Chapter 2
Topics in Stability

Pura potenza tenne la parte ima;
nel mezzo strinse potenza con atto
tal vime, che giammai non si divima.
34, XXIX, Paradiso A. Dante

The flows that occur in Nature must not only obey the equations of fluid dynamics, but also be stable. L.D. Landau & E.M. Lifschitz (1959)

2.1 Introduction

In this chapter we introduce some definitions and qualitative methods useful in the study of nonlinear stability with respect to the initial data of a basic fluid motion. The aim of the chapter is to recall the energy and Dirichlet methods used to study distinctive properties of nonlinear stability for incompressible fluids (parabolic equations) and for elastic bodies (hyperbolic equations), respectively, and to give an overview of the results obtained in the book.

The central part of this chapter is a *generalization of the Dirichlet method*, achieved using the free work equation, that we call the *modified energy method*. Specifically, we will study the asymptotic behavior in time of perturbations to a basic state, introducing an auxiliary equation called *the free work equation*. To give a preview of the two main technical tools we will use some simple examples.

It is worth emphasizing that our method is intended to be naive and straightforward, and does not require complicated analysis.

We will first present a short survey of the *energy and Dirichlet methods* used to study nonlinear stability of parabolic and hyperbolic systems. We

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recall that in mechanics the energy and Dirichlet methods are used for incompressible and elastic media, using Eulerian and Lagrangian coordinate systems, respectively.

Next we will introduce two key tools which will be used in the proofs of nonlinear stability and the asymptotic decay to steady compressible flows, which are central to the thesis of this book.

The first tool concerns an *extension of the Dirichlet method* in the wake of the work by Arnold, cf. [4]. In order not to obscure the main idea, we will explain the Dirichlet method by studying the stability of steady flows particular to inviscid fluids both incompressible and compressible.

The second tool is represented by the *free work equation* (FWE), and appears useful for systems of mixed parabolic-hyperbolic type. The FWE allows the transfer of asymptotic behavior in time characteristic of the parabolic part to the hyperbolic one.

For pedagogical reasons, in this introductive chapter we limit ourselves to simple examples; generalizations will be considered in the remaining chapters.

Section 2.2 Basic definitions of nonlinear stability of steady fluids motions will be reviewed. The classical "Energy Method" will be employed to study the nonlinear stability of a steady viscous incompressible flow.

Section 2.3 The Lagrange–Dirichlet method is outlined, and four applications are explained. The first two applications are known stability theorems for the rest state of both inviscid, incompressible fluids and elastic continua; cf. [2, 4]. The second two applications concern the stability of a basic flow for both inviscid and viscous, barotropic gases.

Section 2.4 Nonlinear stability and instability theorems to be proved in the next three chapters are listed.

2.2 Nonlinear Stability

Let us begin with some abstract settings. If we set by Y a scalar Banach space, then $X = [Y]^n$ will denote the vector Banach space given by the Cartesian product of Y n times. In this section we introduce some definitions of stability.

Given a steady flow S_b, the physical concept of the *stability* of S_b is linked to the concept of *observability*. Assume, as a qualitative definition of **stability**, the following proposition:

> *The stability of a given motion is its capacity to 'hold' (to be observed) in the presence of the small perturbations present in any physical system.*

This definition allows us to introduce the correct definition of stability of a given steady solution S_b to the equations of motion that will be referred to as *stability of the basic motion* S_b. At time $t = 0$ we perturb the basic motion S_b, and call $\widetilde{S}_0 = \widetilde{S}(0)$ the perturbed initial data that produce the perturbed motion $\widetilde{S}(t)$. Correspondingly, the **perturbation** $\widetilde{S}_0 - S_b$ at initial time produces the evolution in time of the perturbation $\widetilde{S}(t) - S_b$. The stability question ask:

Is it possible to control $\widetilde{S}(t) - S_b$, in a given spatial norm $\|.\|_X$, for $t \in (0, \infty)$, provided $\widetilde{S}_0 - S_b$ is sufficiently small in the same norm $\|.\|_X$?

Definition 2.2.1 ***Stability*** *The motion* S_b *is said* ***stable in the fixed norm***[1] $\|.\|_X$***, with respect to initial data*** *if and only if for all numbers* $\epsilon > 0$*, there exists* $\delta > 0$ *such that, for all initial perturbations* $\widetilde{S}_0 - S_b$ *having norm in* X *less than* δ*, i.e.* $\|\widetilde{S}_0 - S_b\|_X < \delta$*, the corresponding perturbations* $\widetilde{S}(t) - S_b$ *in the norm* $\|.\|_X$*, remain less that* ϵ*, i.e.* $\|\widetilde{S}(t) - S_b\|_X < \epsilon$ *for all time* t*.*

Basic flows that verify Definition 2.2.5 are sometimes called *stable in the mean*, but this notation is not generally accepted.

Definition 2.2.2 ***Asymptotic Stability*** *If perturbations come back to zero as time goes to infinity,*

$$\lim_{t \to \infty} \|\widetilde{S}(t) - S_b\|_X = 0,$$

we say that the basic motion is ***asymptotically stable****.*

Definition 2.2.3 ***Instability*** *A motion* $S_b(t)$ *is said to be* ***unstable*** *in the* $X-$*norm if it is not stable; that is, if there exists* $\epsilon > 0$*, a sequence of initial data* $\{S_i(0)\}$ *approaching* S_b*, and a sequence of times* t_i*, such that* $\|S_i(t) - S_b\|_X \geq \epsilon$ *for any* $i \in \mathbb{N}$*.*

Under nonlinear stability hypotheses, if we can physically control that at initial time the norm in X of $\widetilde{S}_0 - S_b$ is less than δ, then the basic flow S_b will also be experimentally observable in the class of perturbed flows having initial data sufficiently close to S_b.

2.2.1 Abstract Settings

We denote by f a C^1 vector function defined in the vector Banach space X, with values in X, $f : V \in X \to f(V) \in X$. We study the *abstract* **autonomous evolution problem**

$$\begin{aligned} \frac{dV}{dt} &= f(V); \\ V(0) &= V_0. \end{aligned} \tag{2.2.1}$$

[1] The distance between two motions may be calculated by a measure, instead that by a norm. This will be not considered here.

Usually, (2.2.1) represents an evolution law of a physical quantity of the system S. In our case, $V(t, V_0)$ represents the motion of S, corresponding to initial state V_0. The notation $V(t) = V(t; V_0)$ may also be used. We assume that system (2.2.1) has existence and uniqueness theorems of global in time regular solutions in correspondence of large initial data. In general, the problem of finding explicit solutions to (2.2.1) is very difficult, if not impossible, to solve. As such, it is worth applying qualitative information, such as the uniqueness or the asymptotic behavior of a solution in time, in a given norm $\|.\|_X$. We are trying to frame the concept of time control for a solution (stability) according to a rigorous mathematical formulation. Such arguments were developed in the second half of the nineteenth century, mainly by the French mathematician H. Poincare' (1854–1912), and by the Russian mathematician A.M. Liapunov (1857–1918). In this section we will study the direct method developed by Lyapunov. In referring to a *direct method*, we mean to describe a mechanism that would allow for the direct deduction of certain types of qualitative data once external data are given, and without integrating the equations.

Definition 2.2.4 *A value $V_b \in X$ is said to be a* **critical point** *of* (2.2.1) *if $f(V_b) = 0$.*

Note that the existence of critical points infers knowledge of steady solutions $V(t; V_b) = V_b$.

If there exists a critical point for (2.2.1), then by a simple subtraction we deduce that the function $W(t, W_0)) = V(t; V_0) - V_b \in X$ with $W_0 = V_0 - V_b$ solves the problem

$$\frac{dW}{dt} = g(W) \qquad W(0) = W_0, \tag{2.2.2}$$

where

$$g(W) = f(V_b + W) - f(V_b).$$

In our case we are directly studying the difference of motions, hence S_b is the critical point of (2.2.2), and verifies $W_b(t) = W(t; 0) = 0$.

Definition 2.2.5 ***Nonlinear Stability*** *The zero solution $W_b(t) = 0$ to* (2.2.2) *is said to be* ***nonlinearly stable*** *in the $X-$norm if*

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0 : \ \|W_0\|_X < \delta \quad \Rightarrow \quad \|W(t; W_0)\|_X < \epsilon, \qquad \forall t \in (0, \infty). \tag{2.2.3}$$

A solution $W_b(t) = 0$ to (2.2.2) *is said to be* ***unstable*** *in the $X-$norm if it is not stable; that is, if there exists $\epsilon > 0$, a sequence of initial data $\{W_i\}$ approaching zero, and a sequence of times t_i, such that $\|W(t_i, W_i)\|_X \geq \epsilon$ for any $i \in \mathbb{N}$.*

The difference between continuous dependence and stability lies within the times intervals $(0, T)$ and $(0, \infty)$, where control occurs.

Definition 2.2.6 ***Nonlinear Exponential Stability*** *The solution $W_b(t,0) = 0$ to (2.2.2) is said to be* ***nonlinearly unconditionally stable*** *in the $X-$norm if there is control of perturbations in terms of initial data in the $X-$norm, however large are the perturbations at initial time in the $X-$norm.*

The solution $W_b(t,0) = 0$ to (2.2.2) is said to be ***nonlinearly exponentially stable*** *in the $X-$norm if it is stable and, for any initial datum W_0, it holds*

$$\lim_{t\to\infty} \|W(t;W_0)\|_X = 0, \qquad \|W(t;W_0)\|_X < c\exp^{-\beta t}, \qquad \textit{resp.},$$

with c, β suitable constants, β is the time decay constant.

Notice that *stability Definitions 2.2.6 are not intrinsic properties of the critical point V_b, but rather depend on the norm $\|.\|_X(t)$ and on the radius δ of the ball in the space X, on the difference W_0 between the basic motion V_b, and the initial data $V(0)$; specifically, it is a local statement.*

The term '*unconditional*' means without the condition of smallness for initial data, however this adjective is not generally accepted.

Definition 2.2.7 *The rest state is said to be* ***unstable*** *if it is not nonlinearly stable.*

For the linearized problem associated with (2.2.2), all definitions are simplified.

Definition 2.2.8 *The rest state S_b is said to be* ***linearly stable*** *in the norm $\|\cdot\|_X(t)$ if it is stable in the system obtained by linearizing around zero the term $g(W)$ at right hand side of (2.2.2). Namely, setting*

$$\frac{dW'}{dt} = f'(V_b)\, W',$$

if there exists a constant $\beta > 0$ such that

$$\|W'(t;W_0)\|_X < \|W_0\|_X \exp^{-\beta t}, \qquad \forall t > 0.$$

If $\beta = 0$ then we have ***marginal stability.***[2]

Definition 2.2.9 *The rest state S_b is said to be* ***unstable*** *in the norm $\|\cdot\|_X(t)$ if there exists a constant $\beta > 0$ such that*

$$\|W(t;W_0)\|_X \geq \|W_0\|_X\, e^{\beta t}, \qquad \forall t > 0.$$

[2]Definitions for linear stability are given in the complex plane, employing the eigenvalues of the linearized operator. Thus the marginal stability involves only the real part of the eigenvalues. However, boundedness can be proven.

Notice that the Definitions 2.2.8 and 4.3.4 are independent of the size of the initial data; namely they are global statements.

Once more we wish to outline that *the **difference between linear and nonlinear stability** with respect to initial data in classical fluid-dynamics is due to the **size** of the distance between the initial data $S(0)$ and S_b*, and to the **decay rate** of perturbations.

*To questions of linear and nonlinear stability we require regularity on steady and unsteady flows, and such regularity **do depend** on external forces.*

Below we recall the **Linearization Principle.** The *linearization principle* refers to a theorem proving that stability properties of the exact steady solution S_b to the nonlinear system, are deduced from those of the system linearized around S_b, provided that the initial data are sufficiently close to S_b. Therefore, if a *linearization principle* holds for the rest state S_b, any linearly stable or unstable S_b is also nonlinearly stable or unstable, respectively, for a class of suitable small initial data.

If a linearization principle holds, then any linearly stable state S_b is also nonlinearly stable. This means that solutions corresponding to initial data in a sufficiently small neighborhood of 0 remain close to 0 for all time. In reality we may prescribe large initial data, so let us study what will happen in this circumstance.

