Chapter 2
The Commutation's Theorem

We show that for a locally compact unimodular group G, every $T \in CV_2(G)$ is the limit of convolution operators associated to bounded measures.

2.1 The Convolution Operator $T\lambda_G^p(f)$

Theorem 1. *Let G be a locally compact group, $1 < p < \infty$, $T \in CV_p(G)$, $f \in \mathcal{M}_{00}^{\infty}(G)$, $r \in T[f]$, $\varphi \in \mathcal{L}^p(G)$ and $\psi \in \mathcal{L}^{p'}(G)$. Then:*

1. $\psi * \tilde{r} \in \mathcal{L}^{p'}(G)$,
2. $N_{p'}(\psi * \tilde{r}) \leq |||T|||_p \, N_{p'}(\psi) \int_G |f(x)|\Delta_G(x)^{-\frac{1}{p'}} dx$,
3. $\left\langle T\lambda_G^p(\tau_p f)[\varphi], [\psi] \right\rangle = \int_G \varphi(x)\overline{(\psi * \tilde{r})(x)} dx$.

Proof. To begin with suppose $\varphi \in C_{00}(G)$. We have

$$\left\langle T\lambda_G^p(\tau_p f)[\varphi], [\psi] \right\rangle = \langle [\varphi * r], [\psi] \rangle.$$

From $(|\varphi| * |r|)|\psi| \in \mathcal{L}^1(G)$ we get

$$\int_G (\varphi * r)(x)\overline{\psi(x)} dx = \int_G \varphi(x)\overline{(\psi * \tilde{r})(x)} dx.$$

The inequalities

$$\left| \int_G \varphi(x)\overline{(\psi * \tilde{r})(x)} dx \right| \leq N_p(\varphi * r) N_{p'}(\psi) \leq |||T|||_p \, N_p(\varphi) N_{p'}(\psi) \int_G f(x)|\Delta_G(x)^{-\frac{1}{p'}} dx$$

prove (1) and (2).

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Suppose now that $\varphi \in \mathcal{L}^p(G)$.

There is a sequence (φ_n) of $C_{00}(G)$ with $N_p(\varphi_n - \varphi) \to 0$. We have

$$\lim \int_G \varphi_n(x)\overline{(\psi * \tilde{r})(x)}dx = \left\langle T\lambda_G^p(\tau_p f)[\varphi], [\psi]\right\rangle$$

and

$$\left| \int_G \varphi(x)\overline{(\psi * \tilde{r})(x)}dx - \int_G \varphi_n(x)\overline{(\psi * \tilde{r})(x)}dx \right| \leq N_p(\varphi_n - \varphi)N_{p'}(\psi * \tilde{r}).$$

Consequently

$$\int_G \varphi(x)\overline{(\psi * \tilde{r})(x)}dx = \left\langle T\lambda_G^p(\tau_p f)[\varphi], [\psi]\right\rangle.$$

Remark. Even for $p = 2$, we are unable to decide whether $|\psi| * |\check{r}|$ is in $\mathcal{L}^{p'}(G)$.

We now show that every $T \in CV_p(G)$ can be approximated by $T\lambda_G^p(f)$.

Proposition 2. *Let G be a locally compact group, $1 < p < \infty$ and I the set of all $f \in C_{00}(G)$ with $f(x) \geq 0$ for every $x \in G$, $f(e) \neq 0$ and*

$$\int_G f(x)\Delta_G(x)^{-1/p'}dx = 1.$$

Then:

1. *on I the relation* supp $f' \subset$ supp f *is a filtering partial order,*
2. *for $f \in I$ we have $\left|\left|\left|\lambda_G^p(\tau_p f)\right|\right|\right|_p \leq 1$,*
3. *for every $T \in CV_p(G)$ the net $\left(T\lambda_G^p(\tau_p f)\right)_{f \in I}$ converges strongly to T.*

Proof. Let $T \in CV_p(G)$, $\varphi \in \mathcal{L}^p(G)$ and $\varepsilon > 0$. Let U be a neighborhood of e in G such that for $y \in U$

$$N_p\left(\varphi - (\varphi)_{y^{-1}}\Delta_G(y^{-1})^{1/p}\right) < \frac{\varepsilon}{(1 + |||T|||_p)}.$$

Let also $f \in I$ with supp $f \subset U$. From

$$\|T[\varphi] - T\lambda_G^p(\tau_p f)[\varphi]\|_p \leq |||T|||_p \int_G N_p\left(\varphi - (\varphi)_{y^{-1}}\Delta_G(y^{-1})^{1/p}\right) f(y)\Delta_G(y^{-1})^{1/p'}dy$$

we get $\|T[\varphi] - T\lambda_G^p(\tau_p f)[\varphi]\|_p < \varepsilon$.

