Chapter II
Manifolds

We continue to fix the nonarchimedean field $(K, |\ |)$. But we change our system of notations insofar as from now on we will denote K-Banach spaces by letters like E whereas we reserve letters like U and V for open subsets in a topological space.

7 Charts and Atlases

Let M be a Hausdorff topological space.

Definition. i. *A chart for M is a triple (U, φ, K^n) consisting of an open subset $U \subseteq M$ and a map $\varphi : U \longrightarrow K^n$ such that*:

(a) $\varphi(U)$ *is open in K^n,*

(b) $\varphi : U \xrightarrow{\simeq} \varphi(U)$ *is a homeomorphism.*

ii. *Two charts $(U_1, \varphi_1, K^{n_1})$ and $(U_2, \varphi_2, K^{n_2})$ for M are called compatible if both maps*

$$\varphi_1(U_1 \cap U_2) \underset{\varphi_1 \circ \varphi_2^{-1}}{\overset{\varphi_2 \circ \varphi_1^{-1}}{\rightleftarrows}} \varphi_2(U_1 \cap U_2)$$

are locally analytic.

We note that the condition in part ii. of the above definition makes sense since $\varphi_1(U_1 \cap U_2)$ is open in K^{n_i}. If (U, φ, K^n) is a chart then the open subset U is called its *domain of definition* and the integer $n \geq 0$ its *dimension*. Usually we omit the vector space K^n from the notation and simply write (U, φ) instead of (U, φ, K^n). If x is a point in U then (U, φ) is also called a *chart around x*.

Lemma 7.1. *Let $(U_i, \varphi_i, K^{n_i})$ for $i = 1, 2$ be two compatible charts for M; if $U_1 \cap U_2 \neq \emptyset$ then $n_1 = n_2$.*

Proof. Let $x \in U_1 \cap U_2$ and put $x_i := \varphi_i(x)$. We consider the locally analytic maps

$$\varphi_1(U_1 \cap U_2) \underset{g := \varphi_1 \circ \varphi_2^{-1}}{\overset{f := \varphi_2 \circ \varphi_1^{-1}}{\rightleftarrows}} \varphi_2(U_1 \cap U_2).$$

P. Schneider, *p-Adic Lie Groups*,
Grundlehren der mathematischen Wissenschaften 344,
DOI 10.1007/978-3-642-21147-8_2,

They are differentiable and inverse to each other, and $x_2 = f(x_1)$. Hence, by the chain rule, the derivatives

$$\begin{tikzcd} K^{n_1} \arrow[r, shift left, "D_{x_1} f"] & K^{n_2} \arrow[l, shift left, "D_{x_2} g"] \end{tikzcd}$$

are linear maps inverse to each other. It follows that $n_1 = n_2$. $\square$

Definition. i. *An atlas for M is a set $\mathcal{A} = \{(U_i, \varphi_i, K^{n_i})\}_{i \in I}$ of charts for M any two of which are compatible and which cover M in the sense that $M = \bigcup_{i \in I} U_i$.*

ii. *Two atlases $\mathcal{A}$ and $\mathcal{B}$ for M are called equivalent if $\mathcal{A} \cup \mathcal{B}$ also is an atlas for M.*

iii. *An atlas $\mathcal{A}$ for M is called maximal if any equivalent atlas $\mathcal{B}$ for M satisfies $\mathcal{B} \subseteq \mathcal{A}$.*

Remark 7.2. i. *The equivalence of atlases indeed is an equivalence relation.*

ii. *In each equivalence class of atlases there is exactly one maximal atlas.*

Proof. i. Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ be three atlases such that $\mathcal{A}$ is equivalent to $\mathcal{B}$ and $\mathcal{B}$ is equivalent to $\mathcal{C}$. Then $\mathcal{A}$ is equivalent to $\mathcal{C}$ if we show that any chart (U_1, φ_1) in $\mathcal{A}$ is compatible with any chart (U_2, φ_2) in $\mathcal{C}$. By symmetry it suffices to show that the map $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \longrightarrow \varphi_2(U_1 \cap U_2)$ is locally analytic in a sufficiently small open neighbourhood of $\varphi_1(x)$ for any point $x \in U_1 \cap U_2$. Since $\mathcal{B}$ covers M we find a chart (V, ψ) around x in $\mathcal{B}$. By assumption (V, ψ) is compatible with both (U_1, φ_1) and (U_2, φ_2). Then $\varphi_1(U_1 \cap V \cap U_2)$ is an open neighbourhood of $\varphi_1(x)$ in $\varphi_1(U_1 \cap U_2)$ on which the map $\varphi_2 \circ \varphi_1^{-1}$ is the composite of the two locally analytic maps $\varphi_2 \circ \psi^{-1}$ and $\psi \circ \varphi_1^{-1}$. Hence it is locally analytic by Lemma 6.3.

ii. If the given equivalence class consists of the atlases $\mathcal{A}_j$ for $j \in J$ then $\mathcal{A} := \bigcup_{j \in J} \mathcal{A}_j$ is the unique maximal atlas in this class. $\square$

Lemma 7.3. *If $\mathcal{A}$ is a maximal atlas for M the domains of definition of all the charts in $\mathcal{A}$ form a basis of the topology of M.*

Proof. Let $U \subseteq M$ be an open subset. We have to show that U is the union of the domains of definition of the charts in some subset of $\mathcal{A}$, or equivalently that for any point $x \in U$ we find a chart (U_x, φ_x) around x in $\mathcal{A}$ such that

$U_x \subseteq U$. Since $\mathcal{A}$ covers M we at least find a chart (U'_x, φ'_x) around x in $\mathcal{A}$. We put $U_x := U'_x \cap U$ and $\varphi_x := \varphi'_x | U_x$. Clearly (U_x, φ_x) is a chart around x for M such that $U_x \subseteq U$. We claim that (U_x, φ_x) is compatible with any chart (V, ψ) in $\mathcal{A}$. But we do have the locally analytic maps

$$\varphi'_x(U'_x \cap V) \underset{\varphi'_x \circ \psi^{-1}}{\overset{\psi \circ \varphi'^{-1}_x}{\rightleftarrows}} \psi(U'_x \cap V)$$

which restrict to the locally analytic maps

$$\varphi_x(U_x \cap V) \underset{\varphi_x \circ \psi}{\overset{\psi \circ \varphi_x^{-1}}{\rightleftarrows}} \psi(U_x \cap V).$$

Hence $\mathcal{B} := \mathcal{A} \cup \{(U_x, \varphi_x)\}$ is an atlas equivalent to $\mathcal{A}$. The maximality of $\mathcal{A}$ then implies that $\mathcal{B} \subseteq \mathcal{A}$ and a fortiori $(U_x, \varphi_x) \in \mathcal{A}$. $\square$

Definition. *An atlas $\mathcal{A}$ for M is called n-dimensional if all the charts in $\mathcal{A}$ with nonempty domain of definition have dimension n.*

Remark 7.4. *Let $\mathcal{A}$ be an n-dimensional atlas for M; then any atlas $\mathcal{B}$ equivalent to $\mathcal{A}$ is n-dimensional as well.*

Proof. Let (V, ψ) be any chart in $\mathcal{B}$ and choose a point $x \in V$. We find a chart (U, φ) in $\mathcal{A}$ around x. Since $\mathcal{A}$ and $\mathcal{B}$ are equivalent these two charts have to be compatible. It then follows from Lemma 7.1 that both have the same dimension u. $\square$

8 Manifolds

Definition. *A (locally analytic) manifold $(M, \mathcal{A})$ (over K) is a Hausdorff topological space M equipped with a maximal atlas $\mathcal{A}$. The manifold is called n-dimensional (we write $\dim M = n$) if the atlas $\mathcal{A}$ is n-dimensional.*

By abuse of language we usually speak of a manifold M while considering $\mathcal{A}$ as given implicitly. A chart for M will always mean a chart in $\mathcal{A}$.

Example. *K^n will always denote the n-dimensional manifold whose maximal atlas is equivalent to the atlas $\{(U, \subseteq, K^n) : U \subseteq K^n \text{ open}\}$.*

Remark 8.1. *Let (U, φ, K^n) be a chart for the manifold M; if $V \subseteq U$ is an open subset then $(V, \varphi|V, K^n)$ also is a chart for M.*

Proof. This was shown in the course of the proof of Lemma 7.3. □

Let $(M, \mathcal{A})$ be a manifold and $U \subseteq M$ be an open subset. Then

$$\mathcal{A}_U := \{(V, \psi, K^n) \in \mathcal{A} : V \subseteq U\},$$

by Lemma 7.3, is an atlas for U. We claim that $\mathcal{A}_U$ is maximal. Let (V_0, ψ_0) be a chart for U which is compatible with any chart in $\mathcal{A}_U$. To see that $(V_0, \psi_0) \in \mathcal{A}_U$ it suffices, by the maximality of $\mathcal{A}$, to show that (V_0, ψ_0) is compatible with any chart (V, ψ) in $\mathcal{A}$. The Remark 8.1 implies that $(V \cap U, \psi|V \cap U)$ is a chart in $\mathcal{A}$ and hence in $\mathcal{A}_U$. By assumption (V_0, ψ_0) is compatible with $(V \cap U, \psi|V \cap U)$. Since $V_0 \cap V \subseteq V \cap U$ the compatibility of (V_0, ψ_0) with (V, ψ) follows trivially. The manifold $(U, \mathcal{A}_U)$ is called an *open submanifold* of $(M, \mathcal{A})$.

As a nontrivial example of a manifold we discuss the d-dimensional projective space $\mathbb{P}^d(K)$ over K. We recall that $\mathbb{P}^d(K) = (K^{d+1} \setminus \{0\})/\sim$ is the set of equivalence classes in $K^{d+1} \setminus \{0\}$ for the equivalence relation

$$(a_1, \ldots, a_{d+1}) \sim (ca_1, \ldots, ca_{d+1}) \text{ for any } c \in K^\times.$$

As usual we write $[a_1 : \ldots : a_{d+1}]$ for the equivalence class of $(a_1, \ldots, a_{d+1})$. With respect to the quotient topology from $K^{d+1} \setminus \{0\}$ the projective space $\mathbb{P}^d(K)$ is a Hausdorff topological space. For any $1 \leq j \leq d+1$ we have the open subset

$$U_j := \{[a_1 : \ldots : a_{d+1}] \in \mathbb{P}^d(K) : |a_i| \leq |a_j| \text{ for any } 1 \leq i \leq d+1\}$$

together with the homeomorphism

$$\begin{aligned} \varphi_j : \qquad & U_j \xrightarrow{\simeq} B_1(0) \subseteq K^d \\ [a_1 : \ldots : a_{d+1}] & \longmapsto \left(\frac{a_1}{a_j}, \ldots, \frac{a_{j-1}}{a_j}, \frac{a_{j+1}}{a_j}, \ldots, \frac{a_{d+1}}{a_j}\right). \end{aligned}$$

The (U_j, φ_j, K^d) are charts for $\mathbb{P}^d(K)$ such that $\bigcup_j U_j = \mathbb{P}^d(K)$. We claim that they are pairwise compatible. For $1 \leq j < k \leq d+1$ the composite

$$f : V := \{x \in B_1(0) : |x_{k-1}| = 1\} \xrightarrow{\varphi_j^{-1}} U_j \cap U_k \xrightarrow{\varphi_k} \{y \in B_1(0) : |y_j| = 1\}$$

is given by

$$f(x_1, \ldots, x_d) = \left(\frac{x_1}{x_{k-1}}, \ldots, \frac{x_{j-1}}{x_{k-1}}, \frac{1}{x_{k-1}}, \frac{x_j}{x_{k-1}}, \ldots, \frac{x_{k-2}}{x_{k-1}}, \frac{x_k}{x_{k-1}}, \ldots, \frac{x_d}{x_{k-1}}\right).$$

Let $a \in V$ be a fixed but arbitrary point and choose a $0 < \varepsilon < 1$. Then $B_\varepsilon(a) \subseteq V$. We consider the power series

$$F_j(X) := \frac{1}{a_{k-1}} \sum_{n \geq 0} \left(-\frac{1}{a_{k-1}} \right)^n X_{k-1}^n$$

and

$$F_i(X) := F_j(X) \cdot \begin{cases} (X_i + a_i) & \text{if } 1 \leq i < j \text{ or } k \leq i \leq d, \\ (X_{i-1} + a_{i-1}) & \text{if } j < i < k. \end{cases}$$

Because of $|a_{k-1}| = 1$ we have $F := (F_1, \ldots, F_d) \in \mathcal{F}_\varepsilon(K^d; K^d)$. For $x \in B_\varepsilon(a)$ we compute

$$F_j(x - a) = \frac{1}{a_{k-1}} \sum_{n \geq 0} \left(-\frac{x_{k-1} - a_{k-1}}{a_{k-1}} \right)^n = \frac{1}{a_{k-1}} \cdot \frac{1}{1 + \frac{x_{k-1} - a_{k-1}}{a_{k-1}}} = \frac{1}{x_{k-1}}$$

and then

$$f(x) = F(x - a).$$

Hence f is locally analytic. In case $j > k$ the argument is analogous. The above charts therefore form a d-dimensional atlas for $\mathbb{P}^d(K)$.

Exercise. *Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be two manifolds. Then*

$$\mathcal{A} \times \mathcal{B} := \{(U \times V), \varphi \times \psi, K^{m+n}) : (U, \varphi, K^m) \in \mathcal{A}, (V, \psi, K^n) \in \mathcal{B}\}$$

is an atlas for $M \times N$ with the product topology. We call $M \times N$ equipped with the equivalent maximal atlas the product manifold of M and N.

Let M be a manifold and E be a K-Banach space.

Definition. *A function $f : M \longrightarrow E$ is called locally analytic if $f \circ \varphi^{-1} \in C^{\mathrm{an}}(\varphi(U), E)$ for any chart (U, φ) for M.*

Remark 8.2. i. *Every locally analytic function $f : M \longrightarrow E$ is continuous.*

ii. *Let $\mathcal{B}$ be any atlas consisting of charts for M; a function $f : M \longrightarrow E$ is locally analytic if and only if $f \circ \varphi^{-1} \in C^{\mathrm{an}}(\varphi(U), E)$ for any $(U, \varphi) \in \mathcal{B}$.*

The set

$$C^{\mathrm{an}}(M, E) := \text{all locally analytic functions } f : M \longrightarrow E$$

is a K-vector space with respect to pointwise addition and scalar multiplication. It is easy to see that a list of properties 1)–6) completely analogous to the one given in Sect. 6 holds true. In a later section we will come back to a more detailed study of this vector space.