2.2.2 Initial Data Control

In reality, we may prescribe initial data far from the stable state S_b, and we may wish to study what happens under these circumstances. *Previous definitions of nonlinear stability say nothing about the control of solutions with finite initial data.* Indeed, in nonlinear phenomena with large initial data, a solution $S(t)$ may lose its control from initial data, even though S_b is nonlinearly stable *(for small initial perturbations)*. This situation occurs frequently, and it constitutes the real discrepancy between linear and nonlinear stability. To this day it appears that there are no rigorous definitions for this phenomenon, thus we introduce here two new definitions:

Definition 2.2.10 *A perturbation $W(t; W_0)$ to the rest state S_b is said to be **controlled by initial data** in the range $\mathcal{I}_{2a}/\mathcal{I}_a$ if, and only if, for all initial data W_0 satisfying*

$$a < \|W_0\|_Y < 2a, \tag{2.2.4}$$

the solution $W(t; W_0)$ is bounded for all time; that is, there exists a suitable constant $\alpha = \alpha(a) > 0$ such that

$$\|W(t; W_0)\|_X(t) \leq \alpha, \qquad \forall t > 0. \tag{2.2.5}$$

Definition 2.2.11 *The rest state is said to* ***lose the control of the initial data*** *if there exists a positive number a and initial data* W_0 *satisfying* (2.2.4), *such that the corresponding solution* $W(t; W_0)$ *is not controlled by the initial data; that is, whenever given* $\alpha > 0$, *there exists* $T > 0$ *such that the solution* $W(t; W_0)$ *to problem* (2.2.2) *with initial data satisfying* (2.2.4) *satisfies the inequality*

$$\|W(t; W_0)\|_X(T) \geq \alpha. \tag{2.2.6}$$

Definitions 2.2.10, 2.2.11 are meaningful only for nonlinear systems, because the definition of linear stability is valid for all initial data.

2.2.3 Lyapunov Method

We begin with the abstract settings (2.2.1). Let V_b denote a critical point of (2.2.1), namely

$$f(V_b) = 0. \tag{2.2.7}$$

Stability studies the evolution in time of the disturbance

$$W(t; V_0 - V_b) = V(t; V_0) - V_b, \qquad W(0; V_0 - V_b) = V_0 - V_b,$$

in some prescribed norm $|.|_X$.

Definition 2.2.12 *Let* $W \in X$ *be a solution to* (2.2.2). *A smooth function*

$$\mathcal{F} : W \in X \longrightarrow \mathcal{F}(W) \in R,$$

is said to be a ***Lyapunov functional*** *for the null solution* $W_b = 0$ *in the abstract space* X *if:*

(1) It is positive definite in the neighborhood of the origin $\mathcal{I}$, *i.e.*

$$\mathcal{F}(0) = 0, \qquad \mathcal{F}(W) > 0, \qquad W \neq 0, \qquad \forall W \in \mathcal{I};$$

(2) It is continuous in the neighborhood of 0 *of radius* R, *i.e.*

$$\forall R > \epsilon > 0, \quad \exists \delta > 0 : \qquad \|W\|_X < \delta \longrightarrow \quad \mathcal{F}(W) < \epsilon;$$

(3) It is decreasing along the solution to (2.2.2), *i.e.*

$$\frac{d\mathcal{F}(W(t))}{dt} \leq 0, \qquad \forall t > 0. \tag{2.2.8}$$

Theorem 2.2.1 ***Lyapunov Theorem*** *If there exists a Lyapunov functional for system* (2.2.2), *then the zero solution* $W_b(t; 0) = 0$ *to* (2.2.2) *is stable in the* X *norm.*

Theorem 2.2.2 *If there exists a Lyapunov functional for system (2.2.1), then the stationary solution $V(t; V_b) = V_b$ to (2.2.1) is stable.*

Remark 2.2.1 *For stable motions of fixed space X, there exist infinite Lyapunov functionals. One problem lies in the construction of the most appropriate Lyapunov functional $\mathcal{F}$.*

As such, we will now limit ourselves to describing the energy method which proposes **one choice** of a Lyapunov functional. Note that other generalized energy methods have been proposed to construct more refined Lyapunov functionals; cf. [37, 53].

The stability result given below provides nonlinear stability results in the class of regular motions.

2.2.4 Energy Method

The energy method takes as a starting point the initial value problem described in (2.2.2), the "perturbations system." It deduces Lyapunov functions by operating on system (2.2.2). A typical operation of this method is the multiplication of $(2.2.2)_1$ by a function of W in X, the latter of which is usually a Hilbert space $X = H$ where (2.2.2) is defined.[3]

The energy method is customarily used to study the nonlinear stability of incompressible viscous flows, governed by "generalized" parabolic systems, see Orr (1842–1912) [87], Reynolds (1842–1912) [124], etc. Notice that the '**energy method**' generally doesn't use physical energy.

Let $\mathbf{v}, p$ be a solution to the **Navier–Stokes unsteady equations** in a fixed domain Ω that satisfies

$$\begin{aligned} \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} - \nu \Delta \mathbf{v} &= -\nabla p + \mathbf{f} \\ \nabla \cdot \mathbf{v} &= 0, \qquad (x,t) \in \Omega \times (0,\infty), \\ \mathbf{v}(x,0) &= \mathbf{v}_0(x), \qquad x \in \Omega, \\ \mathbf{v}|_{\partial\Omega} &= \mathbf{v}_\Sigma, \end{aligned} \tag{2.2.9}$$

with $\mathbf{f}$ being external force, $\mathbf{v}_\Sigma$ being boundary velocity, and $\mathbf{v}_0$ the initial data.

[3] If $W \in L^p$, we could multiply (2.2.2) by $|W|^{p-2}W$ and integrate over Ω to get

$$\frac{d}{dt} \int_\Omega \frac{W^p}{p} dx = \int_\Omega W^{p-2} W \, g(W) dx.$$

In the energy method one usually takes $p = 2$.

In this case, a critical point of (2.2.9) represents a solution to the **Navier–Stokes steady equations** given by

$$\begin{aligned} \mathbf{v}_b \cdot \nabla \mathbf{v}_b - \nu \Delta \mathbf{v}_b &= -\nabla p_b + \mathbf{f} \\ \nabla \cdot \mathbf{v}_b &= 0, \qquad x \in \Omega, \\ \mathbf{v}_b|_{\partial\Omega} &= \mathbf{v}_\Sigma, \end{aligned} \tag{2.2.10}$$

with the same external force and the same boundary velocity.

Here we wish to study the **stability with respect initial data** $\mathbf{v}_0 = \mathbf{v}_b + \mathbf{u}_0$. Then, the perturbation $\mathbf{u} = \mathbf{v} - \mathbf{v}_b$ satisfies the equation

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{v} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} &= -\nabla(p - p_b) - \mathbf{u} \cdot \nabla \mathbf{v}_b \\ \nabla \cdot \mathbf{u} &= 0, \qquad (x,t) \in \Omega \times (0,\infty), \\ \mathbf{u}(x,0) &= \mathbf{u}_0(x), \qquad x \in \Omega, \\ \mathbf{u}|_{\partial\Omega} &= 0. \end{aligned} \tag{2.2.11}$$

Multiplying $(2.2.11)_1$ by $\mathbf{u}$ and integrating over Ω, we deduce the following **energy equation**

$$\frac{d}{dt}\int_\Omega \frac{|\mathbf{u}|^2}{2} dx = -\nu \int_\Omega |\nabla \mathbf{u}|^2 dx - \int_\Omega \mathbf{u} \cdot \nabla \mathbf{v}_b \mathbf{u}\, dx, \qquad t \in (0,\infty). \tag{2.2.12}$$

From (2.2.12) one can easily prove a continuous dependence theorem for suitable regularity classes of solutions. In general nothing can be said about the stability of the stationary solution $\mathbf{v}_b$ of (2.2.11), except that unsteady perturbations $\mathbf{u}(x,t)$ do depend on the size of $\mathbf{v}_b$. Furthermore, it is clear that a candidate Lyapunov functional $\mathcal{F}(\mathbf{u})$ coincides with the spatial L^2-norm of $\mathbf{u}$. $\|\mathbf{u}\|_{L^2}$ will become a Lyapunov functional for system (2.2.11) if

$$-\nu \int_\Omega |\nabla \mathbf{u}|^2 dx - \int_\Omega \mathbf{u} \cdot \nabla \mathbf{v}_b \mathbf{u}\, dx \le 0. \tag{2.2.13}$$

Drawback $\nu \neq 0$ To prove stability one must prove that the right hand side of (2.2.12) is less than zero. To this end, we notice that the second integral at l.h.s. of (2.2.13) $A := -\int_\Omega \mathbf{u} \cdot \nabla \mathbf{v}_b \cdot \mathbf{u}\, dx$, has no definite sign. Hence one requires the dissipative term $D := -\nu \int_\Omega |\nabla \mathbf{u}|^2 dx$ be larger than A. This in turn requires that $\nu \neq 0$, thus the fluid must be viscous. Under this assumption, in order to control the term A, we require that the basic motion $\mathbf{v}_b$ and the domain be such that there exists a constant $c = c(\Omega, \mathbf{v}_b)$ such that

$$\left| \int_\Omega \mathbf{u} \cdot \nabla \mathbf{v}_b \cdot \mathbf{u}\, dx \right| \le c \int_\Omega |\nabla \mathbf{u}|^2 dx, \qquad \forall \mathbf{u} \in J_1(\Omega).$$

Finally, to prove stability it is enough to assume that $c \le \nu$.

2.3 Lagrange–Dirichlet Method

The Lagrange–Dirichlet method considers the abstract initial value problem (2.2.1), and constructs Lyapunov functionals by using direct physical balance laws (in particular conservation laws).

The Lagrange–Dirichlet method is customarily used to study nonlinear stability of the rest state of elastic and thermodynamic systems; cf. [24]. For the stability study one computes the first and second variation of the total energy. More generally, the Lagrange–Dirichlet Method states that those V_b which are the minima of total energy are stable steady states.

Nonlinear stability has been investigated by Dirichlet using general conservation laws.

2.3.1 Hyperbolic First Order Systems

Arnold [4] has extended the Dirichlet method to study the nonlinear stability of potentially non-smooth steady solutions of two-dimensional Euler equations; for incompressible fluids in symmetric bounded domains, see also [127]. He used as a Lyapunov functional a linear combination of conservation laws, first integrals, due to the symmetry of the problem. Using as prototype a functional $\mathcal{E}$, the basic idea was to look for conditions ensuring the vanishing of the first variation of $\mathcal{E}$ and the positivity of its second variation. We will give an outline of the stability theorem proved by **Arnold** for a class of steady solutions to **Euler equations**.

Consider a two-dimensional domain Ω, bounded by two smooth, fixed, closed, non-intersecting curves C_0, C_1, or the internal and external boundaries, respectively. Let us denote by (x, y) the independent space variables in the plane π containing Ω, and by $\mathbf{k}$ a direction orthogonal to π.

For the local angular velocity we set

$$\omega = curl\mathbf{v} \cdot \mathbf{k} = \partial_x v_y - \partial_y v_x,$$

where $(v_x,\, v_y)$ denotes the components of $\mathbf{v}$ along π.

Let the velocity $\mathbf{v}_b$, p_b and the pressure be a **steady solution of Euler equations** to the Boundary Value Problem (BVP)

$$\begin{aligned} &\mathbf{v}_b \cdot \nabla \mathbf{v}_b = -\nabla p_b, && \\ &\nabla \cdot \mathbf{v}_b = 0, && \text{in } \Omega; \\ &\mathbf{v}_b \cdot \mathbf{n}|_{C_i} = v_i, && \\ &\int_{C_i} v_i ds = 0, && i = 0, 1. \end{aligned} \tag{2.3.1}$$

Denoted by ψ_b the stream function $\mathbf{v}_b = \nabla\psi_b \times \mathbf{k}$, it is well known, cf. [65], Sect. 10 in Chap. 1, that any function of $\Delta\psi_b = 0$ is constant through the paths of fluid particles. Furthermore, by the motion equation $(2.3.1)_1$ we also know that

$$\mathbf{v}_b \cdot \nabla\omega_b = \nabla\psi_b \times \mathbf{k} \cdot \nabla\Delta\psi_b = 0. \tag{2.3.2}$$

The parallelism between $\nabla\psi_b \times \nabla\Delta\psi_b$ and $\mathbf{k}$, combined with (2.3.2) implies

$$\nabla\psi_b \times \nabla\Delta\psi_b \; = \; 0.$$

The parallelism between the gradients of ψ_b, and of $\Delta\psi_b$, if $\nabla\Delta\psi_b \neq 0$, infers a functional dependence between these two functions, expressed by

$$\psi_b = \aleph(\Delta\psi_b). \tag{2.3.3}$$

Therefore, all basic flows must satisfy (2.3.3).