The investigation of $CV_2(G)$ requires the study of those continuous operators S of $L^2(G)$ for which $S(\varphi_a) = (S\varphi)_a$.

In full analogy with Sect. 1.2 we have

$$(\mu * \varphi)(x) = \int_G \varphi(y^{-1}x)d\mu(y)$$

for $\mu \in M^1(G)$, $\varphi \in C_{00}(G)$ and $x \in G$. We also have $\mu * \varphi \in C(G) \cap \mathcal{L}^p(G)$ and

$$N_p(\mu * \varphi) \le \|\mu\| N_p(\varphi)$$

for $1 < p < \infty$. There is a unique continuous operator S of $L^p(G)$ with $S[\varphi] = [\mu * \varphi]$ for $\varphi \in C_{00}(G)$. We have $S(f_a) = (Sf)_a$ for $f \in L^p(G)$ and $a \in G$. This operator S is denoted $\rho_G^p(\mu)$. For $f \in \mathcal{L}^1(G)$ we set $\rho_G^p(f) = \rho_G^p(fm_G)$ and $\rho_G^p([f]) = \rho_G^p(f)$.

Definition 1. Let G be a locally compact group, $1 < p < \infty$ and $S \in \mathcal{L}(L^p(G))$. We say that S belongs to the set $CV_p^d(G)$ if $S(\varphi_a) = (S\varphi)_a$ for every $a \in G$ and for every $\varphi \in L^p(G)$.

Proposition 3. *Let G be a locally compact group and $1 < p < \infty$. Then $CV_p^d(G)$ is a Banach subalgebra of $\mathcal{L}(L^p(G))$.*

Proposition 4. *Let G be a locally compact group and $1 < p < \infty$. Then:*

1. *ρ_G^p is a linear injective contraction of the Banach space $M^1(G)$ into the Banach space $CV_p^d(G)$,*
2. *for every $a \in G$ and every $\varphi \in L^p(G)$ we have*

$$\rho_G^p(\delta_a)\varphi = {}_{a^{-1}}\varphi$$

   and $\left\|\left|\rho_G^p(\delta_a)\right|\right\|_p = 1$,
3. *$\rho_G^p(\delta_{ab}) = \rho_G^p(\delta_a)\rho_G^p(\delta_b)$ for every $a, b \in G$,*
4. *for $f \in L^1(G)$ and $\varphi \in C_{00}(G)$ we have $\rho_G^p(f)[\varphi] = f * [\varphi]$.*

Theorem 5. *Let G be a locally compact group $1 < p < \infty$ and $S \in \mathcal{L}(L^p(G))$. Then $S \in CV_p^d(G)$ if and only if*

$$S\big(\varphi * \Delta_G^{1/p'} f\big) = (S\varphi) * \big(\Delta_G^{1/p'} f\big)$$

for every $f \in L^1(G)$ and every $\varphi \in L^p(G)$.

Remarks. 1. The map $x \mapsto \rho_G^2(\delta_x)$ is the left regular representation of G.

2. The proofs of Proposition 4 and Theorem 5 are entirely similar to those of the corresponding results concerning $CV_p(G)$ and λ_G^p (cf Sect. 1.2).

Similarly to Theorem 1 and Proposition 2 the following two results are verified.

Proposition 6. *Let G be a locally compact group, $1 < p < \infty$ and I the set of all $f \in C_{00}(G)$ with $f(x) \geq 0$ for every $x \in G$, $f(e) \neq 0$ and $\int_G f(x)dx = 1$. Then:*

1. *on I the relation* $\operatorname{supp} f' \subset \operatorname{supp} f$ *is a filtering partial order,*
2. *For $f \in I$ we have $\left\|\left|\rho_G^p(f)\right|\right\|_p \leq 1$,*
3. *For every $S \in CV_p^d(G)$ the net $\left(S\rho_G^p(f)\right)_{f \in I}$ converges strongly to S.*

Theorem 7. *Let G be a locally compact group, $1 < p < \infty$, $S \in CV_p^d(G)$, $g \in \mathcal{M}_{00}^{\infty}(G)$, $s \in S[g]$, $\varphi \in \mathcal{L}^p(G)$ and $\psi \in \mathcal{L}^{p'}(G)$. Then:*

1. $s^* * \psi \in \mathcal{L}^{p'}(G)$,
2. $N_{p'}(s^* * \psi) \leq \|\|S\|\|_p N_{p'}(\psi) N_1(g)$,
3. $\left\langle S\rho_G^p(g)[\varphi], [\psi]\right\rangle = \int_G \varphi(x)\overline{(s^* * \psi)(x)}dx.$

2.2 A Commutation Property of $CV_2(G)$

For $S \in CV_p^d(G)$ and $T \in CV_p(G)$, we first obtain integral formulas for $T\lambda_G^p(f)S\rho_G^p(g)$ and for $S\rho_G^p(g)T\lambda_G^p(f)$.

