\rangle = \sqrt{\frac{\alpha T_{\rm c} h}{2m}}\begin{pmatrix} \sqrt{l}\delta_{l,n+1} + \sqrt{n}\delta_{n,l+1} \\ {\rm i}\sqrt{l}\delta_{l,n+1} - {\rm i}\sqrt{n}\delta_{n,l+1} \end{pmatrix}. \quad (1.70)$$

Using these relations, the necessary product of matrix elements can be calculated:

$$\langle l|\widehat{\mathbf{v}}^{x}|n\rangle\langle n|\widehat{\mathbf{v}}^{x}|l\rangle = \frac{\alpha T_{\rm c} h}{2m}(l\delta_{l,n+1} + n\delta_{n,l+1}). \quad (1.71)$$

Its substitution to the expression (1.64) gives for the diagonal in-plane component of the paraconductivity tensor

$$\sigma^{\mathrm{xx}}(\epsilon, h)=\frac{\pi\alpha^2 T_{\mathrm{c}}^2 \mathrm{e}^2}{4m} h \sum_{\{n,l\}=0}^{\infty} \Re \frac{(l\delta_{l,n+1}+n\delta_{n,l+1})}{\varepsilon_{\{l\}}\varepsilon_{\{n\}}\left[\varepsilon_{\{l\}}+\varepsilon_{\{n\}}\right]}.$$

Summation over the subscript $\{l\}$ and accounting of the degeneracy of the Landau levels $H/\Phi_0=2m\alpha T_{\mathrm{c}}h/\pi$ (the layer area we assume to be equal one) gives for the diagonal component of the in-plane paraconductivity tensor:

$$\sigma^{\mathrm{xx}}(\epsilon, H)=\frac{\mathrm{e}^2\left(\alpha T_{\mathrm{c}}\right)^3 h^2}{2} \int_{-\frac{\pi}{s}}^{\frac{\pi}{s}} \frac{\mathrm{d}k_z}{2\pi} \sum_{n=0}^{\infty} \Re \frac{n+1}{\varepsilon_{n+1}\varepsilon_n\left(\varepsilon_{n+1}+\varepsilon_n\right)}. \tag{1.72}$$

1.5.5 Out-of Plane Conductivity

The situation with the out-of plane component of paraconductivity turns out to be even simpler because of the diagonal structure of the

$$\widehat{\mathbf{v}}_{\{in\}}^{z}=-\frac{\alpha r s}{2} T_{\mathrm{c}} \sin(k_z s)\times\delta_{\mathrm{in}}\times\delta(k_z-k_{z'}).$$

Taking into account that the Landau state degeneracy, we write

$$\sigma^{\mathrm{zz}}(\epsilon, H)=\frac{\pi\mathrm{e}^2\left(\alpha T_{\mathrm{c}}\right)^3}{32}\left(\frac{sr}{\xi_{\mathrm{xy}}}\right)^2 h \sum_{n=0}^{\infty} \int_{-\frac{\pi}{s}}^{\frac{\pi}{s}} \frac{\mathrm{d}k_z}{2\pi} \Re\left[\frac{\sin^2(k_z s)}{\varepsilon_n^2(k_z)[\varepsilon_n(k_z)]}\right].$$

1.5.6 Analysis of the Limiting Cases

In principle, the expressions derived above give an exact solution for the d.c. paraconductivity tensor of a layered superconductor in a perpendicular magnetic field $H \ll H_{\mathrm{c2}}$ ($h \ll 1$) in the vicinity of the critical temperature ($\epsilon \ll 1$). The interplay of the parameters r, ϵ, h, as we have seen in the example of fluctuation magnetization yields a variety of crossover phenomena.

The simplest and most important results which can be derived are the components of the d.c. paraconductivity of a layered superconductor in the absence of magnetic field. Setting $h \to 0$ one can change the summations over Landau levels into integration and find

$$\sigma^{\mathrm{xx}}(\epsilon, h\to 0, \omega=0)=\frac{\mathrm{e}^2}{16s}\frac{1}{\sqrt{[\epsilon(r+\epsilon)]}}, \tag{1.73}$$

$$\sigma^{zz}(\epsilon, h \to 0, \omega = 0) = \frac{e^2 s}{32\xi_{xy}^2}\left(\frac{\epsilon + r/2}{[\epsilon(\epsilon + r)]^{1/2}} - 1\right). \tag{1.74}$$

For $2D$ case, the sum can be calculated exactly in terms of the ψ-functions:and one finds the expression for the 2D magnetoconductivity:

$$\sigma_{(2)}^{xx}(\epsilon, h) = \frac{e^2}{2s}\frac{1}{\epsilon}F\left(\frac{\epsilon}{2h}\right) = \frac{e^2}{16s}\begin{cases} 1/\epsilon, & h \ll \epsilon \\ 2/h, & \epsilon \ll h \\ 4/(\epsilon + h), & \epsilon + h \to 0 \end{cases}, \tag{1.75}$$

where

$$F(x) = x^2\left[\psi\left(\frac{1}{2} + x\right) - \psi(x) - \frac{1}{2x}\right]. \tag{1.76}$$

The $2D$ AL theory was extended [16, 74] to the high temperature region by taking into account the short-wavelength and dynamic fluctuations. The following universal formula for paraconductivity of a $2D$ superconductor as a function of the generalized reduced temperature $\epsilon = \ln T/T_c$ and magnetic field was obtained [16]:

$$\delta\sigma_{xx}^{AL}(t, h) = \frac{e^2}{\pi}\sum_{m=0}^{\infty}(m+1)\int_{-\infty}^{\infty}\frac{dx}{\sinh^2 \pi x} \times \left\{\frac{\operatorname{Im}^2\mathcal{E}_m}{|\mathcal{E}_m|^2} + \frac{\operatorname{Im}^2\mathcal{E}_{m+1}}{|\mathcal{E}_{m+1}|^2} + \frac{\operatorname{Im}^2\mathcal{E}_{m+1} - \operatorname{Im}^2\mathcal{E}_m}{|\mathcal{E}_m|^2|\mathcal{E}_{m+1}|^2}\operatorname{Re}[\mathcal{E}_m\mathcal{E}_{m+1}]\right\} \tag{1.77}$$

with

$$\mathcal{E}_m \equiv \mathcal{E}_m(\epsilon, h, iz) = \epsilon + \psi\left[\frac{1 + iz}{2} + \frac{2h}{t}\frac{(2m+1)}{\pi^2}\right] - \psi\left(\frac{1}{2}\right), \tag{1.78}$$

and

$$h = \frac{\pi^2}{8\gamma_E}\frac{H}{H_{c2}(0)}. \tag{1.79}$$

In the limit of zero fields, one can find [74]:

$$\sigma_{xx}^{(fl)} = \frac{e^2}{16}\begin{cases} \frac{1}{\epsilon} & \epsilon \ll 1 \\ \frac{0.11}{\epsilon^3} & \epsilon \gtrsim 1 \end{cases}.$$

Here, it is worth making an important comment. The proportionality of the fluctuation magnetoconductivity to h^2 is valid when using the parametrization $\epsilon = (T - T_{c0})/T_{c0}$ only. Often the analysis of the experimental data is carried out by choosing as the reduced temperature parameter $\epsilon_h = (T - T_c(H))/T_c(H)$. At that point, it is important to recognize that the effect of a weak magnetic field on the fluctuation conductivity cannot be reduced to a simple replacement of T_{c0} by $T_c(H)$ in the appropriate formula without the field. In this parametrization, one can get a term in the magnetoconductivity linear in h. Point is that besides the cases of the special specimen geometry, a weak magnetic field shifts the critical temperature linearly. Such linear correction is exactly compensated by the change in the functional dependence of the paraconductivity in magnetic field, and finally it contains the negative quadratic contribution only.