Let $\mathbf{v}$, p be solutions to the **incompressible Euler unsteady equations**, namely let $\mathbf{v}$, p solve the Initial Boundary Value Problem (IBVP)

$$\begin{aligned} \partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} &= -\nabla p; \\ \nabla\cdot\mathbf{v} &= 0; \\ \mathbf{v}(x,0) &= \mathbf{v}_0; \\ \mathbf{v}\cdot\mathbf{n}|_{C_i} &= v_i, \qquad i = 0,1. \end{aligned} \tag{2.3.4}$$

Let $d\mathbf{l}$ be the infinitesimal element of the line tangent to oriented curves C_i, $i = 1, 2$. It is not difficult to show that the **modified energy** functional $\mathcal{E}(\mathbf{v})$ remains constant along the motion

$$\mathcal{E}(\mathbf{v}) = \frac{1}{2}\int_\Omega v^2 dxdy + \int_\Omega \Phi(\omega)dxdy + \sum_{i=0}^{2} a_i \int_{C_i} \mathbf{v}\cdot d\mathbf{l}, \tag{2.3.5}$$

where $\Phi : R \longrightarrow R$ is any smooth function with a_i real numbers.

Set $\mathbf{u} = \mathbf{v} - \mathbf{v}_b$. Then, as **possible Lyapunov functional** we may choose

$$\mathcal{F}(\mathbf{u}) = \mathcal{E}(\mathbf{v}) - \mathcal{E}(\mathbf{v}_b). \tag{2.3.6}$$

Choice of Φ. For functionals Φ in (2.3.5), we take any function having a derivative coinciding with

$$\Phi'(\cdot) = \aleph((\cdot).$$

It's worth checking under which conditions $\mathcal{F}$ becomes a Lyapunov functional.

Let J and J_1 denote the subspaces of $L^2(\Omega)$ and $W^{1,2}(\Omega)$, respectively, of solenoidal vector fields. Conditions on the first and second variations of $\mathcal{F}$ are given by (2.3.6),

$$\begin{aligned} &\delta\mathcal{F}(\psi_b)[\varphi] = 0, \qquad \forall\varphi \in J, \\ &\delta^2\mathcal{F}(\psi_b)[\varphi]^2 > 0, \qquad \forall\varphi \in J_1, \end{aligned} \tag{2.3.7}$$

and ensure that $\mathcal{F}$ is a right Lyapunov functional.

Conditions (2.3.7) are conditions on the basic motion ψ_b. Therefore, we are lead to select the stable steady flows $\mathbf{v}_b$ in the set of functions that satisfy (2.3.7).

The second variation introduces the norm, called the **natural norm** in which the perturbation is controlled. In this case, the *natural norm* with which to study the problem of stability is

$$\delta^2\mathcal{E}(\mathbf{v}_b)[\mathbf{u}]^2 = \int_\Omega u^2 dxdy + \int_\Omega \aleph(\bar{\omega})\omega^2 dxdy,$$

with $\bar{\omega}$ between ω_b and ω. Finally, the hypotheses on $\aleph$ ensuring $\delta^2\mathcal{E}$ is positive definitely ensure that $\mathcal{F}$ is a good Lyapunov function.

2.3.2 Second Order ODE

We wish to explore the **Dirichlet method** by studying the solutions $x = x(t, x_0, \dot{x}_0)$ of an ordinary second order system

$$\begin{aligned} &\frac{d^2}{dt^2}x \;=\; f(x) - h\dot{x}, \\ &x(0) = x_0, \qquad \frac{d}{dt}x(0) = \dot{x}_0. \end{aligned} \tag{2.3.8}$$

In (2.3.8) h is a positive constant, and sometime we have set $\frac{dx}{dt} = \dot{x}$. Let x_b be a critical point of f that satisfies $f(x_b) = 0$. This leads to the knowledge of an equilibrium solution $x(t, x_b, 0) = x_b$ for (2.3.8).

The **energy equation** is written as

$$\frac{d}{dt}\mathcal{E}(x) = -h\dot{x}^2, \tag{2.3.9}$$

with

$$\mathcal{E}(x) = \frac{1}{2}\dot{x}^2 - F(x), \qquad \frac{d}{dx}F(x) =: F'(x) = f(x). \tag{2.3.10}$$

Therefore, for $y = x - x_b$, $\dot{y} = \dot{x}$ we see that

$$\mathcal{F}(y, \dot{y}) = \mathcal{E}(x_b + y) - \mathcal{E}(x_b)$$

becomes a Lyapunov functional once $(x_b, 0)$ is a minimum for $\mathcal{E}$. A sufficient condition then is that the Hessian $\delta^2\mathcal{F}(x_b, 0)$ defines a positive definite quadratic form. Since the Hessian of $\mathcal{F}$ is in diagonal form, it is enough to compute the two eigenvalues $\partial^2_{\dot{x}} F(x_b, 0)$, $\partial^2_x F(x_b, 0)$. The derivative $\partial^2_{\dot{x}} F(x_b, 0)$ is given by 1/2, and is always positive. The derivative $\partial^2_x F(x_b, 0)$ is $-f'(x_b)$, hence we must assume that $-f'(x_b) > 0$, that is $f'(x_b) < 0$.

To give an **asymptotic result**, we first observe that $\dot{x} = \dot{y}$. Thus we multiply (2.3.8) times $y = x - x_b$, recalling that x_b is a critical point. So we find the **free work equation**

$$\frac{d}{dt}\Big(y\dot{y} + \frac{hy^2}{2}\Big) - \dot{y}^2 = \Big(f(x_b + y) - f(x_b)\Big)\, y = f'(x_b + \overline{y})y^2, \tag{2.3.11}$$

where $\overline{y}$ is a point between y and 0. The hypothesis $f'(x_b) < 0$, together with the regularity of solutions, implies that there exists a neighborhood of x_b in which $f'(x) < 0$. Notice that by applying the Taylor expansion to the variable x with initial point x_b up to the second order, and recalling that x_b is a critical point, if $F'(x_b) = f(x_b) = 0$, we obtain

$$\mathcal{E}(x) - \mathcal{E}(x_b) = \frac{1}{2}\dot{y}^2 - \Big(F(x) - F(x_b)\Big) = \frac{1}{2}\dot{y}^2 - \frac{1}{2}f'(\overline{x})y^2. \tag{2.3.12}$$

Multiplying (2.3.11) by an arbitrary positive constant ϵ and adding (2.3.9), one obtains

$$\frac{d}{dt}\mathbb{E}(t) = -(h - \epsilon)\dot{y}^2 \; + \epsilon\, f'(x_b + \overline{y})y^2$$

where we have introduced the **modified energy**

$$\mathbb{E} = \frac{1}{2}\dot{y}^2 - \frac{1}{2}f'(x_b + \overline{y})y^2 + \epsilon\, y\dot{y} + \epsilon\;\frac{h}{2}y^2.$$

It is trivial to check that there exists a b such that

$$\mathbb{E}(t) \le \frac{1}{b}(y^2 + \dot{y}^2). \tag{2.3.13}$$

Furthermore we observe that, for $f'(\overline{x}) < 0$, it is $h - \sqrt{h^2 - 4f'(\overline{x})} < 0$, hence condition

$$(h - \sqrt{h^2 - 4f'(\overline{x})}) < 2\epsilon < (h + \sqrt{h^2 - 4f'(\overline{x})}),$$

is satisfied for

$$0 < 2\epsilon < (h + \sqrt{h^2 - 4f'(\overline{x})}),$$

namely for ϵ sufficiently small. Under this assumption the modified energy $\mathbb{E}$ is always a positive definite quadratic form in the variables y, $\dot{y}$. Moreover, it holds

$$\frac{d}{dt}\mathbb{E}(t) = -a^2(\dot{y}^2 + y^2), \tag{2.3.14}$$

where

$$a^2 = min\{(h-\epsilon), -\epsilon f'(x_b + \overline{y})\}.$$

Hence (2.3.13) in (2.3.14) yields the following modified energy differential inequality

$$\frac{d}{dt}\mathbb{E}(t) \leq -a^2 b\mathbb{E}(t), \tag{2.3.15}$$

which integrated infers

$$\mathbb{E}(t) \leq \mathbb{E}(0)\exp^{-a^2 b\, t}.$$

If $f'(x_b) < 0$, then nonlinear stability holds and asymptotic decay has been proved.

Analogous calculations have been developed for elastic media; cf. [2, 24, 106, 108]. In particular, we note that the method is applicable to hyperbolic systems.

Question:
Is the Dirichlet method applicable to coupled hyperbolic-parabolic equations? In particular, where the rest state of a compressible fluid is asymptotically stable? In the following chapters we shall give a partial answer to this question. Below we apply the Dirichlet method to compressible fluids, studying two model problems.

2.3.3 Stability of Barotropic Inviscid Fluids $\mathbf{v}_b \neq \mathbf{0}$

Here we give an example of natural norms in which the perturbation is controlled by initial data, for all times, for zero viscosity and non-zero basic velocity.

We recall that the nonsteady flows of a barotropic inviscid fluid in a bounded domain are governed by the **compressible Euler equations**. The unknown velocity $\mathbf{v}$ and density ρ are governed by the following initial boundary value problem in a bounded fixed domain Ω:

$$\begin{aligned}
&\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0, && \\
&\rho(\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla p(\rho) + \rho\nabla U, && (x,t)\in\Omega\times(0,T), \\
&\mathbf{v}(x,0) = \mathbf{v}_0(x), && \rho(x,0) = \rho_0(x), x\in\Omega, \\
&\int_\Omega \rho(x,t)dx = M, && \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0.
\end{aligned} \tag{2.3.16}$$

If the pressure $p(\rho)$ is a convex function of ρ, system (2.3.16) is strictly hyperbolic.

Set $x \in \Omega$, $x \equiv (x, y, z)$, and denote by $\mathbf{v}_b$, ρ_b the steady solution to the system (2.3.16). We give sufficient conditions for nonlinear stability of steady potential flows, with respect to three dimensional perturbations. We work in Eulerian coordinates.