Let now M and N be two manifolds. The following result is immediate.

Lemma 8.3. *For a map $g : M \longrightarrow N$ the following assertions are equivalent:*

i. *g is continuous and $\psi \circ g \in C^{\mathrm{an}}(g^{-1}(V), K^n)$ for any chart (V, ψ, K^n) for N;*

ii. *for any point $x \in M$ there exist a chart (U, φ, K^m) for M around x and a chart (V, ψ, K^n) for N around $g(x)$ such that $g(U) \subseteq V$ and $\psi \circ g \circ \varphi^{-1} \in C^{\mathrm{an}}(\varphi(U), K^n)$.*

Definition. *A map $g : M \longrightarrow N$ is called locally analytic if the equivalent conditions in Lemma 8.3 are satisfied.*

Lemma 8.4. i. *If $g : M \longrightarrow N$ is a locally analytic map and E is a K-Banach space then*

$$\begin{aligned} C^{\mathrm{an}}(N, E) &\longrightarrow C^{\mathrm{an}}(M, E) \\ f &\longmapsto f \circ g \end{aligned}$$

is a well defined K-linear map.

ii. *With $L \xrightarrow{f} M \xrightarrow{g} N$ also $g \circ f : L \longrightarrow N$ is a locally analytic map of manifolds.*

Proof. This follows from Lemma 6.3. $\square$

Examples 8.5. 1) *For any open submanifold U of M the inclusion map $U \xrightarrow{\subseteq} M$ is locally analytic.*

2) *Let $g : M \longrightarrow N$ be a locally analytic map; for any open submanifold $V \subseteq N$ the induced map $g^{-1}(V) \xrightarrow{g} V$ is locally analytic.*

3) *The two projection maps*

$$\mathrm{pr}_1 : M \times N \longrightarrow M \quad \textit{and} \quad \mathrm{pr}_2 : M \times N \longrightarrow N$$

are locally analytic.

4) *For any pair of locally analytic maps $g : L \longrightarrow M$ and $f : L \longrightarrow N$ the map*

$$\begin{aligned}(g, f) : L &\longrightarrow M \times N \\ x &\longmapsto (g(x), f(x))\end{aligned}$$

is locally analytic.

For the remainder of this section we will discuss a certain technical but useful topological property of manifolds. First let X be an arbitrary Hausdorff topological space. We recall:

– Let $X = \bigcup_{i\in I} U_i$ and $X = \bigcup_{j\in J} V_j$ be two open coverings of X. The second one is called a *refinement* of the first if for any $j \in J$ there is an $i \in I$ such that $V_j \subseteq U_i$.

– An open covering $X = \bigcup_{i\in I} U_i$ of X is called *locally finite* if every point $x \in X$ has an open neighbourhood U_x such that the set $\{i \in I : U_x \cap U_i \neq \emptyset\}$ is finite.

– The space X is called *paracompact*, resp. *strictly paracompact*, if any open covering of X can be refined into an open covering which is locally finite, resp. which consists of pairwise disjoint open subsets.

Remark 8.6. i. *Any ultrametric space X is strictly paracompact.*

ii. *Any compact space X is paracompact.*

Proof. i. This follows from Lemma 1.4. ii. This is trivial. □

Proposition 8.7. *For a manifold M the following conditions are equivalent*:

i. *M is paracompact*;

ii. *M is strictly paracompact*;

iii. *the topology of M can be defined by a metric which satisfies the strict triangle inequality.*

Proof. The implication iii. $\Longrightarrow$ ii. is Lemma 1.4, and the implication ii. $\Longrightarrow$ i. is trivial.

i. $\Longrightarrow$ ii. We suppose that M is paracompact. From general topology we recall the following property of paracompact Hausdorff spaces (cf. [B-GT] Chap. IX §4.4 Cor. 2). Let $A \subseteq U \subseteq M$ be subsets with A closed and U open. Then there is another open subset $V \subseteq M$ such that

$$A \subseteq V \subseteq \bar{V} \subseteq U.$$

Step 1: We show that the open and closed subsets of M form a basis of the topology. Given a point x in an open subset $U \subseteq M$ we have to find an open and closed subset $W \subseteq M$ such that $x \in W \subseteq U$. By Lemma 7.3 we may assume that U is the domain of definition of a chart (U, φ, K^n) for M. As recalled above there is an open neighbourhood $V \subseteq M$ of x such that $\bar{V} \subseteq U$. We then have the vertical homeomorphisms

$$\begin{tikzcd}
V \arrow[r, "\subseteq"] \arrow[d, "\simeq"] & \bar{V} \arrow[r, "\subseteq"] \arrow[d, "\simeq"] & U \arrow[r, "\subseteq"] \arrow[d, "\simeq"] & M \\
\varphi(V) \arrow[r, "\subseteq"] & \varphi(\bar{V}) \arrow[r, "\subseteq"] & \varphi(U) \arrow[r, "\subseteq"] & K^n.
\end{tikzcd}$$

Since $\varphi(V)$ is open in K^n there is a ball $B := B_\varepsilon(\varphi(x)) \subseteq \varphi(V)$ around $\varphi(x)$. We put $W := \varphi^{-1}(B) \subseteq V$. Clearly $x \in W \subseteq U$. The ball B is open and hence B is open in V and M. But the ball B also is closed in K^n. Hence W is closed in $\bar{V}$ and therefore in M. This finishes step 1.

Let now $M = \bigcup_{i \in I} U_i$ be a fixed but arbitrary open covering. By Lemma 7.3 we may assume, after refinement, that any U_i is the domain of definition of some chart for M. By the first step and Remark 8.1 we may even assume, after a further refinement, that each U_i is open and closed in M and is the domain of definition of some chart for M. In particular, each U_i has the topology of an ultrametric space. By assumption we may pick a locally finite refinement $(V_j)_{j \in J}$ of $(U_i)_{i \in I}$. So we have the locally finite open covering $M = \bigcup_{j \in J} V_j$, and for each $j \in J$ there is an $i(j) \in I$ such that $V_j \subseteq U_{i(j)}$.

Step 2: We construct a covering $M = \bigcup_{j \in J} W_j$ by open and closed subsets $W_j \subseteq M$ such that $W_j \subseteq V_j$ for any $j \in J$. For this purpose we equip J with a well-order $\leq$ (recall that this is a total order on J with the property that each nonempty subset of J has a minimal element—by the axiom of choice such a well-order always exists). We now use transfinite induction to find open and closed subsets $W_j \subseteq M$ such that

(a) $W_j \subseteq V_j$ for any $j \in J$, and

(b) $M = \big(\bigcup_{j \leq k} W_j\big) \cup \big(\bigcup_{j > k} V_j\big)$ for any $k \in J$.

We fix a $k \in J$ and suppose that the W_j for $j < k$ are constructed already.

Claim: $M = \big(\bigcup_{j<k} W_j\big) \cup \big(\bigcup_{j \geq k} V_j\big)$.

Let $x \in M$. Since the covering $(V_j)_j$ is locally finite the set

$$\{j \in J : x \in V_j\} = \{j_1 < \cdots < j_r\}$$

is finite. If $j_r \geq k$ then $x \in V_{j_r} \subseteq \bigcup_{j \geq k} V_j$. If $j_r < k$ then $x \notin V_j$ for any $j > j_r$ and the induction hypothesis (property (b) for j_r) implies $x \in \bigcup_{j \leq j_r} W_j \subseteq \bigcup_{j<k} W_j$. This establishes the claim.

We see that the closed subset

$$W := M \setminus \left(\left(\bigcup_{j<k} W_j \right) \cup \left(\bigcup_{j>k} V_j \right) \right)$$

of M satisfies

$$W \subseteq V_k \subseteq U_{i(k)}.$$

Claim: Let (X, d) be an ultrametric space; for any subsets $A \subseteq U \subseteq X$ with A closed and U open there exists an open and closed subset $V \subseteq X$ such that

$$A \subseteq V \subseteq U.$$

For any subset $D \subseteq X$ and any $x \in X$ we put

$$d(x, D) := \inf_{y \in D} d(x, y).$$

The strict triangle inequality implies that the function $d(., D)$ on X is continuous and that

$$D(\varepsilon) := \{x \in X : d(x, D) = \varepsilon\},$$

for any $\varepsilon > 0$, is open in X. Moreover, $D(0) = \bar{D}$. The closed subsets A and $B := X \setminus U$ of X satisfy $A \cap B = \emptyset$. By the continuity of the functions $d(., A)$ and $d(., B)$ the subset

$$V := \{x \in X : d(x, A) < d(x, B)\}$$

therefore is open in X and satisfies $A \subseteq V \subseteq U$. Similarly $V' := \{x \in X : d(x, A) > d(x, B)\}$ is open in X. It follows that V as the complement in X of the open subset $V' \cup \big(\bigcup_{\varepsilon > 0} (A(\varepsilon) \cap B(\varepsilon))\big)$ is closed. This establishes the claim.

We apply this claim to $W \subseteq V_k \subseteq U_{i(k)}$ and obtain an open and closed subset $W_k \subseteq U_{i(k)}$ such that $W \subseteq W_k \subseteq V_k$. With $U_{i(k)}$ also W_k is open and closed in M. As $W \subseteq W_k$ the index k has the property (b). It remains to show that the W_j for $j \in J$ actually cover M. Let $x \in M$. As argued before the set $\{j \in J : x \in V_j\} = \{j_1 < \cdots < j_r\}$ is finite. Then $x \notin V_j$ for any $j > j_r$. The property (b) for the index j_r therefore implies that $x \in \bigcup_{j \le j_r} W_j$. This finishes step 2.

Step 3: At this point we have constructed a locally finite refinement $(W_j)_{j\in J}$ of our initial covering which consists of open and closed subsets $W_j \subseteq M$.

Claim: $W'_L := \bigcup_{j\in L} W_j$, for any subset $L \subseteq J$, is open and closed in M.

Obviously W'_L is open. To see that its complement $M \setminus W'_L$ is open as well let $x \in M \setminus W'_L$ be any point. In particular, $x \notin W_j$ for any $j \in L$. Since the covering $(W_j)_j$ is locally finite we find an open neighbourhood $U_x \subseteq M$ of x such that the set $\{j \in L : U_x \cap W_j \neq \emptyset\} = \{j_1, \ldots, j_s\}$ is finite. Then $U_x \setminus (W_{j_1} \cup \cdots \cup W_{j_s})$ is an open neighbourhood of x in $M \setminus W'_L$. This establishes the claim.

We finally define a new index set P by

$$P := \text{ all nonempty finite subsets of } J,$$

and for any $L \in P$ we put

$$W_L := \left(\bigcap_{j\in L} W_j \right) \setminus \left(\bigcup_{j\in J\setminus L} W_j \right) = \left(\bigcap_{j\in L} W_j \right) \setminus W'_{J\setminus L}.$$

Clearly any W_L is contained in some W_j. By the above claim each W_L is open and closed in M. To check that

$$M = \bigcup_{L\in P} W_L$$

holds true let $x \in M$ be any point. Then $x \in W_L$ for the finite set $L := \{j \in J : x \in W_j\}$. Moreover, the W_L are pairwise disjoint: Let $L_1 \neq L_2$ be two different indices in P. By symmetry we may assume that there is a $j \in L_1 \setminus L_2$. Then $W_{L_1} \subseteq W_j$ and $W_{L_2} \subseteq M \setminus W_j$. It follows that $(W_L)_{L\in P}$ is a refinement of our initial covering by pairwise disjoint open subsets. This proves that M is strictly paracompact.

ii. $\Longrightarrow$ iii. We start with an open covering of M by domains of definition of charts for M. By assumption we may refine it into a covering $M = \bigcup_{i\in I} U_i$

by pairwise disjoint open subsets. According to Remark 8.1 each U_i also is the domain of definition of some chart for M. In particular, the topology of U_i can be defined by a metric d_i' which satisfies the strict triangle inequality. We put

$$d_i(x,y) := \frac{d_i'(x,y)}{1+d_i'(x,y)} \qquad \text{for any } x,y \in U_i.$$

Obviously we have $d_i(x,y) = d_i(y,x)$ and $d_i(x,y) = 0$ if and only if $x = y$. To see that d_i satisfies the strict triangle inequality we compute

$$\begin{aligned} d_i(x,z) = \frac{d_i'(x,z)}{1+d_i'(x,z)} &\le \frac{\max(d_i'(x,y), d_i'(y,z))}{1+\max(d_i'(x,y), d_i'(y,z))} \\ &= \max\left(\frac{d_i'(x,y)}{1+d_i'(x,y)}, \frac{d_i'(y,z)}{1+d_i'(y,z)}\right) \\ &= \max(d_i(x,y), d_i(y,z)). \end{aligned}$$

Here we have used the simple fact that $t \ge s \ge 0$ implies $t(1+s) = t + ts \ge s + st = s(1+t)$ and hence $\frac{t}{1+t} \ge \frac{s}{1+s}$. For trivial reasons we have $d_i \le d_i'$. On the other hand

$$d_i' = \frac{d_i}{1-d_i}$$

and hence, for $0 < \varepsilon \le 1$,

$$d_i'(x,y) \le \varepsilon \quad \text{if} \quad d_i(x,y) \le \frac{\varepsilon}{2}.$$

This shows that the metrics d_i' and d_i define the same topology on U_i. We note that

$$d_i(x,y) < 1 \qquad \text{for any } x,y \in U_i.$$

We now define

$$\begin{aligned} d : M \times M &\longrightarrow \mathbb{R}_{\ge 0} \\ (x,y) &\longmapsto \begin{cases} d_i(x,y) & \text{if } x,y \in U_i \text{ for some } i \in I, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

This is a metric on d. The strict triangle equality

$$d(x,z) \le \max(d(x,y), d(y,z))$$

only needs justification if not all three points lie in the same subset U_i. But then the right hand side is ≥ 1 whereas the left hand side is ≤ 1. We claim that this metric d defines the topology of M. First consider any ball $B_\varepsilon(x)$

with respect to d in M. If $\varepsilon \geq 1$ then $B_\varepsilon(x) = M$, and if $\varepsilon < 1$ then $B_\varepsilon(x)$ is open in some U_i. Hence $B_\varepsilon(x)$ is open in M. Vice versa let $V \subseteq M$ be any open subset and let $x \in V$. We choose an $i \in I$ such that $x \in U_i$. Then $V \cap U_i$ is an open neighbourhood of x in U_i. Hence, for some $0 < \varepsilon < 1$, the ball $B_\varepsilon(x)$ with respect to d (or equivalently d_i) is contained in $V \cap U_i \subseteq V$. □

Corollary 8.8. *Open submanifolds and product manifolds of paracompact manifolds are paracompact.*

9 The Tangent Space

Let M be a manifold, and fix a point $a \in M$. We consider pairs (c, v) where

- $c = (U, \varphi, K^m)$ is a chart for M around a and
- $v \in K^m$.

