Chapter 2
Homogeneous Pinning Systems: A Class of Exactly Solved Models

Abstract We introduce a class of statistical mechanics non-disordered models – the homogeneous pinning models – starting with the particular case of random walk pinning. We solve the model in the sense that we compute the precise asymptotic behavior of the partition function of the model. In particular, we obtain a formula for the free energy and show that the model exhibits a phase transition, in fact a localization/delocalization transition. We focus in particular on the critical behavior, that is on the behavior of the system close to the phase transition. The approach is then generalized to a general class of Markov chain pinning, which is more naturally introduced in terms of (discrete) renewal processes. We complete the chapter by introducing the crucial notion of correlation length and by giving an overview of the applications of pinning models. Ising models are presented at this stage because pinning systems appear naturally as limits of two dimensional Ising models with suitably chosen interaction potentials. In spite of the fact that these lecture notes may be read focusing exclusively on pinning, the physical literature on disordered systems and Ising models cannot be easily disentangled. So a full appreciation of some physical arguments/discussions in these notes does require being acquainted with Ising models.

2.1 What Happens if We Reward a Random Walk When it Touches the Origin?

2.1.1 The Random Walk Pinning Model

We start rather abruptly by making more precise the question in the title and by answering it. So let us give ourselves a random walk $S = \{S_0, S_1, \ldots\}$ with $S_0 = 0$ and such that the increment variables $\{S_n - S_{n-1}\}_{n \in \mathbb{N}}$, that form an IID sequence, take values -1, 0 and $+1$. More precisely we consider a symmetric walk and set

G. Giacomin, *Disorder and Critical Phenomena Through Basic Probability Models*, Lecture Notes in Mathematics 2025, DOI 10.1007/978-3-642-21156-0_2,

$\mathbf{P}(S_1 = +1) = \mathbf{P}(S_1 = -1) =: p/2$ and $\mathbf{P}(S_1 = 0) = q$. Of course $p+q = 1$: we exclude the trivial case $q = 1$ and the simple random walk $q = 0$ for its somewhat unpleasant periodic character. For every $N \in \mathbb{N}$ we introduce the *local time* $L_N(S) = \sum_{n=1}^{N} \mathbf{1}_{S_n=0}$ and the probability measure $\mathbf{P}_{N,h}$ ($h \in \mathbb{R}$) such that

$$\frac{\mathrm{d}\mathbf{P}_{N,h}}{\mathrm{d}\mathbf{P}}(S) = \frac{1}{Z_{N,h}} \exp\left(hL_N(S)\right) \mathbf{1}_{S_N=0}, \tag{2.1}$$

where $Z_{N,h}$ is typically called *partition function* and it is just the normalization that makes $\mathbf{P}_{N,h}$ a probability. Of course

$$Z_{N,h} = \mathbf{E}\left[\exp\left(hL_N(S)\right); S_N = 0\right]. \tag{2.2}$$

A word about an abuse of notation that, in different forms, will be ubiquitous in these notes: in (2.1) S is a trajectory of the random walk, rather than the sequence of random variables. Note moreover that we have introduced $\mathbf{P}_{N,h}$ as a measure on the full trajectory and not just for the part of the trajectory that we have really modified. This has plenty of almost irrelevant advantages that, added up, largely overcome (in the eyes of the author, of course) the disadvantage of a rather abstract formulation in terms of the relative density of measures. Note in fact that for every $s_1, s_2, \ldots, s_N$

$$\begin{aligned}\mathbf{P}_{N,h}&\left(S_1 = s_1, S_2 = s_2, \ldots, S_N = s_N\right) \\ &= \frac{\mathbf{1}_{s_N=0}}{Z_{N,h}} \exp\left(h\sum_{n=1}^{N} \mathbf{1}_{s_n=0}\right) \mathbf{P}\left(S_1 = s_1, S_2 = s_2, \ldots, S_N = s_N\right), \end{aligned} \tag{2.3}$$

and we could have used the right-hand side of this expression to define the process, at the expense of having a family of processes living on different spaces and of a less compact notation. A last observation on notation is that $\mathbf{1}_{S_N=0}$ is used in place of the more precise, but less *expressive*, $\mathbf{1}_{\{0\}}(S_N)$.