We study nonlinear stability of the steady flow of a ***barotropic inviscid*** *fluid, governed by the* ***compressible Euler equations***. Begin by recalling the Boundary Value Problem governing steady flows

$$
\begin{aligned}
&\nabla \cdot (\rho_b \mathbf{v}_b) = 0, \\
&(\rho \mathbf{v}_b \cdot \nabla)\ \mathbf{v}_b = -\nabla p(\rho_b) + \rho_b \mathbf{f}, \qquad x \in \Omega\,, \\
&\mathbf{v}_b \cdot \mathbf{n}\Big|_{\partial\Omega} = 0, \qquad \int_\Omega \rho_b(x)\, dx = M, \quad \rho_b \geq 0,
\end{aligned}
\tag{2.3.17}
$$

where $\mathbf{n}$ is the normal to the boundary, and $p = p(\rho_b)$. As we know, the rest state $S_b = \{\mathbf{v} = 0, \quad \rho_b = \rho_b(x)\}$ exists only when forces $\mathbf{f}$ are positional and derived from a uniform potential $\mathbf{f} = \nabla U$. Indeed in this case, the rest state is the exact solution to (2.3.17), with ρ_b implicitly given by

$$
\int^{\rho_b} \frac{p'(s)}{s} ds = U + c, \qquad \int_\Omega \rho_b\, dx = M. \tag{2.3.18}
$$

Notice that the constant c is given by the condition that the total mass is prescribed $(2.3.17)_4$, and therefore (2.3.18) may furnish a complex value for the density. In order to have real positive solutions ρ_b (densities), we are led to assume the following:

Hypotheses on the basic flow $(\rho_b, \mathbf{v}_b)$

(i) The flow $(\rho_b, \mathbf{v}_b)$ satisfies the boundary problem (2.3.17).
(ii) The velocity $\mathbf{v}_b$ is *potential*.
(iii) The momentum $\rho_b \mathbf{v}_b$ is *potential*

$$
\rho_b \mathbf{v}_b = \nabla \chi.
$$

Assumptions (i) and (ii) allow us to write (for regular flows) **Bernoulli (1700-1782) equation**

$$
\begin{aligned}
&\nabla\Big(\frac{1}{2}\mathbf{v}_b^2 + \Phi(\rho_b) - U\Big) = 0, \qquad x \in \Omega \\
&\Phi(\rho) = \int^\rho \frac{p'(s)}{s} ds, \qquad p'(\rho) = \frac{dp}{d\rho},
\end{aligned}
\tag{2.3.19}
$$

where Φ is the enthalpy; cf. (1.5.24). We define

$$\mathcal{E}(\mathbf{v}_b, \rho_b) = \int_\Omega \rho_b\Big(\frac{1}{2}\mathbf{v}_b^2 + \Phi(\rho_b) - U\Big)dx.$$

We rewrite (1.7.7) for inviscid fluids and find

$$\frac{d}{dt}\int_\Omega \rho\Big(\frac{1}{2}|\mathbf{v}|^2 + \int^\rho \frac{p(s)}{s^2}ds\Big)dx = -\int_{\partial\Omega} p(\rho)\mathbf{v}\cdot\mathbf{n}dS + \int_\Omega \rho\mathbf{v}\cdot\nabla U dx. \tag{2.3.20}$$

In this case, the total energy is given by

$$\begin{aligned} E(\mathbf{v},\rho) &= \int_\Omega \big\{\frac{1}{2}\rho\mathbf{v}^2 + \rho\Psi(\rho) - \rho U(x)\big\}dx, \\ \Psi(\rho) &= \int^\rho \frac{p(s)}{s^2}ds, \end{aligned} \tag{2.3.21}$$

where Ψ is the Helmholtz free energy; cf. (1.5.25). Now, set

$$\mathbf{u} = \mathbf{v} - \mathbf{v}_b, \qquad \sigma = \rho - \rho_b,$$

Solutions $\mathbf{v}_b + \mathbf{u}$, $\rho_b + \sigma$ to (2.3.16) satisfy the side condition

$$\int_\Omega \sigma dx = 0.$$

The Lyapunov functional is

$$\begin{aligned} \mathcal{F}(\mathbf{u},\sigma) &= E(\mathbf{v}_b + \mathbf{u}, \rho_b + \sigma) - \mathcal{E}(\mathbf{v}_b,\rho_b) \\ = \int_\Omega &\big\{\frac{1}{2}(\rho_b+\sigma)(\mathbf{v}_b+\mathbf{u})^2 + (\rho_b+\sigma)(\Psi(\rho_b+\sigma) - U(x))\big\}dx \\ &- \int_\Omega \big\{\frac{1}{2}\rho_b\mathbf{v}_b^2 + \rho_b(\Phi(\rho_b) - U(x))\big\}dx. \end{aligned} \tag{2.3.22}$$

We notice that the time derivative of $\mathcal{E}(\mathbf{v}_b,\rho_b)$ is zero, thus from (2.3.19) we get

$$\frac{d}{dt}\Big(E(\mathbf{v},\rho) - \mathcal{E}(\mathbf{v}_b,\rho_b)\Big) = \frac{d}{dt}\int_\Omega \big\{\frac{1}{2}\rho(\mathbf{v}^2 - \mathbf{v}_b^2) + \rho(\Psi(\rho) - \Phi(\rho_b))\big\}dx = 0. \tag{2.3.23}$$

We have now proven that $\mathcal{F}(\mathbf{u},\sigma)$ is a good Lyapunov functional.

Applying the **Taylor (1685-1731) polynomial** formula, with initial point $(\mathbf{v}_b, \rho_b)$, we get:

$$\begin{aligned}\mathcal{F}(\mathbf{u},\sigma) &= E(\mathbf{v}_b+\mathbf{u},\rho_b+\sigma)-\mathcal{E}(\mathbf{v}_b,\rho_b)\\ &= E(\mathbf{v}_b+\mathbf{u},\rho_b+\mathbf{u})-E(\mathbf{v}_b,\rho_b)+E(\mathbf{v}_b,\rho_b)-\mathcal{E}(\mathbf{v}_b,\rho_b)\\ &= \delta E(\mathbf{v}_b,\rho_b)[\mathbf{u},\sigma]+\frac{1}{2}\delta^2 E(\bar{\mathbf{v}},\bar{\rho})[\mathbf{u},\sigma]^2+\int_\Omega \rho_b(\Psi(\rho_b)-\Phi(\rho_b))dx,\end{aligned} \tag{2.3.24}$$

where $\bar{\mathbf{v}}, \bar{\rho}$ is a point between $(\mathbf{v}_b, \rho_b)$ and $(\mathbf{v}, \rho)$.

Since the antiderivatives are defined up to a constant, the definitions of Ψ, and Φ yield

$$\begin{aligned}\int_\Omega \rho_b(\Psi(\rho_b)-\Phi(\rho_b))dx &= \int_\Omega \rho_b \int^{\rho_b} \frac{d}{ds}\Big(\frac{p(s)}{s}\Big) ds\, dx\\ &= \int_\Omega \rho_b\Big(\frac{p(\rho_b)}{\rho_b}+c\Big)\, dx \qquad = \int_\Omega p(\rho_b)dx + cM.\end{aligned} \tag{2.3.25}$$

The left hand side of (2.3.25) vanishes for c suitable

$$c = -\frac{\int_\Omega p(\rho_b)dx}{M}.$$

If such a choice is made for c, the function $\mathcal{F}$ reduces to

$$\mathcal{F}(\mathbf{u},\sigma) = \delta E(\mathbf{v}_b,\rho_b)[\mathbf{u},\sigma]+\frac{1}{2}\delta^2 E(\bar{\mathbf{v}},\bar{\rho})[\mathbf{u},\sigma]^2. \tag{2.3.26}$$

Let'a now compute the first and second order variations of E taking as our initial point $(\mathbf{v}_b, \rho_b)$. We get

$$\begin{aligned}\delta E(\mathbf{v}_b,\rho_b)[\mathbf{u},\sigma] &= \int_\Omega \frac{\partial}{\partial\rho}\Big[\rho(\mathbf{v}^2-\mathbf{v}_b^2)+\rho\Psi(\rho)-\rho\Phi(\rho_b)\Big]_{\rho_b,\mathbf{v}_b}\sigma dx+\int_\Omega \rho_b\mathbf{v}_b\cdot\mathbf{u}dx,\\ \frac{1}{2}\delta^2 E(\bar{\mathbf{v}},\bar{\rho})[\mathbf{u},\sigma]^2 dx &= \int_\Omega \Big\{\frac{1}{2}\rho\mathbf{u}^2+\frac{1}{2}\frac{\partial^2(\rho\Psi(\rho))}{\partial\rho^2}\Big|_{(\overline{u},\bar{\rho})}\sigma^2+\bar{\mathbf{v}}_b\cdot\mathbf{u}\sigma\Big\}dx.\end{aligned} \tag{2.3.27}$$

Concerning the first integral in the first variation of E from the basic state, we notice that,

$$\begin{aligned}&\frac{\partial}{\partial\rho}\Big[\rho(\mathbf{v}^2-\mathbf{v}_b^2)+\rho\Psi(\rho))-\rho\Phi(\rho_b)\Big]_{(\mathbf{v}_b,\rho_b)} = \rho_b\Psi'(\rho_b)+(\Psi(\rho_b)-\Phi(\rho_b))\\ &= \frac{p(\rho_b)}{\rho_b}+\int^{\rho_b}\frac{p(s)}{s^2}ds-\int^{\rho_b}\frac{p'(s)}{s}ds = \frac{p(\rho_b)}{\rho_b}-\int^{\rho_b}\frac{d}{ds}\Big(\frac{p(s)}{s}\Big)ds = const = C.\end{aligned} \tag{2.3.28}$$

Finally, $\int_\Omega C\sigma\, dx = 0$ because $\int_\Omega \sigma dx = 0$, thus the first integral in δE is zero.

Concerning the second integral in the first variation of E, using the **Helmholtz decomposition** for $\mathbf{u}$, $\mathbf{u} = \mathbf{w} + \nabla\xi$, with $\mathbf{w}$ solenoidal and with zero normal component at the boundary, recalling the second hypothesis (ii) together with the boundary conditions and $(2.3.17)_1$, we get

$$\int_\Omega \rho_b \mathbf{v}_b \cdot \mathbf{u}\, dx = \int_\Omega \rho_b \mathbf{v}_b \cdot (\mathbf{w} + \nabla\xi)\, dx = \int_\Omega \nabla\chi \cdot \mathbf{w}\, dx + \int_\Omega \rho_b \mathbf{v}_b \cdot \nabla\xi\, dx = 0.$$

Thus, the first variation of E is zero at $(\mathbf{v}_b, \rho_b)$. Now, we need to calculate the second order variation of E given by $(2.3.27)_2$. We recall that the point $(\mathbf{v}_b, \rho_b)$ is a minimum for the Helmhotz free energy, and therefore it holds that

$$\frac{\partial^2(\rho\Psi(\rho))}{\partial\rho^2}\Big|_{(\mathbf{v}_b,\rho_b)} > 0,$$

which infers by continuity that it remains positive for all values of $I_R(\mathbf{v}_b, \rho_b)$, for R suitably small, and we can deduce

$$\frac{\partial^2(\rho\Psi(\rho))}{\partial\rho^2}\Big|_{(\overline{u},\bar{\rho})} > 0, \qquad (\overline{u}, \bar{\rho}) \in I_R(\mathbf{v}_b, \rho_b).$$

If $\mathbf{v}_b$ is not too large as compared to basic density[4]

$$esssup_\Omega|\bar{\mathbf{v}}_b| \; < \; essinf_{\,\Omega\times(0,infty)}\rho(x,t)\, \frac{\partial^2\rho\Psi}{\partial\rho^2}\Big|_{\bar{\rho}}$$

$(2.3.27)_2$ is equivalent to the L^2 norm of perturbations $\mathbf{u}$ and σ. Finally, integrating (2.3.23) in time yields

$$\begin{aligned} a\Big(\|\mathbf{u}\|^2_{L^2} + \|\sigma\|^2_{L^2}\Big) &\le \int_\Omega \Big\{\frac{1}{2}\rho\mathbf{u}^2 + \frac{1}{2}\frac{\partial^2(\rho\Psi(\rho))}{\partial\rho^2}\Big|_{(\overline{u},\bar{\rho})}\sigma^2 + \bar{\mathbf{v}}_b\cdot\mathbf{u}\sigma\Big\}dx \\ &= \int_\Omega \Big\{\frac{1}{2}\rho_0\mathbf{u}_0^2 + \frac{1}{2}\frac{\partial^2(\rho_0\Psi(\rho_0))}{\partial\rho_0^2}\Big|_{(\overline{u_0},\bar{\rho}_0)}\sigma_0^2 + \bar{\mathbf{v}}_b\cdot\mathbf{u}_0\sigma_0\Big\}dx, \end{aligned} \tag{2.3.29}$$

which delivers control of perturbations $\mathbf{u}$ and σ in the L^2 norm, for all times $t > 0$.

[4]For isothermal flows it is enough to assume

$$esssup_\Omega|\bar{\mathbf{v}}_b| \; < \; k,$$

where $k = R_*\theta_b$ and R_* is the universal gas constant.

2.3.4 Isothermal Viscous Fluids $\mathbf{v}_b = \mathbf{0}$

In this subsection we use the energy method in an unorthodox manner to find the behavior in time of the difference between the energies $E(t)$ of the non steady motion and E_b of the rest state. We observe that $E(t) - E_b$ dominates the spatial L^2-norm of the difference of two solutions $(\rho, \mathbf{v})$, $(\rho_b, \mathbf{v}_b)$. Next, to get the decay to zero in time, we apply the free work equation, which furnishes dissipative terms for the perturbation in the density. This will be achieved by using suitable test functions whose existence is ensured by Lemma 3.7.5.