Two such pairs (c, v) and (c', v') are called equivalent if we have

$$D_{\varphi(a)}(\varphi' \circ \varphi^{-1})(v) = v'.$$

It follows from the chain rule that this indeed defines an equivalence relation.

Definition. *A tangent vector of M at the point a is an equivalence class $[c, v]$ of pairs (c, v) as above.*

We define

$$T_a(M) := \text{set of all tangent vectors of } M \text{ at } a.$$

Lemma 9.1. *Let $c = (U, \varphi, K^m)$ and $c' = (U', \varphi', K^m)$ be two charts for M around a; we then have:*

i. *The map*

$$\begin{aligned} \theta_c : K^m &\xrightarrow{\sim} T_a(M) \\ v &\longmapsto [c, v] \end{aligned}$$

is bijective.

ii. $\theta_{c'}^{-1} \circ \theta_c : K^m \xrightarrow{\cong} K^m$ *is a K-linear isomorphism.*

Proof. (We recall from Lemma 7.1 that the dimensions of two charts around the same point necessarily coincide.) i. Surjectivity follows from

$$[c'', v''] = [c, D_{\varphi''(a)}(\varphi \circ \varphi''^{-1})(v'')].$$

If $[c, v] = [c, v']$ then $v' = D_{\varphi(a)}(\varphi \circ \varphi^{-1})(v) = v$. This proves the injectivity.

ii. From $[c, v] = [c', D_{\varphi(a)}(\varphi' \circ \varphi^{-1})(v)]$ we deduce that

$$\theta_{c'}^{-1} \circ \theta_c = D_{\varphi(a)}(\varphi' \circ \varphi^{-1}).$$

□

The set $T_a(M)$, by Lemma 9.1.i., has precisely one structure of a topological K-vector space such that the map θ_c is a K-linear homeomorphism. Because of Lemma 9.1.ii. this structure is independent of the choice of the chart c around a.

Definition. *The K-vector space $T_a(M)$ is called the tangent space of M at the point a.*

Remark. *The manifold M has dimension m if and only if $\dim_K T_a(M) = m$ for any $a \in M$.*

Let $g : M \longrightarrow N$ be a locally analytic map of manifolds. By Lemma 8.3.ii. we find charts $c = (U, \varphi, K^m)$ for M around a and $\tilde{c} = (V, \psi, K^n)$ for N around $g(a)$ such that $g(U) \subseteq V$. The composite

$$T_a(g) : T_a(M) \xrightarrow{\theta_c^{-1}} K^m \xrightarrow{D_{\varphi(a)}(\psi \circ g \circ \varphi^{-1})} K^n \xrightarrow{\theta_{\tilde{c}}} T_{g(a)}(N)$$

is a continuous K-linear map. We claim that $T_a(g)$ does not depend on the particular choice of charts. Let $c' = (U', \varphi')$ and $\tilde{c}' = (V', \psi')$ be other charts around a and $g(a)$, respectively. Using the identity in the proof of Lemma 9.1.ii. as well as the chain rule we compute

$$\begin{aligned}
&\theta_{\tilde{c}} \circ D_{\varphi(a)}(\psi \circ g \circ \varphi^{-1}) \circ \theta_c^{-1} \\
&\quad = \theta_{\tilde{c}'} \circ D_{\psi(g(a))}(\psi' \circ \psi^{-1}) \circ D_{\varphi(a)}(\psi \circ g \circ \varphi^{-1}) \circ D_{\varphi(a)}(\varphi' \circ \varphi^{-1})^{-1} \circ \theta_{c'}^{-1} \\
&\quad = \theta_{\tilde{c}'} \circ D_{\varphi'(a)}(\psi' \circ g \circ \varphi'^{-1}) \circ \theta_{c'}^{-1}.
\end{aligned}$$

Definition. *$T_a(g)$ is called the tangent map of g at the point a.*

Remark. $T_a(\mathrm{id}_M) = \mathrm{id}_{T_a(M)}$.

Lemma 9.2. *For any locally analytic maps of manifolds* $L \xrightarrow{f} M \xrightarrow{g} N$ *we have*

$$T_a(g \circ f) = T_{f(a)}(g) \circ T_a(f) \qquad \text{for any } a \in L.$$

Proof. This is an easy consequence of the chain rule. □

Proposition 9.3. (*Local invertibility*) *Let* $g : M \longrightarrow N$ *be a locally analytic map of manifolds, and suppose that* $T_a(g) : T_a(M) \xrightarrow{\cong} T_{g(a)}(N)$ *is bijective for some* $a \in M$*; then there are open neighbourhoods* $U \subseteq M$ *of* a *and* $V \subseteq N$ *of* $g(a)$ *such that* g *restricts to a locally analytic isomorphism*

$$g : U \xrightarrow{\simeq} V$$

of open submanifolds.

Proof. By Lemma 8.3.ii. we find charts $c = (U', \varphi, K^m)$ for M around a and $\tilde{c} = (V', \psi, K^n)$ for N around $g(a)$ such that $g(U') \subseteq V'$. We consider the locally analytic function

$$\varphi(U') \xrightarrow{\varphi^{-1}} U' \xrightarrow{g} V' \xrightarrow{\psi} \psi(V') \subseteq K^n.$$

By assumption the derivative

$$D_{\varphi(a)}(\psi \circ g \circ \varphi^{-1}) = \theta_{\tilde{c}}^{-1} \circ T_a(g) \circ \theta_c$$

is bijective. Prop. 6.4 therefore implies the existence of open neighbourhoods $W_0 \subseteq \varphi(U')$ of $\varphi(a)$ and $W_1 \subseteq \psi(V')$ of $\psi(g(a))$ such that

$$\psi \circ g \circ \varphi^{-1} : W_0 \xrightarrow{\simeq} W_1$$

is a locally analytic isomorphism. Hence

$$g : U := \varphi^{-1}(W_0) \xrightarrow{\simeq} V := \psi^{-1}(W_1)$$

is a locally analytic isomorphism as well (observe the subsequent exercise). □

Exercise. *Let* (U, φ, K^m) *be a chart for the manifold* M*; then* $\varphi : U \xrightarrow{\simeq} \varphi(U)$ *is a locally analytic isomorphism between the open submanifolds* U *of* M *and* $\varphi(U)$ *of* K^m*.*

Let M be a manifold, E be a K-Banach space, $f \in C^{\mathrm{an}}(M, E)$, and $a \in M$. If $c = (U, \varphi, K^m)$ is a chart for M around a then $f \circ \varphi^{-1} \in C^{\mathrm{an}}(\varphi(U), E)$. Hence

$$\begin{aligned} d_a f : T_a(M) &\xrightarrow{\theta_c^{-1}} K^m \xrightarrow{D_{\varphi(a)}(f\circ\varphi^{-1})} E \\ [c, v] &\longmapsto D_{\varphi(a)}(f \circ \varphi^{-1})(v) \end{aligned}$$

is a continuous K-linear map. If $c' = (U', \varphi', K^m)$ is another chart around a then

$$\begin{aligned} D_{\varphi(a)}(f \circ \varphi^{-1}) \circ \theta_c^{-1} &= D_{\varphi(a)}(f \circ \varphi^{-1}) \circ D_{\varphi(a)}(\varphi' \circ \varphi^{-1})^{-1} \circ \theta_{c'}^{-1} \\ &= D_{\varphi'(a)}(f \circ \varphi'^{-1}) \circ \theta_{c'}^{-1}. \end{aligned}$$

This shows that $d_a f$ does not depend on the choice of the chart c.

Definition. *$d_a f$ is called the derivative of f in the point a.*

Remark 9.4. *For $E = K^r$ viewed as a manifold and for the chart $c_0 = (K^r, \mathrm{id}, E)$ for E we have*

$$T_a(f) = \theta_{c_0} \circ d_a f.$$

Obviously the map

$$\begin{aligned} C^{\mathrm{an}}(M, E) &\longrightarrow \mathcal{L}(T_a(M), E) \\ f &\longmapsto d_a f \end{aligned}$$

is K-linear.

Lemma 9.5. (*Product rule*)

i. *Let $u : E_1 \times E_2 \longrightarrow E$ be a continuous bilinear map between K-Banach spaces; if $f_i \in C^{\mathrm{an}}(M, E_i)$ for $i = 1, 2$ then $u(f_1, f_2) \in C^{\mathrm{an}}(M, E)$ and*

$$d_a(u(f_1, f_2)) = u(f_1(a), d_a f_2) + u(d_a f_1, f_2(a)) \qquad \text{for any } a \in M.$$

ii. *For $g \in C^{\mathrm{an}}(M, K)$ and $f \in C^{\mathrm{an}}(M, E)$ we have*

$$d_a(gf) = g(a) \cdot d_a f + d_a g \cdot f(a) \qquad \text{for any } a \in M.$$

Proof. i. It is a straightforward consequence of Prop. 5.2 that the function $u(f_1, f_2)$ is locally analytic (compare the property 5) of the list in Sect. 6). Let $c = (U, \varphi)$ be a chart of M around a. Using the product rule in Remark 4.1.iv. we compute

$$\begin{aligned}
d_a(u(f_1, f_2))([c, v]) &= D_{\varphi(a)}(u(f_1, f_2) \circ \varphi^{-1})(v) \\
&= D_{\varphi(a)}(u(f_1 \circ \varphi^{-1}, f_2 \circ \varphi^{-1}))(v) \\
&= u(f_1 \circ \varphi^{-1}(\varphi(a)), D_{\varphi(a)}(f_2 \circ \varphi^{-1})(v)) \\
&\qquad + u(D_{\varphi(a)}(f_1 \circ \varphi^{-1})(v), f_2 \circ \varphi^{-1}(\varphi(a))) \\
&= u(f_1(a), d_a f_2([c, v])) + u(d_a f_1([c, v]), f_2(a)).
\end{aligned}$$

ii. This is a special case of the first assertion. $\square$

Let $c = (U, \varphi, K^m)$ be a chart for M. On the one hand, by definition, we have $d_a\varphi = \theta_c^{-1}$ for any $a \in U$; in particular

$$d_a\varphi : T_a(M) \xrightarrow{\cong} K^m$$

is a K-linear isomorphism. On the other hand viewing $\varphi = (\varphi_1, \ldots, \varphi_m)$ as a tuple of locally analytic functions $\varphi_i : U \longrightarrow K$ we have

$$d_a\varphi = (d_a\varphi_1, \ldots, d_a\varphi_m).$$

This means that $\{d_a\varphi_i\}_{1 \le i \le m}$ is a K-basis of the dual vector space $T_a(M)'$. Let

$$\left\{ \left(\frac{\partial}{\partial \varphi_i} \right)(a) \right\}_{1 \le i \le m}$$

denote the corresponding dual basis of $T_a(M)$, i. e.,

$$d_a\varphi_i \left(\left(\frac{\partial}{\partial \varphi_j} \right)(a) \right) = \delta_{ij} \qquad \text{for any } a \in U$$

where δ_{ij} is the Kronecker symbol. For any $f \in C^{\mathrm{an}}(M, E)$ we define the functions

$$\begin{aligned}
\frac{\partial f}{\partial \varphi_i} : U &\longrightarrow E \\
a &\longmapsto d_a f\left(\left(\frac{\partial}{\partial \varphi_i} \right)(a) \right).
\end{aligned}$$

Lemma 9.6. $\frac{\partial f}{\partial \varphi_i} \in C^{\mathrm{an}}(U, E)$ *for any* $1 \leq i \leq m$*, and*

$$d_a f = \sum_{i=1}^{m} d_a\varphi_i \cdot \frac{\partial f}{\partial \varphi_i}(a) \qquad \textit{for any } a \in U.$$

Proof. We have

$$\begin{aligned}\frac{\partial f}{\partial \varphi_i}(a) &= D_{\varphi(a)}(f \circ \varphi^{-1}) \circ \theta_c^{-1}\left(\left(\frac{\partial}{\partial \varphi_i}\right)(a)\right) \\ &= D_{\varphi(a)}(f \circ \varphi^{-1})(e_i)\end{aligned}$$

where $e_1, \ldots, e_m$ denotes the standard basis of K^m. Hence $\frac{\partial f}{\partial \varphi_i}$ is the composite

$$U \xrightarrow{\varphi} \varphi(U) \xrightarrow{x \mapsto D_x(f \circ \varphi^{-1})} \mathcal{L}(K^m, E) \xrightarrow{D \mapsto D(e_i)} E.$$

The function in the middle is locally analytic by Prop. 6.1. Clearly, $D \longmapsto D(e_i)$ is a continuous K-linear map. Hence the composite of the right two maps is locally analytic by the property 6) in Sect. 6. That the full composite $\frac{\partial f}{\partial \varphi_i}$ is locally analytic now follows from Lemma 8.4.i. Let

$$t = \sum_{i=1}^{m} c_i \left(\frac{\partial}{\partial \varphi_i}\right)(a) \in T_a(M)$$

be an arbitrary vector. By the definition of the dual basis we have $c_i = d_a\varphi_i(t)$. We now compute

$$d_a f(t) = \sum_{i=1}^{m} c_i \cdot d_a f\left(\left(\frac{\partial}{\partial \varphi_i}\right)(a)\right) = \sum_{i=1}^{m} d_a\varphi_i(t) \cdot \frac{\partial f}{\partial \varphi_i}(a).$$

$\square$

In a next step we want to show that the disjoint union

$$T(M) := \bigcup_{a \in M} T_a(M)$$

in a natural way is a manifold again. We introduce the projection map

$$\begin{aligned} p_M : T(M) &\longrightarrow M \\ t &\longmapsto a \quad \text{if} \quad t \in T_a(M). \end{aligned}$$

Hence $T_a(M) = p_M^{-1}(a)$.