Remark 2.1. Why constraining to $S_N = 0$? $S_N = 0$ is a boundary condition and a priori it is more natural to introduce the *free*, or *unconstrained*, model

$$\frac{\mathrm{d}\mathbf{P}^{\mathrm{f}}_{N,h}}{\mathrm{d}\mathbf{P}}(S) = \frac{1}{Z^{\mathrm{f}}_{N,h}} \exp\left(hL_N(S)\right), \tag{2.4}$$

but the *constrained* model often turns out to be more manageable. We anticipate that, even if for most of the main results there will be little or no difference between the two models, the interplay between them plays a role in several proofs.

2.1.2 *Visits to the Origin and the Computation of the Partition Function*

The epochs $\tau = \{\tau_0, \tau_1, \dots\}$ of successive visits to the origin

$$\tau_0 := 0 \ \text{ and } \ \tau_n \overset{n\in\mathbb{N}}{=} \{j > \tau_{n-1} : S_j = 0\}, \tag{2.5}$$

is a natural random walk (another one! With positive increments this time) associated to the problem, in the sense that $\{\tau_n - \tau_{n-1}\}_{n\in\mathbb{N}}$ is an IID sequence: this is a direct consequence of the strong Markov property and of the recurrent character of S that guarantees that $\mathbf{P}(\tau_j < \infty$ for every $j) = 1$. In a more customary terminology, τ is a *renewal process* with *inter-arrival law* $K(n) := \mathbf{P}(\tau_1 = n)$. Two basic facts on this inter-arrival law are

$$\sum_{n=1}^{\infty} K(n) = 1 \ \text{ and } \ \lim_{n\to\infty} n^{3/2} K(n) =: \mathsf{c}_K > 0, \tag{2.6}$$

where the first fact is just a restatement of $\mathbf{P}(\tau_1 < \infty) = 1$, but the second requires a bit more work (see e.g. [22, Appendix A.6] where the value of c_K is computed: $\sqrt{p/2\pi}$).

Remark 2.2. We will soon encounter other renewal processes (i.e. random walks with positive increments: Appendix A offers an introduction to these processes). So we introduce some (more or less) standard terminology: a renewal τ with inter-arrival law $K(\cdot)$ will be called $K(\cdot)$-renewal. If $\sum_n K(n) = 1$ then a.s. $|\{j : \tau_j < \infty\}| = \infty$ and the renewal is said *persistent*. It is *positive persistent* if also $\mathbf{E}[\tau_1] = \sum_n nK(n) < \infty$. If instead $\sum_n K(n) < 1$, $K(\cdot)$ can be extended to a probability distribution by setting $K(\infty) := 1 - \sum_{n\in\mathbb{N}} K(n)$ and each realization of τ, still defined as the sequence of partial sums of the IID sequence of variables with distribution $K(\cdot)$ on $\mathbb{N}\cup\{\infty\}$, contains only a finite number of finite numbers (points, epochs,....). In this case we say that the renewal is *terminating*: after a finite number of bounded jumps, the process jumps to infinity and stays there (in a sense, it leaves the space once for all). In general, it is very practical to look at τ as a subset of $\mathbb{N}$, rather than a sequence (in the terminating case we neglect the repeated ∞ and a typical realization of τ is therefore just a finite subset of $\mathbb{N}$, while in the persistent case it is an infinite subset). This convention leads to particularly compact notations: for example $n \in \tau$ means that there exists $j \in \mathbb{N}\cup\{0\}$ such that $\tau_j = n$. It is customary to call $n \mapsto \mathbf{P}(n \in \tau)$ *renewal function*.

By repeated use of the total probability formula we obtain

$$\begin{aligned} Z_{N,h} &= \sum_{n=1}^{N} \mathbf{E}\left[\exp\left(hL_N(S)\right); S_N = 0, L_N(S) = n\right] \\ &= \sum_{n=1}^{N} \exp(hn)\, \mathbf{P}\left(S_N = 0, L_N(S) = n\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} \exp(hn) \sum_{\ell \in \mathbb{N}^n : |\ell| = N} \mathbf{P}\left(\tau_1 = \ell_1, \tau_2 - \tau_1 = \ell_2, \ldots, \tau_n - \tau_{n-1} = \ell_n\right) \\
&= \sum_{n=1}^{N} \exp(hn) \sum_{\ell \in \mathbb{N}^n : |\ell| = N} \prod_{j=1}^{n} K(\ell_j),
\end{aligned} \tag{2.7}
$$

where $|\ell| = \sum_{i=1}^{n} \ell_i$. The net outcome is:

$$
Z_{N,h} = \sum_{n=1}^{N} \sum_{\ell \in \mathbb{N}^n : |\ell| = N} \prod_{j=1}^{n} \exp(h) K(\ell_j) . \tag{2.8}
$$