Let us consider the system

$$
\begin{aligned}
\partial_t \rho + \mathbf{u} \cdot \nabla \rho &= -\rho \nabla \cdot \mathbf{u}, \\
\rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} &= -k \nabla \rho, \qquad (x,t) \in \Omega \times (0,T), \\
\mathbf{u}(x,0) &= \mathbf{u}_0(x), \qquad \rho(x,0) = \rho_0(x), \qquad x \in \Omega, \\
\mathbf{u}|_{\partial\Omega} &= 0, \qquad \int_\Omega \rho dx = M,
\end{aligned}
\tag{2.3.30}
$$

with $k = R_* \theta_b$ a positive constant. System (2.3.30) describes isothermal fluids moving in the absence of external force. It is easy to verify that $\mathbf{v}_b = 0$, $\rho_b = M/|\Omega|$ is a solution to (2.3.30). Furthermore, it is also standard to verify that the **energy equation** holds

$$
\frac{d}{dt} \Big\{ \int_\Omega \rho \frac{u^2}{2} + k \rho \ln \rho \Big\} dx + \mu \, D_u(t) \; = 0, \tag{2.3.31}
$$

with

$$
\mu \, D_u(t) = \int_\Omega \Big[(\lambda + \mu)(\nabla \cdot \mathbf{u})^2 \, dx + \mu |\nabla \; \mathbf{u}|^2 \Big] \, dx.
$$

Notice that (2.3.31) can be equally rewritten in the following form

$$
\frac{d}{dt} \mathcal{F}(\mathbf{u}, \sigma) + \mu \, D_u(t) \; = 0. \tag{2.3.32}
$$

where $\mathcal{F}$ is the energy of perturbations $\mathbf{u}$, σ:

$$
\mathcal{F}(\mathbf{u}, \sigma) = \int_\Omega \Big\{ \rho \frac{u^2}{2} + k \rho \ln \rho - k \rho \ln \rho_b - k(\rho - \rho_b) \Big\} dx = \int_\Omega \left(\rho \frac{u^2}{2} + k \frac{\sigma^2}{2 \overline{\rho}} \right) dx,
$$

with $\overline{\rho}$ between ρ and ρ_b. It follows from the definition and from (2.3.32) that $\mathcal{F}$ is a Lyapunov functional, hence the rest state is stable.

Remark 2.3.1 *The stability in the mean continues to hold for inviscid fluids, for example if $\mu = 0$. In fact, $\mathcal{F}$ is a Lyapunov functional despite the viscosity term.*

For f and g, two vector functions in dual spaces, we use the notation (f, g) to denote the integral over Ω of the scalar product between these two functions.

Let $\mu > 0$; we wish to analyze the asymptotic behavior of a perturbation in time. To this end we construct a dissipative term for σ.

We recall that in (2.3.2) to get asymptotic decay we have multiplied the equation (2.3.8), that coincides with the equation of perturbation, times the free displacement $x(t) - x_b$ obtaining the free work equation (2.3.11). Here, in order to provide a dissipative term for the perturbation $\sigma = \rho - \rho_b$, we also construct a free work equation where now the free displacement is given by a suitable test function $\mathbf{V}$.

Let us multiply (2.3.30) by an auxiliary function $\mathbf{V}$, having a dimension of displacement "**free displacement**," and integrate over Ω. We obtain the **free work equation**

$$\frac{d}{dt}(\rho \mathbf{u}, \mathbf{V}) + k(\nabla \sigma, \mathbf{V}) = \mathcal{I}, \tag{2.3.33}$$

with

$$\mathcal{I} = (\rho \mathbf{u}, \partial_t \mathbf{V} + \mathbf{u} \cdot \nabla \mathbf{V}) - (\lambda + \mu)\Big(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{V}\Big) - \mu\Big(\nabla\ \mathbf{u}, \nabla\ \mathbf{V}\Big).$$

We call equation (2.3.33) the **free work equation** because it equates the time derivative of $(\rho \mathbf{u}, \mathbf{V})$ to an appropriate work. Notice that the *displacement* $\mathbf{V}$ is a free vector field *to be suitably chosen.*

Given σ as smooth function $\sigma = \sigma(x, t)$, now we can choose the displacement $\mathbf{V}$ as a solution to the boundary value problem; cf. Lemma 3.7.5 of Chap. 3,

$$\begin{aligned} &\nabla \cdot \mathbf{V} = \sigma, \qquad\qquad (x, t) \in \Omega \times (0, \infty), \\ &\mathbf{V}|_{\partial\Omega} = 0. \end{aligned} \tag{2.3.34}$$

We note that, since $\int_\Omega \sigma dx = 0$, the compatibility condition is satisfied.

Furthermore, for solutions to (2.3.34) there exist constants c' $\bar{c}$, c_* such that the following estimates hold, cf. (3.7.4) Lemma 3.7.5 of Chap. 3,

$$\begin{aligned} \|\mathbf{V}\|_{L^2} &\leq c' \|\sigma\|_{L^2}, \\ \|\nabla \mathbf{V}\|_{L^2} &\leq \bar{c} \|\sigma\|_{L^2}, \\ \|\partial_t \mathbf{V}\|_{L^2} &\leq c_* \|\nabla \mathbf{u}\|_{L^2}. \end{aligned} \tag{2.3.35}$$

Notice that (2.3.34) furnishes the **free work equation**

$$-\frac{d}{dt}(\rho \mathbf{u}, \mathbf{V}) + k\|\sigma\|^2_{L^2} = -\mathcal{I}. \tag{2.3.36}$$

Equation (2.3.36) appears in the form of the time derivative of a term, having the dimensions of the integral in time of an energy, plus a dissipative term for σ that equals the functional $\mathcal{I}$ having the dimension of a work.

If we multiply (2.3.33) by $-\nu$, where ν is an arbitrary positive constant having the dimension of inverse of time 1/sec, and add it to the energy equation (2.3.31), we can deduce, as an alternative Lyapunov functional $\mathcal{F}$, the **modified energy** $\mathbb{E}$

$$\mathbb{E} = \Big\{ \int_\Omega \frac{u^2}{2} + k\frac{\sigma^2}{2\overline{\rho}} - \nu\,\rho\mathbf{u}\cdot\mathbf{V}\Big\} dx. \tag{2.3.37}$$

that satisfies the **modified energy equation**

$$\frac{d\mathbb{E}}{dt} = -\nu\|\nabla\mathbf{u}\|^2_{L^2}dx - \nu k\|\sigma\|^2_{L^2} - \nu\mathcal{I} =: -\mathcal{D} - \nu\mathcal{I}. \tag{2.3.38}$$

Employing (2.3.35), it is trivial to verify that the modified energy function (2.3.37) becomes equivalent to a norm of perturbations for ν sufficiently small

$$m_1(\|\mathbf{u}\|^2_{L^2}dx + \|\sigma\|^2_{L^2}) \leq \mathbb{E} \leq m_2(\|\mathbf{u}\|^2_{L^2}dx + \|\sigma\|^2_{L^2}).$$

We now use the Dirichlet generalized method, and analyze $\mathcal{I}$, which for small ν as well as for regular solutions, constitutes a positive quadratic form in the L^2 norm of perturbations $\mathbf{u}$, σ. From the estimates enjoyed by $\mathbf{V}$, it follows that

$$|\mathcal{I}| \leq c(\|\nabla\mathbf{u}\|_{L^2}\|\sigma\|_{L^2} + \|\nabla\mathbf{u}\|^2_{L^2}),$$

where $c = c(\mathbf{u}, \sigma, \rho_b)$. Hence the term

$$\mathcal{D} - \nu\mathcal{I} = \int_\Omega \Big(\mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)(\nabla\cdot\mathbf{u})^2\Big)dx + \nu k\|\sigma\|^2_{L^2} - \nu\mathcal{I}$$

is a positive definite quadratic form in the L^2 norm in the perturbations $\mathbf{u}$, σ. It follows that for ν small enough, using the Poincaré inequality, it is possible to prove that there exists a constant $\beta > 0$ such that

$$\mathcal{D} \geq \beta\mathbb{E}.$$

Substituting this information in (2.3.38) we deduce the differential inequality for the modified energy

$$\frac{d\mathbb{E}}{dt} + \beta\mathbb{E} \leq 0,$$

that implies exponential decay to zero of the L^2 norm of the perturbations $\mathbf{u}$, σ. The decay constant depends upon the value of ν thus in general is very small.

Open problem *To extend the method to study the stability of a general steady flow.*

2.4 Main Theorems

Here we list the main theorems proved in next three chapters:

(a) The first set of theorems will be proven in Chap. 3. They concern the steady flows S_b of *barotropic viscous fluids* filling a domain Ω, with a **rigid boundary**. Once the basic flow S_b is given, there are proven theorems of uniqueness of S_b in the class of *steady motions*, stability and asymptotic stability of S_b in a suitable regularity class of solutions.
(b) The second set of theorems will be proven in Chap. 4. They concern the *rest state* of a horizontal layer of *isothermal viscous fluids* filling a periodicity cell Ω, which has as bottom a rigid flat surface, and above a **free boundary**. There are proven uniqueness theorems for the rest state S_b in the class of steady motions, stability, and asymptotic stability theorems of S_b in a suitable regularity class of solutions. Next, for fluids having a free boundary below the rigid flat surface, we introduce the concept of initial data control, thus proving one instability theorem, and one theorem relative the loss of initial data control.
(c) The third set of theorems will be proven in Chap. 5. They concern the *rest state* of a horizontal layer of *polytropic viscous fluids* in a periodicity cell Ω between **rigid** flat horizontal **boundaries**, heated from below.

There are proven uniqueness theorems of the rest state S_b in the class of steady motions, stability and asymptotic stability theorems of S_b, in a suitable regularity class of solutions.

Remark 2.4.1 *In cases (b) and (c), in order to simplify the problem, we have considered the infinite horizontal plane as union of rectangular cells periodic in two horizontal directions.*

All the proofs employ the "free work equation" FWE, which is one essential tool of the book.

2.4.1 Case (a) Barotropic Fluid, Rigid Boundary

We study the following initial boundary value problem IBVP

$$
\begin{array}{lll}
\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) & = 0, & \\
\rho\partial_t\mathbf{v} + (\rho\mathbf{v}\cdot\nabla)\mathbf{v} & = -\nabla p + (\lambda+\mu)\nabla\nabla\cdot\mathbf{v} + \mu\Delta\mathbf{v} + \rho\,\mathbf{f}, & (x,t)\in\Omega\times(0,T),\\
\mathbf{v}(x,0) & = \mathbf{v}_0(x), \qquad \rho(x,0) = \rho_0(x), \qquad x\in\Omega, & \\
\mathbf{v}|_{\partial\Omega}(x,t) & = \mathbf{w}(x,t), \qquad\qquad x\in\partial\Omega, & \\
\int_\Omega \rho & = M, \quad \rho\geq 0, &
\end{array}
\tag{2.4.1}
$$

where $\mathbf{w}$ is the velocity of the point of boundary, M the total mass of the fluid, and we have assumed

$$p = p(\rho), \qquad \mu \geq 0, \qquad (2\mu + 3\lambda) \geq 0.$$

Associated to (2.4.1), we consider the Boundary Value Problem

$$\begin{aligned} &\nabla \cdot (\rho_b \mathbf{v}_b) = 0, \\ &\rho_b \mathbf{v}_b \cdot \nabla)\mathbf{v}_b = -\nabla p(\rho_b) + (\lambda + \mu)\nabla\nabla \cdot \mathbf{v}_b + \mu \Delta \mathbf{v}_b + \rho_b \mathbf{f}, \qquad x \in \Omega, \\ &\mathbf{v}_b\Big|_{\partial\Omega} = \mathbf{w}(x), \qquad \int_\Omega \rho_b \, dx = M, \quad \rho_b \geq 0, \end{aligned} \tag{2.4.2}$$

with $p = p(\rho)$ a smooth function.

(a_1) Rest State

As we know, the rest state exists only when external forces derive from a uniform potential $\mathbf{f} = \nabla U$, and $\mathbf{w} = 0$. Indeed, in this case, the rest state $\mathbf{v} = 0$, $\rho_b = \rho_b(x)$ is the exact solution to (2.4.2), with ρ_b implicitly given by

$$\int^{\rho_b} \frac{p'(s)}{s} ds = U + c, \qquad \int_\Omega \rho_b \, dx = M. \tag{2.4.3}$$

For the steady case, in order to have real positive solutions ρ_b (densities), we are lead to assume

Hypothesis R *The force is such that there exists a positive real solution to* (2.4.3).

Our uniqueness result will hold for **large potential forces** satisfying only **Hypothesis R**.