Consider any chart $c = (U, \varphi, K^m)$ for M. By Lemma 9.1.i. the map

$$\begin{aligned} \tau_c : U \times K^m &\xrightarrow{\sim} p_M^{-1}(U) \\ (a, v) &\longmapsto [c, v] \ \text{ viewed in } \ T_a(M) \end{aligned}$$

is bijective. Hence the composite

$$\varphi_c : p_M^{-1}(U) \xrightarrow{\tau_c^{-1}} U \times K^m \xrightarrow{\varphi \times \mathrm{id}} K^m \times K^m = K^{2m}$$

is a bijection onto an open subset in K^{2m}. The idea is that

$$c_T := \big(p_M^{-1}(U), \varphi_c, K^{2m}\big)$$

should be a chart for the manifold $T(M)$ yet to be constructed. Clearly we have

$$T(M) = \bigcup_{c=(U,\varphi)} p_M^{-1}(U).$$

Let $\tilde{c} = (V, \psi, K^m)$ be another chart for M such that $U \cap V \neq \emptyset$. The composed map

$$\varphi(U \cap V) \times K^m \xrightarrow{\varphi_c^{-1}} p_M^{-1}(U \cap V) = p_M^{-1}(U) \cap p_M^{-1}(V) \xrightarrow{\psi_{\tilde{c}}} \psi(U \cap V) \times K^m$$

is given by

$$(x, v) \longmapsto (\psi \circ \varphi^{-1}(x), D_x(\psi \circ \varphi^{-1})(v)). \tag{8}$$

The first component $\psi \circ \varphi^{-1}$ of this map is locally analytic on $\varphi(U \cap V)$ since M is a manifold. The second component can be viewed as the composite

$$\begin{array}{ccccc} \varphi(U \cap V) \times K^m & \longrightarrow & \mathcal{L}(K^m, K^m) \times K^m & \longrightarrow & K^m \\ (x, v) & \longmapsto & (D_x(\psi \circ \varphi^{-1}), v) & & \\ & & (u, v) & \longmapsto & u(v). \end{array}$$

The left function is locally analytic by Prop. 6.1. The right bilinear map is continuous. Hence the composite is locally analytic by Lemma 9.5.i. This shows that, once c_T and $\tilde{c}_T$ are recognized as charts for $T(M)$ with respect to a topology yet to be defined, they in fact are compatible, and hence that the set $\{c_T : c \text{ a chart for } M\}$ is an atlas for $T(M)$.

We have shown in particular that the composed map

$$(U \cap V) \times K^m \xrightarrow{\tau_c} p_M^{-1}(U \cap V) \xrightarrow{\tau_{\tilde{c}}^{-1}} (U \cap V) \times K^m \tag{9}$$

is a homeomorphism.

Definition. *A subset $X \subseteq T(M)$ is called open if $\tau_c^{-1}(X \cap p_M^{-1}(U))$ is open in $U \times K^m$ for any chart $c = (U, \varphi, K^m)$ for M.*

This defines the finest topology on $T(M)$ which makes all composed maps

$$U \times K^m \xrightarrow{\tau_c} p_M^{-1}(U) \xrightarrow{\subseteq} T(M)$$

continuous.

Lemma 9.7. i. *The map $\tau_c : U \times K^m \xrightarrow{\simeq} p_M^{-1}(U)$ is a homeomorphism with respect to the subspace topology induced by $T(M)$ on $p_M^{-1}(U)$.*

ii. *The map p_M is continuous.*

iii. *The topological space $T(M)$ is Hausdorff.*

Proof. i. The continuity of τ_c holds by construction. Let $Y \subseteq U \times K^m$ be an open subset. We will show that $\tau_c(Y)$ is open in $T(M)$, i. e., that $\tau_{\tilde{c}}^{-1}(\tau_c(Y) \cap p_M^{-1}(V))$ is open in $V \times K^m$ for any chart $\tilde{c} = (V, \psi, K^n)$ for M. We may of course assume that $U \cap V \neq \emptyset$ so that $n = m$. Clearly the subset $Y \cap ((U \cap V) \times K^m)$ is open in $(U \cap V) \times K^m$. By (9) the subset

$$\begin{aligned} \tau_{\tilde{c}}^{-1}(\tau_c(Y \cap ((U \cap V) \times K^m))) &= \tau_{\tilde{c}}^{-1}(\tau_c(Y) \cap p_M^{-1}(U \cap V)) \\ &= \tau_{\tilde{c}}^{-1}(\tau_c(Y) \cap p_M^{-1}(U) \cap p_M^{-1}(V)) \\ &= \tau_{\tilde{c}}^{-1}(\tau_c(Y) \cap p_M^{-1}(V)) \end{aligned}$$

is open in $(U \cap V) \times K^m$ and therefore in $V \times K^m$.

ii. The above reasoning for $Y = U \times K^m$ shows that $\tau_c(Y) = p_M^{-1}(U)$ is open in $T(M)$ where U is the domain of definition of any chart for M. It then follows from Lemma 7.3 that p_M is continuous.

iii. Let s and t be two different points in $T(M)$. *Case* 1: We have $p_M(s) \neq p_M(t)$. Since M is Hausdorff we find open neighbourhoods $U \subseteq M$ of $p_M(s)$ and $V \subseteq M$ of $p_M(t)$ such that $U \cap V = \emptyset$. By ii. then $p_M^{-1}(U)$ and $p_M^{-1}(V)$ are open neighbourhoods of s and t, respectively, such that $p_M^{-1}(U) \cap p_M^{-1}(V) = p_M^{-1}(U \cap V) = \emptyset$. *Case* 2: We have $a := p_M(s) = p_M(t)$. We choose a chart $c = (U, \varphi, K^m)$ for M around a. The two points s and t lie in the open (by ii.) subset $p_M^{-1}(U)$ of $T(M)$. But by i. the subspace $p_M^{-1}(U)$ is homeomorphic, via the map τ_c, to the Hausdorff space $U \times K^m$. Hence $p_M^{-1}(U)$ is Hausdorff and s and t can be separated by open neighbourhoods in $p_M^{-1}(U)$ and a fortiori in $T(M)$. □

The Lemma 9.7 in particular says that c_T indeed is a chart for $T(M)$. Altogether we now have established that $\{c_T : c \text{ a chart for } M\}$ is an atlas for $T(M)$. We always view $T(M)$ as a manifold with respect to the equivalent maximal atlas.

Definition. *The manifold $T(M)$ is called the tangent bundle of M.*

Remark. *If M is m-dimensional then $T(M)$ is $2m$-dimensional.*

Lemma 9.8. *The map $p_M : T(M) \longrightarrow M$ is locally analytic.*

Proof. Let $c = (U, \varphi, K^m)$ be a chart for M. It suffices to contemplate the commutative diagram

$$\begin{tikzcd}
T(M) \arrow[d, "p_M"] & p_M^{-1}(U) \arrow[l, "\supseteq"'] \arrow[r] \arrow[d] & \varphi_c(p_M^{-1}(U)) = \varphi(U) \times K^m \arrow[r, "\subseteq"] \arrow[d, "\mathrm{pr}_1"] & K^{2m} \\
M & U \arrow[l, "\supseteq"'] \arrow[r, "\varphi"] & \varphi(U) \arrow[r, "\subseteq"] & K^m .
\end{tikzcd}$$

$\square$

Let $g : M \longrightarrow N$ be a locally analytic map of manifolds. We define the map

$$T(g) : T(M) \longrightarrow T(N)$$

by

$$T(g)|T_a(M) := T_a(g) \qquad \text{for any } a \in M.$$

In particular, the diagram

$$\begin{tikzcd}
T(M) \arrow[r, "T(g)"] \arrow[d, "p_M"'] & T(N) \arrow[d, "p_N"] \\
M \arrow[r, "g"] & N
\end{tikzcd}$$

is commutative.

Proposition 9.9. i. *The map $T(g)$ is locally analytic.*

ii. *For any locally analytic maps of manifolds $L \xrightarrow{f} M \xrightarrow{g} N$ we have*

$$T(g \circ f) = T(g) \circ T(f).$$

Proof. i. We choose charts $c = (U, \varphi, K^m)$ for M and $\tilde{c} = (V, \psi, K^n)$ for N such that $g(U) \subseteq V$. The composite

$$\varphi(U) \times K^m \xrightarrow{\varphi_c^{-1}} p_M^{-1}(U) \xrightarrow{T(g)} p_N^{-1}(V) \xrightarrow{\psi_{\tilde{c}}} \psi(V) \times K^n$$

is given by

$$(x, v) \longmapsto (\psi \circ g \circ \varphi^{-1}(x), D_x(\psi \circ g \circ \varphi^{-1})(v)).$$

It is locally analytic by the same argument as for (8).
ii. This is a restatement of Lemma 9.2. $\square$

The following is left to the reader as an exercise.

Remark 9.10. i. *If $U \subseteq M$ is an open submanifold then $T(\subseteq)$ induces an isomorphism between $T(U)$ and the open submanifold $p_M^{-1}(U)$.*

ii. *For any two manifolds M and N the map*

$$T(\mathrm{pr}_1) \times T(\mathrm{pr}_2) : T(M \times N) \xrightarrow{\simeq} T(M) \times T(N)$$

is an isomorphism of manifolds.

Now let M be a manifold and E be a K-Banach space. For any $f \in C^{\mathrm{an}}(M, E)$ we define

$$\begin{aligned} df : T(M) &\longrightarrow E \\ t &\longmapsto d_{p_M(t)} f(t). \end{aligned}$$

Lemma 9.11. *We have $df \in C^{\mathrm{an}}(T(M), E)$.*

Proof. Let $c = (U, \varphi, K^m)$ be a chart for M. The composed map

$$\varphi(U) \times K^m \xrightarrow{\varphi_c^{-1}} p_M^{-1}(U) \xrightarrow{df} E$$

is given by

$$(x, v) \longmapsto D_x(f \circ \varphi^{-1})(v)$$

and hence is locally analytic by the same argument as for (8). $\square$

Lemma 9.12. *Let $g : M \longrightarrow N$ be a locally analytic map of manifolds; for any $f \in C^{\mathrm{an}}(N, E)$ we have*

$$d(f \circ g) = df \circ T(g).$$

Proof. This is a consequence of the chain rule. □

Exercise. *The map*

$$\begin{aligned} d : C^{\mathrm{an}}(M,E) &\longrightarrow C^{\mathrm{an}}(T(M),E) \\ f &\longmapsto df \end{aligned}$$

is K-linear.

Remark 9.13. *If K has characteristic zero then a function $f \in C^{\mathrm{an}}(M,E)$ is locally constant if and only if $df = 0$.*

Proof. Let $c = (U,\varphi)$ be any chart for M. As can be seen from the proof of Lemma 9.11 we have $df|p_M^{-1}(U) = 0$ if and only if $D_x(f \circ \varphi^{-1}) = 0$ for any $x \in \varphi(U)$. By Remark 6.2 the latter is equivalent to $f \circ \varphi^{-1}$ being locally constant on $\varphi(U)$ which, of course, is the same as f being locally constant on U. □

Definition. *Let $U \subseteq M$ be an open subset; a vector field ξ on U is a locally analytic map $\xi : U \longrightarrow T(M)$ which satisfies $p_M \circ \xi = \mathrm{id}_U$.*

We define

$$\Gamma(U,T(M)) := \text{ set of all vector fields on } U.$$

It follows from Remark 9.10.i. that

$$\Gamma(U,T(M)) = \Gamma(U,T(U)).$$

Suppose that U is the domain of definition of some chart $c = (U,\varphi,K^m)$ for M. Because of the commutative diagram

$$\begin{tikzcd} p_M^{-1}(U) \arrow[rr, "\simeq", "\varphi_c"'] \arrow[dr, "p_M"'] & & \varphi(U) \times K^m \arrow[dl, "(x,v)\mapsto\varphi^{-1}(x)"] \\ & U & \end{tikzcd}$$

the map

$$\begin{aligned} C^{\mathrm{an}}(U,K^m) &\xrightarrow{\sim} \Gamma(U,T(M)) \\ f &\longmapsto \xi_f(a) := \varphi_c^{-1}((\varphi(a), f(a))) = \tau_c(a, f(a)) \end{aligned}$$

is bijective. The left hand side is a K-vector space. On the right hand side this vector space structure corresponds to the pointwise addition and scalar multiplication of maps which makes sense since each $T_a(M)$ is a K-vector space. The latter we can define on any open subset $U \subseteq M$. For any $c \in K$ and $\xi, \eta \in \Gamma(U, T(M))$ we define

$$(c \cdot \xi)(a) := c \cdot \xi(a) \quad \text{and} \quad (\xi + \eta)(a) := \xi(a) + \eta(a).$$

Obviously the result are again maps $? : U \longrightarrow T(M)$ satisfying $p_M \circ ? = \mathrm{id}_U$. But since U can be covered by domains of definition of charts for M the above discussion actually implies that these maps are locally analytic again. We see that

$$\Gamma(U, T(M)) \text{ is a } K\text{-vector space.}$$

We have the bilinear map

$$\begin{aligned} \Gamma(M, T(M)) \times C^{\mathrm{an}}(M, E) &\longrightarrow C^{\mathrm{an}}(M, E) \\ (\xi, f) &\longmapsto D_\xi(f) := df \circ \xi. \end{aligned}$$

Lemma 9.14. *Let $u : E_1 \times E_2 \longrightarrow E$ be a continuous bilinear map between K-Banach spaces; for any $\xi \in \Gamma(M, T(M))$ and $f_i \in C^{\mathrm{an}}(M, E_i)$ we have*

$$D_\xi(u(f_1, f_2)) = u(D_\xi(f_1), f_2) + u(f_1, D_\xi(f_2)).$$

Proof. This follows from the product rule in Lemma 9.5.i. □

Corollary 9.15. *For any vector field $\xi \in \Gamma(M, T(M))$ the map*

$$D_\xi : C^{\mathrm{an}}(M, K) \longrightarrow C^{\mathrm{an}}(M, K)$$

is a derivation, i. e.:

(a) *D_ξ is K-linear,*

(b) *$D_\xi(fg) = D_\xi(f)g + fD_\xi(g)$ for any $f, g \in C^{\mathrm{an}}(M, K)$.*

Proposition 9.16. *Suppose that M is paracompact; then for any derivation D on $C^{\mathrm{an}}(M, K)$ there is a unique vector field ξ on M such that $D = D_\xi$.*

The proof requires some preparation. In the following we always assume M to be paracompact. At first we fix a point $a \in M$. A K-linear map $\Delta : C^{\mathrm{an}}(M, K) \longrightarrow K$ will be called an *a-derivation* if

$$\Delta(fg) = \Delta(f)g(a) + f(a)\Delta(g) \qquad \text{for any } f, g \in C^{\mathrm{an}}(M, K).$$

The a-derivations form a K-vector subspace

$$\mathrm{Der}_a(M, K)$$

of the dual vector space $C^{\mathrm{an}}(M, K)^*$.

