Chapter 2
Commutative Theory

2.1 Maximizing Matrices

We invent a class of infinite matrices $A = (a_{jk})_{j,k=0}^{\infty}$ called (p,q)-maximizing; its definition (see Definition 1 in Sect. 2.1.3) is motivated by a number of classical maximal inequalities intimately related with almost sure summation of orthogonal series with respect to Cesàro, Riesz, and Abel summation. The main examples (given in the next section) are matrix products $A = S\Sigma$ and their "diagonal perturbations" $S\Sigma D_{1/\omega}$, where S is a summation process (see (1.5)), $\Sigma = (\sigma_{jk})$ the so-called sum matrix defined by

$$\sigma_{jk} = \begin{cases} 1 & k \leq j \\ 0 & k > j\,, \end{cases} \tag{2.1}$$

and $D_{1/\omega}$ the diagonal matrix with respect to a Weyl sequence ω. Recall that an increasing and unbounded sequence (ω_k) of positive scalars is said to be a Weyl sequence with respect to a summation method $S = (s_{jk})$ whenever for each orthonormal series in $L_2(\mu)$ we have that

$$\sum_k \alpha_k x_k = \lim_j \sum_k s_{jk} \sum_{\ell=0}^{k} \alpha_\ell x_\ell \quad \mu\text{-a. e.} \tag{2.2}$$

provided $\sum_k |\alpha_k \omega_k|^2 < \infty$; as already explained in (1.6) we call a theorem of this type a coefficient test.

Based on Nagy's dilation lemma we in Theorem 1 characterize $(2,2)$-maximizing matrices in terms of orthonormal series in $L_2(\mu)$, a result which later in Sect. 2.2 will turn out to be crucial in order to derive non trivial examples of maximizing matrices from classical coefficient tests. Theorem 2 shows that for $q < p$ every matrix product $S\Sigma$ is (p,q)-maximizing, whereas for $q \geq p$ an

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additional log-term is needed. By Theorem 7 we have that $S\Sigma D_{(1/\log n)}$ is (p,q)-maximizing whenever $q \geq p$. In this context a characterization of (p,q)-maximizing matrices in terms of p-summing and p-factorable operators (Theorems 3 and 4) in combination with Grothendieck's fundamental theorem of the metric theory of tensor products leads to a powerful link between the theory of general orthogonal series and its related L_p-theory (Theorem 5).

Let us once again mention that this first section was very much inspired by Bennett's seminal papers [2] and [3]. Finally, note that some of our proofs at a first glance may look cumbersome (see e.g. Lemma 2), but we hope to convince the reader that our special point of view later, in the noncommutative part of these notes, will be very helpful.

2.1.1 Summation of Scalar Series

For a scalar matrix $S = (s_{jk})_{j,k\in\mathbb{N}_0}$ with positive entries we call a scalar- or Banach space-valued sequence (x_k) S-summable whenever the sequence

$$\Big(\sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} x_\ell\Big)_j \tag{2.3}$$

of linear means of the partial sums of $\sum_k x_k$ (is defined and) converges. The matrix S is said to be a summation method or a summation process if for each convergent series $s = \sum_k x_k$ the sequence of linear means from (2.3) converges to s,

$$s = \lim_j \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} x_\ell\,. \tag{2.4}$$

All results and examples we need on summation methods are contained in the monographs of Alexits [1] and Zygmund [98]. The following simple characterization of summation methods is due to Toeplitz [91].

Proposition 1. *Let $S = (s_{jk})$ be a scalar matrix with positive entries. Then S is a summation method if and only if*

(1) $\lim_j \sum_{k=0}^{\infty} s_{jk} = 1$
(2) $\lim_j s_{jk} = 0$ *for all k*

Moreover, for each Banach space X and each convergent series $s = \sum_k x_k$ in X we have (2.4)*, the limit taken in X.*

Here we will only prove the fact that (1) and (2) are sufficient conditions for S to be a summation method, or more generally, that (1) and (2) imply (2.4) for every series $\sum_k x_k$ in a Banach space X (the necessity of (1) and (2) will not be needed in the following).

Proof. Take a series $s = \sum_{k=1}^{\infty} x_k$ in a Banach space X, and fix some $\varepsilon > 0$. Then there is k_0 such that we have $\| \sum_{\ell=0}^{k} x_\ell - s\| \leq \varepsilon$ for all $k \geq k_0$. Then for any j we have

$$\begin{aligned}
\Big\| \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} x_\ell - s \Big\| \quad &\leq \Big\| \sum_{k=0}^{\infty} s_{jk} \big(\sum_{\ell=0}^{k} x_\ell - s \big) \Big\| + \Big\| s \sum_{k=0}^{\infty} s_{jk} - s \Big\| \\
&\leq \sum_{k=0}^{k_0} s_{jk} \| \sum_{\ell=0}^{k} x_\ell - s \| + \sum_{k=k_0+1}^{\infty} s_{jk} \| \sum_{\ell=0}^{k} x_\ell - s \| + \Big\| s \sum_{k=0}^{\infty} s_{jk} - s \Big\| \\
&\leq \sum_{k=0}^{k_0} s_{jk} \| \sum_{\ell=0}^{k} x_\ell - s \| + \varepsilon \sum_{k=0}^{\infty} s_{jk} + \Big\| s \sum_{k=0}^{\infty} s_{jk} - s \Big\|,
\end{aligned}$$

and hence the conclusion follows from (1) and (2). □

The following are our basic examples:

(1) The identity matrix $\mathrm{id} = (\delta_{jk})$ is trivially a summation method, and obviously (x_k) is summable if and only if it is id-summable.

(2) The matrix $C = (c_{jk})$ given by

$$c_{jk} := \begin{cases} \dfrac{1}{j+1} & k \leq j \\ 0 & k > j \end{cases}$$

is called Cesàro matrix, and for each series $\sum_k x_k$ (in a Banach space X)

$$\sum_{k=0}^{\infty} c_{jk} \sum_{\ell=0}^{k} x_\ell = \frac{1}{j+1} \sum_{k=0}^{j} \sum_{\ell=0}^{k} x_\ell$$

is its jth Cesàro mean. C-summable sequences are said to be Cesàro summable.

(3) For $r \in \mathbb{R}$ define $A_0^r = 1$, and for $n \in \mathbb{N}$

$$A_n^r := \binom{n+r}{n} = \frac{(r+1)\ldots(r+n)}{n!};$$

in particular, we have $A_n^1 = n+1$ and $A_n^0 = 1$. Then for $r > 0$ the matrix $C^r = (c_{jk}^r)$ defined by

$$c_{jk}^r := \begin{cases} \dfrac{A_{j-k}^{r-1}}{A_j^r} & k \leq j \\ 0 & k > j \end{cases}$$

is said to be the Cesàro matrix of order r. Obviously, we have that $C^1 = C$. All entries of C^r are positive, and on account of the well-known formula $\sum_{k=0}^{n} A_k^{r-1} = A_n^r$ and the fact that $A_n^r = O(n^r)$ (see also (2.44) and (2.48)) we have

$$\sum_{k=0}^{j} c_{jk}^r = 1 \quad \text{and} \quad c_{jk}^r \le c\,\frac{(j-k)^{r-1}}{j^r}\,.$$

Hence, by the preceding proposition the matrices C^r form a scale of summation processes. Sequences which are C^r-summable are said to be Cesàro summable of order r.

(4) Let $(\lambda_k)_{k=0}^{\infty}$ be a strictly increasing sequence of positive scalars which converges to ∞, and such that $\lambda_0 = 0$. Then the so-called Riesz matrix R^λ defined by

$$r_{jk}^\lambda := \begin{cases} \dfrac{\lambda_{k+1}-\lambda_k}{\lambda_{j+1}} & k \le j \\ 0 & k > j \end{cases}$$

forms a summation process; indeed

$$\sum_{k=0}^{j} \frac{\lambda_{k+1}-\lambda_k}{\lambda_{j+1}} = \frac{1}{\lambda_{j+1}}\,(\lambda_{j+1}-\lambda_0) = 1\,,$$

and

$$\lim_j \frac{\lambda_{k+1}-\lambda_k}{\lambda_{j+1}} = 0\,.$$

We call R^λ-summable sequences Riesz summable. Note that for $\lambda_j = j$ we have $R^\lambda = C$. Moreover, it is not difficult to see that for $\lambda = (2^j)$ Riesz-summation means nothing else than ordinary summation.

(5) Take a positive sequence (ρ_j) which increases to 1. Then the matrix A^ρ given by

$$a_{jk}^\rho := \rho_j^k(1-\rho_j)$$

obviously defines a summation process. These matrices are called Abel matrices. Recall that a sequence (x_k) is said to be Abel summable whenever the limit

$$\lim_{r\to 1} \sum_{k=0}^{\infty} x_k r^k$$

exists. For $0 < r < 1$ we have

$$\sum_{k=0}^{\infty} x_k r^k = \sum_{k=0}^{\infty} r^k (1-r) \sum_{\ell=0}^{k} x_\ell$$

which justifies our name for A^ρ.

2.1.2 Maximal Inequalities in Banach Function Spaces

As usual $L_p(\mu)$, $1 \le p \le \infty$ denotes the Banach space of all (equivalence classes of) p-integrable functions over a (in general σ-finite and complete) measure space (Ω, Σ, μ) (with the usual modification for $p = \infty$). We write $\ell_p(\Omega)$ whenever Ω is a set with the discrete measure, and ℓ_p for $\Omega = \mathbb{N}_0$ and ℓ_p^n for $\Omega = \{0, \dots, n\}$. The canonical basis vectors are then denoted by $e_i, i \in \Omega$. More generally, we will consider Banach function spaces (sometimes also called Köthe function spaces) $E = E(\mu)$, i.e. Banach lattices of (μ-almost everywhere equivalence classes of) scalar-valued μ-locally integrable functions on Ω which satisfy the following two conditions:

- If $|x| \le |y|$ with $x \in L_0(\mu)$ and $y \in E(\mu)$, then $x \in E(\mu)$ and $\|x\| \le \|y\|$.
- For every $A \in \Sigma$ of finite measure the characteristic function χ_A belongs to $E(\mu)$.

Examples are L_p-, Orlicz, Lorentz, and Marcinkiewicz spaces.

Recall that a vector-valued function $f : \Omega \to X$, where X now is some Banach space, is μ-measurable whenever it is an almost everywhere limit of a sequence of vector-valued step functions. Then

$$E(X) = E(\mu, X)$$

consists of all (μ-equivalence classes of) μ-measurable functions $f : \Omega \to X$ such that $\|f\|_X \in E(\mu)$, a vector space which together with the norm

$$\|f\|_{E(\mu,X)} = \big\| \|f(\cdot)\|_X \big\|_{E(\mu)}$$

forms a Banach space. For $E(\mu) = L_p(\mu)$ this construction leads to the space $L_p(X) = L_p(\mu, X)$ of Bochner integrable functions; as usual $\ell_p(X)$ and $\ell_p^n(X)$ stand for the corresponding spaces of sequences in X.

We now invent two new spaces of families of integrable functions which will give a very comfortable setting to work with the maximal inequalities we are interested in. Let I be a partially ordered and countable index set, $E = E(\mu)$ a Banach function space, and X a Banach space. Then

$$E(X)[\ell_\infty] = E(\mu, X)[\ell_\infty(I)]$$

denotes the space of all families $(f_i)_{i \in I}$ in $E(\mu, X)$ having a maximal function which again belongs to $E(\mu)$,

$$\sup_{i \in I} \|f_i(\cdot)\|_X \in E(\mu) .$$

Together with the norm

$$\|(f_i)\|_{E(X)[\ell_\infty]} := \big\| \sup_{i \in I} \|f_i(\cdot)\|_X \big\|_{E(\mu)}$$

$E(\mu,X)[\ell_\infty(I)]$ forms a Banach space. The following simple characterization will be extremely useful.

Lemma 1. *Let $(f_i)_{i\in I}$ be a family in $E(\mu,X)$. Then $(f_i)_{i\in I}$ belongs to $E(\mu,X)[\ell_\infty(I)]$ if and only if there is a bounded family $(z_i)_{i\in I}$ of functions in $L_\infty(\mu,X)$ and a scalar-valued function $f\in E(\mu)$ such that*

$$f_i = z_i f \text{ for all } i$$

(the pair $((z_i),f)$ is then said to be a factorization of (f_i)). In this case, we have

$$\|(f_i)\|_{E(X)[\ell_\infty]} = \inf \sup_{i\in I} \|z_i\|_\infty \, \|f\|_{E(\mu)}\,,$$

the infimum taken over all possible factorizations.

For the sake of completeness we include the trivial

Proof. Let $(f_i)\in E(\mu,X)[\ell_\infty(I)]$. Put $f:=\sup_i \|f_i(\cdot)\|_X \in E(\mu)$ and define $z_i(w):= f_i(w)/f(w)$ whenever $f(w)\neq 0$, and $z_i(w):=0$ whenever $f(w)=0$. Obviously, $f_i = z_i f$ and $\sup_i \|z_i\|_\infty \le 1$, hence $\|f\|_{E(\mu)} \sup_i \|z_i\| \le \|(f_i)\|_{E(X)[\ell_\infty]}$. Conversely, we have

$$\sup_i \|f_i(\cdot)\|_X \le \sup_i \|z_i\|_\infty \, \|f(\cdot)\|_X \in E(\mu)\,,$$

and hence

$$\|(f_i)\|_{E(X)[\ell_\infty]} \le \sup_i \|z_i\|_\infty \|f\|_{E(\mu)}\,,$$

which completes the argument. $\square$

We will also need the closed subspace

$$E(\mu,X)[c_0(I)] \subset E(\mu,X)[\ell_\infty(I)]\,,$$

all families $(f_i)\in E(\mu,X)[\ell_\infty(I)]$ for which there is a factorization $f_i = z_i f$ with $\lim_i \|z_i\|_{L_\infty(X)} = 0$ and $f\in E(\mu)$; this notation seems now natural since we as usual denote the Banach space of all scalar zero sequences $(x_i)_{i\in I}$ by $c_0(I)$, and $c_0 = c_0(\mathbb{N}_0)$. The following lemma is a simple tool linking the maximal inequalities we are interested in with almost everywhere convergence.

Lemma 2. *Each family $(f_i)\in E(\mu,X)[c_0(I)]$ converges to 0 μ-almost everywhere.*

Again we give the obvious

Proof. Let $f_i = z_i f$ be a factorization of (f_i) with $\lim_i \|z_i\|_{L_\infty(X)} = 0$ and $f\in E(\mu)$, and let (ε_i) be a zero sequence of positive scalars. Clearly, for each i there is a μ-null set N_i such that $\|z_i(\cdot)\|_X \le \|z_i\|_{L_\infty(X)} + \varepsilon_i$ on the complement of N_i. Take an element w in the complement of the set $N := [|f|=\infty] \cup (\cup_i N_i)$. Then for $\varepsilon > 0$ there is i_0 such that $\|z_i\|_{L_\infty(X)} + \varepsilon_i \le \frac{\varepsilon}{|f(w)|}$ for each $i \ge i_0$, and hence $|f_i(w)| = \|z_i(w)\|_X |f(w)| \le (\|z_i\|_{L_\infty(X)} + \varepsilon_i)|f(w)| \le \varepsilon$. $\square$

2.1.3 (p,q)-Maximizing Matrices

Recall that a sequence (x_k) in a Banach space X is said to be unconditionally summable (or equvialently, the series $\sum_k x_k$ is unconditionally convergent) whenever every rearrangement $\sum_k x_{\pi(k)}$ of the series converges. It is well-known that the vector space $\ell_1^{\mathrm{unc}}(X)$ of all unconditionally convergent series in X together with the norm

$$w_1((x_k)) := \sup_{\|\alpha\|_\infty \leq 1} \Big\| \sum_{k=0}^{\infty} \alpha_k x_k \Big\| < \infty .$$

forms a Banach space. More generally, for $1 \leq p \leq \infty$ a sequence (x_k) in a Banach space X is said to be weakly p-summable if for every $\alpha \in \ell_{p'}$ the series $\sum_k \alpha_k x_k$ converges in X, and by a closed graph argument it is equivalent to say that

$$w_p((x_k)) = w_p((x_k), X) := \sup_{\|\alpha\|_{p'} \leq 1} \Big\| \sum_{k=0}^{\infty} \alpha_k x_k \Big\| < \infty .$$

The name is justified by the fact that (x_k) is weakly p-summable if and only if $(x'(x_k)) \in \ell_p$ for each $x' \in X'$, and in this case we have

$$w_p((x_k)) = \sup_{\|x'\| \leq 1} \Big(\sum_k |x'(x_k)|^p \Big)^{\frac{1}{p}} < \infty.$$

The vector space of all weakly p-summable sequences in X together with the norm w_p forms the Banach space $\ell_p^w(X)$ (after the usual modification the case $p = \infty$ gives all bounded sequences). A sequence (x_k) is weakly summable (= weakly 1-summable) whenever the series $\sum_k x_k$ is unconditionally convergent, and the converse of this implication characterizes Banach spaces X which do not contain a copy of c_0. This is e.g. true for the spaces $L_p(\mu)$, $1 \leq p < \infty$.

The following definition is crucial – let $A = (a_{jk})_{j,k \in \mathbb{N}_0}$ be an infinite matrix which satisfies that $\|A\|_\infty := \sup_{jk} |a_{jk}| < \infty$, or equivalently, A defines a bounded and linear operator from ℓ_1 into ℓ_∞ with norm $\|A\|_\infty$.

Definition 1. We say that A is (p,q)-maximizing, $1 \leq p < \infty$ and $1 \leq q \leq \infty$, whenever for each measure space (Ω, μ), each weakly q'-summable sequence (x_k) in $L_p(\mu)$ and each $\alpha \in \ell_q$ we have that

$$\sup_j \Big| \sum_{k=0}^{\infty} a_{jk} \alpha_k x_k \Big| \in L_p(\mu) ,$$

or in other terms

$$\Big(\sum_{k=0}^{\infty} a_{jk} \alpha_k x_k \Big)_{j \in \mathbb{N}_0} \in L_p(\mu)[\ell_\infty] .$$

Note that here all series $\sum_{k=0}^{\infty} a_{jk}\alpha_k x_k$ converge in $L_p(\mu)$. Clearly, by a closed graph argument a matrix A is (p,q)-maximizing if and only if the following maximal inequality holds: For all sequences (x_k) and (α_k) as above

$$\Big\| \sup_j \Big| \sum_{k=0}^{\infty} a_{jk}\alpha_k x_k \Big| \Big\|_p \leq C\|\alpha\|_q w_{q'}((x_k));$$

here $C \geq 0$ is a constant which depends on A, p, q only, and the best of these constants is denoted by

$$\mathrm{m}_{p,q}(A) := \inf C.$$

Our main examples of maximizing matrices are generated by classical summation processes, and will be given in Sect. 2.2. Most of them are of the form

$$A = S\Sigma D_{1/\omega}, \quad a_{jk} := \frac{1}{\omega_k}\sum_{\ell=k}^{\infty} s_{j\ell}, \tag{2.5}$$

where S is a summation process as defined in Sect. 2.1.1, Σ is the so-called sum matrix defined by

$$\sigma_{jk} := \begin{cases} 1 & k \leq j \\ 0 & k > j, \end{cases}$$

and $D_{1/\omega}$ a diagonal matrix with respect to a Weyl sequence ω for S (see again (2.2)). Since each such S can be viewed as an operator on ℓ_∞ (see Proposition 1,(1)), matrices of the form $S\Sigma D_{1/\omega}$ define operators from ℓ_1 into ℓ_∞.

Note that by definition such a matrix $A = S\Sigma D_{1/\omega}$ is (p,q)-maximizing whenever for each measure space (Ω,μ), each weakly q'-summable sequence (x_k) in $L_p(\mu)$ and each $\alpha \in \ell_q$ we have that

$$\sup_j \Big| \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \frac{\alpha_\ell}{\omega_\ell} x_\ell \Big| \in L_p(\mu), \tag{2.6}$$

or in other terms

$$\Big(\sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \frac{\alpha_\ell}{\omega_\ell} x_\ell \Big)_j \in L_p(\mu)[\ell_\infty].$$

Let us once again repeat that by an obvious closed graph argument $A = S\Sigma D_{1/\omega}$ is (p,q)-maximizing if and only if for all sequences (x_k) and (α_k) as in (2.6) we have

$$\Big\| \sup_j \Big| \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \frac{\alpha_\ell}{\omega_\ell} x_\ell \Big| \Big\|_p \leq C\|\alpha\|_q w_{q'}((x_k)),$$

$C \geq 0$ a constant which depends on A, p, q only.

It is not difficult to check (see also Sect. 2.2.6,(6)) that for the transposed A^t of an infinite matrix A the duality relation

$$\mathrm{m}_{p,q}(A) = \mathrm{m}_{q',p'}(A^t) \tag{2.7}$$

holds, and that $\mathrm{m}_{p,q}(A)$ is decreasing in p and increasing in q, i.e. for $p_2 \le p_1$ and $q_1 \le q_2$

$$\mathrm{m}_{p_1,q_1}(A) \le \mathrm{m}_{p_2,q_2}(A) \le \mathrm{m}_{1,\infty}(A) \tag{2.8}$$

(this will also be obtained as a by-product from Theorem 3). Finally, we include a simple lemma which helps to localize some of our coming arguments.

