Chapter 1
Introduction

Disorder enters modeling in a very natural way: interacting "units" (spins, particles, circuits, cells, individuals,...) are not identical, media (solvents, lattices, environments,...) are not homogeneous or regular, and so on. In many instances it is of course very reasonable to assume that heterogeneities, irregularities, impurities,... can be neglected, and even more for toy models. But it is at least as reasonable to wonder about the stability of the results one obtains for homogeneous systems if disorder is introduced. This concern is omnipresent in the scientific literature and several instances in which small impurities have a drastic effect have been exposed. Four examples for all:

1. Anderson localization in one-dimensional systems, where an arbitrary amount of disorder transforms a conductor into an insulator, see [12] and references therein, and one can find a vast physical literature on the analogous phenomenon in dimension two.
2. Sinai's random walk in random environment, where strongly sub-diffusive behavior sets in for arbitrarily weak randomness in the environment ([13] and references therein).
3. Directed polymers in random environments in dimension one and two, where once again arbitrarily small amounts of disorder lead to drastic phenomena, like, at least in dimension one, super-diffusivity and non Gaussian fluctuations (see [1, 3, 5] and references therein also for the link to other classes of models in the same "universality class").
4. The phase transition in the Ising model can be strongly affected by arbitrarily weak disorder: it may even disappear.

These notes are very close in spirit to all these four remarkable examples – for example, they share with the first three the central keyword "localization" – but the reader will hardly find traces of examples one to three in the sequel. The situation is radically different for the fourth example: these notes contain several discussions and references on disordered Ising models (that is why we have put no reference here!), even if the Ising model is not their main subject. Rather, pinning models are: let us explain why (and what pinning is).

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The main purpose of these notes is to explore the effect of disorder in the framework of equilibrium statistical mechanics of lattice models, that is for Gibbs measures with random interactions or random external fields. These issues have been first developed for the reference statistical mechanics model – the Ising model – and soon after Onsager's celebrated exact solution of the two dimensional case researchers started wondering about the stability of Onsager's result when impurities are introduced. We want to give an overview of the remarkable ideas developed in this context, but trying to follow in detail the Ising model literature would be rather pretentious (and beyond the possibilities of the author): we invite the reader to have a look at the references at the end of Chap. 5 to get a first idea of the body of literature available and we refer to [2] for a comprehensive recent reference on disordered models. The reason why a full account of the Ising literature would be a daunting enterprise is not only due to the amount and depth of the results available, but also to the fact that a consistent part of the physical predictions for the moment are not on firm mathematical grounds. This is the case in particular for the work of Harris [10] on the diluted Ising model and of the various works on disorder relevance/irrelevance that followed. In fact, Harris' idea – that yields the so called Harris criterion – is that one should be able to predict whether a small amount of disorder – "impurities" in Harris' terminology – changes or not the critical behavior of a system, that is the behavior of a system near the phase transition, by simply looking at the critical behavior of the "pure" system. This is very much in the spirit of perturbation theory, but it is a delicate issue because one is dealing with infinite systems and even if locally one adds a small amount of impurities, one always does it in a statistically translation invariant fashion, so in the end the amount of disorder is infinite anyway.

However, if for Ising the full mathematical picture is still escaping, there is a class of models – the pinning models – in which Harris' ideas have been fully understood. Moreover, homogeneous pinning models display phase transitions of "all orders": in fact, by playing on a parameter of the model, one can observe any type of critical behavior of the order parameter at the transition, while for the Ising model one has a discrete spectrum of possible behaviors (the parameter is the dimension), not to speak of the fact that the precise Ising critical behavior is still an open problem in some dimensions. So the basic scheme of the lectures can be summed up to

- Developing in detail the analysis of pinning models
- Discussing, in a less technical fashion, what happens, or what is expected to happen, for the Ising model