1.5.7 Comparison with the Experiment

Although the in-plane and out-of-plane components of the fluctuation conductivity tensor of a layered superconductor contain the same fluctuation contributions, their temperature behavior may be qualitatively different. In fact, for $\sigma_{xx}^{(fl)}$, the negative contributions are considerably less than the positive ones in the entire experimentally accessible temperature range above the transition, and it is a positive monotonic function of the temperature. Moreover, for HTS compounds, where the pair-breaking is strong and the anomalous MT contribution is in the saturated regime, it is almost always enough to take into account only the paraconductivity to fit experimental data. Some examples of the experimental findings for in-plane fluctuation conductivity of HTS materials can be seen in [75–82].

In Fig. 1.6, the fluctuation part of in-plane conductivity $\sigma_{xx}^{(fl)}$ is plotted as a function of $\epsilon = \ln T/T_c$ on a double logarithmic scale for three HTS samples (the solid line represents the $2D$ AL behavior $(1/\epsilon)$, the dotted line represents the $3D$ one: $3.2/\sqrt{\epsilon}$) [83]. One can see that paraconductivity of the less anisotropic YBCO compound asymptotically tends to the $3D$ behavior $(1/\epsilon^{1/2})$ for $\epsilon < 0.1$, showing the LD crossover at $\epsilon \approx 0.07$; the curve for more anisotropic 2223 phase of BSCCO starts to bend for $\epsilon < 0.03$ while the most anisotropic 2212 phase of BSCCO shows a $2D$ behavior in the whole temperature range investigated. All three compounds show a universal $2D$ temperature behavior above the LD crossover up to the limits of the GL region. It is interesting that around $\epsilon \approx 0.24$ all the curves bend down and in accordance to [74], follow the same asymptotic $1/\epsilon^3$ behavior (dashed line). Finally at the value $\epsilon \approx 0.45$, all the curves fall down indicating the end of the observable fluctuation regime.

In the case of the out-of-plane conductivity, the situation is quite different. Both positive contributions (AL and anomalous MT) are suppressed here by the necessity of the interlayer tunneling, what results in a competition between positive and negative terms. Such concurrence can lead to formation of a maximum in the temperature dependence of the c-axis resistivity. This nontrivial effect of fluctuations on the transverse resistance of a layered superconductor allows a successful fit to the data observed on optimally doped and overdoped HTS samples (see, e.g., Fig. 1.7), where the growth of the resistance still can be treated as a correction. The fluctuation mechanism of the growth of the transverse resistance

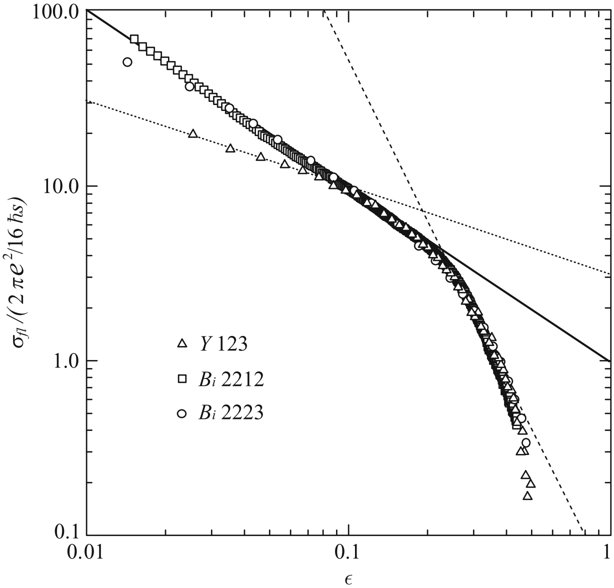

Fig. 1.6 The normalized excess conductivity for samples of YBCO-123 (*triangles*), BSSCO-2212 (*squares*) and BSSCO-2223 (*circles*) plotted against $\epsilon = \ln T/T_c$ on a ln-ln plot as described in [83]. The *dotted and solid lines* are the AL theory in $3D$ and $2D$ respectively. The *dashed line* is the extended theory of [74]

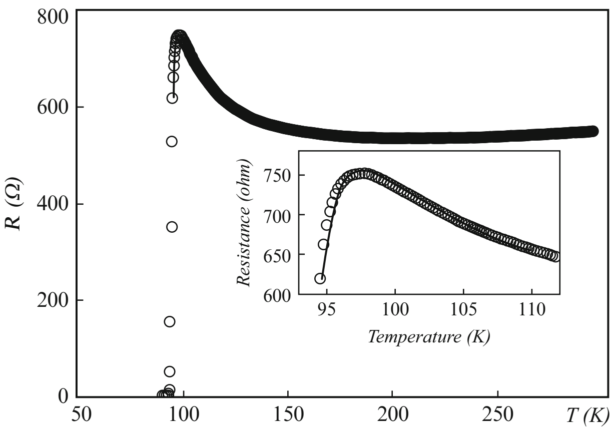

Fig. 1.7 Fit of the temperature dependence of the transverse resistance of a slightly underdoped BSCCO c-axis oriented film with the results of the fluctuation theory [84]. The inset shows the details of the fit in the temperature range between T_c and 110 K

can be easily understood in a qualitative manner. Indeed, to modify the in-plane result for the case of c-axis paraconductivity, one has to take into account the hopping character of the electronic motion in this direction. If the probability of one-electron interlayer hopping is $\mathcal{P}_1$, then the probability of coherent hopping for two electrons during the fluctuation Cooper pair lifetime τ_{GL} is the conditional probability of these two events: $\mathcal{P}_2 = \mathcal{P}_1(\mathcal{P}_1\tau_{\mathrm{GL}})$. The transverse paraconductivity may thus be estimated as $\sigma_{\perp}^{\mathrm{AL}} \sim \mathcal{P}_2\sigma_{\|}^{\mathrm{AL}} \sim \mathcal{P}_1^2\frac{1}{\epsilon^2}$, in complete accordance with the result of microscopic theory. We see that the temperature singularity of $\sigma_{\perp}^{\mathrm{AL}}$ turns out to be stronger than that in $\sigma_{\|}^{\mathrm{AL}}$, however, for a strongly anisotropic layered superconductor $\sigma_{\perp}^{\mathrm{AL}}$ is considerably suppressed by the square of the small probability of interlayer electron hopping, which enters in the prefactor. It is this suppression which leads to the necessity of taking into account the DOS contribution to the transverse conductivity. The latter is less singular in temperature but, in contrast to the paraconductivity, manifests itself in the first, not the second, order in the interlayer transparency $\sigma_{\perp}^{\mathrm{DOS}} \sim -\mathcal{P}_1 \ln(1/\epsilon)$. The DOS fluctuation correction to the one-electron transverse conductivity is negative and, being proportional to the first order of $\mathcal{P}_1$, can completely change the traditional picture of fluctuations just rounding the resistivity temperature dependence around transition. The shape of the temperature dependence of the transverse resistance mainly is determined by the competition between the opposite sign contributions: the paraconductivity and MT term, which are strongly temperature dependent but are suppressed by the square of the barrier transparency and the DOS contribution, which has a weaker temperature dependence but depends only linearly on the barrier transparency.