Of course $Z_{N,0} = \mathbf{P}(N \in \tau)$, that is the partition function is just the renewal function of τ (see Appendix A), and the right-hand side of (2.8), still for $h = 0$, is a more explicit version of such a function: the point is to apply this observation also when $h \neq 0$. The obstacle is of course that $\exp(h)K(\cdot)$ is no longer a probability distribution if $h \neq 0$: this is not really a serious problem if $h < 0$ since we have seen that it suffices to work on $\mathbb{N} \cup \{\infty\}$, but for $h > 0$ we have to do something different. The idea is to introduce the function $\mathrm{F} : \mathbb{R} \to [0, \infty)$ defined by

$$
\sum_{n \in \mathbb{N}} \exp(-n\mathrm{F}(h) + h) K(n) = 1 , \tag{2.9}
$$

when such a solution exists, that is for $h \geq 0$ (the solution is of course unique by the monotonicity of $x \mapsto \sum_n \exp(-xn)K(n)$). When we cannot solve such a problem, that is for $h < 0$, we set $\mathrm{F}(h) = 0$. Now, for every h we set for $n \in \mathbb{N}$

$$
\widetilde{K}_h(n) := \exp(-\mathrm{F}(h)n + h) K(n) , \tag{2.10}
$$

and, adding $\{\infty\}$ if needed, $\widetilde{K}_h(\cdot)$ is a probability distribution.

Remark 2.3. The function $\mathrm{F}(\cdot)$ is called *free energy* and it plays a central role in these notes. A number of properties of $\mathrm{F}(\cdot)$ can be obtained with moderate effort. First of all $\mathrm{F}(\cdot)$ is real analytic except at the origin. The analyticity on the positive semi-axis follows by the Implicit Function Theorem (e.g. [13, Chap. 3, Proposition 2.20]), since $z \mapsto \sum_n K(n) \exp(-zn)$ is analytic on $\{z \in \mathbb{C} : \Re(z) > 0\}$ and its derivative does not vanish on $(0, \infty)$. One verifies directly also that $F(\cdot)$ is convex and that it is increasing on the positive semi-axis: by taking a derivative of the expression in (2.9) and by using the notation $\widetilde{\tau}^{(h)}$ for the $\widetilde{K}_h(\cdot)$-renewal, we see that for $h > 0$

$$
\mathrm{F}'(h) = \frac{1}{\sum_n n \widetilde{K}_h(n)} = \frac{1}{\mathbf{E} \widetilde{\tau}_1^{(h)}} > 0 , \tag{2.11}
$$

and that $\mathrm{F}''(h) = \mathrm{F}'(h)^3 \mathrm{var}(\widetilde{\tau}_1^{(h)}) > 0$.

We can therefore write

$$Z_{N,h} = \exp(\mathrm{F}(h)N) \sum_{n=1}^{N} \sum_{\ell \in \mathbb{N}^n : |\ell| = N} \prod_{j=1}^{n} \widetilde{K}_h(\ell_j), \tag{2.12}$$

a formula that can be made much more compact by using the $\widetilde{K}_h(\cdot)$-renewal $\widetilde{\tau}^{(h)}$:

$$Z_{N,h} = \exp(\mathrm{F}(h)N)\, \mathbf{P}\left(N \in \widetilde{\tau}^{(h)}\right), \tag{2.13}$$

and from such a formula one extracts.

Proposition 2.4. *For the partition functions $Z_{N,h}$ and $Z^{\mathtt{f}}_{N,h}$ defined in* (2.2) *and* (2.4) *we have*

$$\lim_{N\to\infty} \frac{1}{N} \log Z_{N,h} = \lim_{N\to\infty} \frac{1}{N} \log Z^{\mathtt{f}}_{N,h} = \mathrm{F}(h). \tag{2.14}$$

Moreover

$$Z_{N,h} \overset{N\to\infty}{\sim} c_{h,K(\cdot)} \exp(\mathrm{F}(h)N) \times \begin{cases} N^0 & \text{if } h > 0, \\ N^{-1/2} & \text{if } h = 0, \\ N^{-3/2} & \text{if } h < 0, \end{cases} \tag{2.15}$$

with

$$c_{h,K(\cdot)} := \begin{cases} \frac{1}{\sum_n n \widetilde{K}_h(n)} & \text{if } h > 0, \\ \frac{1}{\sqrt{2\pi p}} & \text{if } h = 0, \\ \frac{\mathtt{c}_K \exp(h)}{(1-\exp(h))^2} & \text{if } h < 0. \end{cases} \tag{2.16}$$