Lemma 3. *Let A be an infinite matrix with $\|A\|_\infty < \infty$, $E(\mu,X)$ a vector-valued Banach function space, and $1 \le p < \infty$, $1 \le q \le \infty$. Then the following are equivalent:*

(1) For each $\alpha \in \ell_q$ and each weakly q'-summable sequence (x_k) in $E(\mu,X)$ we have that

$$\sup_j \Big\| \sum_{k=0}^{\infty} a_{jk}\alpha_k x_k(\cdot) \Big\|_X \in E(\mu).$$

(2) There is a constant $C > 0$ such that for each choice of finitely many scalars $\alpha_0,\dots,\alpha_n$ and functions $x_0 \dots, x_n \in E(\mu,X)$ we have

$$\Big\| \sup_j \big\| \sum_{k=0}^{n} a_{jk}\alpha_k x_k(\cdot) \big\|_X \Big\|_E \le C\|\alpha\|_q w_{q'}(x).$$

In particular, A is (p,q)-maximizing if and only if $\sup_n \mathrm{m}_{p,q}(A_n) < \infty$ where A_n equals A for all entries a_{jk} with $1 \le j,k \le n$ and is zero elsewhere; in this case

$$\mathrm{m}_{p,q}(A) = \sup_n \mathrm{m}_{p,q}(A_n).$$

Proof. Clearly, if (1) holds, then by a closed graph argument (2) is satisfied. Conversely, assume that (2) holds. First we consider the case $q < \infty$. Fix a weakly q'-summable sequence (x_k) in $E(\mu,X)$. By assumption we have

$$\sup_n \big\| \Phi_n : \ell_q^n \longrightarrow E(\mu,X)[\ell_\infty] \big\| = D < \infty,$$

where $\Phi_n\alpha := \big(\sum_k a_{jk}\alpha_k x_k\big)_j$. Hence, by continuous extension we find an operator $\Phi : \ell_q \to E(\mu,X)[\ell_\infty]$ of norm $\le D$ which on all ℓ_q^n's coincides with Φ_n. On the other hand, since (x_k) is weakly q'-summable, the operator

$$\Psi : \ell_q \longrightarrow \prod_{\mathbb{N}_0} E(\mu,X), \ \Psi(\alpha) = \big(\sum_k a_{jk}\alpha_k x_k\big)_j$$

is defined and continuous. Clearly, we have $\Psi = \Phi$ which concludes the proof. If $q = \infty$, then for fixed $\alpha \in \ell_\infty$ there is $D > 0$ such that for all n we have

$$\big\| \Phi_n : (\ell_1^n)^w(E(\mu,X)) \longrightarrow E(\mu,X)[\ell_\infty] \big\| \le D,$$

where now $\Phi_n((x_k)) := \big(\sum_k a_{jk}\alpha_k x_k\big)_j$ (here $(\ell_1^n)^w(E(\mu,X))$ of course stands for the Banach space of all sequences of length $n+1$ endowed with the weak ℓ_1-norm w_1). Since the union of all $(\ell_1^n)^w(E(\mu,X))$ is dense in the Banach space $\ell_1^w(E(\mu,X))$, all weakly summable sequences (x_k) in $E(\mu,X)$, we can argue similarly to the first case. Finally, note that the last equality in the statement of the lemma follows from this proof. □

The definition of (p,q)-maximizing matrices appears here the first time. But as we have already mentioned several times this notion is implicitly contained in Bennett's fundamental work on (p,q)-Schur multipliers from [3]; this will be outlined more carefully in Sect. 2.2.6.

2.1.4 Maximizing Matrices and Orthonormal Series

In this section we state our main technical tool to derive examples of (p,q)-maximizing matrices from classical coefficient tests on almost everywhere summation of orthonormal series and their related maximal inequalities (see (1.6) and (1.9)). This bridge is mainly based on dilation, a technique concentrated in the following lemma. Obviously, every orthonormal system in $L_2(\mu)$ is weakly 2-summable, but conversely each weakly 2-summable sequence is the "restriction" of an orthonormal system living on a larger measure space.

The following result due to Nagy is known under the name dilation lemma; for a proof see e.g. [94, Sect. III.H.19.]. It seems that in the context of almost everywhere convergence of orthogonal series this device was first used in Orno's paper [68].

Lemma 4. *Let (x_k) be a weakly 2-summable sequence in some $L_2(\Omega,\mu)$ with weakly 2-summable norm $w_2(x_k) \le 1$. Then there is some measure space (Ω',μ') and an orthonormal system (y_k) in $L_2(\mu \oplus \mu')$ ($\mu \oplus \mu'$ the disjoint sum of both measures) such that each function x_k is the restriction of y_k.*

The following characterization of $(2,2)$-maximizing matrices in terms of orthonormal series is an easy consequence of this lemma.

Theorem 1. *Let $A = (a_{jk})$ be an infinite matrix such that $\|A\|_\infty < \infty$. Then A is $(2,2)$-maximizing if and only if for each $\alpha \in \ell_2$, for each measure μ and each orthonormal system (x_k) in $L_2(\mu)$*

$$\sup_j \Big| \sum_k a_{jk}\alpha_k x_k \Big| \in L_2(\mu)\,. \tag{2.9}$$

Moreover, in this case $\mathrm{m}_{2,2}(A)$ *equals the best constant* C *such that for each orthonormal series* $\sum_k \alpha_k x_k$ *in an arbitrary* $L_2(\mu)$

$$\Big\| \sup_j \Big| \sum_k a_{jk} \alpha_k x_k \Big| \Big\|_2 \leq C \|\alpha\|_2 . \tag{2.10}$$

Proof. Clearly, if A is $(2,2)$-maximizing, then (2.9) holds and the infimum over all $C > 0$ as is (2.10) is $\leq \mathrm{m}_{2,2}(A)$. Conversely, take $\alpha \in \ell_2$ and a weakly 2-summable sequence (y_k) in $L_2(\Omega, \mu)$; we assume without loss of generality that $w_2(y_k) \leq 1$. By the dilation lemma 4 there is some orthonormal system (x_k) in $L_2(\mu \oplus \mu')$ such that $x_k | \Omega = y_k$ for all k (μ' some measure on some measure space Ω'). We know by assumption that

$$\Big(\sum_k a_{jk} \alpha_k x_k \Big)_j \in L_2(\mu \oplus \mu')[\ell_\infty] .$$

Hence by Lemma 1 there is a bounded sequence (z_j) in $L_\infty(\mu \oplus \mu')$ and some $f \in L_2(\mu \oplus \mu')$ for which $\sum_k a_{jk} \alpha_k x_k = z_j f$ for all j. But then as desired

$$\sup_j \Big| \sum_k a_{jk} \alpha_k y_k \Big| = \sup_j \big| z_j|_\Omega f|_\Omega \big| \in L_2(\mu) .$$

If moreover the constant C satisfies (2.10), then we have

$$\Big\| \sup_j \Big| \sum_k a_{jk} \alpha_k y_k \Big| \Big\|_2 \leq \Big\| \sup_j \Big| \sum_k a_{jk} \alpha_k x_k \Big| \Big\|_2 \leq C \|\alpha\|_2 ,$$

hence $\mathrm{m}_{2,2}(A) \leq C$. □

2.1.5 Maximizing Matrices and Summation: The Case $q < p$

Recall that Σ denotes the sum matrix defined by

$$\sigma_{jk} := \begin{cases} 1 & k \leq j \\ 0 & k > j . \end{cases}$$

The study of (p,q)-maximizing matrices of type $S\Sigma$, where S is a summation process, shows two very different cases – the case $q < p$ and the case $p \leq q$. The next theorem handles the first one, for the second see Theorem 7.

Theorem 2. *Let* $1 \leq q < p < \infty$, *and let* S *be a summation process. Then the matrix* $A = S\Sigma$ *given by*

$$a_{jk} = \sum_{\ell = k}^{\infty} s_{j\ell}$$

is (p,q)*-maximizing.*

This theorem is due to Bennett [2, Theorem 3.3] (only formulated for the crucial case, the sum matrix itself) who points out that the technique used for the proof goes back to Erdös' article [15].

Lemma 5. *Let $1 < q < \infty$, and assume that $c_0, \cdots, c_n$ are scalars such that*

$$|c_0|^q + \ldots + |c_n|^q = s > 0\,.$$

Then there is an integer $0 \le k \le n$ such that

$$|c_0|^q + \ldots + |c_{k-1}|^q + |c'_k|^q \le s/2$$

$$|c''_k|^q + |c_{k+1}|^q + \ldots + |c_n|^q \le s/2\,,$$

where $c_k = c'_k + c''_k$ and $\max\{|c'_k|\,|c''_k|\} \le |c_k|$.

Proof. We start with a trivial observation: Take scalars c, d', d'' where d', d'' are positive and such that $d' \le |c| \le d' + d''$. Then there is a decomposition $c = c' + c''$ such that $|c'| \le d'$ and $|c''| \le d''$; indeed, decompose first the positive number $|c|$, and then look at the polar decomposition of c. Take now k such that

$$|c_0|^q + \ldots + |c_{k-1}|^q \le s/2 < |c_0|^q + \ldots + |c_k|^q\,,$$

and define

$$d'_k := \big(s/2 - |c_0|^q - \ldots - |c_{k-1}|^q\big)^{1/q}$$

$$d''_k := \big(|c_0|^q + \ldots + |c_k|^q - s/2\big)^{1/q} = \big(s/2 - |c_{k+1}|^q - \ldots - |c_n|^q\big)^{1/q}.$$

Since $q > 1$ we deduce from the starting observation that there is a decomposition $c_k = c'_k + c''_k$ with $|c'_k| \le d'_k \le |c_k|$ and $|c''_k| \le d''_k \le |c_k|$ which completes the proof. □

Now we proceed with the proof of Theorem 2.

Proof. Let us first reduce the case of a general S to the special case $S = \mathrm{id}$: since S defines a bounded operator on ℓ_∞, we have that

$$\sup_j \Big|\sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \alpha_\ell x_\ell\Big| \le \|S : \ell_\infty \to \ell_\infty\| \sup_k \Big|\sum_{\ell=0}^{k} \alpha_\ell x_\ell\Big|\,, \tag{2.11}$$

hence we only show that the matrix Σ is (p,q)-maximizing. We may assume that $1 < q < p < \infty$. By Lemma 3 it suffices to prove that there is a constant $c(p,q) > 0$ such that for each n

$$\mathrm{m}_{p,q}(\Sigma_n) \le c(p,q)\,.$$

Fix n, and take $x_0,\dots,x_n$ in some $L_p(\mu)$ with $w_{q'}(x_k)=1$ and scalars $\alpha_0,\dots,\alpha_n$ with $\|\alpha\|_q=1$. We show that

$$\int \sup_j \Big|\sum_{k=0}^{j}\alpha_k x_k\Big|^p d\mu \le c(p,q)\,.$$

To do so use the preceding lemma to split the sum

$$\alpha_0 x_0(\omega)+\ldots+\alpha_n x_n(\omega)$$

into two consecutive blocks

$$B_1^{(1)}=\alpha_0 x_0(\omega)+\ldots+\alpha_{k'}x_{k'}(\omega)$$

$$B_2^{(1)}=\alpha_{k''}x_{k''}(\omega)+\ldots+\alpha_n x_n(\omega)$$

such that each of the q-sums of the coefficients of these blocks is dominated by $1/2$ (split $\|\alpha\|_q^q=1$). Applying the lemma we split each of the blocks into two further blocks $B_1^{(2)},B_2^{(2)}$ and $B_3^{(2)},B_4^{(2)}$, respectively. Repeating this process ν times gives a decomposition of the original sum into 2^ν blocks $B_\lambda^{(\nu)}$, $1\le\lambda\le 2^\nu$, each having coefficient q-sums dominated by $2^{-\nu}$. By choosing ν sufficiently large, we may ensure that

$$2^{-\nu-1}<\min\{|\alpha_k|\,|\,\alpha_k\neq 0\}\,,$$

so that each block $B_\lambda^{(\nu)}$ contains at most two non-zero terms (indeed, otherwise $2\cdot 2^{-\nu-1}<2^{-\nu}$). We then have for each $1\le j\le n$ and all ω that

$$\Big|\sum_{k=0}^{j}\alpha_k x_k(\omega)\Big|\le\sum_{\mu=1}^{\nu}\max_{1\le\lambda\le 2^\mu}\big|B_\lambda^{(\mu)}(\omega)\big|+\max_{0\le k\le n}|\alpha_k x_k(\omega)|\,.$$

Hence, for each r (which will be specified later) we obtain from Hölder's inequality that

$$\begin{aligned}
&\Big|\sum_{k=0}^{j}\alpha_k x_k(\omega)\Big|\\
&\le\sum_{\mu=1}^{\nu}\Big(\sum_{\lambda=1}^{2^\mu}\big|B_\lambda^{(\mu)}(\omega)\big|^p\Big)^{1/p}+\Big(\sum_{k=0}^{n}|\alpha_k x_k(\omega)|^p\Big)^{1/p}\\
&\le\Big(\sum_{\mu=1}^{\nu}2^{-r\mu p'}\Big)^{1/p'}\Big(\sum_{\mu=1}^{\nu}2^{r\mu p}\sum_{\lambda=1}^{2^\mu}\big|B_\lambda^{(\mu)}(\omega)\big|^p\Big)^{1/p}+\Big(\sum_{k=0}^{n}|\alpha_k x_k(\omega)|^p\Big)^{1/p},
\end{aligned}$$

and with $d(p,r) = \left(\sum_{\mu=1}^{\infty} 2^{-r\mu p'}\right)^{1/p'}$ we conclude

$$\Big\| \sup_j \Big| \sum_{k=0}^{j} \alpha_k x_k \Big| \Big\|_p$$

$$\leq d(p,r) \Big(\sum_{\mu=1}^{\nu} 2^{r\mu p} \sum_{\lambda=1}^{2^\mu} \|B_\lambda^{(\mu)}\|_p^p \Big)^{1/p} + \Big(\sum_{k=0}^{n} |\alpha_k|^p \Big)^{1/p};$$

use the Minkowski inequality in $L_p(\mu)$, the obvious fact that for each choice of finitely many functions $y_k \in L_p(\mu)$

$$\Big\| \Big(\sum_k |y_k|^p \Big)^{1/p} \Big\|_p = \Big(\sum_k \|y_k\|_p^p \Big)^{1/p},$$

and finally that all $\|x_k\|_p \leq 1$. By assumption we have that for every choice of finitely many scalars $\beta_0, \cdots, \beta_n$

$$\Big\| \sum_k \beta_k x_k \Big\|_p \leq \|(\beta_k)\|_q,$$

and that $1 \leq q < p < \infty$, hence

$$\Big\| \sup_j \Big| \sum_{k=0}^{j} \alpha_k x_k(\omega) \Big| \Big\|_p \leq d(p,r) \Big(\sum_{\mu=1}^{\nu} 2^{r\mu p} \sum_{\lambda=1}^{2^\mu} 2^{-\mu p/q} \Big)^{1/p} + \Big(\sum_{k=0}^{n} |\alpha_k|^q \Big)^{1/q}$$

$$\leq d(p,r) \Big(\sum_{\mu=1}^{\nu} 2^{r\mu p} \sum_{\lambda=1}^{2^\mu} 2^{-\mu p/q} \Big)^{1/p} + 1$$

$$\leq d(p,r) \Big(\sum_{\mu=1}^{\infty} 2^{(rp+1-p/q)\mu} \Big)^{1/p} + 1.$$

Since this latter term converges for each $0 < r < 1/q - 1/p$, the proof completes.

□

As already mentioned, the counterpart of this result for $q \geq p$ will be stated in Theorem 7.

2.1.6 Banach Operator Ideals: A Repetitorium

A considerably large part for our conceptional approach to almost everywhere summation theorems of unconditionally convergent series in L_p-spaces together with their maximal inequalities will be based on the theory of Banach operator ideals.

We give, without any proofs, a brief summary of the results needed – in particular, we recall some of the ingredients from the theory of p-summing and p-factorable operators. Notes, remarks, and references are given at the end of this section.

An operator ideal $\mathscr{A}$ is a subclass of the class of all (bounded and linear) operators $\mathscr{L}$ between Banach spaces such that for all Banach spaces X and Y its components

$$\mathscr{A}(X,Y) := \mathscr{L}(X,Y) \cap \mathscr{A}$$

satisfy the following two conditions: $\mathscr{A}(X,Y)$ is a linear subspace of $\mathscr{L}(X,Y)$ which contains all finite rank operators, and for each choice of appropriate operators $u,w \in \mathscr{L}$ and $v \in \mathscr{A}$ we have $wvu \in \mathscr{A}$ (the ideal property). A (quasi) Banach operator ideal $(\mathscr{A},\alpha)$ is an operator ideal $\mathscr{A}$ together with a function $\alpha : \mathscr{A} \longrightarrow \mathbb{R}_{\geq 0}$ such that every component $(\mathscr{A}(X,Y), \alpha(\cdot))$ is a (quasi) Banach space, $\alpha(\mathrm{id}_{\mathbb{K}}) = 1$, and for each choice of appropriate operators w, v, u we have that

$$\alpha(wvu) \leq \|w\| \alpha(v) \|u\| .$$

If $(\mathscr{A},\alpha)$ is a Banach operator ideal, then it can be easily shown that

$$\|u\| \leq \alpha(u) \quad \text{for all } u \in \mathscr{A},$$

and for all one dimensional operators $x' \otimes y$ with $x' \in X'$, $y \in Y$

$$\alpha(x' \otimes y) = \|x'\| \, \|y\|.$$

We will only consider maximal Banach operator ideals $(\mathscr{A},\alpha)$, i.e. ideals which in the following sense are determined by their components on finite dimensional Banach spaces: An operator $u : X \longrightarrow Y$ belongs to $\mathscr{A}$ if (and only if)

$$\sup_{M,N} \alpha(M \overset{I_M}{\hookrightarrow} X \xrightarrow{u} Y \xrightarrow{Q_N} Y/N) < \infty, \tag{2.12}$$

where the supremum is taken over all finite dimensional subspaces M of X, all finite codimensional subspaces N of X and I_M, Q_N denote the canonical mappings. The duality theory of operator ideals is ruled by the following two notions, the trace tr for finite rank operators and the so-called adjoint operator ideals $\mathscr{A}^*$. If $(\mathscr{A},\alpha)$ is a Banach operator ideal, then its adjoint ideal $(\mathscr{A}^*,\alpha^*)$ is given by: $u \in \mathscr{A}^*(X,Y)$ if

$$\alpha^*(u) := \sup_{M,N} \sup_{\|v:Y/N \to M\| \leq 1} \mathrm{tr}(Q_M \, u I_M \, v) < \infty$$

(M and N as above); note that this ideal by definition is maximal. If $(\mathscr{A},\alpha)$ and $(\mathscr{B},\beta)$ are two quasi Banach operator ideals, then $\mathscr{A} \circ \mathscr{B}$ denotes the operator ideal of all compositions $u = vw$ with $v \in \mathscr{A}$ and $w \in \mathscr{B}$, together with the quasi norm $\alpha \circ \beta(u) := \inf \alpha(u)\beta(w)$. This gives a quasi Banach operator ideal $(\mathscr{A} \circ \mathscr{B}, \alpha \circ \beta)$, the product of $(\mathscr{A},\alpha)$ and $(\mathscr{B},\beta)$. Let us finally recall the meaning of a transposed ideal $(\mathscr{A}^{\text{dual}},\alpha^{\text{dual}})$: It consists of all $u \in \mathscr{L}$ such that its transposed $u' \in \mathscr{A}$, and $\alpha^{\text{dual}}(u) := \alpha(u')$.

Now we collect some of the most prominent examples of Banach operator ideals. Clearly, all operators on Banach spaces together with the operator norm $\|\cdot\|$ form the largest Banach operator ideal, here denoted by $\mathscr{L}$. The Banach ideal of p-summing operators is one of the fundamental tools of these notes. An operator $u : X \longrightarrow Y$ is said to be p-summing, $1 \leq p < \infty$, whenever there is a constant $c \geq 0$ such that for all weakly p-summable sequences (x_k) in X we have

$$\Big(\sum_{k=1}^{\infty}\|u(x_k)\|^p\Big)^{\frac{1}{p}} \leq c \sup_{\|x'\|\leq 1}\Big(\sum_{k=1}^{\infty}|x'(x_k)|^p\Big)^{\frac{1}{p}} = w_p((x_k)), \tag{2.13}$$

and the best constant c is denoted by $\pi_p(u)$. It can be seen easily that the class Π_p of all such operators together with the norm π_p forms a maximal Banach operator ideal (Π_∞ by definition equals $\mathscr{L}$).

There is also a non-discrete variant of (2.13): *An operator $u : X \longrightarrow Y$ is p-summing if and only if there is a constant $c \geq 0$ such that for any function $v \in L_p(\mu, X)$ (the Bochner p-integrable functions with values in X) we have*

$$\int \|u(v(\omega))\|^p d\mu(\omega) \leq c \sup_{\|x'\|\leq 1}\Big(\int |x'(v(\omega))|^p d\mu(\omega)\Big)^{\frac{1}{p}}, \tag{2.14}$$

and in this case again the best c equals $\pi_p(u)$.

The whole theory of p-summing operators is ruled by Pietsch's domination theorem: *Let X and Y be Banach spaces, and assume that X is a subspace of some $C(K)$, where K is a compact Hausdorff space. Then $u : X \longrightarrow Y$ is p-summing if and only if there is a constant $c \geq 0$ and a Borel probability measure μ on K such that for all $x \in X$*

$$\|u(x)\| \leq c\Big(\int_K |x(w)|^p d\mu(\omega)\Big)^{\frac{1}{p}}, \tag{2.15}$$

and in this case the infimum over all possible c is a minimum and equals $\pi_p(u)$.

This result has many equivalent formulations in terms of factorization – we will need the following particular case: *For every p-summing operator $u : c_0 \longrightarrow Y$ there is a factorization*

$$\begin{tikzcd} c_0 \arrow[r, "u"] \arrow[dr, "D_\alpha"'] & Y \\ & \ell_p \arrow[u, "v"'] \end{tikzcd} \tag{2.16}$$

with a diagonal operator D_α and an operator v satisfying $\|\alpha\|_p\|v\| \leq \pi_p(u)$.