1.6 Quantum Superconductive Fluctuations Above $H_{c2}(0)$

1.6.1 Dynamic Clustering of FCPs

The qualitative picture for SF in the quantum region at very low temperatures and close to $H_{c2}(0)$ drastically differs from the Ginzburg–Landau one, valid close to T_{c0}. As we saw above, the latter can be described in terms of the set of long-wavelength fluctuation modes (with $\lambda \gtrsim \xi_{\mathrm{GL}}(T) \gg \xi_{\mathrm{BCS}}$) of the order parameter, with the characteristic lifetime $\tau_{\mathrm{GL}} = \pi\hbar/8k_{\mathrm{B}}(T - T_{c0})$. In the former, the order parameter oscillates in much smaller scale, the fluctuation modes with the wave-lengths up ξ_{BCS} are excited. One can imagine that FCPs here rotate in magnetic field with the Larmor radius $\sim \xi_{\mathrm{BCS}}$ and cyclotron frequency $\omega_c \sim \Delta_{\mathrm{BCS}}^{-1}$. The microscopic theory shows below that close to $H_{c2}(0)$ these FCPs form some kind of quantum liquid with the long coherence length $\xi_{\mathrm{QF}} \sim \xi_{\mathrm{BCS}}/\widetilde{h}^{1/2}$ and slow relaxation with the characteristic time

$$\tau_{\mathrm{QF}} \sim \hbar\left(\Delta_{\mathrm{BCS}}\widetilde{h}\right)^{-1}, \widetilde{h} = (H - H_{c2}(0))/H_{c2}(0) \tag{1.80}$$

One sees that the functional form of τ_{QF} is completely analogous to that of τ_{GL}: $\Delta_{BCS} \sim T_{c0}$ and the reduced field $\widetilde{h}$ plays the role of reduced temperature ϵ. Equation (1.80) can also be obtained also from the uncertainty principle. Indeed, the energy, characterizing the proximity to the quantum phase transition is $\Delta E = \hbar\omega_c(H) - \hbar\omega_c(H_{c2}(0)) \sim \Delta_{BCS}\widetilde{h}$ and namely this value should be used in the Heisenberg relation instead of $k_B(T - T_{c0})$, as was done in the vicinity of T_{c0}. The spatial coherence scale $\xi_{QF}\left(\widetilde{h}\right)$ can be estimated from the value of τ_{QF} analogously to consideration near T_{c0}. Namely, two electrons with the coherent phase starting from the same point after the time τ_{QF} get separated by the distance

$$\xi_{QF}\left(\widetilde{h}\right) \sim \left(\mathcal{D}\tau_{QF}\right)^{1/2} \sim \xi_{BCS}/\sqrt{\widetilde{h}}.$$

To clarify the physical meaning of τ_{QF} and ξ_{QF}, note that near the quantum phase transition at zero temperature, where $H \to H_{c2}(0)$, the fluctuations of the order parameter $\Delta^{(fl)}(\mathbf{r}, t)$ become highly inhomogeneous, contrary to the situation near T_{c0}. Indeed, below $H_{c2}(0)$, the spatial distribution of the order parameter at finite magnetic field reflects the existence of Abrikosov vortices with average spacing (close to $H_{c2}(0)$ but in the region where the notion of vortices is still adequate) equal to

$$a(H) = \xi_{BCS}/\sqrt{H/H_{c2}(0)} \to \xi_{BCS}.$$

Therefore, one expects that close to and above $H_{c2}(0)$ the fluctuation order parameter $\Delta^{(fl)}(\mathbf{r}, t)$ also has "vortex-like" spatial structure and varies over the scale of ξ_{BCS} being preserved over the time scale τ_{QF}. In the language of FCPs, one describes this situation in the following way. The FCPs at zero temperature and in magnetic field close to $H_{c2}(0)$ rotates with the Larmor radius $r_L \sim v_F/\omega_c(H_{c2}(0)) \sim v_F/\Delta_{BCS} \sim \xi_{BCS}$, which presents their effective size. During the time τ_{QF}, two initially selected electrons participate in the multiple fluctuating Cooper pairings maintaining their coherence. The coherence length $\xi_{QF}\left(\widetilde{h}\right) \gg \xi_{BCS}$ is thus a characteristic size of a cluster of such coherently rotating FCPs, and τ_{QF} estimates the lifetime of such flickering cluster. One can view the whole system as an ensemble of flickering domains of coherently rotating FCPs, precursors of vortices (see Fig. 1.8).

In view of described qualitative picture of SF in the regime of QPT, let us resume the scenario of Abrikosov lattice defragmentation. Approaching to $H_{c2}(0)$ from below, the paddles of fluctuating vortices, which are nothing else as rotating in magnetic field FCPs, are formed. Their characteristic size is $\xi_{QF}\left(|\widetilde{h}|\right)$, and they flicker with the characteristic time $\tau_{QF}\left(|\widetilde{h}|\right)$. At this stage, the supercurrent still can flow through the sample until these paddles do not break the last percolating superconductive channel. Corresponding field determines the value of the renormalized by QF second critical field: $H_{c2}^{*}(0) = H_{c2}(0)\left[1 - 2\mathrm{Gi}\ln(1/\mathrm{Gi})\right]$ (see [1]). Above this field, no supercurrent can flow through the sample more, that is it is the normal state. Nevertheless, as demonstrate our above estimations its properties are strongly

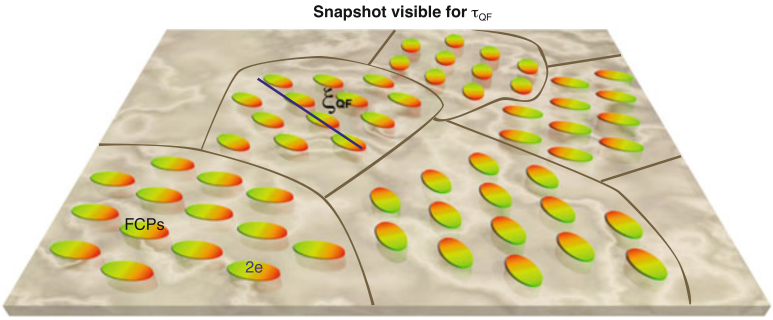

Fig. 1.8 The sketch of cluster structure of fluctuation Cooper pairs above the upper critical field

affected by the QF. The fragments of Abrikosov lattice can be still observed here by the following gedanken experiment. The clusters of rotating FCPs (ex-vortices) of size ξ_{QF} with some kind of the superconductive order should be found at the background of normal metal when one takes the picture with the exposure time shorter than τ_{QF}. When the exposure time is chosen longer than τ_{QF} the picture is smeared out and no traces of Abrikosov vortex state can be found. What kind of the order can be detected is still unclear. It would be attractive to identify these clusters with the splinters of Abrikosov lattice, but more probably this is some kind of quantum FCPs liquid. Indeed, presence of the structural disorder can result in formation close to $H^{*}_{\mathrm{c}2}(0)$ of the hexatic phase, where the translational invariance no longer exists, although it still conserves the oriental order in the vortex positioning.