Proposition 1. *Let G be a locally compact group, $1 < p < \infty$, $S \in CV_p^d(G)$, $T \in CV_p(G)$, $f, g \in \mathcal{M}_{00}^{\infty}(G)$, $s \in S[g]$ and $r \in T[f]$. Then:*

$$\left\langle T\lambda_G^p(\tau_p f)S\rho_G^p(g)[\varphi], [\psi]\right\rangle = \int_G \varphi(x)\overline{(s^* * (\psi * \tilde{r}))(x)}dx$$

for $\varphi \in \mathcal{L}^p(G)$ and $\psi \in \mathcal{L}^{p'}(G)$.

Proof. Let $\varphi_1 \in S\rho_G^p(g)[\varphi]$ and

$$I = \left\langle T\lambda_G^p(\tau_p f)S\rho_G^p(g)[\varphi], [\psi]\right\rangle.$$

Then by Theorem 1 of Sect. 2.1 $\psi * \tilde{r} \in \mathcal{L}^{p'}(G)$ and $I = \int_G \varphi_1(x)\overline{(\psi * \tilde{r})(x)}dx.$

Consequently

$$I = \left\langle S\rho_G^p(g)[\varphi], [\psi * \tilde{r}]\right\rangle.$$

Then by Theorem 7 of Sect. 2.1 $s^* * (\psi * \tilde{r}) \in \mathcal{L}^{p'}(G)$ and

$$\left\langle S\rho_G^p(g)[\varphi], [\psi * \tilde{r}]\right\rangle = \int_G \varphi(x)\overline{(s^* * (\psi * \tilde{r}))(x)}dx.$$

Proposition 2. *Let G be a locally compact group, $1 < p < \infty$, $S \in CV_p^d(G)$, $T \in CV_p(G)$, $f, g \in \mathcal{M}_{00}^\infty(G)$, $s \in S[g]$ and $r \in T[f]$. Then:*

$$\left\langle S\rho_G^p(g)T\lambda_G^p(\tau_p f)[\varphi], [\psi]\right\rangle = \int_G \varphi(x)\overline{((s^* * \psi) * \tilde{r})(x)}dx$$

for every $\varphi \in \mathcal{L}^p(G)$ and every $\psi \in \mathcal{L}^{p'}(G)$.

Proof. Let $\varphi_1 \in T\lambda_G^p(\tau_p f)[\varphi]$ and

$$I = \left\langle S\rho_G^p(g)T\lambda_G^p(\tau_p f)[\varphi], [\psi]\right\rangle.$$

Then by Theorem 7 of Sect. 2.1 $s^* * \psi \in \mathcal{L}^{p'}(G)$ and

$$I = \int_G \varphi_1(x)\overline{(s^* * \psi)(x)}dx$$

and therefore

$$I = \left\langle T\lambda_G^p(\tau_p f)[\varphi], [s^* * \psi]\right\rangle.$$

We finally apply Theorem 1 of Sect. 2.1: we have $(s^* * \psi) * \tilde{r} \in \mathcal{L}^{p'}(G)$ and

$$I = \int_G \varphi(x)\overline{(s^* * (\psi * \tilde{r}))(x)}dx.$$

In the following it will be decisive to assume the unimodularity of the locally compact group G. With this assumption, we have $\tau_p f = \check{f}$.

