

Introduction

This book presents a complete account of the foundations of the theory of p -adic Lie groups. It moves on to some of the important more advanced aspects. Although most of the material is not new, it is only in recent years that p -adic Lie groups have found important applications in number theory and representation theory. These applications constitute, in fact, an increasingly active area of research. The book is designed to give to the advanced, but not necessarily graduate, student a streamlined access to the basics of the theory. It is almost self contained. Only a few technical computations which are well covered in the literature will not be repeated. My hope is that researchers who see the need to take up p -adic methods also will find this book helpful for quickly mastering the necessary notions and techniques. The book comes in two parts. Part A on the analytic side grew out of a course which I gave at Münster for the first time during the summer term 2001, whereas part B on the algebraic side is the content of a course given at the Newton Institute during September 2009.

The original and proper context of p -adic Lie groups is p -adic analysis. This is the point of view in Part A. Of course, in a formal sense the notion of a p -adic Lie group is completely parallel to the classical notion of a real or complex Lie group. It is a manifold over a nonarchimedean field which carries a compatible group structure. The fundamental difference is that the p -adic notion has no geometric content. As we will see, a paracompact p -adic manifold is topologically a disjoint union of charts and therefore is, from a geometric perspective, completely uninteresting. The point instead is that, like for real Lie groups, manifolds and Lie groups in the p -adic world are a rich source, through spaces of functions and distributions, of interesting group representations as well as various kinds of important topological group algebras. We nevertheless find the geometric language very intuitive and therefore will use it systematically. In the first chapter we recall what a nonarchimedean field is and quickly discuss the elementary analysis over such fields. In particular, we carefully introduce the notion of a locally analytic function which is at the base for everything to follow. The second chapter then defines manifolds and establishes the formalism of their tangent spaces. As a more advanced topic we include the construction of the natural topology on vector spaces of locally analytic functions. This is due to Féaux de Lacroix in his thesis. It is the starting point for the representation theoretic applications of the theory. In the third chapter we finally introduce p -adic Lie groups and we construct the corresponding Lie algebras. The main purpose of this chapter then is to understand how much informa-

tion about the Lie group can be recovered from its Lie algebra. Here again lies a crucial difference to Lie groups over the real numbers. Since p -adic Lie groups topologically are totally disconnected they contain arbitrarily small open subgroups. Hence the Lie algebra determines the Lie group only locally around the unit element which is formalized by the notion of a Lie group germ. As the length of the chapter indicates this relation between Lie groups and Lie algebras is technically rather involved. It requires a whole range of algebraic concepts which we all will introduce. As said before, only for a few computations the reader will be referred to the literature. The key result is contained in the discussion of the convergence of the Hausdorff series.

There are three existing books on the material in Part A: “Variétés différentielles et analytiques. Fascicule de résultats” and “Lie Groups and Lie Algebras” by Bourbaki and Serre’s lecture notes on “Lie Algebras and Lie groups”. The first one contains no proofs, the nature of the second one is encyclopedic, and the last one some times is a bit short on details. All three develop the real and p -adic case alongside each other which has advantages but makes a quick grasp of the p -adic case alone more difficult. The presentation in the present book places its emphasis instead on a streamlined but still essentially self contained introduction to exclusively the p -adic case.

Lazard discovered in the 1960s a purely algebraic approach to p -adic Lie groups. Unfortunately his seminal paper is notoriously difficult to read. Part B of this book undertakes the attempt to give an account of Lazard’s work again in a streamlined form which is stripped of all inessential generalities and ramifications. Lazard proceeds in an axiomatic way starting from the notion of a p -valuation ω on a pro- p -group G . After some preliminaries in the fourth chapter this concept is explained in chapter five. It will not be too difficult to show that any p -adic Lie group has an open subgroup which carries a p -valuation. Lazard realized that, vice versa, any pro- p -group with a p -valuation (and satisfying an additional mild condition of being “of finite rank”) is a compact p -adic Lie group in a natural way. The technical tool to achieve this important result is the so called completed group ring $\Lambda(G)$ of a profinite group G . It is the appropriate analog of the algebraic group ring of a finite (or, more generally, discrete) group in the context of profinite groups. In the presence of a p -valuation ω Lazard develops a technique of computation in $\Lambda(G)$, which as such is a highly complicated and in general noncommutative algebra. All of this will be presented in the sixth chapter. In the last chapter seven we go back to Lie algebras. Being a p -adic Lie group a pro- p -group G with a p -valuation of finite rank ω has a Lie algebra $\text{Lie}(G)$ over the field of p -adic numbers \mathbb{Q}_p . By inverting p and a further completion process the completed group ring $\Lambda(G)$ can be enlarged to a \mathbb{Q}_p -Banach

algebra $\Lambda_{\mathbb{Q}_p}(G, \omega)$ which turns out to be naturally isomorphic to a certain completion of the universal enveloping algebra of $\text{Lie}(G)$. This is another one of Lazard's important results. It provides us with a different route to construct $\text{Lie}(G)$ which is independent of any analysis. In fact, it does better than that since it leads to a natural Lie algebra over the ring over p -adic integers \mathbb{Z}_p associated with the pair (G, ω) . This means that the algebraic theory, via this notion of a p -valuation, makes the connection between Lie group and Lie algebra much more precise than the analytic theory was able to do. The final question addressed in the last chapter is the question on the possibility of varying the p -valuation on the same group G . Using the newly established direct connection to the Lie algebra this problem can be transferred to the latter. There it eventually becomes a problem of convexity theory which is much easier to solve. This, in particular, allows to prove the very useful technical fact that there always exists a p -valuation with rational values. Its most important consequence is the result that the completed group ring $\Lambda(G)$ of any (G, ω) of finite rank is a noetherian ring of finite global dimension. This is why completed group rings of p -adic Lie groups have become important in number theory (where they are applied to Galois groups G), and why they deserve further systematic study in noncommutative algebra.

This is the first textbook in the proper sense on Lazard's work. The book "Analytic Pro- p -Groups" by Dixon, du Sautoy, Mann, and Segal has a completely different perspective. It is written entirely from the point of view of abstract group theory. Moreover, it does not mention Lazard's concept of a p -valuation at all but replaces it by an alternative axiomatic approach based on the notion of a uniformly powerful pro- p -group. This approach is very conceptual as well but also less flexible and more restrictive than the one by Lazard which we follow.

It is a pleasure to thank J. Coates for persuading me to undertake this lecture series at the Newton Institute and to write it up in this book, the audience for the valuable feedback, the Newton Institute for its hospitality and support, and T. Schoeneberg for a careful reading of Part B.

Münster, February 2011

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<http://www.springer.com/978-3-642-21146-1>

p-Adic Lie Groups

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2011, XII, 256 p., Hardcover

ISBN: 978-3-642-21146-1