1.6.2 Manifestation of QF Above $H_{\mathrm{c2}}(0)$

At zero temperature and fields above $H_{\mathrm{c}2}(0)$, the systematics of the fluctuation contributions to the conductivity considerably changes with respect to that close to $T_{\mathrm{c}0}$. The collisionless rotation of FCPs (they do not “feel” the presence of elastic impurities, all information concerning electron scattering is already included in the effective mass of the Cooper pair) results in the lack of their direct contribution to the longitudinal (along the applied electric field) electric transport (analogously to the suppression of one-electron conductivity in strong magnetic fields ($\omega_c\tau \gg 1$): $\delta\sigma^{(\mathrm{e})}_{\mathrm{xx}} \sim (\omega_c\tau)^{-2}$, see [29]) and the AL contribution to $\delta\sigma^{(\mathrm{tot})}_{(2)}$ becomes zero. The anomalous MT and DOS contributions turn zero as well but due to different reasons. Namely, the former vanishes since magnetic fields as large as $H_{\mathrm{c}2}(0)$ completely destroy the phase coherence, whereas the latter disappears since magnetic field suppresses the fluctuation gap in the one-electron spectrum. Therefore, the effect of fluctuations on the conductivity at zero temperature is reduced to the renormalization of the one-electron diffusion coefficient. FCPs here occupy the lowest Landau level, but all the dynamic fluctuations in the interval of frequencies from 0 to Δ_{BCS} should be taken into account:

$$\delta\sigma_{\mathrm{xx}}^{\mathrm{DCR}} \sim -\frac{\mathrm{e}^2}{\Delta_{\mathrm{BCS}}}\int_0^{\Delta_{\mathrm{BCS}}} \frac{\mathrm{d}\omega}{\widetilde{h}+\frac{\omega}{\Delta_{\mathrm{BCS}}}} \sim -\frac{\mathrm{e}^2}{\hbar}\ln\frac{1}{\widetilde{h}}. \tag{1.81}$$

In terms of introduced above QF characteristics τ_{QF} and ξ_{QF} one can understand the meaning of QF contributions to different physical values in the vicinity of $H_{c2}(0)$ and derive others which are required. For example, the physical meaning of (1.86) can be understood as follows: one could estimate the FCPs conductivity by mere replacing $\tau_{\mathrm{GL}} \to \tau_{\mathrm{QF}}$ in the classical AL formula, which would give $\delta\widetilde{\sigma}^{\mathrm{AL}} \sim \mathrm{e}^2\tau_{\mathrm{QF}}$. Nevertheless, as we already noticed, the FCPs at zero temperature cannot drift along the electric field but only rotate around the fixed centers. As temperature deviates from zero, the FCPs can change their state due to the interaction with the thermal bath, that is their hopping to an adjacent rotation trajectory along the applied electric field becomes possible. This means that FCPs now can participate in longitudinal charge transfer. This process can be mapped onto the paraconductivity of a granular superconductors [85] at temperatures above T_{c0}, where the FCPs tunneling between grains occurs in two steps: first one electron jumps, then the second follows. The probability of each hopping event is proportional to the inter-grain tunneling rate $\varGamma$. To conserve the superconductive coherence between both events, the latter should occur during the FCPs lifetime τ_{GL}. The probability of FCPs tunneling between two grains is determined as the conditional probability of two one-electron hopping events and is proportional to $W_{\varGamma} = \varGamma^2\,\tau_{\mathrm{GL}}$. Coming back to the situation of FCPs above $H_{c2}(0)$, one can identify the tunneling rate with temperature T, while τ_{GL} corresponds to τ_{QF}. Therefore, to get a final expression, $\delta\widetilde{\sigma}^{\mathrm{AL}}$ should be multiplied by the probability factor $W_{\mathrm{QF}} = t^2\tau_{\mathrm{QF}}$ of the FCPs hopping to the neighboring trajectory:

$$\delta\sigma_{\mathrm{xx}}^{\mathrm{AL}} \sim \delta\widetilde{\sigma}^{\mathrm{AL}} W_{\mathrm{QF}} \sim \mathrm{e}^2 t^2/\widetilde{h}^2,$$

which corresponds to the asymptotic (1.86).

To estimate the contribution of QF to the fluctuation magnetic susceptibility of the SC in the vicinity of $H_{c2}(0)$, one can apply the Langevin formula to a coherent cluster of FCPs and identifying its average size with the rotator radius to find

$$\chi^{\mathrm{AL}} = \frac{\mathrm{e}^2 n_{\mathrm{c.p.}}}{m_{\mathrm{c.p.}}c}\left\langle \xi_{\mathrm{QF}}^2\left(\widetilde{h}\right)\right\rangle \sim \xi_{\mathrm{BCS}}^2/c\widetilde{h}$$

in complete agreement with the result of [14].

One further reproduces the contribution of QF to the Nernst coefficient. Close to $H_{c2}(0)$ the chemical potential of FCPs can be identified as $\mu_{\mathrm{FCPs}} = \hbar\omega_{\mathrm{c}}(H_{c2}(0)) - \hbar\omega_{\mathrm{c}}(H)$ (as in [15], close to T_{c0}, $\mu_{\mathrm{FCPs}} = k_{\mathrm{B}}(T_{c0} - T)$). Corresponding derivative $\mathrm{d}\mu_{\mathrm{FCPs}}/\mathrm{d}T \sim \mathrm{d}H_{c2}(T)/\mathrm{d}T \sim -T/\Delta_{\mathrm{BCS}}$. Using the relation between the latter and the Nernst coefficient it is possible to reproduce one of the results of [15]:

$$\nu^{\mathrm{AL}} \sim \left[\tau_{\mathrm{QF}}/m_{\mathrm{c.p.}}\right] \mathrm{d}\mu_{\mathrm{FCPs}}/\mathrm{d}T \sim \xi_{\mathrm{BCS}}^{2} t/\widetilde{h}.$$

1.7 Fluctuation Conductivity of 2D Superconductor in Magnetic Field: A Complete Picture

The complete expression for the total fluctuation correction to conductivity $\delta\sigma_{\mathrm{xx}}^{(\mathrm{tot})}(T,H)$ of a disordered 2D SC in a perpendicular magnetic field that holds through all T-H phase diagram above the line $H_{c2}(T)$ is given by the sum [16]:

$$\delta\sigma_{\mathrm{xx}}^{(\mathrm{tot})}(t,h) = \delta\sigma_{\mathrm{xx}}^{\mathrm{AL}} + \delta\sigma_{\mathrm{xx}}^{\mathrm{MT}} + \delta\sigma_{\mathrm{xx}}^{\mathrm{DOS}} + \delta\sigma_{\mathrm{xx}}^{\mathrm{DCR}} \tag{1.82}$$

with $\delta\sigma_{\mathrm{xx}}^{\mathrm{AL}}$ defined by (1.77) and

$$\begin{aligned}
\delta\sigma_{\mathrm{xx}}^{\mathrm{MT(an)}} + \delta\sigma_{\mathrm{xx}}^{\mathrm{MT(reg2)}} &= \frac{\mathrm{e}^2}{\pi}\left(\frac{h}{t}\right)\sum_{m=0}^{M}\frac{1}{\gamma_\phi + \frac{2h}{t}(m+1/2)}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sinh^2\pi x}\frac{\mathrm{Im}^2\mathcal{E}_m}{|\mathcal{E}_m|^2}\\
\delta\sigma_{\mathrm{xx}}^{\mathrm{MT(reg1)}} &= \frac{\mathrm{e}^2}{\pi^4}\left(\frac{h}{t}\right)\sum_{m=0}^{M}\sum_{k=-\infty}^{\infty}\frac{4\mathcal{E}_m''(t,h,|k|)}{\mathcal{E}_m(t,h,|k|)}
\end{aligned} \tag{1.83}$$

$$\delta\sigma_{\mathrm{xx}}^{\mathrm{DOS}} = \frac{4\mathrm{e}^2}{\pi^3}\left(\frac{h}{t}\right)\sum_{m=0}^{M}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sinh^2\pi x}\frac{\mathrm{Im}\mathcal{E}_m\mathrm{Im}\mathcal{E}_m'}{|\mathcal{E}_m|^2} \tag{1.84}$$

$$\delta\sigma_{\mathrm{xx}}^{\mathrm{DCR}} = \frac{4\mathrm{e}^2}{3\pi^6}\left(\frac{h}{t}\right)^2\sum_{m=0}^{M}\left(m+\frac{1}{2}\right)\sum_{k=-\infty}^{\infty}\frac{8\mathcal{E}_m'''(t,h,|k|)}{\mathcal{E}_m(t,h,|k|)}. \tag{1.85}$$