Finally, we mention two basic examples which in view of the preceding two results are prototypical:

(1) *$\pi_p(j : L_\infty(\mu) \hookrightarrow L_p(\mu)) = \mu(\Omega)$, where (Ω,μ) denotes some measure space and j the canonical embedding.*
(2) *$\pi_p(D_\alpha : c_0 \longrightarrow \ell_p) = \|\alpha\|_p$, where D_α denotes the diagonal operator associated to $\alpha \in \ell_p$ (here c_0 can be replaced by ℓ_∞).*

Let us now describe the adjoint ideal Π_p^* of Π_p in the more general context of factorable operators. For $1 \le p \le q \le \infty$ denote by $\Gamma_{p,q}$ the Banach operator ideal of all operators $u : X \longrightarrow Y$ which have a factorization

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "v"] & Y \arrow[r, hook, "\kappa_Y"] & Y'' \\ L_q(\mu) \arrow[r, hook, "j"] & L_p(\mu) \arrow[ur, "w"] & \end{tikzcd} \tag{2.17}$$

where μ is a probability measure and v, w are two operators (clearly, κ_Y and j denote the canonical embeddings). The ideal $\Gamma_{p,q}$ of all so-called (p,q)-factorable operators together with the norm $\gamma_{p,q}(u) := \inf \|w\|\,\|v\|$ forms a maximal Banach operator ideal. For operators $u : X \longrightarrow Y$ between finite dimensional spaces X and Y it can be easily proved that

$$\gamma_{p,q}(u) = \inf \|w\|\,\|D_\mu\|\,\|v\|\,, \tag{2.18}$$

where "the infimum is taken over all possible diagrams" of the form

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "v"] & Y \\ \ell_q^m \arrow[r, "D_\mu"] & \ell_p^m\,. \arrow[u, "w"] \end{tikzcd}$$

Define $\mathscr{I}_p := \Gamma_{p,\infty}$, the class of all p-integral operators, and $\Gamma_p := \Gamma_{p,p}$, the class of all p-factorable operators; note that Γ_2 is the Banach operator ideal of all hilbertian operators, all operators factorizing through a Hilbert space. Then (as a consequence of Pietsch's domination theorem 2.15) *for operators u defined on $C(K)$-spaces or with values in $C(K)$-spaces the p-integral and the p-summing norms coincide:*

$$\pi_p(u) = \iota_p(u)\,. \tag{2.19}$$

Note that $(\mathscr{I}, \iota) := (\mathscr{I}_1, \iota_1)$ is the Banach operator ideal of all integral operators – it is the smallest of all possible maximal Banach operator ideals, and moreover it is the adjoint ideal of $(\mathscr{L}, \|\cdot\|)$. The following important trace formulas hold isometrically:

$$\mathscr{I}_p^* = \Pi_{p'}\,, \tag{2.20}$$

and more generally for $1 \le p \le q \le \infty$

$$\Gamma_{p,q}^* = \Pi_{q'}^{\text{dual}} \circ \Pi_{p'}\,. \tag{2.21}$$

As an easy consequence of the preceding equality the ideal of (p,q)-factorable operators can be rewritten as a sort of quotient of summing operators and integral operators — this "quotient formula" in the future will be absolutely crucial: *An operator $u : X \longrightarrow Y$ is (p,q)-factorable if and only if for each operator $v \in \Pi_q^{dual}(Z,X)$ the composition $uv \in \mathscr{I}_p(X,Y)$, and in this case*

$$\gamma_{p,q}(u) = \sup_{\pi_q(v') \le 1} \iota_p(uv)\,. \tag{2.22}$$

Now we turn to tensor products – the theory of maximal Banach operator ideals and the theory of tensor products in Banach spaces are two in a sense equivalent languages. Recall that the projective norm $\|\cdot\|_\pi$ for an element z in the tensor product $X \otimes Y$ of two Banach spaces is given by

$$\|z\|_\pi = \inf \sum_k \|x_k\|\,\|y_k\|\,,$$

the infimum taken over all finite representation $z = \sum_k x_k \otimes y_k$. Dually, the injective norm $\|\cdot\|_\varepsilon$ for $z = \sum_k x_k \otimes y_k$ (a fixed finite representation) is defined by

$$\|z\|_\varepsilon = \sup_{\|x'\|_{X'},\, \|y'\|_{Y'} \le 1} \Big| \sum_k x'(x_k)\, y'(y_k) \Big|\,.$$

We will need the simple fact: *For each integral operator $u \in \mathscr{L}(X,Y)$*

$$\iota(u) = \sup \|\, \mathrm{id} \otimes u : Z \otimes_\varepsilon X \longrightarrow Z \otimes_\pi Y \|\,, \tag{2.23}$$

where the supremum is taken over all Banach spaces Z.

Let us finish with Grothendieck's *fundamental theorem of the metric theory of tensor products* (his théorème fondamental more or less in its original form) which is in a sense the hidden power in the background of most of the material following: *Every hilbertian operator $u : \ell_1 \longrightarrow \ell_\infty$ is integral, and*

$$\iota(u) = \pi_1(u) \le K_G\, \gamma_2(u)\,, \tag{2.24}$$

where K_G is a universal constant (this best constant is usually called Grothendieck's constant).

An equivalent formulation of this highly non trivial fact is *Grothendieck's theorem* which states that *each operator $u : \ell_1 \to \ell_2$ is 1-summing, and $\pi_1(u) \leq K_G\|u\|$*. We will also need a weaker fact, the so called *little Grothendieck theorem*: *Every operator $u : \ell_1 \to \ell_2$ is 2-summing; in terms of tensor products this means that for each such u and each Hilbert space H we have*

$$\sup_n \|u \otimes \mathrm{id} : \ell_1^n \otimes_\varepsilon H \to \ell_2^n(H)\| \leq K_{LG}\|u\|\,, \tag{2.25}$$

and here (in contrast to Grothendieck's theorem) the precise constant $K_{LG} = 2/\sqrt{\pi}$ (the little Grothendieck constant) is known.

Notes and remarks: Most of the results presented in this section are standard, and can be found in the textbooks [6, 9, 76, 77], or [94]. The characterization of summing operators from (2.14) can be found in [94, Sect. III.F.33]. Pietsch's domination theorem (2.15) and factorization theorems like (2.16) are crucial, and contained in each of the above monographs. The trace duality theory of summing, integral and factorable operators is due to Kwapień, and at least for $p = q$ outlined in detail in the quoted textbooks; all needed properties of the ideal $\Gamma_{p,q}$ for $p \neq q$, in particular its relation with summing and integral norms like (2.19), (2.20), (2.21), and (2.22), are included in [6, Sects. 18, 25]. The estimate (2.24) is the main result in Grothendieck's famous "Résumé" [21] (the original source of all of this material), and together with (2.25) it forms one of the central topics in all monographs cited above.

2.1.7 Maximizing Matrices and Summation: The Case $q \geq p$

The following characterization of (p,q)-maximizing matrices links the classical theory of orthonormal series with modern operator theory in Banach spaces. Recall that by definition every (p,q)-maximizing matrix can be considered as an operator from ℓ_1 into ℓ_∞, and denote for $\alpha \in \ell_q$ by $D_\alpha : \ell_{q'} \to \ell_1$ the diagonal operator associated to α.

Theorem 3. *Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$, and let A be an infinite matrix with $\|A\|_\infty < \infty$. Then the following are equivalent:*

(1) A is (p,q)-maximizing
(2) $\exists\, c \geq 0\ \forall\, \alpha \in \ell_q : \pi_p(AD_\alpha) \leq c\,\|\alpha\|_q$
(3) $\exists\, c \geq 0\ \forall n\ \forall\, u \in \mathscr{L}(\ell_{q'}^n, \ell_1) : \pi_p(Au) \leq c\,\pi_q(u')$
(4) $\exists\, c \geq 0\ \forall$ Banach space $X\ \forall\, u \in \Pi_q^{dual}(X, \ell_1) : \pi_p(Au) \leq c\,\pi_q(u')$.

In this case, $m_{pq}(A) = \sup\limits_{\|\alpha\|_q \leq 1} \pi_p(AD_\alpha) = \sup\limits_{\pi_q(u') \leq 1} \pi_p(Au)$.

We try to make the proof a bit more transparent by proving a lemma first.

Lemma 6. *For every operator $B : \ell_{q'} \longrightarrow \ell_\infty$ the following are equivalent:*

(1) B is p-summing.

(2) $\exists\, c \geq 0\, \forall\, x_0, \ldots, x_m \in L_p(\mu) : \Big\| \sup_j \big| \sum_{k=0}^m b_{jk} x_k \big| \Big\|_p \leq c\, w_{q'}(x_k)$

In this case, $\pi_p(B) := \inf c$.

Proof. Let us first show that (1) implies (2). Take $x_0, \ldots, x_m \in L_p(\mu)$. Then we obtain from (2.14) and the Bochner-integrable function

$$g := \sum_{k=0}^m x_k \otimes e_k \in L_p(\mu, \ell_{q'}^m)$$

that

$$\begin{aligned}
\Big\| \sup_j \Big| \sum_{k=0}^m b_{jk} x_k \Big| \Big\|_p &= \Big(\int \|Bg\|_\infty^p d\mu \Big)^{\frac{1}{p}} \\
&\leq \pi_p(B) \sup_{\|x'\|_{\ell_q^m} \leq 1} \Big(\int |x' \circ g|^p d\mu \Big)^{\frac{1}{p}} \\
&= \pi_p(B) \sup_{\|c\|_{\ell_q^m} \leq 1} \Big\| \sum_{k=0}^m c_k x_k \Big\|_p \\
&= \pi_p(B)\, w_{q'}(x_k, L_p(\mu)).
\end{aligned}$$

Conversely, it suffices to show that for $x_0, \ldots, x_m \in \ell_{q'}^M$

$$\Big(\sum_{k=0}^m \|Bx_k\|_\infty \Big)^{\frac{1}{p}} \leq c \sup_{\|x'\|_{\ell_q^M} \leq 1} \Big(\sum_{k=0}^m |x'(x_k)|^p \Big)^{\frac{1}{p}}.$$

Put $y_\ell := \sum_{n=0}^M x_n(\ell) e_n \in \ell_p^m$, $0 \leq \ell \leq M$. Then we have

$$\begin{aligned}
\Big\| \sup_j \Big| \sum_{\ell=0}^M b_{j\ell} y_\ell \Big| \Big\|_{\ell_p^m} &= \Big(\sum_{k=0}^m \sup_j \Big| \sum_{\ell=0}^M b_{j\ell} y_\ell(k) \Big|^p \Big)^{\frac{1}{p}} \\
&= \Big(\sum_{k=0}^m \sup_j \Big| \sum_{\ell=0}^M b_{j\ell} \sum_{n=0}^m x_n(\ell) e_n(k) \Big|^p \Big)^{\frac{1}{p}} \\
&= \Big(\sum_{k=0}^m \|Bx_k\|_\infty^p \Big)^{\frac{1}{p}}
\end{aligned}$$

and

$$
\begin{aligned}
\sup_{\|x'\|_{\ell_q^M}\le 1}\Big(\sum_{k=0}^{m}|x'(x_k)|^p\Big)^{\frac{1}{p}} &= \sup_{\|x'\|_{\ell_q^M}\le 1}\ \sup_{\|d\|_{\ell_{p'}^m}\le 1}\Big|\sum_{k=0}^{m} d_k x'(x_k)\Big| \\
&= \sup_{\|d\|_{\ell_{p'}^m}\le 1}\ \sup_{\|c\|_{\ell_q^M}\le 1}\Big|\sum_{\ell=0}^{M} c_\ell \sum_{k=0}^{m} d_k x_k(\ell)\Big| \\
&= \sup_{c}\sup_{d}\Big|\sum_{k=0}^{m} d_k \sum_{\ell=0}^{M} c_\ell \sum_{n=0}^{m} x_n(\ell) e_n(k)\Big| \\
&= \sup_{c}\Big\|\sum_{\ell=0}^{M} c_\ell \sum_{n=0}^{m} x_n(\ell) e_n\Big\|_{\ell_p^m} \\
&= \sup_{\|c\|_{\ell_q^M}\le 1}\Big\|\sum_{\ell=0}^{M} c_\ell y_\ell\Big\|_{\ell_p^m} = w_{q'}\big(y_\ell, \ell_p^m\big)\,.
\end{aligned}
$$

Since we assume that (2) holds, these two equalities complete the proof. □

Now we are prepared for the

Proof (of Theorem 3). First assume that A is (p,q)-maximizing, i.e. for every choice of a measure μ, a sequence $\alpha \in \ell_q$ and functions $x_0,\dots,x_m \in L_p(\mu)$ we have

$$
\Big\|\sup_j\Big|\sum_{k=0}^{j} a_{jk}\alpha_k x_k\Big|\Big\|_p \le \mathrm{m}_{p,q}(A)\|\alpha\|_q\, w_{q'}(x_k)\,.
$$

But then the preceding lemma implies that $AD_\alpha : \ell_{q'} \longrightarrow \ell_\infty$ is p-summing, and $\pi_p(AD_\alpha) \le \mathrm{m}_{p,q}(A)\|\alpha\|_q$. Conversely, assume that (2) holds. Then, again by the lemma,

$$
\Big\|\sup_j\Big|\sum_{k=0}^{j} a_{jk}\alpha_k x_k\Big|\Big\|_p \le c\|\alpha\|_q\, w_{q'}(x_k)\,,
$$

which yields (1). Next, we show that (2) implies (3). Take some $u \in \mathscr{L}(\ell_{q'}^n, \ell_1)$. Then by (2.16) there is a factorization

$$
\begin{tikzcd}
c_0 \arrow[r, "u'|c_0"] \arrow[dr, "D_\alpha"'] & \ell_q^n \\
 & \ell_q \arrow[u, "R"']
\end{tikzcd}
$$

with $\|D_\alpha\|\,\|R\| \le \pi_q(u')$. But then (2) implies (3):

$$\pi_p(Au) = \pi_p(AD_\alpha R') \le \pi_p(AD_\alpha)\|R\| \le c\|\alpha\|_q\,\|R\| \le c\,\pi_q(u')\,.$$

Now we prove the implication (3) $\Rightarrow$ (4): Recall that the Banach operator ideal (Π_p, π_p) is maximal (see (2.12)). Hence, we fix some operator $u: X \longrightarrow \ell_1$, and assume without loss of generality that X is finite dimensional. The aim is to show that

$$\pi_p(Au) \le c\,\pi_q(u')\,.$$

It is well-known that there is some finite rank operator S on ℓ_1 such that $\|S\| \le 1+\varepsilon$ and $S|_M = \mathrm{id}$ where $M := uX$ (ℓ_1 has the metric approximation property, see e.g. [6] or [53]). Put

$$v: X \longrightarrow M\,, \quad vx := Sux\,,$$

and let $I_M: M \hookrightarrow \ell_1$ be the canonical embedding. Without loss of generality there is a linear bijection $T: M \longrightarrow \ell_1^{\dim M}$ such that $\|T\|\,\|T^{-1}\| \le 1+\varepsilon$ (ℓ_1 is a $\mathscr{L}_{1,\lambda}$-space, $\lambda > 1$; for this see again [6] or [53]). Again by (2.16) there is a factorization

$$\begin{tikzcd} \ell_\infty^n \arrow[r, "(Tu)'"] \arrow[dr, "R"'] & X' \\ & \ell_q^N \arrow[u, "S"'] \end{tikzcd} \qquad \pi_q(R)\|S\| \le \pi_q((Tu)')\,.$$

Hence, we conclude that

$$\begin{aligned} \pi_p(Au) &= \pi_p(AI_M u) \\ &= \pi_p(AI_M T^{-1}Tu) \\ &\le \pi_p(AI_M T^{-1}R')\|S'\| \\ &\overset{(3)}{\le} c\,\pi_q((I_M T^{-1}R')')\|S\| \\ &\le c\,\pi_q(R)\|T^{-1}\|\,\|S\| \\ &\le c\,\pi_q((Tu)')\|T^{-1}\| \le c\,\pi_q(u')(1+\varepsilon)\,, \end{aligned}$$

the conclusion. This completes the whole proof since (4) trivially implies (2). $\square$

The preceding characterization has some deep consequences.

Theorem 4. *Let A be an infinite matrix such that $\|A\|_\infty < \infty$, and assume that $1 \le p < \infty$, $1 \le q \le \infty$ with $p \le q$.*

(1) A is $(2,2)$*-maximizing if and only if* $A : \ell_1 \longrightarrow \ell_\infty$ *is hilbertian, and in this case* $\mathrm{m}_{2,2}(A) = \gamma_2(A)$.

(2) More generally, A is (p,q)*-maximizing if and only if* $A : \ell_1 \longrightarrow \ell_\infty$ *is* (p,q)*-factorable,*

$$\begin{tikzcd} \ell_1 \arrow[r, "A"] \arrow[d, "v"] & \ell_\infty \\ L_q(\mu) \arrow[r, hook, "j"] & L_p(\mu) \arrow[u, "w"'] \end{tikzcd},$$

and in this case $\mathrm{m}_{p,q}(A) = \gamma_{p,q}(A)$.

(3) In particular, A is (p,∞)*-maximizing if and only if* $A : \ell_1 \longrightarrow \ell_\infty$ *is p-summing (= p-integral by (2.19)), and in this case* $\mathrm{m}_{p,\infty}(A) = \pi_p(A)$.

Proof. It suffices to check (2) since (1) is an immediate consequence of (2), and (3) follows from (2) and (2.19). But (2) obviously is a consequence of the characterization of maximizing operators given in Theorem 3, $(1) \Leftrightarrow (4)$ combined with the quotient formula from (2.22) and the equality from (2.19). □

Note that (1) and (3) in combination with Grothendieck's théorème fondamental from (2.24) show that a matrix A is $(2,2)$-maximizing ($A : \ell_1 \to \ell_\infty$ is hilbertian) if and only A is $(1,\infty)$-maximizing ($A : \ell_1 \to \ell_\infty$ is integral). This is part of the following theorem which together with Theorem 1 is our second crucial tool later used to deduce a commutative and noncommutative L_p-theory of classical coefficient tests.

Theorem 5. *Let A be an infinite matrix such that* $\|A\|_\infty < \infty$*. The following are equivalent:*

(1) A is $(2,2)$*-maximizing.*
(2) A is $(1,\infty)$*-maximizing.*
(3) A is (p,q)*-maximizing for some* $1 \le p \le 2 \le q \le \infty$*.*
(4) A is (p,q)*-maximizing for all* $1 \le p < \infty$*,* $1 \le q \le \infty$*.*

In this case, $K_G^{-1}\, \mathrm{m}_{1,\infty}(A) \le \mathrm{m}_{2,2}(A) \le \mathrm{m}_{1,\infty}(A)$.

Proof. We have already explained that the first two statements are equivalent. All other implications are then either trivial or follow by monotonicity. □

2.1.8 Almost Everywhere Summation

As anounced earlier one aim of this second chapter is to develop an L_p-theory for classical coefficient tests for almost sure summation of orthonormal series. The following theorem links the type of maximal inequalities in L_p-spaces we are interested in (i.e. inequalities induced by maximizing matrices) with almost everywhere convergence.

Proposition 2. *Let $A = (a_{jk})$ be a (p,q)-maximizing matrix which converges in each column, and $E(\mu,X)$ a vector-valued Banach function space. Then for every $\alpha \in \ell_q$ and every weakly q'-summable sequence (x_k) in $E(\mu,X)$ (in the case $q=\infty$ we only consider unconditionally summable sequences) the sequence*

$$\Big(\sum_{k=0}^{\infty} a_{jk}\alpha_k x_k\Big)_j$$

converges μ-almost everywhere.

Our proof will turn out to be a sort of model for the noncommutative case in Chap. 3; see Lemmas 22 and 27. That is the reason why we isolate the following lemma which here appears to be a bit too "heavy" – but obviously it allows to deduce the preceding proposition as an immediate consequence.

Lemma 7. *Let $A = (a_{jk})$ be a matrix with $\|A\|_\infty < \infty$ and such that each column forms a convergent sequence, $E(\mu,X)$ a vector-valued Banach function space, and $1 \le q \le \infty$. Assume that*

$$\Big(\sum_{k=0}^{\infty} a_{jk}\alpha_k x_k\Big)_j \in E(\mu,X)[\ell_\infty]$$

for every sequence $\alpha \in \ell_q$ and every weakly q'-summable sequence (x_k) in $E(\mu,X)$ (in the case $q=\infty$ we only consider unconditionally summable sequences). Then for every such α and (x_k) the sequence

$$\Big(\sum_{k=0}^{\infty} a_{jk}\alpha_k x_k\Big)_j$$

converges μ-almost everywhere.

Proof. We show that for every α and x as in the statement we have

$$\Big(\sum_{k=0}^{\infty} a_{ik}\alpha_k x_k - \sum_{k=0}^{\infty} a_{jk}\alpha_k x_k\Big)_{(i,j)} \in E(\mu,X)[c_0(\mathbb{N}_0^2)]\,; \tag{2.26}$$

then we conclude from Lemma 2 that for each w in the complement of a zero set N

$$\lim_{(i,j)\to\infty}\Big(\sum_k a_{ik}\alpha_k x_k(w) - \sum_k a_{jk}\alpha_k x_k(w)\Big)_{(i,j)} = 0.$$

But this means that in $\complement N$ the sequence $\Big(\sum_k a_{jk}\alpha_k x_k\Big)_j$ is pointwise Cauchy, the conclusion.