Lemma 3. *Let G be a locally compact unimodular group, $S \in CV_2^d(G)$, $T \in CV_2(G)$ and $f, g \in \mathcal{M}_{00}^\infty(G)$. Then $T\lambda_G^2(f)S\rho_G^2(g) = S\rho_G^2(g)T\lambda_G^2(f)$.*

Proof. For $r \in T[f]$, $s \in S[g]$ and $\varphi, \psi \in \mathcal{M}_{00}^\infty(G)$ we have

$$\left\langle T\lambda_G^2(\check{f})S\rho_G^2(g)[\varphi], [\psi]\right\rangle = \int_G \varphi(x)\overline{(s^* * (\psi * \tilde{r}))(x)}dx$$

and

$$\left\langle S\rho_G^2(g)T\lambda_G^2(\check{f})[\varphi], [\psi]\right\rangle = \int_G \varphi(x)\overline{((s^* * \psi) * \tilde{r})(x)}dx.$$

By the unimodularity of G, for every $x \in G$ we have $(|\psi| * |\check{r}|)_x \in \mathcal{L}^2(G)$, and consequently

$$\int_G^* |s(y)| \left(\int_G |\psi(yxz)| r(z) | dz \right) dy < \infty.$$

This implies $s^* * (\psi * \tilde{r}) = (s^* * \psi) * \tilde{r}$.

Theorem 4. *Let G be a locally compact unimodular group. Then $ST = TS$ for $S \in CV_2^d(G)$ and $T \in CV_2(G)$.*

Proof. To begin with we prove that for $S \in CV_2^d(G)$, $T \in CV_2(G)$ and $f \in \mathcal{M}_{00}^\infty(G)$ we have $ST\lambda_G^2(f) = T\lambda_G^2(f)S$.

Let $\varphi \in L^2(G)$ and $\varepsilon > 0$. There is $g \in C_{00}(G)$ with:
$g(x) \geq 0$ for every $x \in G$, $\int_G g(x)dx = 1$,

$$\|S\rho_G^2(g)\varphi - S\varphi\|_2 < \frac{\varepsilon}{2(1 + \|T\lambda_G^2(f)\|_2)} \quad \text{and} \quad \|S\rho_G^2(g)T\lambda_G^2(f)\varphi - ST\lambda_G^2(f)\varphi\|_2 < \frac{\varepsilon}{2}.$$

Now from

$$\|ST\lambda_G^2(f)\varphi - T\lambda_G^2(f)S\varphi\|_2 \leq \|ST\lambda_G^2(f)\varphi - S\rho_G^2(g)T\lambda_G^2(f)\varphi\|_2$$

$$+\|S\rho_G^2(g)T\lambda_G^2(f)\varphi - T\lambda_G^2(f)S\rho_G^2(g)\varphi\|_2 + \|T\lambda_G^2(f)S\rho_G^2(g)\varphi - T\lambda_G^2(f)S\varphi\|_2,$$

Lemma 3 and

$$\|T\lambda_G^2(f)S\rho_G^2(g)\varphi - T\lambda_G^2(f)S\varphi\|_2 < \frac{\varepsilon}{2}$$

we get

$$\|ST\lambda_G^2(f)\varphi - T\lambda_G^2(f)S\varphi\|_2 < \varepsilon.$$

Next let $\varphi \in L^2(G)$ and $\varepsilon > 0$. According to Proposition 2 of Sect. 2.1 there is $f \in C_{00}(G)$ with $f(x) \geq 0$ for every $x \in G$, $\int_G f(x)dx = 1$,

$$\|TS\varphi - T\lambda_G^2(f)S\varphi\|_2 < \frac{\varepsilon}{2} \quad \text{and} \quad \|T\varphi - T\lambda_G^2(f)\varphi\|_2 < \frac{\varepsilon}{2(1 + \|S\|_2)}.$$

From

$$\|TS\varphi - ST\varphi\|_2$$

$$\leq \|TS\varphi - T\lambda_G^2(f)S\varphi\|_2 + \|T\lambda_G^2(f)S\varphi - ST\lambda_G^2(f)\varphi\|_2 + \|ST\lambda_G^2(f)\varphi - ST\varphi\|_2,$$

$$T\lambda_G^2(f)S\varphi = ST\lambda_G^2(f)\varphi \quad \text{and} \quad \|ST\lambda_G^2(f)\varphi - ST\varphi\|_2 < \frac{\varepsilon}{2}$$

we obtain

$$\|TS\varphi - ST\varphi\|_2 < \varepsilon.$$

2.3 An Approximation Theorem for $CV_2(G)$

Using the commutation theorem of Sect. 2.2 (Theorem 4) we show that every $T \in CV_2(G)$ is the limit of $\lambda_G^2(\mu)$ for G a locally compact unimodular group.