Here, $t = T/T_{\mathrm{c0}}$,

$$h = \frac{\pi^2}{8\gamma_E}\frac{H}{H_{\mathrm{c2}}(0)} = 0.69\frac{H}{H_{\mathrm{c2}}(0)},$$

$\gamma_E = \mathrm{e}^{\gamma_e}$ (γ_e is the Euler constant), $M = (tT_{\mathrm{c0}}\tau)^{-1}, \gamma_\phi = \pi/(8T_{\mathrm{c0}}\tau_\phi)$, τ_ϕ is the phase-breaking time, $\mathcal{E}_m(t,h,z)$ is defined by (1.78) and its derivatives $\mathcal{E}_m^{(p)}(t,h,z) \equiv \partial_z^p\mathcal{E}_m(t,h,z)$. All of them, side by side with the asymptotic expressions for $\delta\sigma_{\mathrm{xx}}^{(\mathrm{tot})}$ are shown in Table 1.1.

Let us start its discussion from the first line, corresponding to the Ginzburg–Landau region of fluctuations close to T_{c0} and in zero magnetic field (domain I). One can see our general expression naturally reproduces the well-known AL, MT, and DOS contributions. The only news here is the written in the explicit form contribution $\delta\sigma^{(\mathrm{DCR})}$, which was usually ignored in view of the lack of its divergency close to T_{c0}. Nevertheless, one can see that its constant contribution $\sim \ln\ln(T_{\mathrm{c0}}\tau)^{-1}$ is necessary for matching of the GL results with the neighbor

Table 1.1 Asymptotic expressions for different fluctuation contributions overall phase diagram

	$\delta\sigma_{\mathrm{xx}}^{\mathrm{AL}}$	$\delta\sigma_{\mathrm{xx}}^{\mathrm{MT}}$	$\delta\sigma_{\mathrm{xx}}^{\mathrm{DOS}}$	$\delta\sigma_{\mathrm{xx}}^{\mathrm{DCR}}$	$\delta\sigma_{\mathrm{xx}}^{\mathrm{tot}}$
I	$\frac{\mathrm{e}^2}{16\epsilon}-\frac{7\zeta(3)\,\mathrm{e}^2}{8\pi^4}\ln\frac{1}{\epsilon}$	$\frac{\mathrm{e}^2}{8(\epsilon-\gamma_\phi)}\ln\frac{\epsilon}{\gamma_\phi}-\frac{14\zeta(3)\,\mathrm{e}^2}{\pi^4}\ln\frac{1}{\epsilon}$	$-\frac{14\zeta(3)\,\mathrm{e}^2}{\pi^4}\ln\frac{1}{\epsilon}$	$\frac{\mathrm{e}^2}{3\pi^2}\ln\ln\frac{1}{T_{c0}\tau}+O(\epsilon)$	$\frac{\mathrm{e}^2}{16\epsilon}+\frac{\mathrm{e}^2}{8(\epsilon-\gamma_\phi)}\ln\frac{\epsilon}{\gamma_\phi}+\frac{\mathrm{e}^2}{3\pi^2}\ln\ln\frac{1}{T_{c0}\tau}$
$I-III$	$\frac{\mathrm{e}^2}{2\epsilon}\left(\frac{\epsilon}{2h}\right)^2\left[\psi\left(\frac{1}{2}+\frac{\epsilon}{2h}\right)-\psi\left(\frac{\epsilon}{2h}\right)-\frac{h}{\epsilon}\right]$	$\frac{\mathrm{e}^2}{8}\frac{1}{\epsilon-\gamma_\phi}\left[\psi\left(\frac{1}{2}+\frac{t\epsilon}{2h}\right)-\psi\left(\frac{1}{2}+\frac{t\gamma_\phi}{2h}\right)\right]-\frac{14\zeta(3)\,\mathrm{e}^2}{\pi^4}\left[\ln\left(\frac{t}{2h}\right)-\psi\left(\frac{1}{2}+\frac{t\epsilon}{2h}\right)\right]$	$-\frac{14\zeta(3)\,\mathrm{e}^2}{\pi^4}\left[\ln\left(\frac{t}{2h}\right)-\psi\left(\frac{1}{2}+\frac{t\epsilon}{2h}\right)\right]$	$\frac{\mathrm{e}^2}{3\pi^2}\ln\ln\frac{1}{T_{c0}\tau}+O\left(\max\left[\epsilon,h^2\right]\right)$	
IV	$\frac{4\mathrm{e}^2\gamma_E^2t^2}{3\pi^2\tilde{h}^2}$	$-\frac{2\mathrm{e}^2}{\pi^2}\ln\frac{1}{\tilde{h}}-\frac{2\gamma_E\mathrm{e}^2}{\pi^2}\left(\frac{t}{\tilde{h}}\right)$	$-\frac{4\mathrm{e}^2\gamma_E^2t^2}{3\pi^2\tilde{h}^2}$	$\frac{4\mathrm{e}^2}{3\pi^2}\ln\frac{1}{\tilde{h}}$	$-\frac{2\mathrm{e}^2}{3\pi^2}\left(\ln\frac{1}{\tilde{h}}+\frac{3t}{\tilde{h}}\right)$
V	$\frac{2\gamma_E\mathrm{e}^2}{\pi^2}\left(\frac{t}{\tilde{h}}\right)$	$-\frac{2\mathrm{e}^2}{3\pi^2}\ln\frac{1}{4\gamma_E t}$	$-\frac{2\gamma_E\mathrm{e}^2}{\pi^2}\left(\frac{t}{\tilde{h}}\right)$	$\frac{4\mathrm{e}^2}{3\pi^2}\ln\frac{1}{4\gamma_E t}$	$-\frac{2\mathrm{e}^2}{3\pi^2}\ln\frac{1}{4\gamma_E t}$
$VI-VII$	$\frac{\mathrm{e}^2}{4}\frac{t}{h-h_{c2}(t)}$	$-\frac{2\mathrm{e}^2}{3\pi^2}\ln\frac{2h}{\pi^2t}$	$-\frac{\mathrm{e}^2}{4}\frac{t}{h-h_{c2}(t)}$	$\frac{4\mathrm{e}^2}{3\pi^2}\ln\frac{2h}{\pi^2t}$	$-\frac{2\mathrm{e}^2}{3\pi^2}\ln\frac{h_{c2}(t)}{h-h_{c2}(t)}$
$VIII$	$\frac{\mathrm{e}^2}{6\pi^2}\frac{C_1}{\ln^3t}$	$-\frac{\mathrm{e}^2}{\pi^2}\ln\frac{\ln\frac{1}{T_{c0}\tau}}{\ln t}+\frac{\pi^2\mathrm{e}^2}{192}\frac{\ln\frac{\pi^2}{2\gamma_\phi}}{\ln^2t}$	$-\frac{\pi^2\mathrm{e}^2}{192}\frac{1}{\ln^2t}$	$\frac{\mathrm{e}^2}{3\pi^2}\ln\frac{\ln\frac{1}{T_{c0}\tau}}{\ln t}$	$-\frac{2\mathrm{e}^2}{3\pi^2}\ln\frac{\ln\frac{1}{T_{c0}\tau}}{\ln t}$
IX	$\frac{\pi^2\mathrm{e}^2}{192}\left(\frac{t}{h}\right)^2\frac{C_2}{\ln^3\frac{2h}{\pi^2}}$	$-\frac{\mathrm{e}^2}{\pi^2}\ln\frac{\ln\frac{1}{T_{c0}\tau}}{\ln\frac{2h}{\pi^2}}+\frac{7\zeta(3)\,\pi^2\mathrm{e}^2}{768}\left(\frac{t}{h}\right)^2\frac{1}{\ln^2\frac{2h}{\pi^2}}$	$-\frac{7\zeta(3)\,\pi^2\mathrm{e}^2}{384}\left(\frac{t}{h}\right)^2\frac{1}{\ln^2\frac{2h}{\pi^2}}$	$\frac{\mathrm{e}^2}{3\pi^2}\ln\frac{\ln\frac{1}{T_{c0}\tau}}{\ln\frac{2h}{\pi^2}}$	$-\frac{2\mathrm{e}^2}{3\pi^2}\ln\frac{\ln\frac{1}{T_{c0}\tau}}{\ln\frac{2h}{\pi^2}}-\frac{7\zeta(3)\,\pi^2\mathrm{e}^2}{768}\left(\frac{t}{h}\right)^2\frac{1}{\ln^2\frac{2h}{\pi^2}}$