In order to show (2.26) we first consider the case $1 \le q < \infty$. Fix a weakly q'-summable sequence (x_k) in $E(\mu, X)$. Note first that for $(u_k) \in E(\mu,X)[\ell_\infty]$

$$(u_k - u_l)_{(k,l)} \in E(\mu,X)[\ell_\infty(\mathbb{N}_0^2)]$$

and

$$\Big\| \sup_{k,l} \|u_k(\cdot) - u_l(\cdot)\|_X \Big\|_{E(\mu)} \le 2 \Big\| \sup_k \|u_k(\cdot)\|_X \Big\|_{E(\mu)} ;$$

this is obvious, but for later use in noncommutative settings let us also mention the following argument: if $u_k = z_k f$ is a factorization according to the definition of $E(\mu,X)[\ell_\infty]$, then

$$u_k - u_l = (z_k - z_l) f$$

defines a factorization for $(u_k - u_l)_{(k,l)}$. Hence by assumption the mapping

$$\begin{aligned} \Phi : \ell_q &\longrightarrow E(\mu,X)[\ell_\infty(\mathbb{N}_0^2)] \\ \alpha &\rightsquigarrow \Big(\sum_k a_{ik} \alpha_k x_k - \sum_k a_{jk} \alpha_k x_k \Big)_{(i,j)} \end{aligned}$$

is defined, linear and (by a closed graph argument) bounded. Our aim is to show that Φ has values in the closed subspace $E(\mu,X)[c_0(\mathbb{N}_0^2)]$. By continuity it suffices to prove that, given a finite sequence $\alpha = (\alpha_0, \dots, \alpha_{k_0}, 0, \dots)$ of scalars, $\Phi\alpha \in E(\mu,X)[c_0(\mathbb{N}_0^2)]$. Clearly, $(\alpha_k x_k)_{0 \le k \le k_0} \in E(\mu,X)[\ell_\infty]$, and hence there is a factorization

$$\alpha_k x_k = z_k f, \quad 0 \le k \le k_0$$

with $\|z_k\|_{L_\infty(X)} \le 1$ and $f \in E(\mu)$. But then for all i, j

$$\sum_{k=0}^{k_0} a_{ik} \alpha_k x_k - \sum_{k=0}^{k_0} a_{jk} \alpha_k x_k = \sum_{k=0}^{k_0} (a_{ik} - a_{jk}) \alpha_k x_k = \Big(\sum_{k=0}^{k_0} (a_{ik} - a_{jk}) z_k \Big) f.$$

This means that the right side of this equality defines a factorization of

$$\Big(\sum_{k=0}^{k_0} a_{ik} \alpha_k x_k - \sum_{k=0}^{k_0} a_{jk} \alpha_k x_k \Big)_{(i,j)} .$$

Since

$$\Big\| \sum_{k=0}^{k_0} (a_{ik} - a_{jk}) z_k \Big\|_{L_\infty(X)} \le \sum_{k=0}^{k_0} |a_{ik} - a_{jk}| \|z_k\|_{L_\infty(X)} \le \sum_{k=0}^{k_0} |a_{ik} - a_{jk}| ,$$

and A converges in each column, we even see that as desired

$$\Big(\sum_k a_{ik}\alpha_k x_k - \sum_k a_{jk}\alpha_k x_k\Big)_{(i,j)} \in E(\mu,X)[c_0(\mathbb{N}_0^2)].$$

For the remaining case $q=\infty$ fix $\alpha \in \ell_\infty$ and define

$$\begin{aligned}\Phi : \ell_1^{unc}(E(\mu)) &\longrightarrow E[\ell_\infty(\mathbb{N}_0^2)]\\ (x_k) &\rightsquigarrow \Big(\sum_k a_{ik}\alpha_k x_k - \sum_k a_{jk}\alpha_k x_k\Big)_{(i,j)}.\end{aligned}$$

Like in the first case we see that Φ is well-defined and continuous. Since the finite sequences are dense in $\ell_1^{unc}(E(\mu,X))$, we can finish exactly as above. □

2.2 Basic Examples of Maximizing Matrices

For some fundamental coefficient tests within the theory of pointwise summation of general orthogonal series with respect to classical summation methods, we isolate the maximal inequalities which come along with these results. In view of the results of the preceding section this leads to several interesting scales of (p,q)-maximizing matrices A – the main results are given in the Theorems 7 (ordinary summation), 8 (Riesz summation), 9 and 10 (Cesàro summation), 11 (Kronecker matrices), and 12 (Abel summation).

Let us once again repeat that most of our examples (but not all) have the form $A=(a_{jk})_{j,k\in\mathbb{N}_0} = S\Sigma D_{1/\omega}$, where S is some summation process (see (2.4)), $D_{1/\omega}$ some diagonal matrix with some Weyl sequence ω for S (see (2.2)), and Σ the sum matrix (see (2.1)):

$$a_{jk} := \frac{1}{\omega_k}\sum_{\ell=k}^{\infty} s_{j\ell}\,. \tag{2.27}$$

In the final section, we link our setting of maximizing matrices with Bennett's powerful theory of (p,q)-multipliers. We recall again that $\log x$ always means $\max\{1,\log x\}$.

2.2.1 The Sum Matrix

We already know from Theorem 2 that every matrix $S\Sigma$ is (p,q)-maximizing whenever $q<p$. The aim here is to prove the fundamental inequality of the theory of general orthonormal series – the famous Kantorovitch-Menchoff-Rademacher maximal inequality. This result will then show that every matrix of the form $S\Sigma D_{(1/\log k)}$ in fact is (p,q)-maximizing for arbitrary p,q.

Theorem 6. *Let (x_k) be an orthonormal system in $L_2(\mu)$ and (α_k) a scalar sequence satisfying $\sum_{k=0}^{\infty} |\alpha_k \log k|^2 < \infty$. Then the orthonormal series $\sum_k \alpha_k x_k$ converges almost everywhere, and its maximal function satisfies*

$$\Big\| \sup_j \Big| \sum_{k=0}^{j} \alpha_k x_k \Big| \Big\|_2 \leq C \|(\alpha_k \log k)\|_2 \,, \tag{2.28}$$

where C is an absolute constant.

Improving many earlier results, the statement on almost everywhere convergence was independently discovered by Menchoff [60] and Rademacher [81], and today it is usually called Menchoff-Rademacher theorem (see e.g. [1, 47, 94]). Note that it is best possible in the following sense: Menchoff in [60] constructed an orthonormal system (x_k) such that for every increasing sequence (ω_k) in $\mathbb{R}_{\geq 1}$ with $\omega_k = o(\log k)$ there is an orthonormal series $\sum_k \alpha_k x_k$ which is divergent almost everywhere, but such that $\sum_{k=0}^{\infty} |\alpha_k \omega_k|^2 < \infty$. The maximal inequality (2.28) was isolated by Kantorovitch [46], and the result on almost everywhere convergence is clearly an easy consequence of it (see also Proposition 2). The optimality of the log-term in (2.28) can also be shown by use of the discrete Hilbert transform on ℓ_2 (see e.g. [2, 50, 59]).

The proof of the Kantorovitch-Menchoff-Rademacher maximal inequality (2.28) is done in two steps. First we show the following weaker estimate: Let $(\alpha_k)_{k=0}^n$ be scalars and $(x_k)_{k=0}^n$ an orthonormal system in $L_2(\mu)$. Then

$$\Big\| \max_{0 \leq j \leq n} \Big| \sum_{k=0}^{j} \alpha_k x_k \Big| \Big\|_2 \leq K \log n \|\alpha\|_2 \,, \tag{2.29}$$

where $K > 0$ is an absolute constant.

Although the literature provides many elementary proofs of this inequality, we prefer to present a proof within our setting of maximizing matrices. In view of Theorem 1 the preceding estimate is equivalent to

$$\mathrm{m}_{2,2}(\Sigma_n) \leq K \log n \,,$$

where Σ_n denotes the "finite" sum matrix

$$\sigma_{jk}^n := \begin{cases} 1 & k \leq j \leq n \\ 0 & j < k \leq n \,. \end{cases} \tag{2.30}$$

We show the apparently stronger (but by Theorem 5 equivalent) result

$$\mathrm{m}_{1,\infty}(\Sigma_n) \leq K \log n \,, \tag{2.31}$$

which by Theorem 3, our general characterization of maximizing matrices through summing operators, is an immediate consequence of the following estimate.

Lemma 8. *There is a constant $K > 0$ such that for all n*

$$\pi_1(\Sigma_n : \ell_1^n \to \ell_\infty^n) \leq K \log n \,.$$

This lemma is well-known; see e.g. [2, 3] and [59]; the idea for the proof presented here is taken from[94, Sect. III.H.24]. For the estimate $\pi_1(\Sigma_n : \ell_1^n \to \ell_\infty^n) \leq \pi^{-1} \log n + O(1)$, where π^{-1} is optimal, see [3, Corollary 8.4].

Proof. Consider on the interval $[0, 2\pi]$ the matrix-valued function

$$A(\theta) := D(\theta)\Big(e^{i(j-k)\theta}\Big)_{jk},$$

where $D(\theta) = \sum_{j=0}^n e^{ij\theta}$ as usual denotes the Dirichlet kernel. Since we have that $A(\theta) = D(\theta)\, x \otimes y$ with $x = (e^{ij\theta})_j$ and $y = (e^{-ik\theta})_k$, the matrix $A(\theta)$ represents a one dimensional operator on $\mathbb{C}^n$. Hence

$$\pi_1(A(\theta) : \ell_1^n \longrightarrow \ell_\infty^n) = \|A(\theta) : \ell_1^n \longrightarrow \ell_\infty^n\| = |D(\theta)|\,,$$

and by the triangle inequality this implies that

$$\pi_1\Big(\frac{1}{2\pi}\int_0^{2\pi} A(\theta) d\theta\Big) \leq \frac{1}{2\pi}\int_0^{2\pi} |D(\theta)| d\theta \leq K \log n \,.$$

Since by coordinatewise integration we have

$$\Sigma_n = \frac{1}{2\pi}\int_0^{2\pi} A(\theta) d\theta\,,$$

the conclusion of the lemma follows. □

Now we give the

Proof (of Theorem 6). It suffices to check the following two estimates:

$$\Big\| \sup_n \Big| \sum_{k=0}^{2^n} \alpha_k x_k \Big| \Big\|_2 \leq C_1 \,\|(\alpha_k \log k)\|_2\,, \tag{2.32}$$

$$\sum_n \Big\| \max_{2^n < \ell \leq 2^{n+1}} \Big| \sum_{k=0}^{\ell} \alpha_k x_k - \sum_{k=0}^{2^n} \alpha_k x_k \Big| \Big\|_2^2 \leq C_2 \,\|(\alpha_k \log k)\|_2^2\,; \tag{2.33}$$

indeed, for $2^m < j \leq 2^{m+1}$

$$\Big|\sum_{k=0}^{j}\alpha_k x_k\Big|^2 \le \Big(\Big|\sum_{k=0}^{2^m}\alpha_k x_k\Big| + \Big|\sum_{k=2^m+1}^{j}\alpha_k x_k\Big|\Big)^2$$

$$\le 2\Big(\Big|\sum_{k=0}^{2^m}\alpha_k x_k\Big|^2 + \Big|\sum_{k=2^m+1}^{j}\alpha_k x_k\Big|^2\Big)$$

$$\le 2\Big(\sup_n\Big|\sum_{k=0}^{2^n}\alpha_k x_k\Big|^2 + \sum_{n=0}^{\infty}\max_{2^n<\ell\le 2^{n+1}}|\sum_{k=2^n+1}^{\ell}\alpha_k x_k|^2\Big).$$

Hence we obtain by integration from (2.32) and (2.33) as desired

$$\Big\|\sup_j\Big|\sum_{k=0}^{j}\alpha_k x_k\Big|\Big\|_2 \le C\|(\alpha_k\log k)\|_2.$$

For the proof of (2.32) put $\varphi_0 := \sum_{k=0}^{2}\alpha_k x_k$ and $\varphi_\nu := \sum_{k=2^\nu+1}^{2^{\nu+1}}\alpha_k x_k$, $\nu\ge 1$. Since $\nu+1\le 2\log(2^\nu)$, we have by orthogonality

$$\sum_{\nu=0}^{\infty}(\nu+1)^2\|\varphi_\nu\|_2^2 = \|\varphi_0\|_2^2 + \sum_{\nu=1}^{\infty}(\nu+1)^2\sum_{k=2^\nu+1}^{2^{\nu+1}}|\alpha_k|^2$$

$$\le \|\varphi_0\|_2^2 + 4\sum_{\nu=1}^{\infty}\sum_{k=2^\nu+1}^{2^{\nu+1}}|\alpha_k\log k|^2 \le 4\|(\alpha_k\log k)\|_2^2.$$

On the other hand $\sup_n\Big|\sum_{k=0}^{2^n}\alpha_k x_k\Big| \le \sum_{\nu=0}^{\infty}|\varphi_\nu|$ which now implies (2.32):

$$\Big\|\sup_n\Big|\sum_{k=0}^{2^n}\alpha_k x_k\Big|\Big\|_2 \le \sum_{\nu=0}^{\infty}\|\varphi_\nu\|_2 = \sum_{\nu=0}^{\infty}(\nu+1)\|\varphi_\nu\|_2\frac{1}{\nu+1}$$

$$\le \Big(\sum_{\nu=0}^{\infty}(\nu+1)^2\|\varphi_\nu\|_2^2\Big)^{\frac12}\Big(\sum_{\nu=0}^{\infty}\frac{1}{(\nu+1)^2}\Big)^{\frac12}$$

$$\le C_1\|(\alpha_k\log k)\|_2.$$

Finally, (2.33) is a consequence of (2.29): We have for all m

$$\Big\|\max_{2^m<j\le 2^{m+1}}\Big|\sum_{k=2^m+1}^{j}\alpha_k x_k\Big|\Big\|_2^2 \le C_2(\log 2^m)^2\sum_{k=2^m+1}^{2^{m+1}}|\alpha_k|^2,$$

and hence

$$\begin{aligned}\sum_{n=0}^{\infty}\Big\|\max_{2^n<\ell\le 2^{n+1}}\Big|\sum_{k=2^n+1}^{\ell}\alpha_k x_k\Big|\Big\|_2^2 &\le C_2\sum_{n=0}^{\infty}(\log 2^n)^2\sum_{k=2^n+1}^{2^{n+1}}|\alpha_k|^2\\ &\le C_2\sum_{n=0}^{\infty}\sum_{k=2^n+1}^{2^{n+1}}|\alpha_k\log k|^2\\ &\le C_2\|(\alpha_k\log k)\|_2^2.\end{aligned}$$

This completes the proof of Theorem 6. ☐

Finally, we extend the preceding theorem within our setting of maximizing matrices. In combination with Theorem 1 we conclude from Theorem 6 that the matrix $A=\Sigma D_{(1/\log k)}$ given by

$$a_{jk}=\begin{cases}\dfrac{1}{\log k} & k\le j\\ 0 & k>j\end{cases}\tag{2.34}$$

is $(2,2)$-maximizing, hence Theorem 5 implies that this matrix is even (p,q)-maximizing for all p,q. The following formal extension of this statement complements Theorem 2.

Theorem 7. *Let S be a summation process, and $1\le p<\infty$ and $1\le q\le\infty$. Then the matrix $A=S\Sigma D_{(1/\log k)}$ given by*

$$a_{jk}=\frac{1}{\log k}\sum_{\ell=k}^{\infty}s_{j\ell}$$

is (p,q)-maximizing. Moreover, if $q<p$, then in the preceding statement no log*-term is needed.*

Proof. The matrix S defines a bounded operator on ℓ_∞. Hence we see from the argument already used in (2.11) and Theorem 6 that $S\Sigma D_{(1/\log k)}$ is $(2,2)$-maximizing, and therefore (p,q)-maximizing for all possible p,q by Theorem 5. The final statement is Theorem 2. ☐

The following consequence of Theorem 4 is an interesting by-product on summing operators.

Corollary 1. *Let S be a summation process. Then the matrices $A=S\Sigma D_{(1/\log k)}$ from Theorem 7, if considered as operators from ℓ_1 into ℓ_∞, are 1-summing.*

We finish this section with a result on the "lacunarity" of the sum matrix Σ.

Corollary 2. *Take a strictly increasing unbounded sequence (λ_n) in $\mathbb{R}_{\ge 0}$ and let (ℓ_n) be its inverse sequence, i.e. if $\lambda:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ is linear in the interval $[n,n+1]$ and $\lambda(n):=\lambda_n$, then $\ell_n:=\ell(n)$ with $\ell:=\lambda^{-1}$. Let $\Sigma^o=(\sigma^o_{jk})$ be an infinite matrix* *which equals the sum matrix Σ except that some columns are entirely zero. If for all n we have*

$$\operatorname{card}\{k \,|\, \ell_n \leq k \leq \ell_{n+1}\,, \textit{ the k-th column of } \Sigma^o \textit{ is non-vanishing}\} \leq O(n)\,,$$

then the matrix A defined by

$$a_{jk} := \begin{cases} \dfrac{\sigma^o_{jk}}{\log \lambda(k)} & k \leq j \\ 0 & k > j\,, \end{cases}$$

is (p,q)-maximizing for all p,q.

Proof. By Theorem 5 we only have to show that A is $(2,2)$-maximal, and hence by Theorem 1 we check that for a given orthonormal series $\sum_k \alpha_k x_k$ in $L_2(\mu)$

$$\Big\| \sup_j \Big| \sum_k \sigma^o_{jk} \alpha_k x_k \Big| \Big\|_2 \leq C \|(\alpha_k \log \lambda_k)\|_2\,.$$

The proof is based on the Kantorovitch-Menchoff-Rademacher inequality (2.28), but it also repeats part of its proof. As there, it suffices to check that the sequence of partial sums

$$s_j = \sum_{k=0}^{j} \sigma^o_{jk} \alpha_k x_k\,,\ j \in \mathbb{N}$$

satisfies the following two inequalities:

$$\sum_n \Big\| \max_{\ell_n < \ell \leq \ell_{n+1}} |s_\ell - s_{\ell_n}| \Big\|_2^2 \leq C \|(\alpha_k \log \lambda_k)\|_2^2 \tag{2.35}$$

$$\big\| \sup_n |s_{\ell_n}| \big\|_2 \leq C \|(\alpha_k \log \lambda_k)\|_2\,, \tag{2.36}$$

$C \geq 1$ some constant. We assume without loss of generality that all ℓ_n are natural numbers and all scalars $\beta_n := \big(\sum_{k=\ell_n+1}^{\ell_{n+1}} |\sigma^o_{\ell_n+1\,k} \alpha_k|^2\big)^{1/2} \neq 0$. Define the orthonormal system

$$y_n := \frac{1}{\beta_n} \sum_{k=\ell_n+1}^{\ell_{n+1}} \sigma^o_{\ell_n+1\,k} \alpha_k x_k\,,\ n \in \mathbb{N}\,.$$

Then we obtain (2.36) from (2.28) (note that $\sigma^o_{\ell_n+1\,\ell} = \sigma^o_{\ell_{k+1}\,\ell}$ for $\ell \leq \ell_{k+1}$):

$$\begin{aligned} \Big\| \sup_n \Big| \sum_{k=0}^{\ell_{n+1}} \sigma^o_{\ell_n+1\,k} \alpha_k x_k \Big| \Big\|_2 &= \Big\| \sup_n \Big| \sum_{k=0}^{n} \beta_k y_k \Big| \Big\|_2 \\ &\leq C \Big(\sum_{k=0}^{\infty} \log^2 k\, \beta_k^2 \Big)^{1/2} \end{aligned}$$

$$\leq C\Big(\sum_{k=0}^{\infty}\log^2 k \sum_{\ell=\ell_k+1}^{\ell_{k+1}}|\alpha_\ell|^2\Big)^{1/2}$$

$$\leq C\|(\alpha_k \log\lambda_k)\|_2\,.$$

Moreover, from another application of (2.28) (more precisely, the weaker estimate from (2.29)) and the hypothesis on the number of nonzero columns of A we conclude

$$\Big\|\max_{\ell_n<\ell\leq\ell_{n+1}}\Big|\sum_{k=\ell_n+1}^{\ell}\sigma^o_{\ell k}\alpha_k x_k\Big|\Big\|_2^2 \leq C\log^2 n\sum_{\ell=\ell_n+1}^{\ell_{n+1}}|\alpha_\ell|^2$$

$$\leq C\sum_{k=\ell_n+1}^{\ell_{n+1}}|\alpha_k\log\lambda_k|^2,$$

which after summation over all n yields (2.35). □

2.2.2 Riesz Matrices

For particular summation methods S the log-term in Theorem 7 can be improved. In the following section we handle Riesz matrices $S = R^\lambda$; recall their definition from Sect. 2.1.1.

Theorem 8. *Let $(\lambda_n)_n$ be a strictly increasing unbounded sequence of positive numbers with $\lambda_0 = 0$, and $1 \leq p < \infty, 1 \leq q \leq \infty$. Then the matrix $A = R^\lambda \Sigma D_{(1/\log\log\lambda_k)}$ given by*

$$a_{jk} = \begin{cases}(1-\dfrac{\lambda_k}{\lambda_{j+1}})\dfrac{1}{\log\log\lambda_k} & k \leq j\\ 0 & k > j\end{cases}$$

is (p,q)-maximizing. No log*-term is needed whenever $q < p$.*

Recall that we agreed to write $\log\lambda_k$ for $\max\{1,\log\lambda_k\}$. Clearly, the last statement on the log-term is a consequence of Theorem 2. Moreover, note that for the special case $\lambda_n = 2^n$ this result still contains the Kantorovitch-Mechoff-Rademacher inequality (2.28) together with its (p,q)-variants. From Theorem 4 we deduce the following immediate

Corollary 3. *All matrices $A = R^\lambda \Sigma D_{(1/\log\log\lambda_k)}$ from the preceding Theorem 8, if considered as operators from ℓ_1 into ℓ_∞, are* 1*-summing.*

Theorem 8 is due to Bennett [2, Theorem 6.5] who gives a direct, may be more elementary proof for it, and Corollary 3 was first stated in [3, Corollary 6.4]. Before we enter the proof of Theorem 8 let us recall what it means in terms of a maximal inequality: There is a constant $C > 0$ such that for each sequence (α_k) of scalars and for each sequence (x_k) in $L_p(\mu)$

$$\Big\| \sup_j \Big| \sum_{k=0}^{j} \frac{\lambda_{k+1} - \lambda_k}{\lambda_{j+1}} \sum_{\ell=0}^{k} \alpha_\ell x_\ell \Big| \Big\|_p \leq C \|(\alpha_k \log\log \lambda_k)\|_q \, w_{q'}(x_k) \,. \tag{2.37}$$

In order to prove this inequality we have to check, by what was shown in Theorem 1 and Theorem 5, an a priori weaker estimate for orthonormal series. It suffices to prove that for each orthonormal series $\sum_k \alpha_k x_k$ in $L_2(\mu)$ we have

$$\Big\| \sup_j \Big| \sum_{k=0}^{j} \frac{\lambda_{k+1} - \lambda_k}{\lambda_{j+1}} \sum_{\ell=0}^{k} \alpha_\ell x_\ell \Big| \Big\|_2 \leq C \|(\alpha_k \log\log \lambda_k)\|_2 \,, \tag{2.38}$$

$C > 0$ some universal constant. This maximal inequality for orthonormal series corresponds to a famous almost everywhere summation theorem due to Zygmund [97]; our proof follows from a careful analysis of the proof of Zygmund's result given in Alexits [1, p.141], and it is based on the Kantorovitch-Menchoff-Rademacher inequality (2.28).

