Preface

Roughly speaking a convolution operator T on a group G is a linear operator on complex functions $\varphi : G \to \mathbb{C}$ that commutes with left translations

$$_g(Tf) = T(_g f).$$

Typically convolution by fixed functions gives rise to convolution operators.

To be more precise, one has to specify G and the underlying function space for T. One may suppose that G is a locally compact group with Haar measure m, and choose T to be a continuous linear endomorphism of $L^p(G) = L^p(G; m)$, where $p > 1$ is some fixed real number. It's these convolution operators that will be the subject of this book, individual cases of them as well as, for given p and G, the space $CV_p(G)$ of all of them.

The set $CV_p(G)$ is a sub Banach algebra of the Banach algebra of all continuous linear endomorphisms of $L^p(G)$. If G is abelian, it is possible to define the Fourier transform of every T in $CV_2(G)$. The Fourier transform is a Banach algebra isometry of $CV_2(G)$ onto $L^\infty(\widehat{G})$. Here, $\widehat{G}$ denotes the Pontrjagin dual of G. Moreover, $CV_p(G) \subset CV_2(G)$, this permits to define the Fourier transform of every T in $CV_p(G)$.

The case of $G = \mathbb{R}^n$ involves results of classical Fourier analysis. For instance, the fact that the Heaviside function is the Fourier transform of some $T \in CV_p(\mathbb{R})$ implies Marcel Riesz's famous theorem on the convergence in L^p of Fourier series. This convergence still holds in two variables for square summation, but not for circular summation if $p \neq 2$. This reflects the fact that the indicator function of any square is the Fourier transform of some $T \in CV_p(\mathbb{R}^2)$ but not the indicator function of the disk except if $p = 2$.

In this book, we will be mainly concerned with the investigation of $CV_p(G)$ for noncommutative groups.

If $k \in L^p(G)$ and $l \in L^{p'}(G)$, then $\overline{k} * \check{l} \in C_0(G)$ with $\|\overline{k} * \check{l}\|_\infty \leq \|k\|_p \|l\|_{p'}$.

Forming series of such functions leads to the very important Figà-Talamanca space $A_p(G)$ contained in $C_0(G)$. $A_p(G)$ is an algebra for the pointwise product.

If it is given a norm based on $\|k\|_p\|l\|_{p'}$, it becomes a Banach algebra. There is a natural duality between $CV_p(G)$ and $A_p(G)$ for a large class of locally compact groups. This duality holds for all locally compact groups if $p = 2$. It is conjectured that it holds even for all p. If G is abelian, then $A_2(G)$ turns out to be the space of Fourier transforms of $L^1(\widehat{G})$. Here, again the Fourier transform is a Banach algebra isometry of $L^1(\widehat{G})$ onto $A_2(G)$.

To every integrable function on G, and more generally to every bounded measure on G, there corresponds by convolution an operator in $CV_p(G)$. For finite groups all of $CV_p(G)$ is obtained in this manner. It is not the case for infinite groups like $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{T}$ and probably for all infinite groups. Then we may ask whether every convolution operator may be approximated by operators associated to bounded measures, and in which topology. For $p = 2$ the answer is yes under the weak operator topology. This result was obtained by Murray and von Neumann for discrete groups, by Segal for unimodular groups and finally by Dixmier for general locally compact groups. The duality between $CV_p(G)$ and $A_p(G)$ permits to answer positively for $p \neq 2$ for all amenable groups.

Let I be an ideal of the algebra $A_p(G)$. The set of points of G where all functions in I vanish will be called the cospectrum of I. An elegant formulation of the celebrated tauberian theorem of Wiener is: if G is an abelian group every ideal of $A_2(G)$ with empty cospectrum is necessarily dense in $A_2(G)$. In this book, we will show that this statement holds for every group and also every $p > 1$. The fact that the theorem of Wiener is verified on arbitrary groups is highly surprising: there are papers suggesting the impossibility of such an extension for the group of two by two invertible matrices of complex numbers!

There is a hudge amount of literature concerning the non-commutative version of the Plancherel theorem and the inversion formula for C^∞ functions with compact support on Lie groups. Such questions are, for commutative groups, very simple. An achievement of this book is the extension to non-commutative groups of theorems which are deep and difficult even for $\mathbb{Z}$, $\mathbb{T}$ or $\mathbb{R}$.

An important part of this monograph deals with the relation between $CV_p(H)$ and $CV_p(G)$, where H is a closed subgroup of G. Let i be the inclusion map of H into G. Then i induces a canonical map, also denoted i, of $CV_p(H)$ into $CV_p(G)$. For $G = \mathbb{R}$ and $H = \mathbb{Z}$, this is a famous result due to Karel de Leeuw (1965), and to Saeki (1970) for G abelian and H arbitrary closed subgroup. It is also possible to characterise the image of i in $CV_p(G)$ and to obtain in this way non-commutative analogs of a result of Reiter (1963) concerning the relations between $L^\infty(\widehat{G})$ and $L^\infty(\widehat{H})$ and also to the fact that H is a set of synthesis in G (1956). The characterisation in $CV_p(G)$ of the image of $CV_p(H)$ under the map i, is a deep result due to Lohoué (1980). A large part of Chap. 7 is devoted to a detailed proof of Lohoué's result. As a consequence we obtain the extension of the Kaplansky–Helson theorem to non-abelian groups and to $p \neq 2$: for x in a arbitrary locally compact group G, every ideal of $A_p(G)$ having the cospectrum $\{x\}$ is dense in the set of all functions vanishing in x.

In the last chapter, we prove that for amenable groups $CV_p(G)$ is contained in $CV_2(G)$: this statement, compared to the commutative case, requires an entirely new approach.

The development of harmonic analysis on non-commutative groups is not just a straightforward generalization of the commutative case. It requires new ideas but it also gives rise to new problems which are far from being solved. For instance, the approximation theorem for non-amenable groups and for $p \neq 2$ is still out of reach. The investigation of the noncommutative case gives a better understanding of the commutative case! For example, instead of studying the relations between $L^\infty(\widehat{G})$ and $L^\infty(\widehat{H})$, it is more conceptual and more fruitfull to investigate the relations between the algebras $CV_2(G)$ and $CV_2(H)$.

A large part of the results presented appeared here for the first time in a book's form. The presentation is selfcontained and complete proofs are given. The prerequisities consists mostly with a familiarity with the books of Hewitt and Ross [66, 67]. (Chaps. 4, 6, 8 and 10), Reiter and Stegeman [105] and Rudin [107]. Notes at the end of the volume contain additional information about results of the text.

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