domains YIII, IX. The domains II-III are still described by the GL theory in weak magnetic fields and $\delta\sigma_{\mathrm{xx}}^{(\mathrm{tot})}(t,h)$ reproduces all available in literature asymptotic expressions.

What is really surprising in the Table 1.1 is the domain IV, the region of quantum fluctuations (see Fig. 1.1). Looking at the third line, one finds that the positive AL (anomalous MT contributions here is equal to the AL one) decays with the decrease of temperature as T^2. Moreover, it is exactly canceled by the negative contribution of the four DOS-like diagrams 3–6:

$$\delta\sigma_{\mathrm{xx}}^{\mathrm{AL}} = \delta\sigma_{\mathrm{xx}}^{\mathrm{MT(an)}} = -\delta\sigma_{\mathrm{xx}}^{\mathrm{DOS}} = \frac{4\mathrm{e}^2\gamma_E^2 t^2}{3\pi^2\tilde{h}^2}. \tag{1.86}$$

The total fluctuation contribution to conductivity $\delta\sigma_{\mathrm{xx}}^{(\mathrm{tot})}$ in this important region ($t \ll \tilde{h}$) is *completely determined by the renormalization of the diffusion coefficient*. It turns out to be negative and at zero temperature diverges logarithmically when the magnetic field approaches $H_{c2}(0)$. The nontrivial fact following from (1.82) is that an increase of temperature at a fixed value of the magnetic field in this domain first results in the further decrease of conductivity

$$\delta\sigma_{\mathrm{xx}}^{(\mathrm{tot})} = -\frac{2\mathrm{e}^2}{3\pi^2}\ln\frac{1}{\tilde{h}} - \frac{6\gamma_E\mathrm{e}^2}{\pi^2}\frac{t}{\tilde{h}} + O\left[\left(\frac{t}{\tilde{h}}\right)^2\right]. \tag{1.87}$$

and only at the confine with the domain V, when $t \sim \tilde{h}$, the total fluctuation contribution $\delta\sigma_{\mathrm{xx}}^{(\mathrm{tot})}$ pass through the minimum and starts to grow. Such nonmonotonic behavior of the of the conductivity close to $H_{c2}(0)$ was multiply observed in experiments [86, 87] (see Fig. 1.9).

The domain Y describes the transition regime between quantum and classical fluctuations, while in the domains YI-YII, extended along the line $H_{c2}(T)$, superconductive fluctuations have already classical (but non-Ginzburg–Landau) character. In all these three regions, one observes the same exact cancellation of the AL and DOS contributions as in the domain IY and $\delta\sigma_{\mathrm{xx}}^{(\mathrm{tot})}$ is determined here by the negative DCR contribution.

Finally, in the peripheric domains YIII-IX the direct positive contribution of fluctuation Cooper pairs (AL) to conductivity decays faster than all other: $\sim \ln^{-3}(T/T_{\mathrm{c0}})$. Let us stress that this exact result is in complete agreement with the high temperature asymptotical expression for the paraconductivity of the clean 2D superconductor. Such agreement seems natural: fluctuation Cooper pairs transport is insensitive to the impurity scattering. Anomalous MT contribution in complete accordance [8, 9] decays as $\sim \ln\gamma_\phi^{-1}/\ln^{-2}(T/T_{\mathrm{c0}})$. Contribution of the diagrams 3–6 also decays as $\ln^{-2}(T/T_{\mathrm{c0}})$, but without the large factor $\ln\gamma_\phi^{-1}$. Finally, the regular MT contribution and that one of the diagrams 7–10 decay extremely slow, double logarithmically:

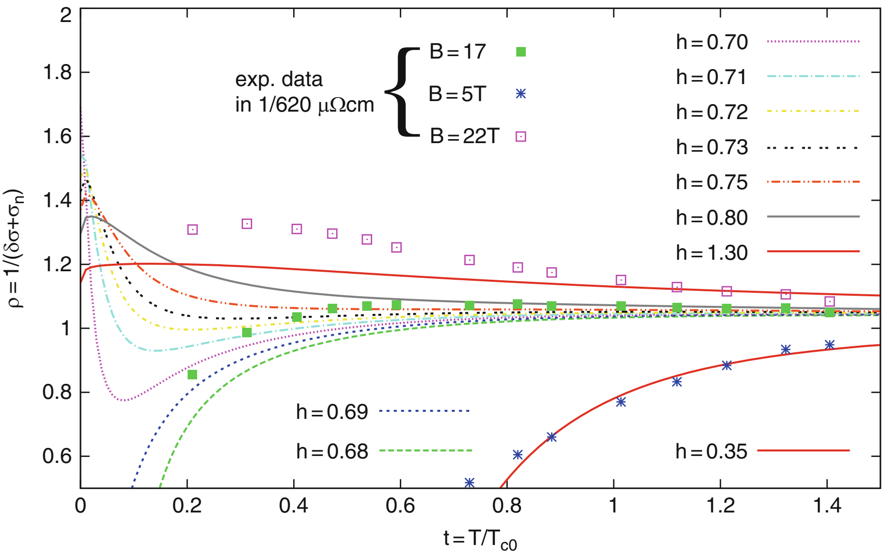

Fig. 1.9 Temperature dependence of the FC at different fields close to $H_{c2}(0)$ and comparison to experimental data for thin films of $La_{2-x}Sr_xCuO_4$ with $T_{c0} \approx 19K$ and $B_{c2}(0) \approx 15T$ (Data is courtesy of B. Leridon, unpublished). Note that for the theoretical curves a fixed $T_{c0}\tau_\phi = 10$ is used, which does not necessarily agree with the experimental value. Nevertheless, the overall behavior can be captured by this rough comparison. All curves are numerically calculated with $T_{c0}\tau = 0.01$

$$\delta\sigma_{\text{xx}}^{(\text{DCR})} = -\frac{2\text{e}^2}{3\pi^2}\left(\ln\ln\frac{1}{T_{\text{c}0}\tau} - \ln\ln\frac{T}{T_{\text{c}0}}\right). \tag{1.88}$$

Up to the numerical factor, this expression coincides with the results [8, 10].