Lemma 9.17. *Suppose that M is paracompact, and let Δ be an a-derivation; if $f \in C^{\mathrm{an}}(M,K)$ is constant in a neighbourhood of the point a then $\Delta(f) = 0$.*

Proof. *Case* 1: We assume that f vanishes in the neighbourhood $U \subseteq M$ of a. By Prop. 8.7 we may assume that U is open and closed in M. Then the function

$$g(x) := \begin{cases} 1 & \text{if } x \notin U, \\ 0 & \text{if } x \in U \end{cases}$$

lies in $C^{\mathrm{an}}(M,K)$ and satisfies $gf = f$. It follows that

$$\Delta(f) = \Delta(gf) = \Delta(g)f(a) + g(a)\Delta(f) = 0.$$

Case 2: We assume that f is constant on M with value c. Let 1_M denote the constant function with value one on M. Then $f = c1_M$ and hence

$$\Delta(f) = c\Delta(1_M) = c\Delta(1_M 1_M) = c\Delta(1_M) + c\Delta(1_M) = 2c\Delta(1_M) = 2\Delta(f)$$

which means $\Delta(f) = 0$.

Case 3: In general we write

$$f = f(a)1_M + (f - f(a)1_M)$$

and use the K-linearity of Δ together with the first two cases. $\square$

As a consequence of the product rule Lemma 9.5.ii. we have the K-linear map

$$\begin{aligned} T_a(M) &\longrightarrow \mathrm{Der}_a(M,K) \\ t &\longmapsto \Delta_t(f) := d_a f(t). \end{aligned} \tag{10}$$

Proposition 9.18. *If M is paracompact then* (10) *is an isomorphism.*

Proof. We fix a chart $c = (U, \varphi, K^m)$ for M around a point a and write $\varphi = (\varphi_1, \ldots, \varphi_m)$. Since M is paracompact we may assume by Prop. 8.7 that U is open and closed in M. Then each φ_i extends by zero to a function $\varphi_{i!} \in C^{\mathrm{an}}(M,K)$. In the discussion before Lemma 9.6 we had seen that

$$d_a\varphi = (d_a\varphi_1, \ldots, d_a\varphi_m) : T_a(M) \xrightarrow{\cong} K^m$$

is an isomorphism, and we had introduced the K-basis $t_i := \frac{\partial}{\partial \varphi_i}(a)$ of $T_a(M)$.

Injectivity: Let $t \in T_a(M)$ such that $\Delta_t(f) = 0$ for any $f \in C^{\mathrm{an}}(M, K)$. In particular

$$0 = \Delta_t(\varphi_{i!}) = d_a\varphi_{i!}(t) = d_a\varphi_i(t) \qquad \text{for any } 1 \leq i \leq m.$$

This means $d_a\varphi(t) = 0$ and hence $t = 0$.

Surjectivity: From the injectivity which we just have established we deduce that the Δ_{t_i} are linearly independent in $\mathrm{Der}_a(M, K)$. It therefore suffices to write an arbitrarily given $\Delta \in \mathrm{Der}_a(M, K)$ as a linear combination of the Δ_{t_i}. In fact, we claim that

$$\Delta = \sum_{i=1}^{m} \Delta(\varphi_{i!}) \cdot \Delta_{t_i}$$

holds true. Let $f \in C^{\mathrm{an}}(M, K)$. We find an open and closed neighbourhood $V \subseteq U$ of a such that $\varphi(V) = B_\varepsilon(\varphi(a))$ and a power series $F(X) = \sum_\alpha c_\alpha X^\alpha \in \mathcal{F}_\varepsilon(K^m; K)$ such that

$$f(x) = F(\varphi(x) - \varphi(a)) \qquad \text{for any } x \in V.$$

We may write

$$f(x) = \sum_\alpha c_\alpha(\varphi(x) - \varphi(a))^\alpha = f(a) + \sum_{i=1}^{m}(\varphi_i(x) - \varphi_i(a))g_i(x)$$

for any $x \in V$ where the $g_i \in C^{\mathrm{an}}(V, K)$ are appropriate functions satisfying $g_i(a) = c_{\underline{i}}$ (recall that $\underline{i} = (0, \ldots, 1, \ldots, 0)$). From the proof of Prop. 6.1 we know that

$$D_{\varphi(a)}(f \circ \varphi^{-1})(e_i) = \frac{\partial F}{\partial X_i}(0) = c_{\underline{i}}$$

where $e_1, \ldots, e_m$ denotes, as usual, the standard basis of K^m. We therefore obtain

$$g_i(a) = c_{\underline{i}} = D_{\varphi(a)}(f \circ \varphi^{-1})(e_i) = d_a f(\theta_c(e_i)).$$

By the construction of the t_i we have $\theta_c(e_i) = t_i$. It follows that

$$g_i(a) = d_a f(t_i).$$

On the other hand we extend each g_i by zero to a function $g_{i!} \in C^{\mathrm{an}}(M, K)$. The function

$$f - \sum_{i=1}^{m}(\varphi_{i!} - \varphi_i(a))g_{i!}$$

is constant (with value $f(a)$) in a neighbourhood of a. Using Lemma 9.17 we compute

$$\begin{aligned}\Delta(f) &= \Delta\left(\sum_{i=1}^{m}(\varphi_{i!} - \varphi_i(a))g_{i!}\right)\\ &= \sum_{i=1}^{m}\Delta\big(\varphi_{i!} - \varphi_i(a)\big)g_i(a)\\ &= \sum_{i=1}^{m}\Delta(\varphi_{i!})\cdot d_af(t_i)\\ &= \sum_{i=1}^{m}\Delta(\varphi_{i!})\cdot \Delta_{t_i}(f).\end{aligned}$$

Since f was arbitrary this establishes our claim. □

Proof of Proposition 9.16: First of all we note that the relation between derivations and a-derivations on $C^{\mathrm{an}}(M,K)$ is given by the formula

$$D_\xi(f)(a) = df(\xi(a)) = d_af(\xi(a)) = \Delta_{\xi(a)}(f). \tag{11}$$

Therefore, if $D_\xi = 0$ then $\Delta_{\xi(a)} = 0$ for any $a \in M$. The Prop. 9.18 then implies that $\xi(a) = 0$ for any $a \in M$, i. e., that $\xi = 0$. This shows that the ξ in our assertion is unique if it exists. For the existence we first fix a point $a \in M$ and consider the a-derivation $\Delta(f) := D(f)(a)$. By Prop. 9.18 there is a tangent vector $\xi(a) \in T_a(M)$ such that $\Delta = \Delta_{\xi(a)}$. For varying $a \in M$ this gives a map $\xi : M \longrightarrow T(M)$ which satisfies $p_M \circ \xi = \mathrm{id}_M$. It remains to show that ξ is locally analytic, since $D = D_\xi$ then is a formal consequence of (11). So let $c = (U, \varphi, K^m)$ be a chart for M. In the proof of Prop. 9.18 we have seen that

$$D(f)(a) = \sum_{i=1}^{m} D(\varphi_{i!})(a)\cdot\Delta_{\theta_c(e_i)}(f).$$

It follows that

$$\xi(a) = \sum_{i=1}^{m} D\big(\varphi_{i!}\big)(a)\cdot\theta_c(e_i) = \theta_c((D(\varphi_{1!})(a), \ldots, D(\varphi_{m!})(a))\big).$$

Using the commutative diagram

$$\begin{array}{ccc} K^m & \xrightarrow{v \longmapsto (a,v)} & U \times K^m \\ {\scriptstyle\theta_c}\downarrow{\scriptstyle\cong} & & {\scriptstyle\simeq}\downarrow{\scriptstyle\tau_c} \\ T_a(M) & \xrightarrow{\subseteq} & p_M^{-1}(U) \end{array}$$

we rewrite this as

$$\xi(a) = \tau_c(a, (D(\varphi_{1!})(a), \ldots, D(\varphi_{m!})(a))).$$

This means that under the identification discussed after the definition of vector fields we have

$$\xi|U = \xi_f \quad \text{with } f := (D(\varphi_{1!}), \ldots, D(\varphi_{m!})) \in C^{\mathrm{an}}(U, K^m).$$

Hence ξ is locally analytic. $\square$

Lemma 9.19. *For any derivations $B, C, D : C^{\mathrm{an}}(M,K) \longrightarrow C^{\mathrm{an}}(M,K)$ we have*:

i. $[B, C] := B \circ C - C \circ B$ *again is a derivation*;

ii. $[\ ,\ ]$ *is K-bilinear*;

iii. $[B, B] = 0$ *and* $[B, C] = -[C, B]$;

iv. (*Jacobi identity*) $[[B, C], D] + [[C, D], B] + [[D, B], C] = 0.$

Proof. These are straightforward completely formal computations. $\square$

Definition. *A K-vector space $\mathfrak{g}$ together with a K-bilinear map*

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

which is antisymmetric (i. e., $[z, z] = 0$ for any $z \in \mathfrak{g}$) and satisfies the Jacobi identity is called a Lie algebra over K.

If M is paracompact then, using Prop. 9.16 and Lemma 9.19, we may define the Lie product $[\xi, \eta]$ of two vector fields $\xi, \eta \in \Gamma(M, T(M))$ by the requirement that

$$D_{[\xi,\eta]} = D_\xi \circ D_\eta - D_\eta \circ D_\xi$$

holds true. This makes $\Gamma(M, T(M))$ into a Lie algebra over K.

Proposition 9.20. *Suppose that M is paracompact, and let E be a K-Banach space and $\xi, \eta \in \Gamma(M, T(M))$ be two vector fields*; *on $C^{\mathrm{an}}(M, E)$ we then have*

$$D_{[\xi,\eta]} = D_\xi \circ D_\eta - D_\eta \circ D_\xi.$$

Proof. Let $f \in C^{\mathrm{an}}(M, E)$. We have to show equality of the functions

$$D_{[\xi,\eta]}(f) = df \circ [\xi, \eta]$$

and

$$\begin{aligned}(D_\xi \circ D_\eta - D_\eta \circ D_\xi)(f) &= d(D_\eta(f)) \circ \xi - d(D_\xi(f)) \circ \eta \\ &= d(df \circ \eta) \circ \xi - d(df \circ \xi) \circ \eta.\end{aligned}$$

This, of course, can be done after restriction to the domain of definition U of any chart $c = (U, \varphi, K^m)$ for M. Since M is paracompact we furthermore need only to consider charts for which U is open and closed in M. Let $\varphi = (\varphi_1, \ldots, \varphi_m)$ and denote, as before, by $\varphi_{i!} \in C^{\mathrm{an}}(M, K)$ the extension by zero of φ_i. We now make use of the following identifications. If ? denotes the restriction to U of any of the vector fields ξ, η, and $[\xi, \eta]$ then, as discussed after the definition of vector fields, we have a commutative diagram

$$\begin{tikzcd} p_M^{-1}(U) \arrow[r, "\simeq", "\varphi_c"'] & \varphi(U) \times K^m \\ U \arrow[u, "?"] \arrow[r, "\simeq", "\varphi"'] & \varphi(U) \arrow[u, "x \mapsto (x, g_?(x))"'] \end{tikzcd}$$

with $g_? \in C^{\mathrm{an}}(\varphi(U), K^m)$. On the other hand, as noticed already in the proof of Lemma 9.11, we have, for any function $? \in C^{\mathrm{an}}(U, E)$, the commutative diagram

$$\begin{tikzcd} & E & \\ p_M^{-1}(U) \arrow[ur, "d?"] \arrow[rr, "\simeq", "\varphi_c"'] & & \varphi(U) \times K^m. \arrow[ul, "(x,v) \mapsto D_x(? \circ \varphi^{-1})(v)"'] \end{tikzcd}$$

These identifications reduce us to proving the equality of the following two functions of $x \in \varphi(U)$ given by

$$D_x(f \circ \varphi^{-1})(g_{[\xi,\eta]}(x)) \tag{12}$$

and

$$\begin{aligned} &D_x(df \circ \eta \circ \varphi^{-1})(g_\xi(x)) - D_x(df \circ \xi \circ \varphi^{-1})(g_\eta(x)) \\ &\quad = D_x\big(D.(f \circ \varphi^{-1})(g_\eta(.))\big)(g_\xi(x)) - D_x\big(D.(f \circ \varphi^{-1})(g_\xi(.))\big)(g_\eta(x)), \end{aligned} \tag{13}$$

respectively. By viewing $D.(f \circ \varphi^{-1})$, resp. $g_\eta(.)$ and $g_\xi(.)$, as functions from $\varphi(U)$ into $\mathcal{L}(K^m, E)$, resp. into K^m, we may apply the product rule Remark 4.1.iv. for the continuous bilinear map

$$\begin{aligned} \mathcal{L}(K^m, E) \times K^m &\longrightarrow E \\ (u, v) &\longmapsto u(v) \end{aligned}$$

to both summands in the last expression for (13) and rewrite it as

$$\begin{aligned} (14) \qquad &= [D_x(D.(f \circ \varphi^{-1}))(g_\xi(x))](g_\eta(x)) + D_x(f \circ \varphi^{-1})[D_x g_\eta(g_\xi(x))] \\ &\quad - [D_x(D.(f \circ \varphi^{-1})(g_\eta(x))](g_\xi(x)) - D_x(f \circ \varphi^{-1})[D_x g_\xi(g_\eta(x))]. \end{aligned}$$

To simplify this further we establish the following general

Claim: *For any open subset $V \subseteq K^m$, any point $x \in V$, any vectors $v = (v_1, \ldots, v_m)$ and $w = (w_1, \ldots, w_m)$ in K^m, and any function $h \in C^{\mathrm{an}}(V, E)$ we have*

$$D_x(D.h(v))(w) = D_x(D.h(w))(v).$$

(Note that the function $D.h(v)$ is the composite

$$V \xrightarrow{y \mapsto D_y h} \mathcal{L}(K^m, E) \xrightarrow{u \mapsto u(v)} E.)$$

We expand h around the point x into a power series

$$h(y) = H(y - x).$$

By the proof of Prop. 6.1 we then have

$$D_y h(v) = \sum_{i=1}^{m} v_i \frac{\partial H}{\partial Y_i}(y - x)$$

and

$$\begin{aligned} D_x(D.h(v))(w) &= \sum_{i=1}^{m} v_i D_x \left(\frac{\partial H}{\partial Y_i}(. - x) \right)(w) \\ &= \sum_{i=1}^{m} v_i \sum_{j=1}^{m} w_j \left(\frac{\partial}{\partial Y_j} \frac{\partial}{\partial Y_i} H \right)(0) \\ &= \sum_{j=1}^{m} w_j \sum_{i=1}^{m} v_i \left(\frac{\partial}{\partial Y_i} \frac{\partial}{\partial Y_j} H \right)(0) \\ &\;\;\vdots \\ &= D_x(D.h(w))(v). \end{aligned}$$

Applying this claim to (14) we see that the expression for the function (13) simplifies to

$$D_x(f \circ \varphi^{-1})[D_x g_\eta(g_\xi(x)) - D_x g_\xi(g_\eta(x))].$$

Comparing this with (12) we are reduced to showing that the identity

$$g_{[\xi,\eta]}(x) = D_x g_\eta(g_\xi(x)) - D_x g_\xi(g_\eta(x)) \tag{15}$$

holds true in $C^{\mathrm{an}}(\varphi(U), K^m)$. But in case $E = K$ our assertion and the whole computation above holds by construction. In particular we have

$$D_x(\varphi_{i!} \circ \varphi^{-1})(g_{[\xi,\eta]}(x)) = D_x(\varphi_{i!} \circ \varphi^{-1})[D_x g_\eta(g_\xi(x)) - D_x g_\xi(g_\eta(x))]$$

for any $1 \leq i \leq m$. Since

$$D_x(\varphi_{i!} \circ \varphi^{-1}) : \quad \begin{aligned} K^m &\longrightarrow K \\ (v_1, \ldots, v_m) &\longmapsto v_i \end{aligned}$$

the identity (15) follows immediately. ☐

Remarks 9.21. i. *The identity* (15) *shows that* $C^{\mathrm{an}}(V, K^m)$, *for any open subset* $V \subseteq K^m$, *is a Lie algebra with respect to*

$$[f, g](x) := D_x f(g(x)) - D_x g(f(x)).$$

ii. *The identity* (15) *can be made into a definition of which one then can show that it is compatible with any change of charts for* M. *In this way a Lie product* $[\xi, \eta]$ *can be obtained and Prop.* 9.20 *can be proved even for manifolds which are not paracompact.*

10 The Topological Vector Space $C^{\mathrm{an}}(M, E)$, Part 1

Throughout this section M is a paracompact manifold and E is a K-Banach space. Following [Fea] we will show that $C^{\mathrm{an}}(M, E)$ in a natural way is a topological vector space.