If $p'(\rho) > 0$, we may introduce the **Orlicz (1903-1990) space** $L_\phi(\Omega)$ with the following convex function

$$\phi(x) = (\rho - \rho_b) \int_{\rho_b(x)}^{\rho(x)} \frac{p'(s)}{s} ds,$$

in $L_1(\Omega)$.

Theorem 2.4.1 ***Uniqueness of the Rest State in the Class of Steady Solutions.*** *Let $\mathbf{f} = \nabla U \in H^{-1}(\Omega)$ satisfies Hypothesis R and $\mathbf{w} = 0$, then the rest state $\mathbf{v}_b = 0$, $\rho_b = \rho_b(x)$, with ρ_b implicitly given by* (2.4.3), *is unique in the class of solutions $\mathbf{v}$, ρ, to* (2.4.2) *where $\mathbf{v} \in H_0^1(\Omega)$, and $\rho - \rho_b$ belongs to $L_\phi(\Omega)$.*

The proof of this theorem is achieved by absurdum procedure, with the help of the FWE.

In order to state our stability theorems we introduce the class of generalized unsteady solutions $\Big(\mathbf{u}(\mathbf{x},t),\rho(\mathbf{x},t)\Big)$ in $\mathcal{W}$

$$\mathcal{W} = L^\infty\left(0,\infty;\, L^3(\Omega)\right)\cap L^2\left(0,\infty;\, W^{1,2}(\Omega)\right)\times L^\infty\left(0,\infty; C^0(\bar\Omega)\right). \quad (2.4.4)$$

Roughly speaking, we prove that in bounded domains Ω the L^2-norm of any *'regular'* perturbation $(\mathbf{u},\,\sigma)$ to the unique basic rest state S_b decays to zero as $t\to\infty$, at an exponential rate.

We begin by deriving an energy equation that represents the energy conservation law of unsteady motions $\mathbf{v}(x,t) = \mathbf{u}(x,t)$ and $\rho(x,t) = \rho_b(x)+\sigma(x,t)$ Dirichlet method. In doing this we shall pay attention on the perturbation terms $\mathbf{u}(x,t)$ and $\sigma(x,t)$.

Theorem 2.4.2 ***Energy Equation.*** *Let* $\mathbf{f}=\nabla U\in L^\infty(0,\infty;H^{-1}(\Omega))$, *and let* $\mathbf{u}=\mathbf{v}$, $\rho=\rho_b+\sigma$ *solve the Initial Boundary Value Problem* (2.4.1) *with* $(\mathbf{u},\rho)\in\mathcal{W}$. *Then, the energy equation holds*

$$\frac{d}{dt}\Big[E_u+E_\sigma\Big]+\mu\,D_u(t)=0, \quad (2.4.5)$$

$$E_u(t)=\frac{1}{2}\int_\Omega \rho\mathbf{u}^2\,dx;$$

$$E_\sigma(t)=\int_\Omega\rho\Big(\int^\rho\frac{p(s)}{s^2}ds-U\Big)\,dx=\frac{1}{2}\int_\Omega\frac{p'(\bar\rho)}{\bar\rho}\sigma^2\,dx;$$

$$\mu\,D_u(t)=\int_\Omega\Big[(\lambda+\mu)(\nabla\cdot\mathbf{u})^2\,dx+\mu|\nabla\ \mathbf{u}|^2\Big]\,dx.$$

We give now the asymptotic result.

Theorem 2.4.3 Nonlinear Exponential Stability. *Let* $\mathbf{f}=\nabla U\in L^\infty(0,\infty;H^{-1}(\Omega))$, *then the rest state* $\mathbf{v}_b=0$, $\rho_b=\rho_b(x)$, *with* ρ_b *given implicitly by* (2.4.3), *is* ***exponentially stable*** *in the class of motions* $\mathbf{u}=\mathbf{v}$, $\rho=\rho_b+\sigma$; *solutions to Initial Boundary Value Problem* (2.4.1) *with* $(\mathbf{u},\sigma)\in\mathcal{W}$.

(a_2) Non-potential Forces

Let us define the regularity class where uniqueness is proved

$$\mathcal{V}=\{(\mathbf{v},\rho)\in L^3(\Omega)\cap W^{1,2}(\Omega)\times(L^\infty(\Omega)\cap W^{1,\infty}(\Omega)):\\ \inf p'(\rho_1)=:m_1>0,\quad \inf\rho>0\},$$

$$\mathcal{V}_b=\{(\mathbf{v}_b,\rho_b)\in(L^3(\Omega)\cap W^{1,2}(\Omega))\times(L^\infty(\Omega)\cap W^{1,\infty}(\Omega));\\ \mathbf{v}_b\cdot\nabla\mathbf{v}_b\in L^3(\Omega);\quad \inf p'(\rho_b)=:m_b>0\}.$$

Notice that $\mathcal{V}_b\subseteq\mathcal{V}$. We remark that *it is reasonable to make assumptions on* $\mathbf{v}_b$ *because it is a given known motion, in this case regularity properties need only be verified, not proved.*

Theorem 2.4.4 Uniqueness of Steady Flows. *Let* $\boldsymbol{f} \in L^\infty(0,\infty;L^3(\Omega))$, *and let* $(\boldsymbol{v}_b, \rho_b) \in \mathcal{V}_b$ *be a given solution to the Boundary Value Problem* (2.4.2). *Then if* $\|\boldsymbol{f}\|_3$ *and constants* c_0, c_1, c_2, C_1, C_2, *defined by* (3.3.16), (3.3.24), *satisfy* (3.3.18), (3.3.27), (3.3.31), $(\boldsymbol{v}_b, \rho_b)$ *is the unique solution to Boundary Value Problem* (2.4.2) *in the regularity class of solutions with* $(\boldsymbol{v}, \rho) \in \mathcal{V}$.

In order to prove stability theorems within the presence of small non-potential forces we now define the following **regularity classes**:
Perturbed unsteady flows

$$\begin{aligned}\mathcal{W} = \{(\mathbf{v},\rho) \in L^\infty(0,\infty;L^3(\Omega)) \cap L^2(0,\infty;W^{1,2}(\Omega)) \times L^\infty(0,\infty;W^{1,\infty}(\Omega));\\ \inf p'(\rho) > 0;\ \inf \rho > 0;\ \partial_t\rho \in L^\infty(0,\infty;L^\infty(\Omega)),\\ \sqrt{\rho}\mathbf{v} \in L^\infty(0,\infty;L^2(\Omega))\}.\end{aligned}$$

Basic steady flows

$$\begin{aligned}\mathcal{V}_b = \{\mathbf{v}_b, \rho_b \in L^3(\Omega) \cap W^{1,2}(\Omega)) \times W^{1,\infty}(\Omega);\\ \mathbf{v}_b \cdot \nabla \mathbf{v}_b \in L^3(\Omega); \quad \inf_x p'(\rho_b) =: m_1 > 0\}.\end{aligned}$$

Assume there exists a steady solution $\mathbf{v}_b$, ρ_b to (2.4.2) in the regularity class $\mathcal{V}_b$, then it is derived an energy inequality, which provides an energy stability result. Next, employing the FWE, it is proved the exponential decay of suitable norms of perturbations to zero.

Theorem 2.4.5 ***Energy Equation.*** *Let* $\boldsymbol{f} \in L^\infty(0,\infty;L^3(\Omega))$, *and* $(\mathbf{v}_b, \rho_b) \in \mathcal{V}_b$. *Let* $\mathbf{v} = \mathbf{v}_b + \mathbf{u}$, $\rho = \rho_b + \sigma$, *solve the Initial Boundary Value Problem* (2.4.1) *with* $(\mathbf{v}, \rho) \in \mathcal{W}$. *Then, the following energy equation holds true*

$$\frac{d}{dt}\Big[E_u + E_\sigma\Big] + \mu\, D_u(t) = \mathcal{I}_1, \tag{2.4.6}$$

$$E_u(t) = \frac{1}{2}\int_\Omega \rho \mathbf{u}^2\, dx, \qquad E_\sigma(t) = \int_\Omega \frac{p'(\bar\rho)}{\rho}\frac{\sigma^2}{2}\, dx,$$

$$\mu\, D_u(t) = \int_\Omega \Big[(\lambda+\mu)(\nabla\cdot\mathbf{u})^2\, dx + \mu|\nabla\ \mathbf{u}|^2\Big]\, dx,$$

$$\begin{aligned}\mathcal{I}_1(t) &= \int_\Omega\ \boldsymbol{u}\cdot\boldsymbol{b}\, dx - \int_\Omega \frac{p'(\bar\rho)}{\rho}\sigma\boldsymbol{u}\cdot\nabla\rho\, dx\\ &\quad - \int_\Omega \Big[\frac{p'(\bar\rho)}{\rho}\nabla\cdot\boldsymbol{v}_b + \frac{\rho_b}{2}\Big(\partial_t + \boldsymbol{v}_b\nabla\Big)\cdot\frac{p'(\bar\rho)}{\rho_b\rho}\Big]\sigma^2\, dx,\end{aligned}$$

$$\boldsymbol{b} = -\rho_b\boldsymbol{u}\cdot\nabla\ \boldsymbol{v}_b + \sigma(\ \boldsymbol{f} -\ \boldsymbol{v}\cdot\nabla\ \boldsymbol{v}_b).$$

Theorem 2.4.6 ***Nonlinear Exponential Stability.*** *Let $\boldsymbol{f} \in L^\infty(0,\infty; L^3(\Omega))$, and let $(\boldsymbol{v}_b, \rho_b) \in \mathcal{V}_b$ be a given solution to the Boundary Value Problem* (2.4.2) *with* $\mathbf{w} = 0$. *If $\|\boldsymbol{f}\|_3$ and constants c_0, c_1, c_2, C_1, C_2, C_3, defined by* (3.3.16), (3.3.24), *satisfy* (3.3.18), (3.3.27) *and* (3.3.31), *then $(\boldsymbol{v}_b, \rho_b)$ is exponentially stable with respect to motions in the regularity class $\mathcal{W}$.*

(a_3) Non Zero Boundary Data

Let us suppose that $\mathbf{f} \in L^\infty(0,\infty; L^3(\Omega))$, $\mathbf{w} \in L^\infty(0,\infty; H^{1/2}(\partial\Omega))$, and let $\mathbf{w} \neq 0$.

Theorem 2.4.7 ***Uniqueness in the Class of Steady Flow*** *Let $\boldsymbol{f} \in L^\infty(0,\infty; L^3(\Omega))$, $\mathbf{w} \in L^\infty(0,\infty; H^{1/2}(\partial\Omega))$, and let $(\boldsymbol{v}_b, \rho_b) \in \mathcal{S}_b$ be a given solution to Boundary Value Problem* (2.4.2). *Then if $\|\boldsymbol{f}\|_3$ and constants c_0, c_1, c_2, C_1, C_2, C_3, defined by* (3.3.16), (3.3.24), *satisfy* (3.3.18), (3.3.27), (3.3.31) *thus $(\boldsymbol{v}_b, \rho_b)$ is the unique solution to Boundary Value Problem* (2.4.2) *in the regularity class of solutions with $(\boldsymbol{v}, \rho) \in \mathcal{V}$.*

Theorem 2.4.8 ***Nonlinear Exponential Stability*** *Let $\boldsymbol{f} \in L^\infty(0,\infty; L^3(\Omega))$, $\mathbf{w} \in L^\infty(0,\infty; H^{1/2}(\partial\Omega))$, and let $(\boldsymbol{v}_b, \rho_b) \in \mathcal{V}_b$ a given solution to* (2.4.2). *Let $\|\boldsymbol{f}\|_3$ and constants c_0, c_1, c_2, C_1, C_2, C_3, defined by* (3.3.16), (3.3.24), *satisfy* (3.3.18), (3.3.27) *and* (3.3.31), *then $(\boldsymbol{v}_b, \rho_b)$ is asymptotically stable with regards to motions in the regularity class W.*

(a_4) Domains Exterior to a Fixed Bounded Body $\mathcal{C}$

Consider the Initial Boundary Value Problem IBVP

$$\begin{aligned} &\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \\ &\partial_t(\rho\,\mathbf{v}) + \nabla(\rho\ \mathbf{v} \otimes\ \mathbf{v}) = -\nabla p(\rho) + \lambda \nabla\nabla \cdot \mathbf{v} + 2\mu \nabla \cdot D(\mathbf{v}) + \rho \mathbf{f}, \qquad (x,t) \in \Omega \times (0,T), \\ &\mathbf{v}|_{\partial\mathcal{C}} = 0, \qquad \lim_{x\to\infty} \mathbf{v} = 0 \qquad \lim_{x\to\infty} \rho = \rho_\infty, \end{aligned} \tag{2.4.7}$$

with $p = p(\rho)$.