Equation (1.82) gives the background for the "fluctuoscopy" of superconductors, that is extraction of its microscopic parameters from the analysis of fluctuation corrections. Indeed, one can see that $\delta\sigma_{\text{xx}}^{(\text{tot})}$ depends on two superconductive parameters: $T_{\text{c}0}, H_{c2}(0)$, the elastic scattering time τ, and magnetic field and temperature dependent phase-breaking time $\tau_\phi(T, H)$. The elastic scattering time can be obtained from the normal state properties of superconductor, while (1.82) can become the instrument of precise definition of the critical temperature $T_{\text{c}0}$ (instead of the often "half width of transition") and $H_{c2}(0)$. Moreover, it can be invaluable tool for the study of the temperature and magnetic field dependencies of the phase-breaking time $\tau_\phi(T, H)$.

The characteristic example of the surface $\delta\sigma_{\text{xx}}^{(\text{tot})}(T, H)$ for $T_{\text{c}0}\tau = 0.1$ and $T_{\text{c}0}\tau_\phi = 0.01$ is presented in Fig. 1.10. The value of τ_ϕ determines the behavior of fluctuation corrections only in the region of low fields. Figure 1.10 is convenient to analyze together with Fig. 1.1 where the lines $\delta\sigma_{\text{xx}}^{(\text{tot})}(T, H) = \text{const}$ through all phase diagram are shown. One sees that FC is positive only in the domain

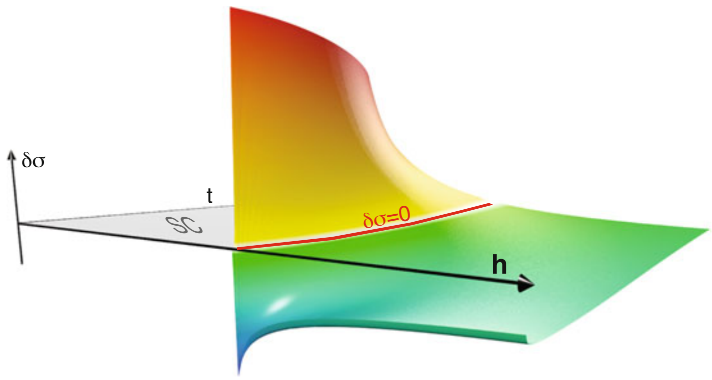

Fig. 1.10 FC as the function of the reduced temperature T/T_{c0} and magnetic field $H/H_{c2}(0)$

restricted by the lines $H_{c2}(T)$ and $\delta\sigma_{xx}^{(tot)}(T, H) = 0$ and is negative through all other parts of the phase diagram. With the growth of the magnetic field, the width of the domain where $\delta\sigma_{xx}^{(tot)}(T, H) > 0$ shrinks and turns zero close to $H_{c2}(0)$. The behavior of FC at low temperatures, in accordance with our asymptotic analysis, becomes nonmonotonic, the surface $\delta\sigma_{xx}^{(tot)}(T, H)$ here has trough-shaped form. It is interesting to note that the numerical analysis of (1.82) shows that the logarithmic asymptotic (1.87) is valid only within the extremely narrow field range $\tilde{h} \lesssim 10^{-6}$.

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References

1. A.I. Larkin, A.A. Varlamov, *Theory of Fluctuations in Superconductors*, 2nd edn (Oxford University Press, USA, 2009)
2. L.G. Aslamazov, A.I. Larkin, Soviet Solid State Phys. **10**, 875 (1968)
3. K. Maki, Prog. Theor. Phys. **39**, 897
4. K. Maki, Prog. Theor. Phys. **40**, 193 (1968)
5. R.S. Thompson, Phys. Rev. B **1**, 327 (1970)
6. L.B. Ioffe, A.I. Larkin, A.A. Varlamov, Yu. Lu, Phys. Rev. B **47**, 8936 (1993)
7. V.V. Dorin, R.A. Klemm, A.A. Varlamov, A.I. Buzdin, D.V. Livanov, Phys. Rev. B **48**, 12591 (1993)
8. L.G. Aslamazov, A.A. Varlamov, J. Low Temp. Phys. **38**, 223 (1980)
9. A.I. Larkin, JETP Lett. **31**, 219 (1980)
10. B.L. Altshuler, M.Yu. Reyzer, A.A. Varlamov, Soviet JETP **57**, 1329 (1983)
11. J.M.B. Lopes dos Santos, E. Abrahams, Phys. Rev. B **31**, 172 (1985)
12. I.S. Beloborodov, K.B. Efetov, Phys. Rev. Lett. **82**, 3332 (1999)
13. I.S. Beloborodov, K.B. Efetov, A.I. Larkin, Phys. Rev. B **61**, 9145 (2000)
14. V.M. Galitski, A.I. Larkin, Phys. Rev. Lett. **87**, 087001 (2001)
15. M.A. Skvortsov, M. Serbin, A.A. Varlamov, V. Galitski, Phys. Rev. Lett. **102**, 067001, (2009)