Proof (of (2.38)). Define

$$s_j = \sum_{k=0}^{j} \alpha_k x_k \text{ and } \sigma_j = \sum_{k=0}^{j} \Big(1 - \frac{\lambda_k}{\lambda_{j+1}}\Big) \alpha_k x_k \,.$$

By assumption there is a strictly increasing function $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ being linear in each interval $[n, n+1]$ and satisfying $\lambda(n) = \lambda_n$ for all n. Put $\nu_n := l(2^n)$, where $l : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the inverse function of λ; we assume that all ν_n's are integers (otherwise the proof needs some modifications). It suffices to check the following three estimates:

$$\sum_n \|s_{\nu_n} - \sigma_{\nu_n}\|_2^2 \leq C_1 \|\alpha\|_2^2 \tag{2.39}$$

$$\sum_n \Big\| \max_{\nu_n < \ell \leq \nu_{n+1}} |\sigma_\ell - \sigma_{\nu_n}| \Big\|_2^2 \leq C_2 \|\alpha\|_2^2 \tag{2.40}$$

$$\Big\| \sup_n |s_{\nu_n}| \Big\|_2 \leq C_3 \|(\alpha_k \log\log \lambda_k)_k\|_2 \,; \tag{2.41}$$

indeed, for $\nu_m < j \leq \nu_{m+1}$

$$\begin{aligned}
|\sigma_j|^2 &\leq (|\sigma_j - \sigma_{\nu_m}| + |\sigma_{\nu_m} - s_{\nu_m}| + |s_{\nu_m}|)^2 \\
&\leq 3(|\sigma_j - \sigma_{\nu_m}|^2 + |\sigma_{\nu_m} - s_{\nu_m}|^2 + |s_{\nu_m}|^2) \\
&\leq 3\Big(\sum_{n=0}^{\infty} \max_{\nu_n < \ell \leq \nu_{n+1}} |\sigma_\ell - \sigma_{\nu_n}|^2 + \sum_{n=0}^{\infty} |\sigma_{\nu_n} - s_{\nu_n}|^2 + \sup_n |s_{\nu_n}|^2 \Big) \,,
\end{aligned}$$

and since the right side is independent of j, we have

$$\sup_j |\sigma_j|^2 \leq 3\Big(\sum_{n=0}^{\infty} \max_{\nu_n < \ell \leq \nu_{n+1}} |\sigma_\ell - \sigma_{\nu_n}|^2 + \sum_{n=0}^{\infty} |\sigma_{\nu_n} - s_{\nu_n}|^2 + \sup_n |s_{\nu_n}|^2 \Big) .$$

Hence, we obtain as desired

$$\begin{aligned} \big\| \sup_j |\sigma_j| \big\|_2^2 &\leq 3\Big(\sum_{n=0}^{\infty} \big\| \max_{\nu_n < \ell \leq \nu_{n+1}} |\sigma_\ell - \sigma_{\nu_n}| \big\|_2^2 + \sum_{n=0}^{\infty} \|\sigma_{\nu_n} - s_{\nu_n}\|_2^2 + \big\| \sup_n |s_{\nu_n}| \big\|_2^2 \Big) \\ &\leq 3\big(C_2 \|\alpha\|_2^2 + C_1 \|\alpha\|_2^2 + C_3 \|(\alpha_k \log\log \lambda_k)_k\|_2^2 \big) \leq C \|(\alpha_k \log\log\lambda_k)\|_2^2 . \end{aligned}$$

For the proof of (2.39) note that $\nu_n = l(2^n) \geq k = \lambda^{-1}(\lambda_k)$ iff $n \geq \log \lambda_k$. Therefore, (2.39) is obtained by orthogonality as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} \|s_{\nu_n} - \sigma_{\nu_n}\|_2^2 &= \sum_{n=0}^{\infty} \Big\| \sum_{k=0}^{\nu_n} \frac{\lambda_k}{\lambda_{\nu_n+1}} \alpha_k x_k \Big\|_2^2 \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\nu_n} \Big(\frac{\lambda_k}{\lambda_{\nu_n+1}} \Big)^2 |\alpha_k|^2 \\ &\leq \sum_{k=0}^{\infty} |\alpha_k|^2 \sum_{n:\nu_n \geq k} \Big(\frac{\lambda_k}{2^n} \Big)^2 \\ &= \sum_{k=0}^{\infty} |\alpha_k|^2 \sum_{n \geq \log \lambda_k} \Big(\frac{1}{2^{n - \log \lambda_k}} \Big)^2 \leq C_1 \|\alpha_k\|_2^2 . \end{aligned}$$

In order to show (2.40) choose for a fixed m some n such that $\nu_n < m \leq \nu_{n+1}$. Then

$$\begin{aligned} |\sigma_m - \sigma_{\nu_n}| &\leq \sum_{j=\nu_n}^{m} |\sigma_{j+1} - \sigma_j| \leq \sum_{j=\nu_n}^{\nu_{n+1}} |\sigma_{j+1} - \sigma_j| \\ &= \sum_{j=\nu_n}^{\nu_{n+1}} \Big(\frac{\lambda_{j+1}}{\lambda_{j+2} - \lambda_{j+1}} \Big)^{\frac{1}{2}} |\sigma_{j+1} - \sigma_j| \Big(\frac{\lambda_{j+2} - \lambda_{j+1}}{\lambda_{j+1}} \Big)^{\frac{1}{2}} \\ &\leq \Big(\sum_{j=\nu_n}^{\nu_{n+1}} \frac{\lambda_{j+1}}{\lambda_{j+2} - \lambda_{j+1}} |\sigma_{j+1} - \sigma_j|^2 \Big)^{\frac{1}{2}} \Big(\sum_{j=\nu_n}^{\nu_{n+1}} \frac{\lambda_{j+2} - \lambda_{j+1}}{\lambda_{j+1}} \Big)^{\frac{1}{2}} . \end{aligned}$$

But

$$\sum_{j=\nu_n}^{\nu_{n+1}} \frac{\lambda_{j+2} - \lambda_{j+1}}{\lambda_{j+1}} \leq \frac{1}{\lambda_{\nu_n}} \sum_{j=\nu_n}^{\nu_{n+1}} \lambda_{j+2} - \lambda_{j+1} \leq \frac{1}{2^n} (2^{n+3} - 2^n) = 7$$

and by orthogonality

$$\int |\sigma_{j+1}-\sigma_j|^2 d\mu = \int \Big|\sum_{k=0}^{j+1}\Big(1-\frac{\lambda_k}{\lambda_{j+2}}\Big)\alpha_k x_k - \sum_{k=0}^{j}\Big(1-\frac{\lambda_k}{\lambda_{j+1}}\Big)\alpha_k x_k\Big|^2 d\mu$$

$$= \Big(\frac{\lambda_{j+2}-\lambda_{j+1}}{\lambda_{j+1}\lambda_{j+2}}\Big)^2 \int \Big|\sum_{k=0}^{j+1}\lambda_k\alpha_k x_k\Big|^2 d\mu$$

$$= \Big(\frac{\lambda_{j+2}-\lambda_{j+1}}{\lambda_{j+1}\lambda_{j+2}}\Big)^2 \sum_{k=0}^{j+1}\lambda_k^2|\alpha_k|^2 .$$

Hence

$$\sum_{n=0}^{\infty}\int \max_{\nu_n<\ell\le\nu_{n+1}} |\sigma_m-\sigma_{\nu_n}|^2 d\mu \le 7\sum_{j=0}^{\infty}\frac{\lambda_{j+1}}{\lambda_{j+2}-\lambda_{j+1}}\int|\sigma_{j+1}-\sigma_j|^2 d\mu$$

$$= 7\sum_{j=0}^{\infty}\frac{\lambda_{j+2}-\lambda_{j+1}}{\lambda_{j+1}\lambda_{j+2}^2}\sum_{k=0}^{j+1}\lambda_k^2|\alpha_k|^2$$

$$= 7\sum_{k=0}^{\infty}\lambda_k^2|\alpha_k|^2\sum_{j=k-1}^{\infty}\frac{\lambda_{j+2}-\lambda_{j+1}}{\lambda_{j+1}\lambda_{j+2}^2}.$$

But since

$$\frac{\lambda_{j+2}-\lambda_{j+1}}{\lambda_{j+1}\lambda_{j+2}^2} = \frac{\lambda_{j+2}^2-\lambda_{j+1}^2}{\lambda_{j+1}\lambda_{j+2}^2(\lambda_{j+2}+\lambda_{j+1})} \le \frac{\lambda_{j+2}^2-\lambda_{j+1}^2}{\lambda_{j+1}^2\lambda_{j+2}^2} = \frac{1}{\lambda_{j+1}^2}-\frac{1}{\lambda_{j+2}^2},$$

we now obtain (2.40):

$$\sum_{n=0}^{\infty}\Big\|\max_{\nu_n<\ell\le\nu_{n+1}}|\sigma_\ell-\sigma_{\nu_n}|\Big\|_2^2 \le 7\sum_{k=0}^{\infty}\lambda_k^2|\alpha_k|^2\sum_{j=k-1}^{\infty}\frac{1}{\lambda_{j+1}^2}-\frac{1}{\lambda_{j+2}^2} \le C_2\,\|\alpha\|_2^2.$$

Finally, the proof of (2.41): We may assume without loss of generality that all

$$\beta_n := \begin{cases} \Big(\sum_{k=0}^{\nu_1}|\alpha_k|^2\Big)^{\frac12} & n=0 \\ \Big(\sum_{k=\nu_n+1}^{\nu_{n+1}}|\alpha_k|^2\Big)^{\frac12} & n\ge 1 \end{cases}$$

are $\neq 0$. Then the functions

$$y_n := \begin{cases} \frac{1}{\beta_0}\sum_{k=0}^{\nu_1}\alpha_k x_k & n=0 \\ \frac{1}{\beta_n}\sum_{k=\nu_n+1}^{\nu_{n+1}}\alpha_k x_k & n\ge 1 \end{cases}$$

define an orthonormal system in $L_2(\mu)$. From $\log n = \log\log 2^n = \log\log \lambda_{\nu_n}$ we derive

$$\begin{aligned}\sum_{n=0}^{\infty} |\beta_n \log n|^2 &= \sum_{k=0}^{\nu_1} |\alpha_k|^2 + \sum_{n=1}^{\infty} (\log n)^2 \sum_{k=\nu_n+1}^{\nu_{n+1}} |\alpha_k|^2 \\ &\leq \sum_{k=0}^{\nu_1} |\alpha_k \log\log \lambda_k|^2 + \sum_{n=1}^{\infty} \sum_{k=\nu_n+1}^{\nu_{n+1}} |\alpha_k \log\log \lambda_k|^2 \\ &= \sum_{k=0}^{\infty} |\alpha_k \log\log \lambda_k|^2 ,\end{aligned}$$

which in combination with Theorem 6 gives as desired

$$\begin{aligned}\big\| \sup_n |s_{\nu_{n+1}}| \big\|_2 &= \Big\| \sup_n \Big| \sum_{k=0}^{n} \beta_k y_k \Big| \Big\|_2 \\ &\leq C_3 \|(\beta_k \log k)\|_2^2 = C_3 \|(\alpha_k \log\log \lambda_k)\|_2^2 .\end{aligned}$$

This completes the proof. □

2.2.3 Cesàro Matrices

We deal with Cesàro matrices C^r defined in Sect. 2.1.1. Note first that for $\lambda_n = n$ Theorem 8 reads as follows.

Theorem 9. *The matrix $A = C\Sigma D_{(1/\log\log k)}$ given by*

$$a_{jk} = \begin{cases} \left(1 - \dfrac{k}{j+1}\right) \dfrac{1}{\log\log k} & k \leq j \\ 0 & k > j \end{cases} \tag{2.42}$$

is (p,q)-maximizing for $1 \leq p < \infty$ and $1 \leq q \leq \infty$. No log*-term is needed whenever $q < p$.*

We will now extend this result for Cesàro matrices C^r of order $r > 0$. For all needed facts on Cesàro summation of order r we once more refer to the monographs [1] and [97]. For $r \in \mathbb{R}$ define $A_0^r = 1$, and for $n \in \mathbb{N}$

$$A_n^r := \binom{n+r}{n} = \frac{(r+1)\ldots(r+n)}{n!} ;$$

recall that these numbers are the coefficients of the binomial series

$$\sum_{n=0}^{\infty} A_n^r z^n = \frac{1}{(1-z)^{r+1}} \tag{2.43}$$

($z \in \mathbb{C}$ with $|z| < 1$). In particular, we see that the equality

$$\sum_{n=0}^{\infty} A_n^{r_1+r_2+1} z^n = (1-z)^{-r_1-1}(1-z)^{-r_2-1} = \sum_{n=0}^{\infty} \big(\sum_{k=0}^{n} A_k^{r_1} A_{n-k}^{r_2} \big) z^n$$

implies the formulas

$$A_n^{r_1+r_2+1} = \sum_{k=0}^{n} A_k^{r_1} A_{n-k}^{r_2} . \tag{2.44}$$

For a sequence (x_k) in a Banach space and $r \in \mathbb{R}$ define the Cesàro means

$$s_j^r = \sum_{k=0}^{j} A_{j-k}^{r-1} s_k \quad \text{and} \quad \sigma_j^r = \frac{1}{A_j^r} s_j^r ,$$

where s_k again is the kth partial sum of the series $\sum_k x_k$. Using that $\sum_{k=0}^{j} A_k^{r-1} = A_j^r$ (this follows from (2.44)) we see that

$$s_j^r = \sum_{k=0}^{j} A_{j-k}^{r-1}(x_0 + \ldots x_k) = \sum_{k=0}^{j} x_k (A_0^r + \ldots A_{j-k}^r) = \sum_{k=0}^{j} A_{j-k}^r x_k . \tag{2.45}$$

In particular, we obtain from (2.43) and (2.45) that

$$\frac{1}{(1-z)^{r+1}} \sum_{n=0}^{\infty} x_n z^n = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} A_{n-k}^r x_k \Big) z^n = \sum_{n=0}^{\infty} s_n^r z^n ; \tag{2.46}$$

therefore

$$\begin{aligned} \sum_{n=0}^{\infty} s_n^{r_1+r_2+1} z^n &= \Big(\frac{1}{(1-z)^{r_1+1}} \sum_{n=0}^{\infty} x_n z^n \Big) \frac{1}{(1-z)^{r_2+1}} \\ &= \sum_{n=0}^{\infty} s_n^{r_1} z^n \sum_{n=0}^{\infty} A_n^{r_2} z^n = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} s_k^{r_1} A_{n-k}^{r_2} \Big) z^n , \end{aligned}$$

implying the identities

$$s_n^{r_1+r_2+1} = \sum_{k=0}^{n} A_{n-k}^{r_2} s_k^{r_1} . \tag{2.47}$$

Furthermore, for $r \neq -1, -2, \ldots$ an easy computation shows the following well-known equality

$$A_n^r = \frac{n^r}{\Gamma(r+1)} (1 + o(1)) . \tag{2.48}$$

After this preparation we are able to improve Theorem 9 for Cesàro summation of arbitrary order $r > 0$.

Theorem 10. *Let $r > 0$, and $1 \le p < \infty, 1 \le q \le \infty$. Then the matrix $A = C^r \Sigma D_{(1/\log\log k)}$ given by*

$$a_{jk} = \begin{cases} \dfrac{A^r_{j-k}}{A^r_j} \dfrac{1}{\log\log k} & k \le j \\ 0 & k > j \end{cases}$$

is (p,q)-maximizing. For $q < p$ the log*-term is superfluous.*

Note again that the last statement on the log-term is a special case of (the last statement in) Theorem 7.

Let us prove Theorem 10. As in the preceding section (see Theorem 1 and Theorem 5) we only have to show that there is some constant $C > 0$ such that for each orthonormal system (x_k) in some $L_2(\mu)$ and each sequence (α_k) of scalars we have

$$\Big\| \sup_j \Big| \sum_{k=0}^{j} \frac{A^{r-1}_{j-k}}{A^r_j} \sum_{\ell=0}^{k} \alpha_\ell x_\ell \Big| \Big\|_2 \le C \|(\alpha_k \log\log k)\|_2 . \tag{2.49}$$

Fix such (x_k) and (α_k), and recall from (2.45) that in our special situation

$$s^r_j = \sum_{k=0}^{j} A^{r-1}_{j-k} \sum_{\ell=0}^{k} \alpha_\ell x_\ell = \sum_{k=0}^{j} A^r_{j-k} \alpha_k x_k , \tag{2.50}$$

and

$$\sigma^r_j = \frac{1}{A^r_j} s^r_j .$$

By Theorem 9 the case $r = 1$ in (2.49) is already proved, and the case $r > 1$ is an immediate consequence of the next lemma (see also [1]).

Lemma 9. *Let $r > -1$ and $\varepsilon > 0$. Then*

$$\| \sup_j |\sigma^{r+\varepsilon}_j| \|_2 \le \| \sup_j |\sigma^r_j| \|_2 .$$

Proof. From (2.47) we deduce that $s^{r+\varepsilon}_j = \sum_{k=0}^{j} A^{\varepsilon-1}_{j-k} s^r_k$, and from (2.44) that

$$\frac{1}{A^{r+\varepsilon}_j} \sum_{k=0}^{j} A^r_k A^{\varepsilon-1}_{j-k} = 1 .$$

Hence we conclude

$$|\sigma_j^{r+\varepsilon}| = \Big|\sum_{k=0}^{j} \frac{A_k^r A_{j-k}^{\varepsilon-1}}{A_j^{r+\varepsilon}}\, \sigma_k^r\Big| \leq \sup_{0\leq k\leq j} |\sigma_k^r|,$$

which clearly proves our claim. □

The proof of (2.49) for $1 > r > 0$ is slightly more complicated, and will follow from two Tauberian type results (we analyze proofs from [1, p.77,110]).

Lemma 10.

(1) For $r > -1/2$ and $\varepsilon > 0$

$$\Big\|\sup_j |\sigma_j^{r+\frac{1}{2}+\varepsilon}|\Big\|_2^2 \leq C\Big\|\sup_j \frac{1}{j+1}\sum_{k=0}^{j} |\sigma_k^r|^2\Big\|_1.$$

(2) For $r > 1/2$

$$\Big\|\sup_j \frac{1}{j+1}\sum_{k=0}^{j} |\sigma_k^{r-1}|^2\Big\|_1 \leq C\Big(\|\alpha\|_2^2 + \Big\|\sup_j |\sigma_j^r|\Big\|_2^2\Big).$$

Proof. For (1) note that by (2.47) and the Cauchy-Schwarz inequality

$$\big|\sigma_j^{r+\frac{1}{2}+\varepsilon}\big|^2 \leq \sum_{k=0}^{j} |\sigma_k^r|^2 \frac{1}{(A_j^{r+\frac{1}{2}+\varepsilon})^2}\sum_{k=0}^{j}\big(A_k^r A_{j-k}^{-\frac{1}{2}+\varepsilon}\big)^2,$$

and by (2.48) (for $j \geq 1$)

$$\frac{1}{(A_j^{r+\frac{1}{2}+\varepsilon})^2}\sum_{k=0}^{j}\big(A_k^r A_{j-k}^{-\frac{1}{2}+\varepsilon}\big)^2 \leq C_1 \frac{j^{2r}}{j^{2r+1+2\varepsilon}}\sum_{k=0}^{j} k^{-1+2\varepsilon} \leq C\frac{1}{j+1},$$

the conclusion. For the proof of (2) note first that

$$\frac{1}{j+1}\sum_{k=0}^{j} |\sigma_k^{r-1}|^2 \leq 2\Big(\frac{1}{j+1}\sum_{k=0}^{j} |\sigma_k^{r-1} - \sigma_k^r|^2 + \frac{1}{j+1}\sum_{k=0}^{j} |\sigma_k^r|^2\Big),$$

hence for

$$\delta_j^r := \frac{1}{j+1}\sum_{k=0}^{j} |\sigma_k^{r-1} - \sigma_k^r|^2$$

we get that

$$\begin{aligned}\Big\|\sup_j \frac{1}{j+1}\sum_{k=0}^{j} |\sigma_k^{r-1}|^2\Big\|_1 &\leq 2\Big(\Big\|\sup_j \delta_j^r\Big\|_1 + \Big\|\sup_j \frac{1}{j+1}\sum_{k=0}^{j} |\sigma_k^r|^2\Big\|_1\Big)\\ &\leq 2\Big(\Big\|\sup_j \delta_j^r\Big\|_1 + \Big\|\sup_j |\sigma_j^r|\Big\|_2^2\Big).\end{aligned}$$

It remains to check that $\left\|\sup_j \delta_j^r\right\|_1 \le C\|\alpha\|_2^2$. Since $A_n^r = A_n^{r-1}\frac{r+n}{r}$ we have by (2.50) that

$$\begin{aligned}
\sigma_j^r - \sigma_j^{r-1} &= \sum_{k=0}^{j}\Big(\frac{A_{j-k}^r}{A_j^r} - \frac{A_{j-k}^{r-1}}{A_j^{r-1}}\Big)\alpha_k x_k \\
&= \frac{1}{A_j^r A_j^{r-1}}\sum_{k=0}^{j}\Big(A_{j-k}^r A_j^{r-1} - A_{j-k}^{r-1}A_j^r\Big)\alpha_k x_k \\
&= -\frac{1}{A_j^r}\sum_{k=0}^{j}\frac{k}{r}A_{j-k}^{r-1}\alpha_k x_k,
\end{aligned}$$

hence by orthogonality

$$\begin{aligned}
\left\|\delta_{2^n}^r\right\|_1 &= \frac{1}{2^n+1}\sum_{j=0}^{2^n}\frac{1}{(A_j^r)^2}\sum_{k=0}^{j}\frac{k^2}{r^2}(A_{j-k}^{r-1})^2|\alpha_k|^2 \\
&= \frac{1}{2^n+1}\frac{1}{r^2}\sum_{k=0}^{2^n}k^2|\alpha_k|^2\sum_{j=k}^{2^n}\Big(\frac{A_{j-k}^{r-1}}{A_j^r}\Big)^2.
\end{aligned}$$

From (2.48) we get

$$\begin{aligned}
\sum_{j=k}^{\infty}\Big(\frac{A_{j-k}^{r-1}}{A_j^r}\Big)^2 &\le C_1\sum_{j=k}^{\infty}\frac{(j-k)^{2r-2}}{j^{2r}} \\
&\le C_1\frac{1}{k^{2r}}\sum_{j=k}^{2k}(j-k)^{2r-2} + C_2\sum_{j=2k+1}^{\infty}\frac{j^{2r-2}}{j^{2r}} \le C_3\frac{1}{k}.
\end{aligned}$$

But then

$$\begin{aligned}
\sum_{n=0}^{\infty}\left\|\delta_{2^n}^r\right\|_1 &\le C_4\sum_{n=0}^{\infty}\frac{1}{2^n+1}\sum_{k=0}^{2^n}k|\alpha_k|^2 \\
&\le C_4\sum_{k=0}^{\infty}k|\alpha_k|^2\sum_{n:2^n\ge k}\frac{1}{2^{n+1}} \le C_5\sum_{k=0}^{\infty}|\alpha_k|^2,
\end{aligned}$$

which gives

$$\left\|\sup_n \delta_n^r\right\|_1 \le 2\left\|\sup_n \delta_{2^n}^r\right\|_1 \le C\sum_{k=0}^{\infty}|\alpha_k|^2.$$

This completes the proof of (2). □

Finally, we complete the

Proof (of (2.49) *for* $0 < r < 1$*).* By Theorem 9

$$\Big\| \sup_j \sigma_j^1 \Big\|_2 \leq C \|(\alpha_k \log\log k)\|_2 \,,$$

hence we deduce from Lemma 10 that for all $\varepsilon > 0$

$$\begin{aligned} \Big\| \sup_j \sigma_j^{\frac{1}{2}+\varepsilon} \Big\|_2^2 &\leq C_1 \Big\| \sup_j \frac{1}{j+1} \sum_{k=0}^{j} |\sigma_k^0|^2 \Big\|_1 \\ &\leq C_2 \Big(\|\alpha\|_2^2 + \Big\| \sup_j \sigma_j^1 \Big\|_2^2 \Big) \leq C_3 \|(\alpha_k \log\log k)\|_2^2 \,. \end{aligned}$$

A repetition of this argument gives

$$\begin{aligned} \Big\| \sup_j \sigma_j^{2\varepsilon} \Big\|_2^2 &\leq C_1 \Big\| \sup_j \frac{1}{j+1} \sum_{k=0}^{j} |\sigma_k^{-\frac{1}{2}+\varepsilon}|^2 \Big\|_1 \\ &\leq C_2 \Big(\|\alpha\|_2^2 + \Big\| \sup_j \sigma_j^{\frac{1}{2}+\varepsilon} \Big\|_2^2 \Big) \leq C_3 \|(\alpha_k \log\log k)\|_2^2 \,, \end{aligned}$$

the desired inequality. □

This finishes the proof of Theorem 10, a result which in the form presented here is new – but let us mention again that the inequality (2.49) on orthonormal series behind Theorem 10 corresponds to the fundamental coefficient tests for Cesàro summation proved by Kaczmarz [43] and Menchoff [61, 62] (see also (1.7) and (1.8)). As in the Corollaries 1 and 3 we take advantage to add another natural scale of summing operators.