To motivate the later construction we first consider a fixed function $f \in C^{\mathrm{an}}(M, E)$. Since, by Prop. 8.7, M is strictly paracompact we find a family of charts $(U_j, \varphi_j, K^{m_j})$, for $j \in J$, for M such that the U_j are pairwise disjoint and $M = \bigcup_{j \in J} U_j$. According to Remark 8.2.ii. the function f is locally analytic if and only if all $f \circ \varphi_j^{-1} : \varphi_j(U_j) \longrightarrow E$, for $j \in J$, are locally analytic. For each $\varphi_j(U_j)$ we find balls $B_{\varepsilon_{j,\nu}}(x_{j,\nu}) \subseteq K^{m_j}$ and power series $F_{j,\nu} \in \mathcal{F}_{\varepsilon_{j,\nu}}(K^{m_j};E)$ such that

$$\varphi_j(U_j) = \bigcup_{\nu} B_{\varepsilon_{j,\nu}}(x_{j,\nu}) \tag{16}$$

and

$$f \circ \varphi_j^{-1}(x) = F_{j,\nu}(x - x_{j,\nu}) \qquad \text{for any } x \in B_{\varepsilon_{j,\nu}}(x_{j,\nu}).$$

By Lemma 1.4 the covering (16) can be refined into a covering by pairwise disjoint balls $B_{\delta_{j,\alpha}}(y_{j,\alpha})$. Consider a fixed α. We find a ν such that $B_{\delta_{j,\alpha}}(y_{j,\alpha}) \subseteq B_{\varepsilon_{j,\nu}}(x_{j,\nu})$. In fact we then have

$$B_{\min(\delta_{j,\alpha},\varepsilon_{j,\nu})}(y_{j,\alpha}) = B_{\delta_{j,\alpha}}(y_{j,\alpha}) \subseteq B_{\varepsilon_{j,\nu}}(x_{j,\nu}) = B_{\varepsilon_{j,\nu}}(y_{j,\alpha}).$$

Hence we may assume that $\delta_{j,\alpha} \leq \varepsilon_{j,\nu}$. Using Cor. 5.5 we may change $F_{j,\nu}$ into a power series $F_{j,\alpha} \in \mathcal{F}_{\delta_{j,\alpha}}(K^{m_j};E)$ such that

$$f \circ \varphi_j^{-1}(x) = F_{j,\alpha}(x - y_{j,\alpha}) \qquad \text{for any } x \in B_{\delta_{j,\alpha}}(y_{j,\alpha}).$$

We put $U_{j,\alpha} := \varphi_j^{-1}(B_{\delta_{j,\alpha}}(y_{j,\alpha}))$. The $(U_{j,\alpha}, \varphi_j, K^{m_j})$ again are charts for M such that the $U_{j,\alpha}$ cover M and are pairwise disjoint.

Resume: *Given $f \in C^{\mathrm{an}}(M,E)$ there is a family of charts $(U_i, \varphi_i, K^{m_i})$, for $i \in I$, for M together with real numbers $\varepsilon_i > 0$ such that*:

(a) $M = \bigcup_{i \in I} U_i$, *and the U_i are pairwise disjoint*;

(b) $\varphi_i(U_i) = B_{\varepsilon_i}(x_i)$ *for one (or any) $x_i \in \varphi_i(U_i)$*;

(c) *there is a power series $F_i \in \mathcal{F}_{\varepsilon_i}(K^{m_i};E)$ with*

$$f \circ \varphi_i^{-1}(x) = F_i(x - x_i) \qquad \textit{for any } x \in \varphi_i(U_i).$$

We note that by Cor. 5.5 the existence of F_i as well as its norm $\|F_i\|_{\varepsilon_i}$ is independent of the choice of the point x_i.

Let (c, ε) be a pair consisting of a chart $c = (U, \varphi, K^m)$ for M and a real number $\varepsilon > 0$ such that $\varphi(U) = B_\varepsilon(a)$ for one (or any) $a \in \varphi(U)$. As a consequence of the identity theorem for power series Cor. 5.8 the K-linear map

$$\begin{aligned} \mathcal{F}_\varepsilon(K^m;E) &\longrightarrow C^{\mathrm{an}}(U,E) \\ F &\longmapsto F(\varphi(.) - a) \end{aligned}$$

is injective. Let $\mathcal{F}_{(c,\varepsilon)}(E)$ denote its image. It is a K-Banach space with respect to the norm

$$\|f\| = \|F\|_\varepsilon \quad \text{if} \quad f(.) = F(\varphi(.) - a).$$

By Cor. 5.5 the pair $(\mathcal{F}_{(c,\varepsilon)}(E), \|\ \|)$ is independent of the choice of the point a.

Definition. *An index for M is a family $\mathcal{I} = \{(c_i, \varepsilon_i)\}_{i\in I}$ of charts $c_i = (U_i, \varphi_i, K^{m_i})$ for M and real numbers $\varepsilon_i > 0$ such that the above conditions* (a) *and* (b) *are satisfied.*

For any index $\mathcal{I}$ for M we have

$$\mathcal{F}_{\mathcal{I}}(E) := \prod_{i\in I} \mathcal{F}_{(c_i,\varepsilon_i)}(E) \subseteq \prod_{i\in I} C^{\mathrm{an}}(U_i, E) = C^{\mathrm{an}}(M, E).$$

Our above resume says that

$$C^{\mathrm{an}}(M, E) = \bigcup_{\mathcal{I}} \mathcal{F}_{\mathcal{I}}(E)$$

where $\mathcal{I}$ runs over all indices for M. Hence $C^{\mathrm{an}}(M, E)$ is a union of direct products of Banach spaces. This is the starting point for the construction of a topology on $C^{\mathrm{an}}(M, E)$.

But first we have to discuss the inclusion relations between the subspaces $\mathcal{F}_{\mathcal{I}}(E)$ for varying $\mathcal{I}$. Let $\mathcal{I} = \{(c_i = (U_i, \varphi_i, K^{m_i}), \varepsilon_i)\}_{i\in I}$ and $\mathcal{J} = \{(d_j = (V_j, \psi_j, K^{n_j}), \delta_j)\}_{j\in J}$ be two indices for M.

Definition. *The index $\mathcal{I}$ is called finer than the index $\mathcal{J}$ if for any $i \in I$ there is a $j \in J$ such that*:

(i) $U_i \subseteq V_j$,

(ii) *there is an $a \in \varphi_i(U_i)$ and a power series $F_{i,j} \in \mathcal{F}_{\varepsilon_i}(K^{m_i}; K^{n_j})$ with $\|F_{i,j} - F_{i,j}(0)\|_{\varepsilon_i} \le \delta_j$ and*

$$\psi_j \circ \varphi_i^{-1}(x) = F_{i,j}(x - a) \qquad \textit{for any } x \in \varphi_i(U_i).$$

We observe that if the condition (ii) holds for one point $a \in \varphi_i(U_i)$ then it holds for any other point $b \in \varphi_i(U_i)$ as well. This follows from Cor. 5.5 which implies that $G_{i,j}(X) := F_{i,j}(X + b - a) \in \mathcal{F}_{\varepsilon_i}(K^{m_i}; K^{n_j})$ with

$$\psi_j \circ \varphi_i^{-1}(x) = G_{i,j}(x - b) \qquad \text{for any } x \in \varphi_i(U_i)$$

and

$$\begin{aligned}\|G_{i,j} - G_{i,j}(0)\|_{\varepsilon_i} &= \|(F_{i,j} - F_{i,j}(0))(X + b - a) + F_{i,j}(0) - G_{i,j}(0)\|_{\varepsilon_i} \\ &\leq \max(\|(F_{i,j} - F_{i,j}(0))(X + b - a)\|_{\varepsilon_i}, \delta_j) \\ &= \max(\|F_{i,j} - F_{i,j}(0)\|_{\varepsilon_i}, \delta_j) \\ &= \delta_j.\end{aligned}$$

Lemma 10.1. *If $\mathcal{I}$ is finer than $\mathcal{J}$ then we have $\mathcal{F}_{\mathcal{J}}(E) \subseteq \mathcal{F}_{\mathcal{I}}(E)$.*

Proof. Let $f \in \mathcal{F}_{\mathcal{J}}(E)$. We have to show that $f|U_i \in \mathcal{F}_{(c_i,\varepsilon_i)}(E)$ for any $i \in I$. In the following we fix an $i \in I$. We have $\varphi_i(U_i) = B_{\varepsilon_i}(a)$. By assumption we find a $j \in J$ and an $F_{i,j} \in \mathcal{F}_{\varepsilon_i}(K^{m_i}; K^{n_j})$ such that

– $U_i \subseteq V_j$,

– $\|F_{i,j} - F_{i,j}(0)\|_{\varepsilon_i} \leq \delta_j$, and

– $\psi_j \circ \varphi_i^{-1}(x) = F_{i,j}(x - a)$ for any $x \in \varphi_i(U_i)$.

We put

$$b := \psi_j \circ \varphi_i^{-1}(a) = F_{i,j}(0) \in \psi_j(V_j).$$

Since $f \in \mathcal{F}_{\mathcal{J}}(E)$ we also find a $G_j \in \mathcal{F}_{\delta_j}(K^{n_j}; E)$ such that

$$f \circ \psi_j^{-1}(y) = G_j(y - b) \qquad \text{for any } y \in \psi_j(V_j) = B_{\delta_j}(b).$$

As a consequence of Prop. 5.4 then the power series

$$F_i := G_j \circ (F_{i,j} - F_{i,j}(0)) \in \mathcal{F}_{\varepsilon_i}(K^{m_i}; E)$$

exists and satisfies

$$F_i(x - a) = G_j(F_{i,j}(x - a) - b) = f \circ \psi_j^{-1}(\psi_j \circ \varphi_i^{-1}(x)) = f \circ \varphi_i^{-1}(x)$$

for any $x \in \varphi_i(U_i)$. □

The relation of being finer only is a preorder. If the index $\mathcal{I}$ is finer than the index $\mathcal{J}$ and $\mathcal{J}$ is finer than $\mathcal{I}$ one cannot conclude that $\mathcal{I} = \mathcal{J}$. But it does follow that $\mathcal{F}_{\mathcal{I}}(E) = \mathcal{F}_{\mathcal{J}}(E)$ which is sufficient for our purposes.

Lemma 10.2. *For any two indices $\mathcal{J}_1$ and $\mathcal{J}_2$ for M there is a third index $\mathcal{I}$ for M which is finer than $\mathcal{J}_1$ and $\mathcal{J}_2$.*

Proof. Let $\mathcal{J}_1 = \{((U_i, \varphi_i, K^{n_i}), \varepsilon_i)\}_{i\in I}$ and $\mathcal{J}_2 = \{((V_j, \psi_j, K^{m_j}), \delta_j)\}_{j\in J}$. We have the covering

$$M = \bigcup_{i,j} U_i \cap V_j$$

by pairwise disjoint open subsets. For any pair $(i, j) \in I \times J$ the function

$$\psi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap V_j) \longrightarrow K^{m_j}$$

is locally analytic. Hence we may cover $\varphi_i(U_i \cap V_j)$ by a family of balls $B_{i,j,k} = B_{\beta_{i,j,k}}(a_{i,j,k})$ such that

- $\beta_{i,j,k} \le \min(\varepsilon_i, \delta_j)$, and
- there is a power series $F_{i,j,k} \in \mathcal{F}_{\beta_{i,j,k}}(K^{n_i}; K^{m_j})$ with

$$\psi_j \circ \varphi_i^{-1}(x) = F_{i,j,k}(x - a_{i,j,k}) \qquad \text{for any } x \in B_{i,j,k}.$$

Using the fact that

$$\|F_{i,j,k} - F_{i,j,k}(0)\|_\alpha \le \frac{\alpha}{\beta_{i,j,k}} \|F_{i,j,k} - F_{i,j,k}(0)\|_{\beta_{i,j,k}} \quad \text{for any } 0 < \alpha \le \beta_{i,j,k}$$

together with Cor. 5.5 we may, after possibly decreasing the $\beta_{i,j,k}$, assume in addition that

- $\|F_{i,j,k} - F_{i,j,k}(0)\|_{\beta_{i,j,k}} \le \delta_j$.