16. A. Glatz, A.A. Varlamov, V.M. Vinokur, Europhys. Lett. **94**, 47005 (2011)
17. J. Bardeen, L.N. Cooper, J.R. Schrieffer, Phys. Rev. **106**, 162 (1957)
18. J. Bardeen, L.N. Cooper, J.R. Schrieffer, Phys. Rev. **108**, 1175 (1957)
19. Strictly speaking τ in the majority of future results should be understood as the electron transport scattering time $\tau_{\rm tr}$. Nevertheless, as is well known, in the case of isotropic scattering these values coincide; so for the sake of simplicity we will use hereafter the symbol τ
20. This particle density is defined in the (D)-dimensional space. This means that it determines the normal volume density of pairs in the $3D$ case, the density per square unit in the $2D$ case and the number of pairs per unit length in $1D$. The real $3D$ concentration N can be defined too: $N = N_s^{(2)}/d$, where d is the thickness of the film and $N = N_s^{(1)}/S$, where S is the wire cross-section
21. This formula is valid for the dimensionalities $D = 2, 3$, when the fluctuation Cooper pair has the ability to "rotate" in the applied magnetic field and the average square of the rotation radius is $< R^2 >\sim \xi^2(T)$. "Size" effects, important for low-dimensional samples, will be discussed later on
22. Z.A. Xu, et al., Nature **406**, 486 (2000)
23. P.W. Anderson, arXiv:cond-mat/0603726.
24. D. Podolsky, S. Raghu, A. Vishwanath, Phys. Rev. Lett. **99**, 117004 (2007)
25. A. Pouret, et al., Phys. Rev. Lett. **96**, 176402 (2006)
26. I. Ussishkin, S.L. Sondhi, D.A. Huse, Phys. Rev. Lett. **89**, 287001 (2002)
27. A. Levchenko, M. Norman, A.A. Varlamov, Phys. Rev. B **83**, 020506 (R) (2011)
28. A.A. Varlamov, A. Kavokin, Europhys. Lett. **87**, 47007 (2009)
29. A.A. Abrikosov, *Fundamentals of the Theory of Metals* (North Holland, 1988)
30. Hereafter $\hbar = k_{\rm B} = c = 1$
31. For simplicity in this subsection the magnetic field is assumed to be zero
32. V.V. Schmidt, in *Proceedings of the 10th International Conference on Low Temperature Physics*, **C2**, p. 205, VINITI, Moscow (1967)
33. T. Tsuboi, T. Suzuki, J. Phys. Soc. Jpn **42**, 654 (1977)
34. The precise value of the effective charge $e^* = 2e$ could not be determined in the framework of the GL phenomenology. It was found in the Gor'kov's microscopic rederivation of their equations
35. For a spherical particle $H_{\rm c2(0)}^{\rm sph}(\epsilon) = \frac{\Phi_0}{\pi d\xi}\sqrt{10\epsilon}$
36. E. Bernardi, et al., Phys. Rev. B **74**, 134509 (2006)
37. W.E. Lawrence, S. Doniach, in *Proceedings* of the 12th International Conference on Low Temperature *Physics*, ed. by E. Kanda, p.361 (Academic Press, Japan, Kyoto, 1971)
38. K. Yamaji, Phys. Lett. A **38**, 43 (1972)
39. L.G. Aslamazov and A.I. Larkin, Zhurnal Eksperimentalnoi i Teoreticheskoi Fisiki **67**, 647 (1973) [Soviet Phys. JETP **40**, 321 (1974)]
40. A. Schmid, Phys. Rev. **180**, 527 (1969)
41. H. Schmidt, Zeitschrift für Physik B **216**, 336 (1968)
42. R.E. Prange, Phys. Rev. B **1**, 2349 (1970)
43. B.R. Patton, V. Ambegaokar, J.W. Wilkins, Solid State Commun. **7**, 1287 (1969)
44. J. Kurkijarvi, V. Ambegaokar, G. Eilenberger, Phys. Rev., B **5**, 868 (1972)
45. J.P. Gollub, M.R. Beasley, R.S. Newbower, M. Tinkham, *Phys. Rev. Lett.* **22**, 1288 (1969)
46. J.P. Gollub, M.R. Beasley, M. Tinkham, Phys. Rev. Lett. **25**, 1646 (1970)
47. J.P. Gollub, M.R. Beasley, R. Callarotti, M. Tinkham, Phys. Rev. B **7**, 3039 (1973)
48. P.A. Lee, S.R. Shenoy, Phys. Rev. Lett. **28**, 1025 (1972)
49. S.P. Farrant, C.E. Gough, Phys. Rev. Lett. **34**, 943 (1975)
50. W.J. Skocpol, M. Tinkham, Rep. Progress Phys. **38**, 1094 (1975)
51. R.S. Thompson, V.Z. Kresin, Modern Phys. Lett. B **2**, 1159 (1988)
52. K.F. Quader, E. Abrahams, Phys. Rev. B **38**, 11977 (1988)
53. W.C. Lee, R.A. Klemm, D.C. Johnson, Phys. Rev. Lett. **63**, 1012 (1989)

54. P. Carretta, A. Lascialfari, A. Rigamonti, A. Rosso, A.A. Varlamov, Phys. Rev. B **61**, 12420 (2000)
55. T.M. Mishonov, E.S. Penev, Int. J. Modern Phys. **14**, 3831 (2000)
56. A.I. Buzdin, V.V. Dorin, in *Fluctuation* phenomena in high temperature *superconductors*, ed. by M. Ausloos, A.A. Varlamov, NATO-ASI Series (Kluwer, Dordrecht, 1997)
57. S. Hikami, A. Fujita, A.I. Larkin, Phys. Rev. B **44**, 10400 (1991)
58. C. Baraduc, A.I. Buzdin, J-Y. Henry, J.P. Brison, L. Puech, Phys. C **248**, 138 (1995)
59. A. Junod, J-Y. Genoud, G. Triscone, Phys. C **294**, 115 (1998)
60. A. Lascialfari, A. Rigamonti, P. Tedesco, A.A. Varlamov, Phys. Rev. B **65**, 144523 (2002)
61. This term may have different origins. First of all, evidently, paraconductivity is analogous to paramagnetism and means excess conductivity. Another possible origin is an incorrect onomatopoeic translation from the Russian "paroprovodimost' " that means pair conductivity
62. A. Schmid, Physik Kondensierter Materie **5**, 302 (1966)
63. C. Caroli, K. Maki, Phys. Rev. **159**, 306 (1967)
64. C. Caroli, K. Maki, Phys. Rev. **159**, 316 (1967)
65. E. Abrahams, T. Tsuneto, Phys. Rev. **152**, 416 (1966)
66. J.W.F. Woo, E. Abrahams, Phys. Rev. **169**, 407 (1968)
67. C. Di Castro, W. Young, Il Nuovo Cimento B **62**, 273 (1969)
68. S. Ullah, A.T. Dorsey, Phys. Rev. B **44**, 262 (1991)
69. S. Ullah, A.T. Dorsey, Phys. Rev. Lett. **65**, 2066 (1990)
70. L.P. Gor'kov, G.M. Eliashberg, Zhurnal Eksperimentalnoi i Teoreticheskoi Fiziki **54**, 612 (1968) [Soviet Phys. JETP **27**, 328 (1968)]
71. An equation of this type was considered by Landau and Khalatnikov in connection with the study of superfluid helium dynamics in early 1950s
72. Account for electron- hole asymmetry leads to the appearance of the imaginary part of γ_{GL} proportional to the derivative $\partial \ln(\rho v^2\tau)/\partial E|_{E_F} \sim \mathcal{O}(1/E_F)$. This is important for such phenomena as fluctuation Hall effect or fluctuation thermopower and, having in mind the writing of the most general formula, we will suppose $\gamma_{\mathrm{GL}} = \pi\alpha/8 + i\, \mathrm{Im}\, \gamma_{\mathrm{GL}}$, where necessary
73. L.D. Landau, E.M. Lifshitz, *Quantum Mechanics. Course of Theoretical Physics,* vol. 3 (Pergamon Press, Oxford, 1978)
74. L. Reggiani, R. Vaglio, A.A. Varlamov, Phys. Rev. B **44**, 9541 (1991)
75. M. Ausloos, Ch. Laurent, Phys. Rev. B **37**, 611 (1988)
76. P.P. Frietas, C.C. Tsuei, T.S. Plaskett, Phys. Rev. B **36**, 833 (1987)
77. M. Hikita, M. Suzuki, Phys. Rev. B **41**, 834 (1990)
78. M. Akinaga, D. Abukay, L. Rinderer, Modern Phys. Lett. **2**, 891 (1988)
79. A. Poddar, P. Mandal, A.N. Das, B. Ghosh, P. Choudhury, Phys. C **159**, 231 (1989)
80. D.H. Kim, A.M. Goldman, J.H. Kang, K.E. Gray, R.T. Kampwirth, Phys. Rev. B **39**, 12275 (1989)
81. G. Balestrino, A. Nigro, R. Vaglio, Phys. Rev. B **39**, 12264 (1989)
82. G. Kumm, K. Winzer, Phys. B **165-166**, 1361 (1990)
83. M.R. Cimberle, C. Ferdeghini, D. Marrè, M. Putti, S. Siri, F. Federici, A.A. Varlamov, Phys. Rev. B **55**, R14745 (1997)
84. G. Balestrino, E. Milani, A.A. Varlamov, Phys. C **210**, 386 (1993)
85. I.V. Lerner, A.A. Varlamov, V.M. Vinokur, Phys. Rev. Lett. **100**, 117003, (2008)
86. B. Leridon, J. Vanacken, T. Wambecq, V. Moshchalkov, Phys. Rev. B **76**, 012503 (2007)
87. V.F. Gantmakher, S.N. Ermolov, G.E. Tsydynzhapov, A.A. Zhukov, T.I. Baturina, JETP Lett. **77**, 498 (2003)

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