Corollary 4. *For* $r > 0$ *all matrices* $A = C^r \Sigma D_{(1/\log\log k)}$ *from Theorem 10, if considered as operators from* ℓ_1 *into* ℓ_∞*, are* 1*-summing.*

2.2.4 Kronecker Matrices

We now generate some matrices which later lead to laws of large numbers. The second part of the following simple lemma is usually known as Kronecker's lemma.

Lemma 11. *Let* $A = (a_{jk})$ *be a lower triangular matrix with entries in a Banach space* X*. Then*

(1) $\sum_{k=0}^{j} \frac{k}{j+1} a_{jk} = \sum_{k=0}^{j} a_{jk} - \frac{1}{j+1} \sum_{k=0}^{j} \sum_{\ell=0}^{k} a_{j\ell}$ *for every* j

(2) $\lim_j \frac{1}{j+1} \sum_{k=0}^{j} k a_{jk} = 0$ *whenever* $\Big(\sum_{k=0}^{j} a_{jk} \Big)_j$ *converges*

(3) *Let A be a lower triangle scalar matrix which is (p,q)-maximizing. Then the matrix B defined by*

$$b_{jk} := \begin{cases} \frac{k}{j+1} a_{jk} & k \leq j \\ 0 & k > j \end{cases}$$

is again (p,q)-maximizing.

Proof. Statement (1) is immediate, and implies (2). In order to prove (3) apply (1) to see that for every choice of finitely many scalars $\xi_0, \dots, \xi_j$ we have

$$\sup_j \Big| \sum_{k=0}^{j} \frac{k}{j+1} a_{jk} \xi_k \Big| \leq 2 \sup_j \Big| \sum_{k=0}^{j} a_{jk} \xi_k \Big|,$$

and therefore by definition

$$m_{p,q}(B) \leq 2 m_{p,q}(A),$$

the conclusion. □

It makes sense to call matrices (b_{jk}) like in statement (3) Kronecker matrices – to see a first example, note that by Theorem 7 and the preceding lemma for any lower triangular summation process S the matrix

$$\Big(\frac{k}{j+1} \frac{1}{\log k} \sum_{\ell=k}^{\infty} s_{j\ell} \Big)_{j,k} \tag{2.51}$$

is (p,q)-maximizing. Sometimes the log-term can be improved – for example, for Cesàro summation of order $r > 0$; here we conclude from Theorem 10 that $\log k$ may be replaced by $\log\log k$. But the following theorem shows that in this case in fact no log-term at all is needed.

Theorem 11. *Let $1 \leq p < \infty, 1 \leq q \leq \infty$. The matrix M defined by*

$$m_{jk} := \begin{cases} \frac{k}{j+1} \big(1 - \frac{k}{j+1}\big) & k \leq j \\ 0 & k > j \end{cases} \tag{2.52}$$

is (p,q)-maximizing. More generally, for $r > 0$ the matrix M^r defined by

$$m^r_{jk} := \begin{cases} \frac{k}{j+1} \frac{A^r_{j-k}}{A^r_j} & k \leq j \\ 0 & k > j \end{cases} \tag{2.53}$$

is (p,q)-maximizing.

Let us start with the proof of (2.52). Again we follow our general philosophy – we only show a maximal inequality for orthonormal series: Fix such a series $\sum_k \alpha_k x_k$ in $L_2(\mu)$, and put

$$\mu_j^0 = \sum_{k=0}^{j} \frac{k}{j+1} \alpha_k x_k \text{ and } \mu_j^1 = \sum_{k=0}^{j} \frac{k}{j+1} \Big(1 - \frac{k}{j+1}\Big) \alpha_k x_k \,.$$

In order to prove that M is (p,q)-maximizing, by Theorem 1 and Theorem 5 it suffices to show that

$$\big\| \sup_j |\mu_j^1| \big\|_2 \leq C \, \|\alpha\|_2 \,, \tag{2.54}$$

$C > 0$ some universal constant. The proof of this inequality follows from a careful analysis of Moricz [63, Theorem 1]; similar to the proof of (2.28) and (2.38) we check three estimates:

$$\sum_{n=0}^{\infty} \|\mu_{2^n}^0 - \mu_{2^n}^1\|_2^2 \leq C_1 \|\alpha\|_2^2 \tag{2.55}$$

$$\sum_{n=0}^{\infty} \big\| \max_{2^n < \ell \leq 2^{n+1}} |\mu_\ell^1 - \mu_{2^n}^1| \big\|_2^2 \leq C_2 \|\alpha\|_2^2 \tag{2.56}$$

$$\sum_{n=0}^{\infty} \|\mu_{2^n}^0\|_2^2 \leq C_3 \|\alpha\|_2^2 \,; \tag{2.57}$$

indeed, for $2^m < j \leq 2^{m+1}$

$$\begin{aligned} |\mu_j^1|^2 &\leq \big(|\mu_j^1 - \mu_{2^m}^1| + |\mu_{2^m}^0 - \mu_{2^m}^1| + |\mu_{2^m}^0|\big)^2 \\ &\leq 3\big(|\mu_j^1 - \mu_{2^m}^1|^2 + |\mu_{2^m}^0 - \mu_{2^m}^1|^2 + |\mu_{2^m}^0|^2\big) \\ &\leq 3\Big(\sum_{n=0}^{\infty} \max_{2^n < \ell \leq 2^{n+1}} |\mu_\ell^1 - \mu_{2^n}^1|^2 + \sum_{n=0}^{\infty} |\mu_{2^n}^0 - \mu_{2^n}^1|^2 + \sum_{n=0}^{\infty} |\mu_{2^n}^0|^2\Big) \,. \end{aligned}$$

Since the right side is independent of j, we obtain

$$\big\| \sup_j |\mu_j^1| \big\|_2^2 \leq 3\Big(\sum_{n=0}^{\infty} \big\| \max_{2^n < \ell \leq 2^{n+1}} |\mu_j^1 - \mu_{2^n}^1| \big\|_2^2 + \sum_{n=0}^{\infty} \|\mu_{2^n}^0 - \mu_{2^n}^1\|_2^2 + \sum_{n=0}^{\infty} \|\mu_{2^n}^0\|_2^2\Big),$$

which by (2.55), (2.56), and (2.57) gives the conclusion. (2.55) follows by orthogonality from

$$\begin{aligned} \sum_{n=0}^{\infty} \|\mu_{2^n}^0 - \mu_{2^n}^1\|_2^2 &= \sum_{n=0}^{\infty} \int \Big| \frac{1}{(2^n+1)^2} \sum_{k=0}^{2^n} k^2 \alpha_k x_k \Big|^2 d\mu \\ &= \sum_{n=0}^{\infty} \frac{1}{(2^n+1)^4} \sum_{k=0}^{2^n} k^4 |\alpha_k|^2 \end{aligned}$$

$$\leq \sum_{k=0}^{\infty} |\alpha_k|^2 \sum_{n:2^n \geq k} \Big(\frac{k}{2^n}\Big)^4$$
$$= \sum_{k=0}^{\infty} |\alpha_k|^2 \sum_{n \geq \log k} \Big(\frac{1}{n - 2^{\log k}}\Big)^4 \leq C_1 \|\alpha\|_2^2,$$

and (2.57) from

$$\sum_{n=0}^{\infty} \int \Big|\frac{1}{2^n+1} \sum_{k=0}^{2^n} k\alpha_k x_k\Big|^2 d\mu = \sum_{n=0}^{\infty} \frac{1}{(2^n+1)^2} \sum_{k=0}^{2^n} k^2 |\alpha_k|^2$$
$$\leq \sum_{k=0}^{\infty} |\alpha_k|^2 \sum_{n:2^n \geq k} \Big(\frac{k}{2^n}\Big)^2 \leq C_3 \|\alpha\|_2^2.$$

For the proof of (2.56) note that for $2^n < \ell \leq 2^{n+1}$ by the Cauchy-Schwarz inequality

$$|\mu_\ell^1 - \mu_{2^n}^1| \leq \sum_{j=2^n+1}^{2^{n+1}} 1 \cdot |\mu_j^1 - \mu_{j-1}^1| \leq 2^{n/2} \Big(\sum_{j=2^n+1}^{2^{n+1}} |\mu_j^1 - \mu_{j-1}^1|^2\Big)^{\frac{1}{2}},$$

and hence

$$\max_{2^n < \ell \leq 2^{n+1}} |\mu_\ell^1 - \mu_{2^n}^1|^2 \leq 2^n \sum_{j=2^n+1}^{2^{n+1}} |\mu_j^1 - \mu_{j-1}^1|^2.$$

Since

$$\mu_j^1 - \mu_{j-1}^1 = \sum_{k=0}^{j} \frac{k}{j+1}\Big(1 - \frac{k}{j+1}\Big)\alpha_k x_k - \sum_{k=0}^{j-1} \frac{k}{j}\Big(1 - \frac{k}{j}\Big)\alpha_k x_k$$
$$= \sum_{k=0}^{j} \frac{kj^2(j+1) - k^2 j^2 - kj(j+1)^2 + k^2(j+1)^2}{j^2(j+1)^2} \alpha_k x_k$$
$$= \sum_{k=0}^{j} \Big(\frac{k^2(2j+1)}{j^2(j+1)^2} - \frac{k}{j(j+1)}\Big)\alpha_k x_k$$

and for $k \leq j$

$$\frac{k^2(2j+1)}{j^2(j+1)^2} \leq \frac{k(2j+1)}{j(j+1)^2} \leq \frac{2k}{j(j+1)},$$

we have

$$\int |\mu_j^1 - \mu_{j-1}^1|^2 d\mu \leq \int \Big|\sum_{k=0}^{j} \frac{k}{j(j+1)} \alpha_k x_k\Big|^2 d\mu = \sum_{k=0}^{j} \Big(\frac{k}{j(j+1)}\Big)^2 |\alpha_k|^2.$$

Hence

$$\int \max_{2^n<\ell\le 2^{n+1}} |\mu_\ell^1-\mu_{2^n}^1|^2 d\mu \le 2^n \sum_{j=2^n+1}^{2^{n+1}} \sum_{k=0}^{j} \frac{k^2}{j^2(j+1)^2}|\alpha_k|^2$$

$$\le \frac{1}{(2^n+1)^2} \sum_{j=2^n+1}^{2^{n+1}} \sum_{k=0}^{j} \frac{k^2}{j}|\alpha_k|^2$$

$$\le \frac{1}{(2^n+1)^2} \sum_{k=0}^{2^{n+1}} k^2|\alpha_k|^2 \sum_{j=\max(k,2^n+1)}^{2^{n+1}} \frac{1}{j}$$

$$\le \frac{1}{(2^n+1)^2} \sum_{k=0}^{2^{n+1}} k^2|\alpha_k|^2 \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j}$$

$$\le \frac{1}{(2^n+1)^2} \sum_{k=0}^{2^{n+1}} k^2|\alpha_k|^2 .$$

This finishes the proof of (2.56):

$$\sum_{n=0}^{\infty} \| \max_{2^n<\ell\le 2^{n+1}} |\mu_\ell^1-\mu_{2^n}^1| \|_2^2 \le \sum_{n=0}^{\infty} \frac{1}{(2^n+1)^2} \sum_{k=0}^{2^{n+1}} k^2|\alpha_k|^2$$

$$\le \sum_{k=0}^{\infty} |\alpha_k|^2 \sum_{n:2^n\ge k} \left(\frac{k}{2^n}\right)^2 \le C_2\|\alpha\|_2^2 ,$$

completing the proof of (2.52).

In order to prove (2.53) we follow the method from Sect. 2.2.3. Once again, we fix an orthonormal series $\sum_k \alpha_k x_k$ in $L_2(\mu)$ and show, according to Theorem 1 and Theorem 5, that for

$$s_j^r = \sum_{k=0}^{j} A_{j-k}^{r-1} \sum_{\ell=0}^{k} \ell\, \alpha_\ell x_\ell = \sum_{k=0}^{j} A_{j-k}^{r} k\, \alpha_k x_k \tag{2.58}$$

(for this last inequality see (2.45)) and

$$\mu_j^r = \frac{1}{(j+1)A_j^r} s_j^r ,$$

the following maximal inequality holds:

$$\| \sup_j |\mu_j^r| \|_2 \le C \|\alpha\|_2 . \tag{2.59}$$

Again the proof follows from an analysis of the work of Moricz in [63, Theorem 2]. In fact, the proof is very similar to the one of Theorem 10 (note the similarity of

(2.58) with (2.50)) – for the sake of completeness we give some details. With (2.52) the case $r = 1$ is again already proved, and the case $r > 0$ follows from two analogs of Lemma 9 and Lemma 10.

Lemma 12. *Let $r > -1$ and $\varepsilon > 0$. Then*

$$\big\| \sup_j |\mu_j^{r+\varepsilon}| \big\|_2 \leq \big\| \sup_j |\mu_j^r| \big\|_2 .$$

Proof. By (2.47) (as in the proof of Lemma 9) we have

$$\begin{aligned} \mu_j^{r+\varepsilon} &= \frac{1}{(j+1)A_j^{r+\varepsilon}} \sum_{k=0}^{j} s_k^r A_{j-k}^{\varepsilon-1} \\ &= \frac{1}{(j+1)A_j^{r+\varepsilon}} \sum_{k=0}^{j} \mu_k^r (k+1) A_k^r A_{j-k}^{\varepsilon-1} = \sum_{k=0}^{j} \beta_{jk} \mu_k^r \end{aligned}$$

with $\beta_{jk} = \frac{(k+1)A_k^r A_{j-k}^{\varepsilon-1}}{(j+1)A_j^{r+\varepsilon}} \geq 0$, and for these coefficients (use again (2.44))

$$\sum_{k=0}^{j} \beta_{jk} \leq \frac{1}{A_j^{r+\varepsilon}} \sum_{k=0}^{j} A_k^r A_{j-k}^{\varepsilon-1} = 1 .$$

Hence the conclusion is immediate. □

Lemma 13.

(1) For $r > -1/2$ and $\varepsilon > 0$

$$\Big\| \sup_j |\mu_j^{r+\frac{1}{2}+\varepsilon}| \Big\|_2^2 \leq C \Big\| \sup_j \frac{1}{j+1} \sum_{k=0}^{j} |\mu_k^r|^2 \Big\|_1 .$$

(2) For $r > 1/2$

$$\Big\| \sup_j \frac{1}{j+1} \sum_{k=0}^{j} |\mu_k^{r-1}|^2 \Big\|_1 \leq C \Big(\|\alpha\|_2^2 + \Big\| \sup_j |\mu_j^r| \Big\|_2^2 \Big) .$$

Proof. Observe first (see the preceding proof) that

$$\mu_j^{r+\frac{1}{2}+\varepsilon} = \frac{1}{(j+1)A_j^{r+\frac{1}{2}+\varepsilon}} \sum_{k=0}^{j} \mu_k^r (k+1) A_k^r A_{j-k}^{-\frac{1}{2}+\varepsilon} ,$$

hence as in the proof of Lemma 10 (by the Cauchy-Schwarz inequality) we get

$$|\mu_j^{r+\frac{1}{2}+\varepsilon}|^2 \leq C \frac{1}{j+1} \sum_{k=0}^{j} |\mu_k^r|^2 .$$

For the proof of (2) define

$$\delta_j^r := \frac{1}{j+1} \sum_{k=0}^{j} |\mu_k^{r-1} - \mu_k^r|^2 ,$$

and show as in the proof of Lemma 10 first

$$\Big\| \sup_j \frac{1}{j+1} \sum_{k=0}^{j} |\mu_k^{r-1}|^2 \Big\|_1 \leq 2 \Big(\Big\| \sup_j \delta_j^r \Big\|_1 + \Big\| \sup_j |\mu_j^r|^2 \Big\|_2^2 \Big) ,$$

and then

$$\begin{aligned} \sum_{n=0}^{\infty} \|\delta_{2^n}^r\|_1 &\leq C_1 \sum_{k=0}^{\infty} \frac{1}{2^n+1} \sum_{k=0}^{2^n} k|\alpha_k|^2 \\ &\leq C_1 \sum_{k=0}^{\infty} k|\alpha_k|^2 \sum_{n:2^n \geq k} \frac{1}{2^{n+1}} \leq C_2 \sum_{k=0}^{\infty} |\alpha_k|^2 , \end{aligned}$$

which again implies the conclusion easily. □

Finally, we deduce (2.59) (and hence complete the proof of Theorem 11) word by word as this was done in the proof of Theorem 10 (or better (2.49)) at the end of Sect. 2.2.3.

2.2.5 *Abel Matrices*

The following result on Abel matrices A^ρ (see Sect. 2.1.1 for the definition) is a straight forward consequence of our results on Cesàro summation.

Theorem 12. *Let (ρ_j) be a positive and strictly increasing sequence converging to 1. Then the matrix $A = A^\rho \Sigma D_{(1/\log\log k)}$ given by*

$$a_{jk} = \frac{\rho_j^k}{\log\log k}$$

is (p,q)-maximizing. Again for $q < p$ no log*-term is needed.*

Proof. The proof is standard, we rewrite the matrix A in terms of the Cesàro matrix. We have for all j and every choice of finitely many scalars $x_0,\dots,x_n$ that (use (2.46) for $r=1$)

$$\begin{aligned}\sum_{k=0}^{n}\rho_j^k x_k &= (1-\rho_j)^2\sum_{k=0}^{n} s_k^1\rho_j^k\\ &= (1-\rho_j)^2\sum_{k=0}^{n}\rho_j^k\sum_{\ell=0}^{k} s_\ell\\ &= \sum_{k=0}^{n}(1-\rho_j)^2\rho_j^k\,(k+1)\frac{1}{k+1}\sum_{\ell=0}^{k}\sum_{m=0}^{\ell} x_m\,.\end{aligned}$$

Define the matrix S through $s_{jk}=(1-\rho_j)^2\rho_j^k\,(k+1)$. By (2.43) we know that $\sum_k s_{jk}=1$ so that S defines a bounded operator on ℓ_∞. Since we just proved that $A^\rho = SC$, the conclusion now follows from Theorem 10 (compare the maximal functions as in (2.11)). The last statement is a special case of Theorem 2. $\square$

As a sort of by product we obtain from Theorem 4 a further interesting scale of 1-summing operators from ℓ_1 to ℓ_∞ (see also the Corollaries 1, 3, and 4).

Corollary 5. *All matrices $A=A^\rho\,\Sigma\,D_{(1/\log\log k)}$ form 1-summing operators from ℓ_1 into ℓ_∞.*

2.2.6 *Schur Multipliers*

We sketch without any proofs that our setting of maximizing matrices is equivalent to Bennett's theory of (p,q)-Schur multipliers from [3]; for precise references see the notes and remarks at the end of this section. As mentioned above our theory of maximizing matrices was up to some part modeled along this theory.

An infinite matrix $M=(m_{jk})_{j,k\in\mathbb{N}_0}$ with $\|M\|_\infty<\infty$ is said to be a (p,q)-multiplier $(1\le p,q\le\infty)$ if its Schur product $M*A=(m_{jk}\,a_{jk})_{j,k}$ with any infinite matrix $A=(a_{jk})_{j,k\in\mathbb{N}_0}$ maps ℓ_p into ℓ_q whenever A does. In this case, the (p,q)-multiplier norm of M is defined to be

$$\mu_{p,q}(M)=\sup\|M*A:\ell_p\to\ell_q\|\,,$$

the infimum taken over all matrices A which define operators from ℓ_p into ℓ_q of norm ≤ 1. For $p=q=2$ we simply speak of multipliers; we remark that

$$\mu_{2,2}(M)=\|M\|_{cb}\,, \tag{2.60}$$

where $\|M\|_{cb}$ denotes the completely bounded norm of M which via Schur multiplication is considered as an operator on the operator space $\mathscr{L}(\ell_2)$.