For a complex Hilbert space $\mathcal{H}$, we denote by $\mathcal{L}(\mathcal{H})$ the involutive Banach algebra of all continuous operators of $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, $\|T\|$ is the norm of the operator T. For $\mathcal{E}$ a subset of $\mathcal{L}(\mathcal{H})$ we denote by $\mathcal{E}'$ the set of all $T \in \mathcal{L}(\mathcal{H})$ with $ST = TS$ for every $S \in \mathcal{E}$, and we put $\mathcal{E}'' = (\mathcal{E}')'$.

Theorem 1. *Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}$ an involutive subalgebra of $\mathcal{L}(\mathcal{H})$ with $\left\{Tx \middle| x \in H, T \in \mathcal{B}\right\}$ dense in H. Then $\mathcal{B}''$ coincides with the closure of $\mathcal{B}$ in $\mathcal{L}(\mathcal{H})$ with respect to the strong operator topology.*

Proof. See [36], J. Dixmier, Chap. I, Sect. 3, no. 4, Corollaire 1, p. 42.

The next result is Kaplansky's density theorem.

Theorem 2. *Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}, \mathcal{C}$ two involutive subalgebras of $\mathcal{L}(\mathcal{H})$ with $\mathcal{B} \subset \mathcal{C}$. Suppose that $\mathcal{C}$ is dense in the strong closure of $\mathcal{B}$ in $\mathcal{L}(\mathcal{H})$. Then for every $T \in \mathcal{C}$ there is a net (S_α) of $\mathcal{B}$ such that:*

1. $\lim_\alpha S_\alpha = T$ *strongly,*
2. $\|S_\alpha\| \le \|T\|$ *for every* α.

Proof. See Dixmier, [36], Chap. I, Sect. 3, no. 5, Théorème 3, p. 43–44.

Let G be a locally compact group. In this paragraph, we denote by $\mathcal{A}$ the set of all $\lambda_G^2(\mu)$, where μ is a complex measure with finite support. Clearly $\mathcal{A}$ is an involutive subalgebra $\mathcal{L}(L^2(G))$ with unit: $\lambda_G^2(\mu)^* = \lambda_G^2(\tilde{\mu})$ and $\lambda_G^2(\delta_e) = \mathrm{id}_{L^2_{\mathbb{C}}(G)}$. The following statement is straightforward.

Proposition 3. *Let G be a locally compact group. Then $CV_2^d(G) = \mathcal{A}'$.*

We obtain now the promised approximation theorem for $CV_2(G)$.

Theorem 4. *Let G be a locally compact unimodular group and $T \in CV_2(G)$. There is a net (μ_α) of complex measures with finite support such that:*

1. $\lim_\alpha \lambda_G^2(\mu_\alpha) = T$ *strongly,*
2. $\left|\left|\left|\lambda_G^2(\mu_\alpha)\right|\right|\right|_2 \le |||T|||_2$ *for every* α.

Proof. By Theorem 4 of Sect. 2.2 we have $T \in \mathcal{A}''$. It suffices to apply Theorems 1 and 2 to finish the proof.

Remarks. 1. The fact that $\left\{\lambda_G^2(\delta_x) \middle| x \in G\right\}'' = CV_2(G)$, for G locally compact and unimodular, is due to Segal ([110], Theorem, p. 294). The case of G discrete, was obtained earlier by Murray and von Neumann ([96], Lemma 5.3.3, p. 789).

2. Using different methods, Dixmier obtained $\left\{\lambda_G^2(\delta_x) \middle| x \in G\right\}'' = CV_2(G)$, and consequently Theorem 4, for every locally compact group G ([35], Théorème 1,

p. 280, [36], Chap. I, Sect. 5, p. 71, Théorème 1 and Exercice 5 p. 80). See also Mackey ([90], p. 207, Lemma 3.3.)

Theorem 5. *Let G be a locally compact unimodular group and $T \in CV_2(G)$. There is a net (f_α) of $C_{00}(G)$ such that:*

1. $\lim_\alpha \lambda_G^2(f_\alpha) = T$ *strongly,*
2. $\left|\left|\left|\lambda_G^2(f_\alpha)\right|\right|\right|_2 \leq |||T|||_2$ *for every α.*

Proof. According to Theorem 1 $\left(\lambda_G^2(C_{00}(G))\right)''$ is the strong closure of $\lambda_G^2(C_{00}(G))$. But by Theorem 5 of Sect. 2.1 $\left(\lambda_G^2(C_{00}(G))\right)' = CV_2^d(G)$ and consequently

$$\left(\lambda_G^2(C_{00}(G))\right)'' = CV_2(G).$$

Remark. We will extend this result to $p \neq 2$ for certain classes of locally compact groups. We will also try to give more information on the approximating net (f_α).

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