Consider the Boundary Value Problem BVP

$$\begin{aligned} &\nabla \cdot (\rho \mathbf{v}) = 0, \\ &\nabla(\rho\ \mathbf{v} \otimes\ \mathbf{v}) = -\nabla p + \lambda \nabla\nabla \cdot \mathbf{v} + 2\mu \nabla \cdot D(\mathbf{v}) + \rho \mathbf{f}, \qquad x \in \Omega, \\ &\mathbf{v}|_{\partial\mathcal{C}} = 0, \qquad \lim_{x\to\infty} \mathbf{v} = 0 \qquad \lim_{x\to\infty} \rho = \rho_\infty, \end{aligned} \tag{2.4.8}$$

with $p = p(\rho)$.

Theorem 2.4.9 ***Uniqueness of the Rest State in the Class of Steady Solutions.*** *Let* $\boldsymbol{f} \in L^\infty(0,\infty; L^3(\Omega))$*, and let* $(\boldsymbol{v}_b, \rho_b) \in \mathcal{V}_b$ *be a given solution to* (2.4.8)*. Assume* $\|\boldsymbol{f}\|_3$ *and constants* c_0*,* c_1*,* c_2*,* C_1*,* C_2*,* C_3*, defined by* (3.3.16)*,* (3.3.24)*, satisfy* (3.3.18)*,* (3.3.27) *and* (3.3.31)*, then the solution* $(\boldsymbol{v}_b, \rho_b) \in \mathcal{V}_b$ *to* (2.4.8) *is unique in the regularity class of steady solutions in* $\mathcal{V}$*.*

Notice that for compressible fluids moving in exterior domains, despite the incompressible case, it is still possible to prove exponential stability. Of course, this requires severe hypotheses of regularity on the density $\rho \in L^{3/2}$, that in turn intuitively require that the mass of gas is not too large. We give the proof for isothermal fluids; different state equations for pressure is an open problem.

Theorem 2.4.10 ***Stability*** *Let* $\boldsymbol{f} \in L^\infty(0,\infty; L^3(\Omega))$*, and let* $(\boldsymbol{v}_b, \rho_b) \in \mathcal{V}_b$ *be a given solution to* (2.4.8)*. If* $\|\boldsymbol{f}\|_3$ *and constants* c_0*,* c_1*,* c_2*,* C_1*,* C_2*,* C_3*, defined by* (3.3.16)*,* (3.3.24)*, satisfy* (3.3.18)*,* (3.3.27) *and* (3.3.31)*, then the solution* $(\boldsymbol{v}_b, \rho_b) \in \mathcal{V}_b$ *to* (2.4.8) *is stable in the regularity class of unsteady solutions* $\mathcal{W}$*.*

Theorem 2.4.11 ***Nonlinear Exponential Stability*** *Let the gas be perfect, i.e. let* $p(\rho) = k\rho$*. Let* $\boldsymbol{f} \in L^\infty(0,\infty; L^3(\Omega))$*, and let* $(\boldsymbol{v}_b, \rho_b) \in \mathcal{V}_b$ *be a given solution to* (2.4.8)*. Assume* $\|\boldsymbol{f}\|_3$ *and constants* c_0*,* c_1*,* c_2*,* C_1*,* C_2*,* C_3*, defined by* (3.3.16)*,* (3.3.24)*, satisfy* (3.3.18)*,* (3.3.27) *and* (3.3.31)*, then the solution* $(\boldsymbol{v}_b, \rho_b) \in \mathcal{V}_b$ *to* (2.4.8) *is asymptotically stable in the regularity class of unsteady solutions* $\mathcal{W}$*.*

With the term "**phase change**" we mean a barotropic fluid whose pressure, for some intervals of density ρ, becomes a decreasing function of ρ.

($\boldsymbol{a_5}$) Instability of Rest State in a Phase Change

Hypothesis of instability (HI):
Before dealing with instability questions, let us observe that for given pressure, it is possible to construct more density fields such that the pressure has constant value; cf. footnote in Sect. 3.6.

Let us assume that:

(a) there exists more than one equilibrium configuration, i.e. the condition $p(\rho) = c$, when c is fixed, is satisfied by several values of ρ, say ρ_i, $i = 1, \ldots, N$, corresponding to the same given mass M and different volumes V_i.
(b) in at least one equilibrium configuration, say ρ_1, it holds

$$\frac{\partial^2 \rho\Psi}{\partial \rho^2}\Big|_{\rho_1} < 0. \tag{2.4.9}$$

In this case, we shall prove the following theorem:

Theorem 2.4.12 ***Instability*** *Assume the second order derivative of the* ***Helmholtz free energy*** *per unit of volume $\rho\Psi(\rho)$ satisfies the hypothesis of instability* (2.4.9)*, then the equilibrium position S_1 at ρ_1 is unstable.*

2.4.2 Case (b) Isothermal Fluid, Deformable Boundary

For problems with deformable boundaries we prove a uniqueness theorem of steady fluid motions occurring in a rectangular section of the horizontal layer Σ, having the rigid bottom below and a free upper surface. The problem is described in Cartesian coordinates by $z = \widehat{\zeta}(x)$; $\mathbf{k}$ is upward oriented, directed toward the free surface. We suppose that the domain occupied by the fluid is given by the cartesian representation $\Omega_t = \{\mathbf{x} = (x', z) : \ x' \in \Sigma, 0 < z < \zeta(x', t)\}$, remarking that such a representation is possible only when we exclude the formation of reversal flows. For the sake of simplicity we assume periodicity conditions at lateral walls. We use the notation $\nabla' = (\partial_1, \partial_2), \mathrm{div}' u' = \partial_1 u_1 + \partial_2 u_2$.

Given the system

$$
\begin{aligned}
\rho_t + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
\rho\left(\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v}\right) &= -k\nabla\rho + \nabla \cdot \mu\mathbf{S}(\mathbf{v}) + \rho\nabla U, \qquad (x,t) \in \Omega \times (0,T), \\
\zeta_t\left(x', t\right) &= \mathbf{v} \cdot \tilde{\mathbf{n}}\left(x', \zeta(x',t), t\right), \\
-k\rho\mathbf{n} + \mu\mathbf{S}(\mathbf{v}) &= \alpha\mathcal{H}(\zeta)\mathbf{n} - p_e\mathbf{n}, \quad on\ \Gamma_t, \\
\mathbf{v}\left(x', 0, t)\right) &= 0, \\
\mathbf{v}\left(x', z, 0\right) &= \mathbf{v}_0\left(x', z\right), \\
\rho\left(x', z, 0\right) &= \rho_0\left(x', z\right), \quad \zeta\left(x', 0\right) = \zeta_0\left(x'\right), \\
M = \int_{\Omega_b} \rho_b dx &= \int_{\Omega_0} \rho_0 dx = \int_{\Omega_t} \rho(x,t) dx, \\
\mu\mathbf{S}(\mathbf{v}) &= 2\mu\mathbf{D} + \lambda\nabla \cdot \mathbf{v}\mathbf{I}.
\end{aligned}
\tag{2.4.10}
$$

Here, $\mathcal{H}(\zeta)$ is the **double mean curvature** to Γ_t, and it holds

$$\mathcal{H}(\zeta) = \mathrm{div}'\Big(\frac{\nabla'\zeta}{\sqrt{1 + |\nabla'\zeta|^2}}\Big),$$

and $\mathbf{n}$ denotes the exterior unit normal vector at the point of free surface Γ_t,

$$\mathbf{n} = \frac{1}{\mathcal{G}}(-\partial_1\zeta, -\partial_2\zeta, 1) = \frac{1}{\mathcal{G}}(-\nabla'\zeta, 1), \qquad \mathcal{G} = \sqrt{1 + |\nabla'\zeta|^2}, \quad \tilde{\mathbf{n}} = \mathcal{G}\mathbf{n}.$$

We assume here that the initial density ρ_0, and height ζ_0, are everywhere positive. We state the following

Initial Free Boundary Value Problem

Given a periodicity cell Σ, external potential forces with potential U, and uniform external pressure p_e, initial data $(\mathbf{v}_0,\ \rho_0,\ \zeta_0)$, and total mass M, find the triple of functions $(\mathbf{v}(x', z, t),\ \rho(x', z, t),\ \zeta(x', t))$ defined in Ω_t, $t \in (0, \infty)$ for a solution to system (2.4.10).

To (2.4.10) we associate the steady system

$$\begin{aligned} \nabla \cdot (\rho \mathbf{u}) &= 0 \\ \rho(\mathbf{u} \cdot \nabla)\mathbf{u} - \mu\Delta\mathbf{u} - (\lambda + \mu)\nabla\nabla \cdot \mathbf{u} &= -\nabla p + \rho\nabla U + \rho\mathbf{f}, \\ \mathbf{u} \cdot \mathbf{n} &= 0, \\ \mu S(\mathbf{u})\mathbf{n} - p\mathbf{n} &= (-p_e + \kappa\mathcal{H}(\zeta))\mathbf{n}, \\ \mathbf{u}(x', 0) &= 0, \\ \int_{\Omega_\zeta} \rho(x)dx &= M. \end{aligned} \tag{2.4.11}$$

We also study the boundary value problem below.

Free Boundary Value Problem

Given a periodicity cell Σ, external potential forces with potential U, uniform external pressure p_e, and total mass M, find the triple of functions $(\mathbf{v}(x', z),\ \rho(x', z),\ \zeta(x'))$ defined in $\left(\Omega^2_\zeta \times \Sigma\right)$ for a solution to system (2.4.11).

(b_1) Uniqueness of Rest State

The first theorem is concerned with the uniqueness of the rest state of an isothermal fluid under gravity action, in the class of steady weak solutions $(\mathbf{v}_b(x', z), \rho_b(x', z), \zeta_b(x')) \in \mathcal{V}_b$

$$\mathcal{V}_b = L^2_\sharp(\Omega) \times W^{1,2}_\sharp(\Omega) \times W^{1,1}_\sharp(\Sigma) \cap L^\infty_\sharp(\Sigma),$$

where $\sharp$ means periodicity in the horizontal directions.

Assume that the rigid side of a layer is below the fluid, and that the gravity force $U = -gz$, with z upward oriented, is acting.

In the rectangle $\Omega_b = \Sigma \times (0, h)$ there exists at least the rest state S_b, with

$$S_b = \left\{\mathbf{v}_b = 0, \rho_b = \rho_* \exp\left(-\frac{gh}{k}\right), h = \frac{k}{g}\ln\left(1 + \frac{Mg}{p_e|\Sigma|}\right)\right\},$$

and ρ_* given in (4.4); cf. [6].

We now state the uniqueness theorem of S_b in the class of three-dimensional steady regular solutions to the boundary value problem (2.4.10), corresponding to the same force g, the same total mass M, the same periodicity Σ, and to the same external pressure p_e.

In order to present the main result we introduce the following regularity classes for steady $(\mathbf{u}(x', z), \rho(x', z), \zeta(x'))$, and unsteady $(\mathbf{u}(x', z, t), \rho(x', z, t), \zeta(x', t))$ solutions

$$\mathcal{V} = W_{\sharp}^{1,2}(\Omega) \times C_{\sharp}^{0}(\Omega) \times W_{\sharp}^{1,1}(\Sigma) \cap L_{\sharp}^{\infty}(\Sigma).$$

$$\mathcal{W} = L^2(0, \infty; W_{\sharp}^{1,2}(\Omega)) \times C_{\sharp}^{0}(\Omega \times (0, \infty)) \times L^{\infty}(0, \infty;\, W_{\sharp}^{1,1}(\Sigma)).$$

We prove the following uniqueness theorem.