Moreover, it is known that the (p,q)-multiplier norm has the following formulation in terms of summing norms:

$$\mu_{p,q}(M) = \sup_{\|\alpha\|_{\ell_p} \le 1} \pi_q(M D_\alpha) \tag{2.61}$$

(here M is considered as an operator from ℓ_1 into ℓ_∞, and $D_\alpha : \ell_{p'} \to \ell_1$ denotes again the diagonal operator associated to α). From Theorem 3 we conclude that M is a (p,q)-Schur multplier if and only if M is (q,p)-maximizing – with equal norms:

$$\mu_{p,q}(M) = \mathrm{m}_{q,p}(M)\,. \tag{2.62}$$

This in particular means that all facts of the rich theory of Schur multipliers apply to maximizing operators, and vice versa. We mention some consequences, of course focusing on maximizing matrices:

(1) Obviously, $\mu_{p,q}(M) = \mu_{q',p'}(M^t)$, where M^t is the transposed (or dual) matrix of M, hence by (2.62) we have

$$\mathrm{m}_{p,q}(M) = \mathrm{m}_{q',p'}(M^t)\,.$$

By definition it is obvious that (p,q)-maximizing matrices are insensitive with respect to row repetitions or row permutations, i.e. if A is (p,q)-maximizing, then

$$\mathrm{m}_{p,q}(A) = \mathrm{m}_{p,q}(\tilde{A})$$

where $\tilde{A}$ is obtained from A by repeating or permuting rows. By transposition, we see that $\mathrm{m}_{p,q}$ is insensitive to column repetitions or permutations.

(2) For two (p,q)-maximizing matrices A and B their Schur product $A * B$ is again (p,q)-maximizing, and

$$\mathrm{m}_{p,q}(A * B) = \mathrm{m}_{p,q}(A)\,\mathrm{m}_{p,q}(B)$$

(a fact obvious for Schur multipliers). A similar result holds for tensor products (Kronecker products) of Schur multipliers,

$$\mathrm{m}_{p,q}(A \otimes B) \le \mathrm{m}_{p,q}(A)\,\mathrm{m}_{p,q}(B)\,.$$

(3) Denote by T_n the nth-main triangle projection, i.e. the projection on the vector space of all infinite matrices $A = (a_{jk})_{j,k \in \mathbb{N}_0}$ with $\|A\|_\infty < \infty$ defined by

$$T_n(A) := \sum_{j+k \le n} a_{jk} e_j \otimes e_k\,;$$

obviously, $T_n(A) = A * \Theta_n$, where

$$\Theta_n(j,k) := \begin{cases} 1 & j+k \leq n \\ 0 & \text{elsewhere}. \end{cases}$$

Then we conclude from (2.62) that for arbitrary p,q

$$\mu_{q,p}(\Theta_n) = \mathrm{m}_{p,q}(\Theta_n) = \mathrm{m}_{p,q}(\Sigma_n),$$

where Σ_n again is the sum matrix (see (2.30)); here the last equality is obvious by the definition of the (p,q)-maximizing norm. From Theorem 2 and the estimate from (2.31) (use also (2.8)) we deduce that for some constant C independent of n

$$\mu_{q,p}(\Theta_n) \leq \begin{cases} C \log n & p \leq q \\ C & q < p. \end{cases}$$

(4) Recall that a matrix $M = (m_{jk})_{j,k\in\mathbb{N}_0}$ is said to be a Toeplitz matrix whenever it has the form $m_{jk} = c_{j-k}$ with $c = (c_n)_{n\in\mathbb{Z}}$ a scalar sequence. A Toeplitz matrix is $(2,2)$-maximizing if and only if there exists a bounded complex Borel measure μ on the torus $\mathbb{T}$ such that its Fourier transform $\hat{\mu}$ equals c.

(5) Denote by $\mathscr{C}$ the closed convex hull of the set of all $(2,2)$-maximizing matrices A of the form $a_{jk} = \alpha_j \beta_k$, where α and β are scalar sequences bounded by 1 and the closure is taken in the coordinatewise topology. Then we have that

$$\mathscr{C} \subset \{A \mid \mathrm{m}_{2,2}(A) \leq 1\} \subset K_G \mathscr{C},$$

K_G Grothendieck's constant.

Notes and remarks: The close connection of Schur multipliers and summing operators was observed and elaborated by many authors. See for example Grothendieck [21], Kwapień-Pełczyński [50] and, very important here, Bennett's seminal paper [3] which is full of relevant information for our purpose and motivated large part of this second chapter. Equation (2.61) is Bennett's Theorem [3, Sect. 4.3], and (2.60) is a result due to Haagerup [25]. For Schur multipliers instead of maximizing matrices the Theorems 4 (note that its analog from [3, Theorem 6.4] for multipliers is weaker and contains a wrong statement on the constant) and 5 are well-known; for $p = q$ see [78, Theorem 5.11] and the notes and remarks thereby (Pisier:“ Once Kwapień had extended the factorization theorems to the L_p-case, it is probably fair to say that it was not too difficult to extend the theory of Schur mulipliers …”). Remark (1), (2) and (4) from Sect. 2.2.6 are taken from Bennett [3] (there of course formulated for Schur multipliers instead of maximizing martrices), and Remark (5) from [78]. For the final estimate in 2.2.6, (3) see [2] and [50].

2.3 Limit Theorems in Banach Function Spaces

It is remarkable that most of the classical almost everywhere summation theorems for orthogonal series in $L_2(\mu)$ without too many further assumptions in a natural way extend to vector-valued Banach function spaces $E(\mu,X)$.

We illustrate that our setting of maximizing matrices in a very comfortable way leads not only to the most important classical results, but also to strong new extensions of them. We show, as announced earlier, that most of the classical coefficient tests on pointwise summation of orthogonal series – in particular those for Cesàro, Riesz and Abel summation – together with their related maximal inequalities, have natural analogs for the summation of unconditionally convergent series in vector-valued Banach function spaces $E(\mu,X)$.

The main results are collected in the Theorems 13 and 14, and then later applied to classical summation methods (see the Corollaries 6 and 7). Moreover, we prove that each unconditionally convergent series in $L_p(\mu)$ is Riesz$^\lambda$-summable for some sequence λ; this is an L_p-analog of an important observation on orthonormal series apparently due to Alexits [1, p.142]. We finish this section with a systematic study of laws of large numbers in vector-valued Banach function spaces $E(\mu,X)$ with respect to arbitrary summation methods – in particular we extend some "non logarithmical" laws of large numbers due to Moricz [64].

2.3.1 *Coefficient Tests in Banach Function Spaces*

We start with a description of the situation in L_p-spaces – here the main step is a rather immediate consequence of our general frame of maximizing matrices:

Assume that S is a summation method and ω a Weyl sequence (see (2.2) for the defintion) such that for each orthonormal series $\sum_k \alpha_k x_k$ in $L_2(\mu)$ we have that the maximal function of the linear means

$$\sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \frac{\alpha_\ell}{\omega_\ell} x_\ell\,,\ \ j \in \mathbb{N}_0$$

is square integrable,

$$\sup_j \Big| \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \frac{\alpha_\ell}{\omega_\ell} x_\ell \Big| \in L_2(\mu)\,; \tag{2.63}$$

this implicitly means to assume that we are in one of the classical situations described above. How can this result be transferred to L_p-spaces, $1 \le p < \infty$?

By Theorem 1 our assumption means precisely that the matrix $A = S\Sigma D_{1/\omega}$ is $(2,2)$-maximizing. As a consequence A by Theorem 5 even is (p,∞)-maximizing,

$1 \leq p < \infty$, i.e. for each unconditionally convergent series $\sum_k x_k$ in $L_p(\mu)$ we have that

$$\sup_j \Big| \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \frac{x_\ell}{\omega_\ell} \Big| \in L_p(\mu) \,, \tag{2.64}$$

or equivalently in terms of an inequality, there is a constant $C > 0$ such that for each such series

$$\Big\| \sup_j \Big| \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \frac{x_\ell}{\omega_\ell} \Big| \Big\|_p \leq C w_1(x_k) \,.$$

But then we deduce from Proposition 2 that for each unconditionally convergent series $\sum_k x_k$ in $L_p(\mu)$

$$\sum_{k=0}^{\infty} \frac{x_k}{\omega_k} = \lim_j \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \frac{x_\ell}{\omega_\ell} \;\; \mu\text{-a.e.} \tag{2.65}$$

To summarize, if we start with a classical pointwise summation theorem on orthogonal series and know in addition that the underlying summation method even allows a maximal theorem for these series like in (2.63), then we obtain with (2.64) and (2.65) a strong extension of this result in L_p-spaces. Based on tensor products we now even prove that here $L_p(\mu)$ can be replaced by an arbitrary vector-valued Banach function space $E(\mu, X)$, and this without any further assumption on the function space $E(\mu)$ or Banach space X.

Theorem 13. *Let $E(\mu)$ be a Banach function space, X a Banach space, and $A = (a_{jk})$ a $(2,2)$-maximzing matrix. Then for each unconditionally convergent series $\sum_k x_k$ in $E(\mu, X)$ the following statements hold:*

(1) $\sup_j \Big\| \sum_{k=0}^{\infty} a_{jk} x_k(\cdot) \Big\|_X \in E(\mu)$

(2) The sequence $\big(\sum_{k=0}^{\infty} a_{jk} x_k\big)_j$ *converges μ-a.e. provided (a_{jk}) converges in each column.*

In particular, let S be a summation method and ω a Weyl sequence with the additional property that for each orthonormal series $\sum_k \alpha_k x_k$ in $L_2(\mu)$ we have

$$\sup_j \Big| \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \frac{\alpha_\ell}{\omega_\ell} x_\ell \Big| \in L_2(\mu) \,.$$

Then for each unconditionally convergent series $\sum_k x_k$ in $E(\mu, X)$ the following two statements hold:

(3) $\sup_j \Big\| \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \dfrac{x_\ell(\cdot)}{\omega_\ell} \Big\|_X \in E(\mu)$

(4) $\sum_{k=0}^{\infty} \dfrac{x_k}{\omega_k} = \lim_j \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \dfrac{x_\ell}{\omega_\ell} \quad \mu\text{-a.e.}$

Proof. In order to establish (1) we prove that for all n

$$\| \operatorname{id}_{E(\mu,X)} \otimes A_n : E(\mu,X) \otimes_\varepsilon \ell_1^n \longrightarrow E(\mu,X)[\ell_\infty^n] \| \le K_G \, \mathrm{m}_{2,2}(A) , \tag{2.66}$$

where A_n equals A for all entries a_{jk} with $1 \le j,k \le n$ and is zero elsewhere, and K_G again stands for Grothendieck's constant. Indeed, this gives our conclusion: For a finite sequence $(x_k)_{k=0}^n \in E(\mu,X)^{n+1}$ we have

$$w_1(x_k) = \Big\| \sum_{k=0}^n x_k \otimes e_k \Big\|_{E(\mu,X)\otimes_\varepsilon \ell_1^n}$$

(direct calculation) and

$$\begin{aligned} \big(\operatorname{id}_{E(\mu,X)} \otimes A_n\big)\Big(\sum_k x_k \otimes e_k\Big) &= \sum_k x_k \otimes A_n(e_k) \\ &= \sum_k x_k \otimes \sum_j a_{jk}\alpha_k e_j \\ &= \sum_j \Big(\sum_k a_{jk} x_k\Big) \otimes e_j , \end{aligned}$$

therefore

$$\| \operatorname{id}_{E(\mu,X)} \otimes A_n \Big(\sum_k x_k \otimes e_k\Big) \|_{E(\mu,X)[\ell_\infty^n]} = \Big\| \sup_j \| \sum_k a_{jk} x_k(\cdot) \|_X \Big\|_{E(\mu)} .$$

Hence we have shown that for every choice of scalars $\alpha_0, \ldots, \alpha_n$ and functions $x_0, \ldots, x_n \in E(\mu,X)$

$$\Big\| \sup_j \| \sum_k a_{jk} x_k(\cdot) \|_X \Big\|_{E(\mu)} \le K_G \, \mathrm{m}_{2,2}(A) w_1(x_k) ,$$

which then by Lemma 3 allows to deduce the desired result on infinite sequences. In order to prove (2.66) note first that by (2.24) and again Theorem 4 we have

$$\iota(A_n) \le K_G \, \gamma_2(A_n) = K_G \, \mathrm{m}_{2,2}(A_n) .$$

Hence we deduce from (2.23) and Lemma 3 that

$$\| \operatorname{id}_{E(\mu,X)} \otimes A_n : E(\mu,X) \otimes_\varepsilon \ell_1^n \longrightarrow E(\mu,X) \otimes_\pi \ell_\infty^n \| \le \iota(A_n) \le K_G \, \mathrm{m}_{2,2}(A) ,$$

but since

$$\| \operatorname{id} : E(\mu,X) \otimes_\pi \ell_\infty^n \to E(\mu,X)[\ell_\infty^n] \| \le 1 ,$$

this gives the desired estimate (2.66) and completes the proof of (1). The proof of statement (2) is now a consequence of Proposition 2. For a slightly different argument which avoids Lemma 3 see the proof of Theorem 17. Finally, for the proof of (3) and (4) define the matrix

$$A = S\Sigma D_{1/\omega}\,, \quad a_{jk} := \frac{1}{\omega_k}\sum_{\ell=k}^{\infty} s_{j\ell}\,,$$

and note that for all j

$$\sum_{k=0}^{\infty} a_{jk}x_k = \sum_{k=0}^{\infty} s_{jk}\sum_{\ell=0}^{k}\frac{x_\ell}{\omega_\ell}\,.$$

Then we conclude by the assumption on S and Theorem 1 that A is $(2,2)$-maximizing which allows to deduce (3) from (2). Since by Proposition 1 for all k

$$\lim_j a_{jk} = \lim_j \frac{1}{\omega_k}\sum_{\ell=k}^{\infty} s_{j\ell} = \lim_j \frac{1}{\omega_k}\Big(\sum_{\ell=0}^{\infty} s_{j\ell} - \sum_{\ell=0}^{k-1} s_{j\ell}\Big) = \frac{1}{\omega_k}\,, \tag{2.67}$$

statement (4) is consequence of (2). □

To illustrate the preceding result, we collect some concrete examples on summation of unconditionally convergent series in vector-valued Banach function spaces. Note that in order to start the method one has to find appropriate maximal inequalties, i.e. to make sure that the matrices $S\Sigma D_{1/\omega}$ are $(2,2)$-maximizing (Theorem 1). In the literature most coefficient tests for almost sure summation (with respect to a summation method S and a Weyl sequence ω) do not come jointly with a maximal inequality. As mentioned, the maximal inequality (2.28) joining the Menchoff-Rademacher Theorem 6 was discovered much later by Kantorovitch in [46]. We showed in the preceding Sect. 2.2 that in many concrete situations the needed maximal inequalities follow from a careful analysis of the corresponding coefficient tests; for pure summation $S = \mathrm{id}$ this is the Kantorovitch-Menchoff-Rademacher inequality (2.28) from Theorem 6, for the Riesz method R^λ see (2.38) inducing Theorem 8, for the Cesàro method of order r (2.49) inducing Theorem 10, and finally Theorem 12 for the Abel method.

Corollary 6. *Let $\sum_k x_k$ be an unconditionally convergent series in a vector-valued Banach function space $E(\mu,X)$. Then*

(1) $\sup_j \Big\| \sum_{k=0}^{j} \frac{x_k(\cdot)}{\log k}\Big\|_X \in E(\mu)$

(2) $\sup_j \Big\| \sum_{k=0}^{j} \frac{\lambda_{k+1}-\lambda_k}{\lambda_{j+1}} \sum_{\ell=0}^{k} \frac{x_\ell(\cdot)}{\log\log\lambda_\ell}\Big\|_X \in E(\mu)$ *for every strictly increasing, unbounded and positive sequence* (λ_k) *of scalars*

(3) $\sup_j \Big\| \sum_{k=0}^{j} \frac{A_{j-k}^{r-1}}{A_j^r} \sum_{\ell=0}^{k} \frac{x_\ell(\cdot)}{\log\log\ell}\Big\|_X \in E(\mu)$ *for every* $r>0$

(4) $\sup_j \Big\| \sum_{k=0}^{\infty} \rho_j^k \frac{x_k(\cdot)}{\log\log k}\Big\|_X \in E(\mu)$ *for every positive strictly increasing sequence* (ρ_j) *converging to* 1.

Moreover, in each of these cases

$$\sum_{k=0}^{\infty} \frac{x_k}{\omega_k} = \lim_j \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \frac{x_\ell}{\omega_\ell} \quad \mu - a.e.,$$

where the summation method S is either given by the identity, Riesz$^{\lambda}$, Cesàro^{r}, or Abel$^{\rho}$ matrix, and ω is the related Weyl sequence from (1) *up to* (4).

For $E(\mu,X) = L_p(\mu)$ the origin of statement (1) in Corollary 6 lies in the article [50, Theorem 5.1] of Kwapień and Pełczyński where a slightly weaker result is shown. The final form of (1) in L_p-spaces is due to Bennett [2, Theorem 2.5, Corollary 2.6] and Maurey-Nahoum [59], and was reproved in [68]. Moreover, in this special situation, statement (2) is also due to Bennett [2, Theorem 6.4], whereas both statements (3) and (4) seem to be new. Recall that the underlying four classical coefficient tests for orthogonal series are well-known theorems by Kaczmarz [43], Kantorovitch [46], Menchoff [60, 61, 62], Rademacher [81], and Zygmund [97]. Finally, we mention that by use of Corollary 2 a "lacunary version" of statement (1) can be proved.

We now extend the preceding result considerably. A Banach function space $E(\mu)$ is said to be p-convex if there is some constant $C \geq 0$ such that for each choice of finitely many functions $x_1, \ldots, x_n \in E(\mu)$ we have

$$\Big\| \Big(\sum_k |x_k|^p \Big)^{1/p} \Big\|_{E(\mu)} \leq C \Big(\sum_k \|x_k\|_{E(\mu)}^p \Big)^{1/p}, \tag{2.68}$$

and the best such C is usually denoted by $M^{(p)}(E(\mu))$ (compare also with Sect. 3.1.1). We here only mention that every Banach space $L_p(\mu)$ is p-convex with constant 1, but there are numerous other examples as can be seen e.g. in [53, 54].

Theorem 14. *Let $A = (a_{jk})$ be a (p,q)-maximizing matrix, $E(\mu)$ a p-convex Banach function space, and X a Banach space. Then for every $\alpha \in \ell_q$ and every weakly q'-summable sequence (x_k) in $E(\mu, X)$ we have*

$$\sup_j \Big\| \sum_{k=0}^{\infty} a_{jk} \alpha_k x_k(\cdot) \Big\|_X \in E(\mu),$$

and moreover $\left(\sum_{k=0}^{\infty} a_{jk}\alpha_k x_k\right)_j$ converges μ-a.e. provided each column of A converges; for this latter statement assume that (x_k) is unconditionally summable whenever $q = \infty$.

Note that Theorem 14 still contains Theorem 13 as a special case: If A is $(2,2)$-maximizing, then we conclude from Theorem 5 that the matrix A is even $(1,\infty)$-maximizing. Since every Banach function space E is 1-convex, in this special situation no convexity condition is needed.