After a possible further refinement based on Lemma 1.4 (compare the argument for the resume at the beginning of this section) we finally achieve that the $B_{i,j,k}$ are pairwise disjoint. We put

$$W_{i,j,k} := \varphi_i^{-1}(B_{i,j,k})$$

and obtain the index $\mathcal{I} := \{((W_{i,j,k}, \varphi_i, K^{n_i}), \beta_{i,j,k})\}_{i,j,k}$ for M. By construction $\mathcal{I}$ is finer than $\mathcal{J}_2$. Moreover, observing that $\varphi_i \circ \varphi_i^{-1} : \varphi_i(W_{i,j,k}) \xrightarrow{\subseteq} K^{n_i}$ is the inclusion map and that $\beta_{i,j,k} \le \varepsilon_i$ we see that $\mathcal{I}$ is finer than $\mathcal{J}_1$ for trivial reasons. $\square$

Given any index $\mathcal{I}$ for M we consider $\mathcal{F}_\mathcal{I}(E) = \prod_{i\in I} \mathcal{F}_{(c_i,\varepsilon_i)}(E)$ from now on as a topological K-vector space with respect to the product topology of the Banach space topologies on the $\mathcal{F}_{(c_i,\varepsilon_i)}(E)$. Obviously $\mathcal{F}_\mathcal{I}(E)$ is Hausdorff. But it is not a Banach space if I is infinite. Suppose that the topology of $\mathcal{F}_\mathcal{I}(E)$ can be defined by a norm. The corresponding unit ball

$B_1(0)$ is open. By the definition of the product topology there exist finitely many indices $i_1, \ldots, i_r \in I$ such that

$$\prod_{i \neq i_1, \ldots, i_r} \mathcal{F}_{(c_i, \varepsilon_i)}(E) \times \{0\} \times \cdots \times \{0\} \subseteq B_1(0).$$

As a vector subspace the left hand side then necessarily is contained in any ball $B_\varepsilon(0)$ for $\varepsilon > 0$. The intersection of the latter being equal to $\{0\}$ it follows that I is finite.

Lemma 10.3. *If $\mathcal{I}$ is finer than $\mathcal{J}$ then the inclusion map $\mathcal{F}_{\mathcal{J}}(E) \xrightarrow{\subseteq} \mathcal{F}_{\mathcal{I}}(E)$ is continuous.*

Proof. For any $i \in I$ there exists, by assumption, a $j(i) \in J$ such that the conditions (i) and (ii) in the definition of "finer" are satisfied. The inclusion map in question can be viewed as the map

$$\begin{aligned} \prod_j \mathcal{F}_{(d_j, \delta_j)}(E) &\longrightarrow \prod_i \mathcal{F}_{(c_i, \varepsilon_i)}(E) \\ (f_j)_j &\longmapsto (f_{j(i)} | U_i)_i. \end{aligned}$$

Hence it suffices to show that each individual restriction map

$$\mathcal{F}_{(d_{j(i)}, \delta_{j(i)})}(E) \longrightarrow \mathcal{F}_{(c_i, \varepsilon_i)}(E)$$

is continuous. But we even know from Prop. 5.4 that the operator norm of this map is ≤ 1. $\square$

We point out that, for $\mathcal{I}$ finer than $\mathcal{J}$, the topology of $\mathcal{F}_{\mathcal{J}}(E)$ in general is strictly finer than the subspace topology induced by $\mathcal{F}_{\mathcal{I}}(E)$.

In the present situation there is a certain universal procedure to construct from the topologies on all the $\mathcal{F}_{\mathcal{I}}(E)$ a topology on their union $C^{\mathrm{an}}(M, E) = \bigcup_{\mathcal{I}} \mathcal{F}_{\mathcal{I}}(E)$. Since this construction takes place within the class of locally convex topologies we first need to review this concept in the next section.

11 Locally Convex K-Vector Spaces

This section serves only as a brief introduction to the subject. The reader who is interested in more details is referred to [NFA]. Let E be any K-vector space.

Definition. *A (nonarchimedean) seminorm on E is a function $q : E \longrightarrow \mathbb{R}$ such that for any $v, w \in E$ and any $a \in K$ we have:*

(i) $q(av) = |a| \cdot q(v)$,

(ii) $q(v + w) \leq \max(q(v), q(w))$.

It follows immediately that a seminorm q also satisfies:

(iii) $q(0) = |0| \cdot q(0) = 0$;

(iv) $q(v) = \max(q(v), q(-v)) \geq q(v - v) = q(0) = 0$ for any $v \in E$;

(v) $q(v + w) = \max(q(v), q(w))$ for any $v, w \in E$ such that $q(v) \neq q(w)$ (compare the proof of Lemma 1.1);

(vi) $-q(v - w) \leq q(v) - q(w) \leq q(v - w)$ for any $v, w \in E$.

Let $(q_i)_{i \in I}$ be a family of seminorms on E. We consider the coarsest topology on E such that:

(1) All maps $q_i : E \longrightarrow \mathbb{R}$, for $i \in I$, are continuous,

(2) all translation maps $v + . : E \longrightarrow E$, for $v \in E$, are continuous.

It is called the *topology defined by* $(q_i)_{i \in I}$. For any finitely many $q_{i_1}, \ldots, q_{i_r}$ and any $w \in E$ and $\varepsilon > 0$ we define

$$B_\varepsilon(q_{i_1}, \ldots, q_{i_r}; w) := \{v \in E : q_{i_r}(v - w), \ldots, q_{i_r}(v - w) \leq \varepsilon\}.$$

The following properties are obvious:

(a) $B_\varepsilon(q_{i_1}, \ldots, q_{i_r}; w) = B_\varepsilon(q_{i_1}; w) \cap \cdots \cap B_\varepsilon(q_{i_r}; w)$;

(b) $B_{\varepsilon_1}(q_{i_1}; w_1) \cap B_{\varepsilon_2}(q_{i_2}; w_2) = \bigcup_w B_{\min(\varepsilon_1, \varepsilon_2)}(q_{i_1}, q_{i_2}; w)$ where w runs over all points in the left hand side;

(c) $B_\varepsilon(q_{i_1}, \ldots, q_{i_r}; w) = w + B_\varepsilon(q_{i_1}, \ldots, q_{i_r}; 0)$;

(d) $a \cdot B_\varepsilon(q_{i_1}, \ldots, q_{i_r}; w) = B_{|a|\varepsilon}(q_{i_1}, \ldots, q_{i_r}; aw)$ for any $a \in K^\times$.

Lemma 11.1. *The subsets $B_\varepsilon(q_{i_1}, \ldots, q_{i_r}; w)$ form a basis for the topology on E defined by $(q_i)_{i \in I}$.*

Proof. The $B_\varepsilon(q_{i_1}, \ldots, q_{i_r}; w)$, by (a) and (b), do form a basis for a (unique) topology $\mathcal{T}'$ on E. On the other hand let $\mathcal{T}$ denote the topology defined by $(q_i)_{i \in I}$. We first show that $\mathcal{T}' \subseteq \mathcal{T}$. By (a), (c), and (2) it suffices to check that $B_\varepsilon(q_i;0) \in \mathcal{T}$ for any $i \in I$ and $\varepsilon > 0$. As a consequence of (1) and (2) we certainly have that

$$B_\delta^-(q_i;w) := \{v \in E : q_i(v-w) < \delta\} \in \mathcal{T}$$

for any $w \in E$ and $\delta > 0$. But using (ii) we see that

$$B_\varepsilon(q_i;0) = B_\varepsilon^-(q_i;0) \cup \bigcup_{q_i(w)=\varepsilon} B_\varepsilon^-(q_i;w).$$

To conclude that actually $\mathcal{T}' = \mathcal{T}$ holds true it now suffices to show that $\mathcal{T}'$ satisfies (1) and (2). The continuity property (2) follows immediately from (c). To establish (1) for $\mathcal{T}'$ we have to show that $q_i^{-1}((\alpha,\beta)) \in \mathcal{T}'$ for any $i \in I$ and any open interval $(\alpha,\beta) \subseteq \mathbb{R}$. Because of (iv) we may assume that $\beta > 0$. Let $w \in q_i^{-1}((\alpha,\beta))$ be any point. *Case* 1: We have $q_i(w) > 0$. Choose any $0 < \varepsilon < q_i(w)$. It then follows from (v) that $B_\varepsilon(q_i;w) \subseteq q_i^{-1}(q_i(w)) \subseteq q_i^{-1}((\alpha,\beta))$. *Case* 2: We have $q_i(w) = 0$. Choose any $0 < \varepsilon < \beta$. We obtain $B_\varepsilon(q_i;w) \subseteq q_i^{-1}([0,\varepsilon]) \subseteq q_i^{-1}((\alpha,\beta))$ since necessarily $\alpha < 0$ in this case. □

Lemma 11.2. *E is a topological K-vector space, i. e., addition and scalar multiplication are continuous, with respect to the topology defined by $(q_i)_{i\in I}$.*

Proof. Using Lemma 11.1 this easily follows from the following inclusions:

– $B_\varepsilon(q_{i_1},\ldots,q_{i_r};w_1) + B_\varepsilon(q_{i_1},\ldots,q_{i_r};w_2) \subseteq B_\varepsilon(q_{i_1},\ldots,q_{i_r};w_1+w_2)$;

– $B_\delta(a) \cdot B_{|a|^{-1}\varepsilon}(q_{i_1},\ldots,q_{i_r};w) \subseteq B_\varepsilon(q_{i_1},\ldots,q_{i_r};aw)$ provided $\delta \leq |a|$ and $\delta \cdot \max(q_{i_1}(w),\ldots,q_{i_r}(w)) \leq \varepsilon$;

– $B_\delta(0) \cdot B_\varepsilon(q_{i_1},\ldots,q_{i_r};w) \subseteq B_\varepsilon(q_{i_1},\ldots,q_{i_r};0)$ provided $\delta \leq 1$ and $\delta \cdot \max(q_{i_1}(w),\ldots,q_{i_r}(w)) \leq \varepsilon$.

The details are left to the reader as an exercise. □

Exercise. *The topology on E defined by $(q_i)_{i\in I}$ is Hausdorff if and only if for any vector $0 \neq v \in E$ there is an index $i \in I$ such that $q_i(v) \neq 0$.*

Definition. *A topology on a K-vector space E is called locally convex if it can be defined by a family of seminorms. A locally convex K-vector space is a K-vector space equipped with a locally convex topology.*

Obviously any normed K-vector space and in particular any K-Banach space is locally convex.

Remark 11.3. *Let $\{E_j\}_{j \in J}$ be a family of locally convex K-vector spaces; then the product topology on $E := \prod_{j \in J} E_j$ is locally convex.*

Proof. Let $(q_{j,i})_i$ be a family of seminorms which defines the locally convex topology on E_j. Moreover, let $\mathrm{pr}_j : E \longrightarrow E_j$ denote the projection maps. Using Lemma 11.1 one checks that the family of seminorms $(q_{j,i} \circ \mathrm{pr}_j)_{i,j}$ defines the product topology on E. $\square$

Exercise 11.4. *Let $\{E_j\}_{j \in J}$ be a family of locally convex K-vector spaces and let $E := \prod_{j \in J} E_j$ with the product topology; for any continuous seminorm q on E there is a unique minimal finite subset $J_q \subseteq J$ such that*

$$q\Bigg(\prod_{j \in J \setminus J_q} E_j \times \{0\} \times \cdots \times \{0\} \Bigg) = \{0\}.$$

For our purposes the following construction is of particular relevance. Let E be a any K-vector space, and suppose that there is given a family $\{E_j\}_{j \in J}$ of vector subspaces $E_j \subseteq E$ each of which is equipped with a locally convex topology.

Lemma 11.5. *There is a unique finest locally convex topology $\mathcal{T}$ on E such that all the inclusion maps $E_j \xrightarrow{\subseteq} E$, for $j \in J$, are continuous.*

Proof. Let Q be the set of all seminorms q on E such that $q|E_j$ is continuous for any $j \in J$, and let $\mathcal{T}$ be the topology on E defined by Q. It follows immediately from Lemma 11.1 that all the inclusion maps $E_j \xrightarrow{\subseteq} (E, \mathcal{T})$ are continuous. On the other hand, let $\mathcal{T}'$ be any topology on E defined by a family of seminorms $(q_i)_{i \in I}$ such that $E_j \xrightarrow{\subseteq} (E, \mathcal{T}')$ is continuous for any $j \in J$. Obviously we then have $(q_i)_{i \in I} \subseteq Q$. This implies, using again Lemma 11.1, that $\mathcal{T}' \subseteq \mathcal{T}$. $\square$

The topology $\mathcal{T}$ on E in the above Lemma is called the *locally convex final topology* with respect to the family $\{E_j\}_{j \in J}$. Suppose that the family $\{E_j\}_{j \in J}$ has the additional properties:

- $E = \bigcup_{j \in J} E_j$;
- the set J is partially ordered by $\leq$ such that for any two $j_1, j_2 \in J$ there is a $j \in J$ such that $j_1 \leq j$ and $j_2 \leq j$;
- whenever $j_1 \leq j_2$ we have $E_{j_1} \subseteq E_{j_2}$ and the inclusion map $E_{j_1} \xrightarrow{\subseteq} E_{j_2}$ is continuous.

In this case the locally convex K-vector space $(E, \mathcal{T})$ is called the *locally convex inductive limit* of the family $\{E_j\}_{j \in J}$.