Theorem 2.4.13 ***Uniqueness of the Rest State in the Class of Steady Solutions.*** *The rest state S_b is the unique solution to the system (2.4.11) in the class of steady solutions $(\mathbf{u}, \rho, \zeta)$ to the system (2.4.11) belonging to $\mathcal{V}$, corresponding to the same external data.*

(b_2) Stability of Rest State

We begin by deriving an energy equation that represents the difference between the energy equations of the non-steady motion and of the rest, respectively. Such a method is also known as **Dirichlet method**.

Theorem 2.4.14 ***Energy Equation.*** *Let $\mathbf{u}$, $\rho = \rho_b + \sigma$, $\zeta = h + \eta$ solve (2.4.10) with $(\mathbf{u}, \rho, \zeta) \in \mathcal{W}$. Then, the following energy equation holds*

$$\frac{d}{dt}\Big[E_u + E_\sigma + E_\zeta\Big] + \mu\, D_u(t) = 0, \tag{2.4.12}$$

$$E_u(t) = \frac{1}{2}\int_{\Omega_t} \rho \mathbf{u}^2\, dx,$$

$$E_\sigma(t) = k\int_{\Omega_t} \Big\{\rho(\ln\rho - \ln\rho_b) - (\rho - \rho_b)\Big\}\, dx,$$

$$E_\eta = \kappa\int_{\Sigma}\Big(\sqrt{1 + |\nabla'\eta|^2} \,-\, 1\Big)\, dx' + kM$$

$$- k\int_{\Sigma}\int_h^{\zeta}\Big(\rho_b(z) - \rho_b(h)\Big) dz dx',$$

$$\mu\, D_u(t) = \frac{\mu}{2}\int_{\Omega_t} |S(\mathbf{u})|^2\, dx.$$

It remains to prove exponential decay to the rest state for the L^2 norm of solutions to the full system (2.4.10), under the action of **large potential forces without smallness conditions on initial data.** To this end, we use again the FWE.

Theorem 2.4.15 ***Nonlinear Exponential Stability.*** *Assume that there exist solutions to* (2.4.10) $\mathbf{u}(x', z, t)$, $\rho(x', z, t)$, $\zeta(x', t)$ *in the regularity class* $\mathcal{W}$, *corresponding to initial data*

$$\mathbf{u}_0(x', z), \quad \rho_0(x', z) = \rho_b(x', z) + \sigma_0(x', z), \quad \zeta_0(x') = h + \eta_0(x').$$

Then, for any data $(\mathbf{u}_0, \rho_0, \eta_0)$ *in* $L^2(\Omega_0) \times C^0(\Omega_0) \times W^{1,\infty}(\Sigma)$, *the rest state* S_b *is exponentially stable in the energy norm in the class of solutions in* $\mathcal{W}$.

(b_3) Instability

Before giving our result, we introduce the non-dimensional characteristic number

$$Gr := \frac{g_0 \rho_*}{\kappa}\,, \tag{2.4.13}$$

with $\rho_* = g_0 M/(k|\Sigma|)$.

Theorem 2.4.16 *Assume that for all* $\delta > 0$, *there exists at least one initial data* $\mathbf{u}_0$, σ_0, η_0 *with* $\|(\mathbf{u}_0, \sigma_0, \eta_0)\|_Y < \delta$, *such that the initial energy* E_0 *is negative,*

$$\begin{aligned} 0 < -E_0 = k \int_\Sigma \int_h^\zeta \Big(\rho_b(z) - \rho_b(h)\Big) dz dx' \qquad\qquad & \qquad (2.4.14)\\ - k\, M - \kappa \int_\Sigma \left(\sqrt{1 + |\nabla' \eta_0|^2} - 1\right) dx' - E_{u_0} - E_{\sigma_0}, & \end{aligned}$$

then the rest state S_b *is nonlinearly unstable. More precisely, there exists an* $\epsilon > 0$ *such that for all* $\delta > 0$ *there exists an initial value* $(\mathbf{u}_0, \rho_0 = \rho_b + \sigma_0, \zeta_0 = h + \eta_0) \in W^{1,2}(\Omega_0) \times W^{1,\infty}(\Sigma)$ *less than* δ, *and there exists* $T > 0$ *such that the solution* $(\mathbf{u}, \rho = \rho_b + \sigma, \zeta = h + \eta)$ *of the problem* (2.4.10) *satisfies the inequality*

$$\|(\mathbf{u}, \sigma, \eta)\|_X(T) \geq \epsilon. \tag{2.4.15}$$

(b_4) Loss of Initial Data Control

To deal with full nonlinear instability problems, we give the following

Definition 2.4.1 *The rest state is said to **lose the control from initial data** if there exists a positive large number a and there exists a perturbation $(\mathbf{u}_0, \sigma_0, \eta_0)$ to initial data satisfying*

$$a < \|(\mathbf{u}_0, \sigma_0, \eta_0)\|_Y < 2a, \tag{2.4.16}$$

such that the corresponding perturbation $(\mathbf{u}(x,t), \sigma(x,t), \eta(x,t))$ is not controlled by the initial data. That is, given $\alpha > 0$, there exists $T > 0$ such that the perturbation $(\mathbf{u}, \sigma, \eta)$ with initial data satisfying (2.4.16) satisfies the inequality

$$\|(\mathbf{u}, \sigma, \eta)\|_X(T) := \|\mathbf{u}(T)\|_{W^{1,2}(\Omega_T)} + \|\sigma(T)\|_{L^2(\Omega_T)} + \|\eta(T)\|_{W^{1,\infty}(\Sigma)} \geq \alpha. \tag{2.4.17}$$

Next we construct a solution $(\mathbf{u}(x,t), \sigma(x,t), \eta(x',t))$ that though linearly stable, is not controlled by initial data when the data set is larger than a computable constant A.

Theorem 2.4.17 *We assume that the linear stability hypothesis:*

$$Gr < 1 \tag{2.4.18}$$

holds; cf. see Sect. 4.3.1. We may construct initial values $(\mathbf{u}_0, \sigma_0, \eta_0) \in W^{1,2}(\Omega_0) \times C_0(\Omega_0) \times W^{1,\infty}(\Sigma)$ sufficiently small such that the value of initial energy E_0 is negative.

Following upon Theorem 2.4.16 we may prove:

Theorem 2.4.18 ***Loss of Initial Data Control of Solution*** *There exists a positive large number a, such that the solution $(\mathbf{u}(x,t), \sigma(x,t), \eta(x,t))$, corresponding to initial data $(\mathbf{u}_0, \sigma_0, \eta_0)$ satisfying (2.4.16), is not controlled by the initial data. That is, however fixed $\alpha > 0$, there exists initial data $(\mathbf{u}_0^\alpha, \sigma^\alpha, \eta_0^\alpha)$ satisfying (2.4.16), and an instant $T^\alpha > 0$ such that the perturbation $(\mathbf{u}^\alpha, \sigma^\alpha, \eta^\alpha)$ corresponding to problem (2.4.10) satisfies the inequality*

$$\|(\mathbf{u}^\alpha, \sigma^\alpha, \eta^\alpha)\|_X(T^\alpha) \geq \alpha. \tag{2.4.19}$$

We conjecture that all proofs and considerations of this subsection continue to hold in the case of an infinite layer. We leave it as an open problem.

2.4.3 Case (c) Polytropic Fluid, Rigid Boundary

Let us consider the initial boundary value problem

$$
\begin{aligned}
&\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0,\\
&\rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla)\mathbf{u}) = \nabla\cdot\mathbf{T} - \rho g\mathbf{k}, \qquad (x,t)\in\Omega\times(0,T),\\
&\rho c_v(\Theta_t + \mathbf{u}\cdot\nabla\Theta) = \chi\Delta\Theta - R_*\rho\Theta\nabla\cdot\mathbf{u} + 2\mu\mathbf{S}(\mathbf{u})^2 + \lambda(\nabla\cdot\mathbf{u})^2, \qquad (x,t)\in\Omega\times(0,T),\\
&\mathbf{u}(x',0,t) = \mathbf{u}(x',h,t) = 0, \qquad \Theta(x',0,t) = \Theta_h + \beta h, \quad \Theta(x',h) = \Theta_h,\\
&\mathbf{u}(x',z,0) = \mathbf{u}_0(x',z), \qquad \Theta(x',z,0) = \Theta_0(x',z), \qquad (x',z)\in\Sigma\times(0,h),\\
&\int_\Omega \rho = M.
\end{aligned}
\tag{2.4.20}
$$

where $\mathbf{T} = -p\mathbf{I} + 2\mu D(\mathbf{u}) + \lambda\nabla\cdot\mathbf{u}\mathbf{I}$ is the stress tensor, $p = R_*\rho\Theta$ is the pressure, R_* the universal gas constant, μ is the shear viscosity, λ the bulk viscosity, $D(\mathbf{u}) = (\nabla\mathbf{u} + \nabla^T\mathbf{u})/2$ is the rate-of-strain tensor, c_v is the specific heat at constant volume, and χ is the coefficient of thermal conductivity.

We also give the system

$$
\begin{aligned}
\nabla\cdot(\rho\mathbf{u}) &= 0,\\
\rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} &= -\nabla p - \rho g\nabla z + \rho\mathbf{f}, \qquad x\in\Omega,\\
\rho c_v(\Theta_t + \mathbf{u}\cdot\nabla\Theta) &= \chi\Delta\Theta - R_*\rho\Theta\nabla\cdot\mathbf{u} + 2\mu\mathbf{S}(\mathbf{u})^2 + \lambda(\nabla\cdot\mathbf{u})^2, \quad x\in\Omega,\\
\mathbf{u}(x',0,t) &= \mathbf{u}(x',h,t) = 0,\\
\Theta(x',h) &= \Theta_h, \quad \Theta(x',0,t) = \Theta_h + \beta h, \quad (x',z)\in\Sigma\times(0,h),\\
\int_\Omega \rho &= M.
\end{aligned}
\tag{2.4.21}
$$

In order to present the results we introduce the following regularity classes

$$
\begin{aligned}
\mathcal{V} &= \Big\{(\rho(x',z), \mathbf{u}(x',z), \Theta(x')) \in C^0_\sharp(\Omega)\times W^{1,2}_\sharp(\Omega)\times W^{1,2}_\sharp(\Omega)\Big\}.\\
\mathcal{W} &= \Big\{\rho(x',z,t), \mathbf{u}(x',z,t), \Theta(x',z,t)\\
&\qquad \in C^0_\sharp(\Omega\times(0,\infty))\times L^2(0,\infty; W^{1,2}_\sharp(\Omega))\times L^2(0,\infty; W^{1,2}_\sharp(\Omega)\Big\}.
\end{aligned}
$$

A non-linear stability result for heat conducting fluids in exterior domain has been proven in cf. [111].

Theorem 2.4.19 ***Uniqueness in the Class of Steady Flow*** *The rest state S_b is the unique steady solution to system (2.4.21) in the class of motions $\mathcal{V}$, corresponding to the same data, provided temperature gradient is sufficiently small.*

Theorem 2.4.20 ***Nonlinear Exponential Stability*** *Let the conditions* (5.3.25) *be verified. Then the rest state S_b is asymptotically stable for solutions to system* (2.4.20) *in the class $\mathcal{W}$, corresponding to the same data.*

2.5 Bibliographical Notes

Stability results, compressible fluids : The stability of compressible fluids is a challenging problem related to the study of the structure of linearized problems, which in turn depends upon the basic flow. In the case of parallel fluid flows, it has been studied in cf. [61].

Concerning the existence of regular solutions close to the rest state, there have been existence theorems since 1981 under assumptions of viscosity coefficients, cf. [90, 91, 95], and without restrictions on the coefficients, cf. [13, 56, 69–72, 130, 146].

Since 1986, several stability results have been proven; cf. [50, 94, 147], we quote also [72, 96, 97, 103, 104, 140]. Where bounded and unbounded domains with rigid, compact boundaries and strictly positive, bounded densities are considered. In these papers, stability of steady flows is proven for barotropic gases, either under large potential forces, or under small non-potential ones. Concerning unbounded domains, we cite [102] for isothermal fluids with density bounded from below, and for isothermal fluids with nonnegative density, uniqueness theorems have been proved; cf.[93, 99].

The method of the free work inequality appears to be useful as a new hint in the study of nonlinear asymptotic stability; a similar method has been introduced in the theory of elasticity by Abeyaratne and Knowles, cf. [2]. Concerning the stability and instability results proved with the free work inequality, we quote the papers by the author in cf. [104–109]. The stability proofs are formal, if initial data are large.

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