Proof. Again, it suffices to show that for every choice of finitely many scalars $\alpha_0,\dots,\alpha_n$

$$\begin{aligned}\|\operatorname{id}_{E(\mu,X)}\otimes A_nD_\alpha : E(\mu,X)\otimes_\varepsilon \ell_{q'}^n \longrightarrow E(\mu,X)[\ell_\infty^n]\|\\ \leq M^{(p)}(E(\mu))\,\mathrm{m}_{p,q}(A)\|\alpha\|_q\,,\end{aligned}\tag{2.69}$$

where D_α stands for the induced diagonal operator, and A_n equals A for all entries a_{jk} with $0\leq j,k\leq n$ and is zero elsewhere. Indeed, as above we then obtain the conclusion: For any finite sequence $(x_k)_{k=0}^n\in E(\mu,X)^{n+1}$ we have

$$w_{q'}(x_k)=\Big\|\sum_{k=0}^n x_k\otimes e_k\Big\|_{E(\mu,X)\otimes_\varepsilon\ell_{q'}^n}$$

and

$$\big(\operatorname{id}_{E(\mu,X)}\otimes A_nD_\alpha\big)\Big(\sum_k x_k\otimes e_k\Big)=\sum_j\Big(\sum_k a_{jk}\alpha_k x_k\Big)\otimes e_j\,.$$

Then

$$\|\operatorname{id}_{E(\mu,X)}\otimes A_nD_\alpha\Big(\sum_k x_k\otimes e_k\Big)\|_{E(\mu,X)[\ell_\infty^n]}=\Big\|\sup_j\|\sum_k a_{jk}\alpha_k x_k(\cdot)\|_X\Big\|_{E(\mu)},$$

and hence we obtain the inequality

$$\Big\|\sup_j\|\sum_k a_{jk}\alpha_k x_k(\cdot)\|_X\Big\|_{E(\mu)}\leq M^{(p)}(E(\mu))\,\mathrm{m}_{p,q}(A)\,\|\alpha\|_q\,w_{q'}(x_k)\,.$$

Finally, this inequality combined with Lemma 3 gives the statement of the theorem. For the proof of (2.69) fix scalars $\alpha_0,\dots,\alpha_n$. By the general characterization of (p,q)-maximizing matrices from Theorem 3 as well as (2.18) and (2.19), we obtain a factorization

$$\begin{tikzcd} \ell_{q'}^n \arrow[r, "A_nD_\alpha"] \arrow[d, "R"] & \ell_\infty^n \\ \ell_\infty^m \arrow[r, "D_\mu"] & \ell_p^m \arrow[u, "S"'] \end{tikzcd}$$

with

$$\|R\|\,\|D_\mu\|\,\|S\|\leq(1+\varepsilon)\,\iota_p(AD_\alpha)\leq(1+\varepsilon)\,\mathrm{m}_{p,q}(A)\|\alpha\|_q\,.$$

Tensorizing gives the commutative diagram

$$
\begin{tikzcd}
E(\mu,X)\otimes_\varepsilon \ell_{q'}^n \arrow[rr, "\mathrm{id}_{E(\mu,X)}\otimes A_n D_\alpha"] \arrow[d, "\mathrm{id}_{E(\mu,X)}\otimes R"] & & E(\mu,X)[\ell_\infty^n] \\
E(\mu,X)\otimes_\varepsilon \ell_\infty^m \arrow[rr, "\mathrm{id}_{E(\mu,X)}\otimes D_\mu"] & & \ell_p^m(E(\mu,X))\,. \arrow[u, "\mathrm{id}_{E(\mu,X)}\otimes S"']
\end{tikzcd}
$$

By the metric mapping property of ε we have

$$\left\| \mathrm{id}_{E(\mu,X)}\otimes R \right\| \le \|R\|\,,$$

and moreover

$$
\begin{aligned}
\Big\| \mathrm{id}_{E(\mu,X)}\otimes D_\mu\Big(\sum_{k=0}^m x_k\otimes e_k\Big)\Big\|_{\ell_p^m(E(\mu,X))} &= \Big(\sum_{k=0}^m \|\mu_k x_k\|_{E(\mu,X)}^p\Big)^{1/p}\\
&\le \sup_k \|x_k\|_{E(\mu,X)}\Big(\sum_{k=0}^m |\mu_k|^p\Big)^{1/p}
\end{aligned}
$$

implies

$$\left\| \mathrm{id}_{E(\mu,X)}\otimes D_\mu \right\| \le \|D_\mu\|\,.$$

We show that

$$\left\| \mathrm{id}_{E(\mu,X)}\otimes S \right\| \le M^{(p)}(E(\mu))\,\|S\|\,; \tag{2.70}$$

indeed, as an easy consequence of (2.68) and Hölder's inequality we obtain

$$
\begin{aligned}
\Big\| \sup_{j=1,\dots,n} \Big\| \sum_{k=0}^m s_{jk}x_k(\cdot)\Big\|_X \Big\|_{E(\mu)} &\le \Big\| \sup_{j=1,\dots,n}\Big(\sum_{k=0}^m |s_{jk}|^{p'}\Big)^{1/p'}\Big(\sum_{k=0}^m \|x_k(\cdot)\|_X^p\Big)^{1/p}\Big\|_{E(\mu)}\\
&= \sup_{j=1,\dots,n}\Big(\sum_{k=0}^m |s_{jk}|^{p'}\Big)^{1/p'}\Big\|\Big(\sum_{k=0}^m \|x_k(\cdot)\|_X^p\Big)^{1/p}\Big\|_{E(\mu)}\\
&\le M^{(p)}(E(\mu))\sup_{j=1,\dots,n}\Big(\sum_{k=0}^m |s_{jk}|^{p'}\Big)^{1/p'}\Big(\sum_{k=0}^m \|x_k\|_{E(\mu,X)}^p\Big)^{1/p},
\end{aligned}
$$

and this completes the proof of (2.69). The result on almost everywhere convergence again follows by Proposition 2. □

Corollary 6 presents analogs of classical coefficient theorems with logarithmic Weyl sequences for unconditionally convergent series in vector-valued Banach function spaces, e.g. on Cesàro or Riesz summation. The following result shows that under restrictions on the series and the underlying function space these logarithmic terms are superfluous.

Corollary 7. *Assume that $1 \leq q < p < \infty$, and let $E(\mu)$ be a p-convex Banach function space and X a Banach space. Then for each $\alpha \in \ell_q$ and each weakly q'-summable sequence (x_k) in $E(\mu, X)$ we have*

(1) $\sup_j \left\| \sum_{k=0}^{j} \alpha_k x_k(\cdot) \right\|_X \in E(\mu)$

(2) $\sup_j \left\| \sum_{k=0}^{j} \frac{\lambda_{k+1} - \lambda_k}{\lambda_{j+1}} \sum_{\ell=0}^{k} \alpha_\ell x_\ell(\cdot) \right\|_X \in E(\mu)$ *for every strictly increasing, unbounded and positive sequence (λ_k) of scalars*

(3) $\sup_j \left\| \sum_{k=0}^{j} \frac{A_{j-k}^{r-1}}{A_j^r} \sum_{\ell=0}^{k} \alpha_\ell x_\ell(\cdot) \right\|_X \in E(\mu)$ *for every $r > 0$*

(4) $\sup_j \left\| \sum_{k=0}^{\infty} \rho_j^k \alpha_\ell x_\ell(\cdot) \right\|_X \in E(\mu)$ *for every positive strictly increasing sequence (ρ_j) converging to* 1.

Moreover, in each of these cases

$$\sum_{k=0}^{\infty} \alpha_k x_k = \lim_j \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} \alpha_\ell x_\ell(\cdot) \quad \mu - a.e.,$$

where the summation method S is either given by the identity, Riesz$^\lambda$, Cesàro^r, or Abel$^\rho$ matrix.

Proof. The argument by now is clear: For each of the considered summation methods the matrix $A = S\Sigma$ by Theorem 2 is (p,q)-maximizing. Since

$$\sum_{k=0}^{\infty} a_{jk} x_k = \sum_{k=0}^{\infty} s_{jk} \sum_{\ell=0}^{k} x_\ell,$$

Theorem 14 gives the conclusion. The result on μ-a.e. convergence again follows from Proposition 2. □

Of course, the preceding corollary could also be formulated for arbitrary summation methods instead of the four concrete examples given here. Statement (1) is a far reaching extension of a well-known result of Menchoff [60] and Orlicz [67] for orthonormal series.

Finally, we present a sort of converse of Corollary 6,(2): The sum of every unconditionally convergent series $\sum_k x_k$ in $E(\mu, X)$ (such that $E(\mu)$ and X have finite cotype) can be obtained by almost everywhere summation of its partial sums through a properly chosen Riesz method. Recall that a Banach space X has cotype p, $2 \leq p < \infty$ whenever there is some constant $C \geq 0$ such that for each choice of finitely many vectors $x_1, \ldots, x_n \in X$ we have

$$\Big(\sum_k \|x_k\|^p\Big)^{1/p} \le C\Big(\int_0^1 \big\|\sum_k r_k(t)x_k\big\|^2 dt\Big)^{1/2}; \tag{2.71}$$

here r_k as usual stands for the ith Rademacher function on $[0,1]$. It is well-known that each $L_p(\mu)$ has cotype $\max\{p,2\}$. A Banach space X is said to have finite cotype if it has cotype p for some $2 \le p < \infty$.

Corollary 8. *Let $E(\mu)$ be a Banach function space and X a Banach space, both of finite cotype. Assume that $\sum_k x_k$ is an unconditionally convergent series in $E(\mu,X)$, and f its sum. Then there is a Riesz matrix $R^\lambda = (r^\lambda_{jk})$ such that*

$$\sup_j \Big\| \sum_{k=0}^{\infty} r^\lambda_{jk} \sum_{\ell=0}^{k} x_\ell(\cdot)\Big\|_X \in E(\mu)\,,$$

and μ-almost everywhere

$$\lim_j \sum_{k=0}^{\infty} r^\lambda_{jk} \sum_{\ell=0}^{k} x_\ell = f\,.$$

In the case of orthonormal series this interesting result is a relatively simple consequence on Zygmund's work from [97] (see e.g. [1, p.142]).

Proof. It can be seen easily that $E(\mu,X)$ has finite cotype, say cotype r for $2 \le r < \infty$ (see e.g. [57, Theorem 3.3]). We know that the operator

$$u : c_0 \longrightarrow E(\mu,X),\; ue_k := x_k$$

by a result of Maurey is q-summing for each $r < q < \infty$; indeed, the fact that $E(\mu,X)$ has cotype r implies that u is $(r,1)$-summing, and then it is $r+\varepsilon$-summing for each $\varepsilon > 0$ (see e.g. [6, Sect. 24.7]). Fix such q. Then by (2.16) we get a factorization

$$\begin{tikzcd} c_0 \arrow[r, "u"] \arrow[dr, "D_\alpha"'] & E(\mu,X) \\ & \ell_q \arrow[u, "v"'] \end{tikzcd}$$

where v is some operator and D_α is a diagonal operator with $\alpha \in \ell_q$. In particular, we see that $x_k = \alpha_k y_k$ where the $y_k := v(e_k)$ form a weakly q'-summable sequence in $E(\mu,X)$. Choose a positive sequence (μ_k) which increases to ∞ and which satisfies $\sum_k |\alpha_k \mu_k|^q < \infty$. Define first $\lambda_k := 2^{2^{\mu_k}}$, hence $\sum_k |\alpha_k \log\log\lambda_k|^q < \infty$, and second the desired Riesz matrix R^λ by

$$r^\lambda_{jk} := \begin{cases} \dfrac{\lambda_{k+1}-\lambda_k}{\lambda_{j+1}} & k \le j \\ 0 & k > j\,. \end{cases}$$

By Theorem 8 the matrix product $A = R^\lambda \,\Sigma D_{(1/\log\log\lambda_k)}$ given by

$$a_{jk} = \begin{cases} \left(1 - \dfrac{\lambda_k}{\lambda_{j+1}}\right) \dfrac{1}{\log\log\lambda_k} & k \le j \\ 0 & k > j \end{cases}$$

is (p,q)-maximizing – in particular, we have that

$$\sup_j \Big| \sum_k r^\lambda_{jk} \sum_{\ell=0}^k x_\ell \Big| = \sup_j \Big| \sum_{k=0}^\infty a_{jk}\, \alpha_k \log\log\lambda_k\, y_k \Big| \in E(\mu, X)\,.$$

In order to obtain the second statement we conclude from Proposition 2 that

$$\Big(\sum_k r^\lambda_{jk} \sum_{\ell=0}^k x_\ell \Big)_j = \Big(\sum_{k=0}^\infty a_{jk}\, \alpha_k \log\log\lambda_k\, y_k \Big)_j$$

converges μ-almost everywhere. Since (r^λ_{jk}) is a summation process we finally see – taking the limit first in $E(\mu, X)$ – that

$$f = \sum_k x_k = \lim_j \sum_k r^\lambda_{jk} \sum_{\ell=0}^k x_\ell \quad \mu - \text{a.e.}\,,$$

which completes the proof. □

2.3.2 Laws of Large Numbers in Banach Function Spaces

Given a sequence of random variables X_k on a probability space all with variation 0, a typical law of large numbers isolates necessary conditions under which the arithmetic means

$$\frac{1}{j+1} \sum_{k=0}^j X_k$$

converge to zero almost everywhere. Of course, theorems of this type also make sense if instead of the arithmetic means we take linear means

$$\sum_{k=0}^j s_{jk} \sum_{\ell=0}^k X_\ell$$

with respect to a given lower triangle summation process S. Via Kronecker's Lemma 11 each coefficient test for orthonormal series generates a law of large numbers for orthogonal sequences – this is the content of the following

Lemma 14. *Let S be an lower triangular summation method and ω a Weyl sequence. Then for each orthogonal sequence (x_k) in $L_2(\mu)$ with $\sum_k \frac{\omega_k^2}{k^2}\|x_k\|_2^2 < \infty$ we have*

$$\lim_j \frac{1}{j+1}\sum_{k=0}^{j} s_{jk}\sum_{\ell=0}^{k} x_\ell = 0 \quad \mu - a.e.$$

If S in addition satisfies that for each orthonormal series $\sum_k \alpha_k x_k$ in $L_2(\mu)$

$$\sup_j \Big|\sum_{k=0}^{j} s_{jk}\sum_{\ell=0}^{k}\frac{\alpha_\ell}{\omega_\ell}x_\ell\Big| \in L_2(\mu),$$

then for each orthogonal sequence (x_k) in $L_2(\mu)$ with $\sum_k \frac{\omega_k^2}{k^2}\|x_k\|_2^2 < \infty$

$$\sup_j \Big|\frac{1}{j+1}\sum_{k=0}^{j} s_{jk}\sum_{\ell=0}^{k} x_\ell\Big| \in L_2(\mu).$$

This result in particular applies to ordinary summation, Riesz$^\lambda$ summation, Cesàro^r summation or Abel$^\rho$ summation, and ω in this case is the related optimal Weyl sequence (see Sect. 2.2).

Proof. Take some orthogonal sequence (x_k) in $L_2(\mu)$ such that $\sum_k \frac{\omega_k^2}{k^2}\|x_k\|_2^2 < \infty$. Then $\sum_k \frac{\omega_k\|x_k\|_2}{k}\frac{x_k}{\|x_k\|_2}$ is an orthonormal sequence, and since ω is a Weyl sequence for S we see that

$$\lim_j \sum_{k=0}^{j} s_{jk}\sum_{\ell=0}^{k}\frac{x_\ell}{\ell} = \sum_{k=0}^{\infty}\frac{x_k}{k} \quad \mu - a.e.$$

Define the matrix $A = S\Sigma$ and note that for each choice of finitely many scalars $\xi_0,\dots,\xi_j$

$$\sum_{k=0}^{j} a_{jk}\xi_k = \sum_{k=0}^{j} s_{jk}\sum_{\ell=0}^{k}\xi_\ell\,.$$

Hence by Kronecker's Lemma 11,(2) we see that

$$0 = \lim_j \frac{1}{j+1}\sum_{k=0}^{j} a_{jk}x_k = \lim_j \frac{1}{j+1}\sum_{k=0}^{j} s_{jk}\sum_{\ell=0}^{k} x_\ell \quad \mu - a.e.$$

To prove the second result, note that by assumption we have that

$$\sup_j \Big|\sum_{k=0}^{j} s_{jk}\sum_{\ell=0}^{k}\frac{x_\ell}{\ell}\Big| \in L_2(\mu)\,.$$

Hence we apply Lemma 11,(1) to conclude that

$$\begin{aligned}\sup_j \Big| \frac{1}{j+1} \sum_{k=0}^{j} s_{jk} \sum_{\ell=0}^{k} x_\ell \Big| &= \sup_j \Big| \sum_{k=0}^{j} \frac{1}{j+1} a_{jk} x_k \Big| \\ &\leq 2 \sup_j \Big| \sum_{k=0}^{j} a_{jk} \frac{x_k}{k} \Big| \\ &= 2 \sup_j \Big| \sum_{k=0}^{j} s_{jk} \sum_{\ell=0}^{k} \frac{x_\ell}{\ell} \Big| \in L_2(\mu),\end{aligned}$$

the conclusion. □

To see an example we mention the law of large numbers which in the sense of the preceding result corresponds to the Menchoff-Rademacher theorem 6 (see e.g. [82, p.86-87]): *For each orthogonal system (x_k) in $L_2(\mu)$ with $\sum_k \frac{\log^2 k}{k^2} \|x_k\|_2^2 < \infty$ we have*

$$\lim_j \frac{1}{j+1} \sum_{k=0}^{j} x_k = 0 \quad \mu - a.e.$$

and

$$\sup_j \Big| \frac{1}{j+1} \sum_{k=0}^{j} x_k \Big| \in L_2(\mu). \tag{2.72}$$

The main aim of this section is to show that each such law of large numbers for orthogonal sequences of square integrable random variables which additionally satisfies a maximal inequality like in (2.72), transfers in a very complete sense to a law of large numbers in vector-valued Banach function spaces $E(\mu, X)$.

Theorem 15. *Let S be a lower triangular summation method. Assume that ω is an increasing sequence of positive scalars such that for each orthogonal sequence (x_k) in $L_2(\mu)$ with $\sum_k \frac{\omega_k^2}{k^2} \|x_k\|_2^2 < \infty$ we have*

$$\sup_j \Big| \frac{1}{j+1} \sum_{k=0}^{j} s_{jk} \sum_{\ell=0}^{k} x_\ell \Big| \in L_2(\mu).$$

Then for each unconditionally convergent series $\sum_k \frac{\omega_k}{k} x_k$ in $E(\mu, X)$

(1) $\sup_j \Big\| \frac{1}{j+1} \sum_{k=0}^{j} s_{jk} \sum_{\ell=0}^{k} x_\ell(\cdot) \Big\|_X \in E(\mu)$
(2) $\lim_j \frac{1}{j+1} \sum_{k=0}^{j} s_{jk} \sum_{\ell=0}^{k} x_\ell = 0 \quad \mu$*-a.e.*

Proof. We have to repeat part of the preceding proof. For every orthonormal series $\sum_k \alpha_k x_k$ in $L_2(\mu)$ we have by assumption that

$$\sup_j \Big| \sum_{k=0}^{j} \frac{1}{j+1} s_{jk} \sum_{\ell=0}^{k} \frac{\ell}{\omega_\ell} \alpha_\ell x_\ell \Big| \in L_2(\mu).$$

Moreover for

$$b_{jk} := \begin{cases} \frac{k}{(j+1)\omega_k} \sum_{\ell=k}^{j} s_{j\ell} & k \leq j \\ 0 & k > j \end{cases}$$

(compare with (2.5)) we have for each choice of scalars $\xi_0, \ldots, \xi_j$ that

$$\sum_{k=0}^{j} b_{jk}\xi_k = \sum_{k=0}^{j} \frac{1}{j+1} s_{jk} \sum_{\ell=0}^{k} \frac{\ell}{\omega_\ell} \xi_\ell .$$

Hence, we deduce from Theorem 1 that B is $(2,2)$-maximizing, and obtain (1) from Theorem 13, (1). Moreover, since the kth column of B converges to 0 (compare with (2.67)), we deduce from Theorem 13, (2) that the limit in (2) exists almost everywhere, and it remains to show that this limit is 0 almost everywhere. Define the matrix $A = S\Sigma$. Since S is a summation method and the series $\sum_k \frac{x_k}{k}$ converges in $E(\mu,X)$, we have

$$\sum_{k=0}^{\infty} \frac{x_k}{k} = \lim_j \sum_{k=0}^{j} s_{jk} \sum_{\ell=0}^{k} \frac{x_\ell}{\ell} = \lim_j \sum_{k=0}^{j} a_{jk} \frac{x_k}{k} ,$$

the limits taken in $E(\mu,X)$. Hence by Kronecker's Lemma 11,(2) we see that in $E(\mu,X)$

$$0 = \lim_j \frac{1}{j+1} \sum_{k=0}^{j} a_{jk} x_k = \lim_j \frac{1}{j+1} \sum_{k=0}^{j} s_{jk} \sum_{\ell=0}^{k} x_\ell .$$

As a consequence a subsequence of the latter sequence converges almost everywhere to 0 which clearly gives the claim. □

As a particular case, we deduce from (2.72) the following

Corollary 9. *For sequences (x_k) in $E(\mu,X)$ for which $\sum_k \frac{\log k}{k} x_k$ converges unconditionally we have*

$$\lim_j \frac{1}{j+1} \sum_{k=0}^{j} x_k = 0 \quad \mu - a.e.$$

and

$$\sup_j \Big\| \frac{1}{j+1} \sum_{k=0}^{j} x_k(\cdot) \Big\|_X \in E(\mu) .$$

Applying Theorem 9 to Theorem 15 we obtain in the same way that, given a sequence (x_k) in $E(\mu,X)$ for which $\sum_k \frac{\log\log k}{k} x_k$ converges unconditionally, we have

$$\lim_j \frac{1}{(j+1)^2} \sum_{k=0}^{j} \sum_{\ell=0}^{k} x_\ell = 0 \quad \mu - a.e. \tag{2.73}$$

and

$$\sup_j \Big\| \frac{1}{(j+1)^2} \sum_{k=0}^{j} \sum_{\ell=0}^{k} x_\ell(\cdot) \Big\|_X \in E(\mu)\,.$$

It is surprising and part of the next theorem that in contrast to the situation in Corollary 9 the double logarithmic term in the assumption for (2.73) is superfluous – even for Cesàro summation of arbitrary $r > 0$.

Theorem 16. *Let $\sum_k \frac{x_k}{k}$ be an unconditionally convergent series in some vector-valued Banach function space $E(\mu, X)$. Then for each $r > 0$ we have*

(1) $\sup_j \Big\| \frac{1}{j+1} \sum_{k=0}^{j} \frac{A_{j-k}^{r-1}}{A_j^r} \sum_{\ell=0}^{k} x_\ell \Big\|_X \in E(\mu)$

(2) $\lim_j \frac{1}{j+1} \sum_{k=0}^{j} \frac{A_{j-k}^{r-1}}{A_j^r} \sum_{\ell=0}^{k} x_\ell = 0 \quad \mu - a.e.$

For the very special case of orthogonal sequences (x_k) in some $L_2(\mu)$ statement (2) of this result is due to Moricz [63, Theorem 2]; our proof will use Theorem 11 which after all was a consequences of the maximal inequalities (2.54) and (2.59).

Proof. Recall the definition of Cesàro summation of order r from Sect. 2.1.1:

$$c_{jk}^r := \begin{cases} \dfrac{A_{j-k}^{r-1}}{A_j^r} & k \le j \\ 0 & k > j\,, \end{cases}$$

and that for each choice of scalars $\xi_0, \ldots, \xi_j$ we have

$$\frac{1}{j+1} \sum_{k=0}^{j} \frac{A_{j-k}^{r-1}}{A_j^r} \sum_{\ell=0}^{k} \ell \xi_\ell = \sum_{k=0}^{j} \frac{k}{j+1} \frac{A_{j-k}^r}{A_j^r} \xi_k$$

(see (2.45)). Moreover, we proved in Theorem 11 that the matrix M^r defined by

$$m_{jk}^r := \begin{cases} \dfrac{k}{j+1} \dfrac{A_{j-k}^r}{A_j^r} & k \le j \\ 0 & k > j \end{cases}$$

is $(2,2)$-maximizing. Hence, we know that by the very definition of maximizing matrices for each orthogonal sequence (x_k) in $L_2(\mu)$ with $\sum_k \frac{\|x_k\|_2^2}{k^2} < \infty$ we have

$$\sup_j \Big| \frac{1}{j+1} \sum_{k=0}^{j} \frac{A_{j-k}^{r-1}}{A_j^r} \sum_{\ell=0}^{k} x_\ell \Big| \in L_2(\mu)\,,$$

i.e. the matrix C^r satisfies the assumptions of Theorem 15 which in turn gives the desired result ($\omega_k = 1$). $\square$

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