Lemma 11.6. *A K-linear map $f : E \longrightarrow \tilde{E}$ into any locally convex K-vector space $\tilde{E}$ is continuous (with respect to $\mathcal{T}$) if and only if the restrictions $f|E_j$, for any $j \in J$, are continuous.*

Proof. It is trivial that with f all restrictions $f|E_j$ are continuous. Let us therefore assume vice versa that all $f|E_j$ are continuous. Let $(\tilde{q}_i)_{i \in I}$ be a family of seminorms which defines the topology of $\tilde{E}$. Then all seminorms $q_i := \tilde{q}_i \circ f$, for $i \in I$, lie in the set of seminorms Q which defines the topology $\mathcal{T}$ of E. It follows that

$$f^{-1}(B_\varepsilon(\tilde{q}_{i_1}, \ldots, \tilde{q}_{i_r}; f(w)) = B_\varepsilon(q_{i_1}, \ldots, q_{i_r}; w)$$

is open in E. Because of Lemma 11.1 this means that f is continuous. □

Lemma 11.7. *Let $\{E_j\}_{j \in J}$ be a family of locally convex K-vector spaces and let $E := \prod_{j \in J} E_j$ with the product topology; suppose that each E_j has the locally convex final topology with respect to a family of locally convex K-vector spaces $\{E_{j,k}\}_{k \in I_j}$ and that $E_j = \bigcup_k E_{j,k}$; for any $\underline{k} = (k_j)_j \in I := \prod_{j \in J} I_j$ we put $E_{\underline{k}} := \prod_{j \in J} E_{j,k_j}$ with the product topology; then the topology of E is the locally convex final topology with respect to the family $\{E_{\underline{k}}\}_{\underline{k} \in I}$.*

Proof. By the proof of Lemma 11.5 the locally convex topology of E_j is defined by the set Q_j of all seminorms q such that $q|E_{j,k}$ is continuous for any $k \in I_j$. Let $\mathrm{pr}_j : E \longrightarrow E_j$ denote the projection maps. By Remark 11.3 the topology of E is defined by the set of seminorms $Q := \bigcup_{j \in J} \{q \circ \mathrm{pr}_j : q \in Q_j\}$. For any $q \in Q_j$ and any $\underline{k} \in I$ we have the commutative diagram

$$\begin{tikzcd}
E_{\underline{k}} \arrow[r, "\subseteq"] \arrow[d, "\mathrm{pr}_j"'] & E \arrow[d, "\mathrm{pr}_j"] \arrow[dr, "q \circ \mathrm{pr}_j"] & \\
E_{j,k_j} \arrow[r, "\subseteq"] & E_j \arrow[r, "q"] & \mathbb{R}.
\end{tikzcd}$$

Hence the restriction of any seminorm in Q to any $E_{\underline{k}}$ is continuous. This means that the locally convex final topology on E with respect to the family $\{E_{\underline{k}}\}_{\underline{k}}$ is finer than the product topology. Vice versa, let q be any seminorm on E such that $q|E_{\underline{k}}$, for any $\underline{k} \in I$, is continuous. We have to show that q is continuous. By Exercise 11.4 we find, for any $\underline{k} \in I$, a unique minimal finite subset $J_{q,\underline{k}} \subseteq J$ such that the restriction $q|E_{\underline{k}}$ factorizes into

$$\begin{tikzcd} E_{\underline{k}} \arrow[r, "\mathrm{pr}"] \arrow[dr, "q"'] & \prod_{j\in J_{q,\underline{k}}} E_{j,k_j} \arrow[d] \\ & \mathbb{R} \end{tikzcd} .$$

In particular, $q|E_{j,k_j} \neq 0$ for any $j \in J_{q,\underline{k}}$. We claim that the set

$$J_q := \bigcup_{\underline{k}\in I} J_{q,\underline{k}}$$

is finite. We define $\underline{\ell} = (\ell_j)_j \in I$ in the following way. If $j \in J_q$ we choose a $\underline{k} \in I$ such that $j \in J_{q,\underline{k}}$ and we put $\ell_j := k_j$; in particular, $q|E_{j,\ell_j} = q|E_{j,k_j} \neq 0$. For $j \in J \setminus J_q$ we pick any $\ell_j \in I_j$. By construction we have $J_q \subseteq J_{q,\underline{\ell}}$ so that J_q necessarily is finite. This means that the seminorm q on E factorizes into

$$\begin{tikzcd} E \arrow[r, "\mathrm{pr}"] \arrow[dr, "q"'] & \prod_{j\in J_q} E_j \arrow[d] \\ & \mathbb{R} \end{tikzcd} .$$

It follows that

$$q(v) \leq \max_{j\in J_q} (q|E_j) \circ \mathrm{pr}_j(v) \qquad \text{for any } v \in E.$$

Since each $q|E_j$ is continuous by assumption we conclude that q is continuous. □

12 The Topological Vector Space $C^{\mathrm{an}}(M,E)$, Part 2

As in Sect. 10 we let M be a paracompact manifold and E be a K-Banach space. We have seen that

$$C^{\mathrm{an}}(M,E) = \bigcup_{\mathcal{I}} \mathcal{F}_{\mathcal{I}}(E)$$

where $\mathcal{I}$ runs over all indices for M. Each $\mathcal{F}_{\mathcal{I}}(E)$ by Remark 11.3 is locally convex as a product of Banach spaces. By Lemmas 10.1–10.3 we may and always will view $C^{\mathrm{an}}(M,E)$ as the locally convex inductive limit of the family $\{\mathcal{F}_{\mathcal{I}}(E)\}_{\mathcal{I}}$ (where $\mathcal{I} \leq \mathcal{J}$ if $\mathcal{J}$ is finer than $\mathcal{I}$). All our earlier constructions involving $C^{\mathrm{an}}(M,E)$ are compatible with this topology. In the following we briefly discuss the most important ones.

Proposition 12.1. *For any $a \in M$ the evaluation map*

$$\begin{aligned} \delta_a : C^{\mathrm{an}}(M, E) &\longrightarrow E \\ f &\longmapsto f(a) \end{aligned}$$

is continuous.

Proof. It suffices, by Lemma 11.6, to show that the restriction $\delta_a | \mathcal{F}_{\mathcal{I}}(E)$ is continuous for any index $\mathcal{I}$ for M. Let $\mathcal{I} = \{(c_i = (U_i, \varphi_i, K^{m_i}), \varepsilon_i)\}_{i \in I}$. There is a unique $i \in I$ such that $a \in U_i$. Then $\varphi_i(U_i) = B_{\varepsilon_i}(\varphi_i(a))$, and we have the commutative diagram

$$\begin{tikzcd}
\mathcal{F}_{\mathcal{I}}(E) \arrow[rr, "\delta_a"] \arrow[d, "\mathrm{pr}"'] & & E \\
\mathcal{F}_{(c_i, \varepsilon_i)}(E) & & \mathcal{F}_{\varepsilon_i}(K^{m_i}; E). \arrow[ll, "\cong", "F(\varphi_i(.) - \varphi_i(a)) \leftarrow\!\shortmid F"'] \arrow[u, "F \mapsto F(0)"']
\end{tikzcd}$$

The left vertical projection map clearly is continuous. The lower horizontal map is a topological isomorphism by construction. By Remark 5.1 the right vertical evaluation map is continuous of operator norm ≤ 1. $\square$

Corollary 12.2. *The locally convex vector space $C^{\mathrm{an}}(M, E)$ is Hausdorff.*

Proof. Let $f \neq g$ be two different functions in $C^{\mathrm{an}}(M, E)$. We find a point $a \in M$ such that $f(a) \neq g(a)$. Since E is Hausdorff there are open neighbourhoods V_f of $f(a)$ and V_g of $g(a)$ in E such that $V_f \cap V_g = \emptyset$. Using Prop. 12.1 we see that $U_f := \delta_a^{-1}(V_f)$ and $U_g := \delta_a^{-1}(V_g)$ are open neighbourhoods of f and g, respectively, in $C^{\mathrm{an}}(M, E)$ such that $U_f \cap U_g = \emptyset$. $\square$

Remark 12.3. *With M also its tangent bundle $T(M)$ is paracompact.*

Proof. Since M is strictly paracompact by Prop. 8.7 we find a family of charts $\{c_i = (U_i, \varphi_i, K^{m_i})\}_{i \in I}$ for M such that the U_i are pairwise disjoint and $M = \bigcup_i U_i$. Then the $c_{i,T} = (p_M^{-1}(U_i), \varphi_{i,c_i}, K^{2m_i})$ form a family of charts for $T(M)$ such that $T(M)$ is the disjoint union of the open subsets $p_M^{-1}(U_i)$. Each $p_M^{-1}(U_i)$ being homeomorphic to an open subset in K^{2m_i} carries the topology of an ultrametric space. The construction in the proof of implication ii. $\Longrightarrow$ iii. in Prop. 8.7 then shows that the topology of $T(M)$ can be defined by a metric which satisfies the strict triangle inequality. Hence $T(M)$ is paracompact by Lemma 1.4. $\square$

Proposition 12.4. i. *The map* $d : C^{\mathrm{an}}(M,E) \longrightarrow C^{\mathrm{an}}(T(M),E)$ *is continuous.*

ii. *For any locally analytic map of paracompact manifolds* $g : M \longrightarrow N$ *the map*

$$\begin{aligned} C^{\mathrm{an}}(N,E) &\longrightarrow C^{\mathrm{an}}(M,E) \\ f &\longmapsto f \circ g \end{aligned}$$

is continuous.

iii. *For any vector field* ξ *on* M *the map* $D_\xi : C^{\mathrm{an}}(M,E) \longrightarrow C^{\mathrm{an}}(M,E)$ *is continuous.*

Proof. i. By Lemma 11.6 we have to show that $d|\mathcal{F}_{\mathcal{I}}(E)$ is continuous for any index $\mathcal{I} = \{(c_i = (U_i, \varphi_i, K^{m_i}), \varepsilon_i)\}_{i \in I}$ for M. Let $f \in \mathcal{F}_{\mathcal{I}}(E)$. We have the commutative diagrams

$$\begin{tikzcd} & & E \\ p_M^{-1}(U_i) \arrow[rru, "df"] \arrow[rr, "\varphi_{i,c_i}"] \arrow[d, "p_M"'] & & \varphi_i(U_i) \times K^{m_i} \arrow[u, "(x,v)\mapsto D_x(f\circ\varphi_i^{-1})(v)"'] \arrow[d, "\mathrm{pr}"] \\ U_i \arrow[rr, "\varphi_i"] & & \varphi_i(U_i) \end{tikzcd}$$

(cf. the proof of Lemma 9.11). We also have power series $F_i \in \mathcal{F}_{\varepsilon_i}(K^{m_i}; E)$ such that

$$f \circ \varphi_i^{-1}(x) = F_i(x - a_i) \qquad \text{for any } x \in \varphi_i(U_i) = B_{\varepsilon_i}(a_i).$$

From the proof of Prop. 6.1 we recall the formula

$$D_x(f \circ \varphi_i^{-1})(v) = \sum_{j=1}^{m_i} v_j \frac{\partial F_i}{\partial X_j}(x - a_i)$$

for any $x \in \varphi_i(U_i)$ and any $v = (v_1, \ldots, v_{m_i}) \in K^{m_i}$. We now cover K^{m_i} by pairwise disjoint balls $B_{\varepsilon_i}(w_k^{(i)})$ where $w_k^{(i)} = (w_{k,1}^{(i)}, \ldots, w_{k,m_i}^{(i)})$ runs over an appropriate family of vectors in K^{m_i}, and we put

$$G_{i,k}(X_1, \ldots, X_{m_i}, Y_1, \ldots, Y_{m_i}) := \sum_{j=1}^{m_i} (Y_j + w_{k,j}^{(i)}) \frac{\partial F_i}{\partial X_j} \in \mathcal{F}_{\varepsilon_i}(K^{2m_i}; E).$$

Then

$$df \circ \varphi_{i,c_i}^{-1}(x, v) = D_x(f \circ \varphi_i^{-1})(v) = G_{i,k}(x - a_i, v - w_k^{(i)})$$

for any $(x, v) \in \varphi_i(U_i) \times B_{\varepsilon_i}(w_k^{(i)}) = B_{\varepsilon_i}(a_i, w_k^{(i)})$. This means that $df \in \mathcal{F}_{\mathcal{J}}(E) \subseteq C^{\mathrm{an}}(T(M), E)$ for the index

$$\mathcal{J} := \{((\varphi_{i,c_i}^{-1}(B_{\varepsilon_i}(a_i, w_k^{(i)})), \varphi_{i,c_i}, K^{2m_i}), \varepsilon_i)\}_{i,k}.$$

In other words we have the commutative diagram

$$\begin{tikzcd}
C^{\mathrm{an}}(M, E) \arrow[r, "d"] & C^{\mathrm{an}}(T(M), E) \\
\mathcal{F}_{\mathcal{I}}(E) \arrow[u, "\subseteq"] \arrow[r] & \mathcal{F}_{\mathcal{J}}(E). \arrow[u, "\subseteq"']
\end{tikzcd}$$

Since the vertical inclusion maps are continuous by construction this reduces us to showing the continuity of the lower horizontal map $\mathcal{F}_{\mathcal{I}}(E) \longrightarrow \mathcal{F}_{\mathcal{J}}(E)$. But this easily follows from the inequalities

$$\|G_{i,k}\|_{\varepsilon_i} \leq \max\left(1, \frac{|w_{k,1}^{(i)}|}{\varepsilon_i}, \ldots, \frac{|w_{k,m_i}^{(i)}|}{\varepsilon_i}\right) \cdot \|F_i\|_{\varepsilon_i}.$$

ii. We only sketch the argument and leave the details to the reader. Let $\mathcal{I} = \{((U_i, \varphi_i, K^{n_i}), \varepsilon_i)\}_{i \in I}$ be an index for N. We refine the covering $M = \bigcup_i g^{-1}(U_i)$ into a covering $M = \bigcup_{j \in J} V_j$ which underlies an appropriate index $\mathcal{J} = \{((V_j, \psi_j, K^{m_j}), \delta_j)\}_{j \in J}$ and such that, for any $i \in I$ and $j \in J$ with $V_j \subseteq g^{-1}(U_i)$, there is a power series $G_{i,j} \in \mathcal{F}_{\delta_j}(K^{m_j}; K^{n_i})$ with $\|G_{i,j} - G_{i,j}(0)\|_{\delta_j} \leq \varepsilon_i$ and

$$\varphi_i \circ g \circ \psi_j^{-1}(x) = G_{i,j}(x - a_j) \qquad \text{for any } x \in \psi_j(V_j) = B_{\delta_j}(a_j).$$

In this situation we have the commutative diagram

$$\begin{tikzcd}
C^{\mathrm{an}}(N, E) \arrow[r, "f \mapsto f \circ g"] & C^{\mathrm{an}}(M, E) \\
\mathcal{F}_{\mathcal{I}}(E) \arrow[u, "\subseteq"] \arrow[r] & \mathcal{F}_{\mathcal{J}}(E) \arrow[u, "\subseteq"']
\end{tikzcd}$$

where the lower horizontal arrow in terms of power series is given by the maps

$$\begin{aligned}
\mathcal{F}_{\varepsilon_i}(K^{n_i}; E) &\longrightarrow \mathcal{F}_{\delta_j}(K^{m_j}; E) \\
F &\longmapsto F \circ (G_{i,j} - G_{i,j}(0))
\end{aligned}$$

whose continuity was established in Prop. 5.4.

iii. follows from i. and ii. □

Proposition 12.5. *For any covering $M = \bigcup_{i \in I} U_i$ by pairwise disjoint open subsets U_i we have*

$$C^{\mathrm{an}}(M, E) = \prod_{i \in I} C^{\mathrm{an}}(U_i, E)$$

as topological vector spaces.

Proof. Using Lemma 10.2 one checks that in the construction of $C^{\mathrm{an}}(M, E)$ as a locally convex inductive limit it suffices to consider indices for M whose underlying covering of M refines the given covering $M = \bigcup_i U_i$. Then the assertion is a formal consequence of Lemma 11.7. □

http://www.springer.com/978-3-642-21146-1

p-Adic Lie Groups
Schneider, P.
2011, XII, 256 p., Hardcover
ISBN: 978-3-642-21146-1