But what are pinning models? Just think of an arbitrary, say discrete time, Markov process that visits with positive probability a state (call it 0): for example a random walk on $\mathbb{Z}^d$, with symmetric or asymmetric IID increments, that jumps to nearest neighbor sites. It is well know that an asymmetric walk is transient, so any site, 0 in particular, is visited only a finite number of time if any: moreover, it has a definite drift in a direction. A symmetric walk instead is transient or recurrent according to whether $d \geq 3$ or $d \leq 2$, and in any case there is no drift direction: the walk diffuses. What happens if we reward visits to 0? More precisely, what happens if we modify the law of the walk (up to time N) by weighing the probability of each configuration with the exponential of a constant h times the time spent in 0 up to time N? And we are interested in the limit $N \to \infty$. The answer in general is that there is a delocalization/localization transition, that is for h above a certain h_c the walk localizes at 0 (it becomes positive recurrent), while below h_c the walk visits 0 no more than a finite number of times (transient behavior). The transition can be characterized in terms of the contact fraction, that is the number of returns to 0 in a long stretch of time, divided by the length of the stretch: zero contact density is delocalization and positive contact density is localization. In this case the critical behavior is the way the contact fraction approaches 0 as $h \searrow h_c$ (when it does, because we will see that in some cases it jumps to zero: this is the case of the so-called "first order transitions").

The pinning model we have informally introduced is homogeneous: the reward (or penalty) h is constant ($h > 0$ is a reward, $h < 0$ is a penalty). In the disordered version h is not constant, in fact h is replaced by $h + \beta\omega_n$ with $\beta > 0$ and $\omega := \{\omega_n\}_{n\in\mathbb{N}}$ is a realization of a sequence of random variables (for example, independent and identically distributed: in these notes we will only consider this case, and we will consider ω_1 centered and of variance one). We insist on the fact that we choose a typical realization of ω and we keep it fixed: a walk touching 0 at time n will receive a reward (or penalty) $h + \beta\omega_n$. And here is the main question of these notes: how different are the $\beta = 0$ and the $\beta > 0$ case? We will see that for pinning models the issue is not so much the one of persistence of the phase transition, because the answer is going to be positive in all cases, but the one of whether the disorder has an effect on the critical behavior or not.

Here is an overview of what will follow:

- In Chap. 2 we introduce and solve a general class of homogeneous pinning models. The emphasis is on the renewal process viewpoint because we are mostly interested in (in these notes!) when the process comes back to 0 and not so much on what it does outside of 0. And the return times to 0 form a random sequence that is a renewal sequence. In this chapter the reader will find also a quick review of the modeling aspects: this is definitely not doing justice to the importance of pinning phenomena and pinning modeling, but this is really not the purpose of the notes, so we just refer to [6] and to the monographs [8, 11].
- In Chap. 3 we introduce the disordered pinning models and a number of basic techniques. Notably we introduce the notion of disordered (quenched) free energy.
- In Chap. 4 there is the first contact with the Harris criterion and with the seminal contributions [4, 7], of which we will present the basic ideas. We then give a mathematical proof of disorder irrelevance in agreement with the Harris criterion prediction.
- In Chap. 5 we show that disorder is relevant when predicted by the Harris criterion. We do this by establishing a bound on the quenched free energy that shows that it always possesses a certain minimal amount of smoothness: it is at

least C^1 with Lipschitz derivative. We then present an overview of what is known or is expected to hold for the Ising model.

- In Chap. 6 we study the shift of the critical point. This includes the analysis of the case of marginal disorder, i.e. neither relevant nor irrelevant, a debated issue in the physical literature, solved in [9].
- In Chap. 7 we prove the coarse graining estimates used in Chap. 6.
- In Chap. 8 we talk about path properties and show that they are tightly linked to the free energy.
- The appendix is about discrete renewal processes and it is split into two parts: in the first part we review some basic tools and results of the theory and the second part is about some issues that are more specific to pinning models.

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