Proof. The proof of (2.14) in the constrained case is just a matter of showing that $\log \mathbf{P}(N \in \widetilde{\tau}^{(h)}) = o(N)$. But this is obvious since for $h < 0$ we have $\exp(h)K(n) \le \mathbf{P}(N \in \widetilde{\tau}^{(h)}) \le 1$ and if $h > 0$ (see Fig. 2.1) by the Renewal Theorem (Theorem A.1)

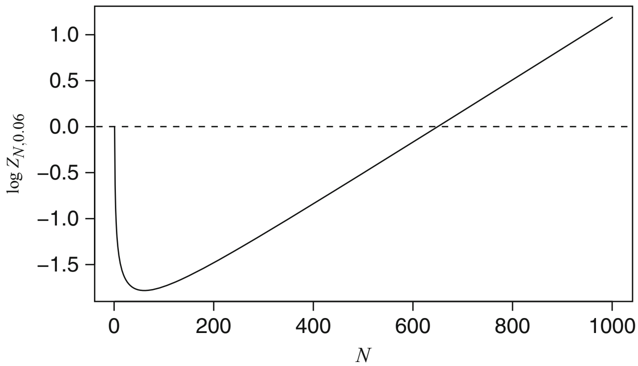

Fig. 2.1 The plot of the logarithm of the partition function for a homogeneous random walk based model with $p = 1/2$ and $q = 1/2$. We have $\mathrm{F}(0.06) \approx 3.4 \times 10^{-3}$

$\mathbf{P}(N \in \widetilde{\tau}^{(h)})$ tends, as $N \to \infty$, to the positive constant $1/\mathbf{E}\widetilde{\tau}_1^{(h)}$: note that this establishes (2.15) for $h > 0$. The sharp estimate (2.15) for $h = 0$ is just the Local Central Limit Theorem for S (which can be established via Stirling's approximation of the factorial), while the case $h < 0$ requires a more delicate analysis, that is however a rather standard result in renewal theory that can be summed up by saying that the leading asymptotic behavior of the renewal function of a terminating renewal differs from the leading asymptotic behavior of the inter-arrival distribution only by a multiplicative constant (see Theorem A.2).

We are left with proving (2.14) in the free case, but this is a direct consequence of the constrained result and of the formula

$$Z^{\mathtt{f}}_{N,h} = \sum_{n=0}^{N} Z^{\mathtt{f}}_{N,h}(\tau \cap [n,N] = \{n\}) = \sum_{n=0}^{N} Z_{n,h}\overline{K}(N-n), \tag{2.17}$$

where we have introduced the notation $Z^{\mathtt{f}}_{N,h}(A)$ (A an event) for $\mathbf{E}[\exp(hL_N(S));A]$ and $\overline{K}(n) = \sum_{j>n} K(j)$ ($n = 0, 1, \ldots$). From (2.17), and (2.15), one can also establish without much effort the analog of (2.15) for $Z^{\mathtt{f}}_{N,h}$, but this is left to the motivated readers. □

2.1.3 *From Partition Function Estimates to Properties of the System*

Proposition 2.4 contains very detailed information on the system: let us spell it out. First of all we have seen in Remark 2.3 that $\mathrm{F}(\cdot)$ is real analytic except at the origin: convexity assures that at least for $h \neq 0$

$$\mathrm{F}'(h) = \lim_{N\to\infty} \frac{1}{N}\frac{\mathrm{d}}{\mathrm{d}h}\log Z_{N,h} = \lim_{N\to\infty}\frac{1}{N}\mathbf{E}_{N,h}\left[L_N(S)\right]. \tag{2.18}$$

Monotonicity and convexity properties can also be inferred directly form the fact that $h \mapsto \log Z_{N,h}$ is increasing and convex (just take derivatives). Here we are interested in the fact that formula (2.18) is already showing that passing from $h < 0$ to $h > 0$ something very drastic is happening in the system: $\mathrm{F}'(h)$ is actually the density of visits to the origin by the random walk path (the *contact density*) and it passes from zero to a positive value (see Fig. 2.3, upper-right inset). This is clearly a transition from what we may call a delocalized to a localized behavior. The transition actually happens in a continuous way – there is no jump in the contact density when h changes sign – and $\mathrm{F}(\cdot)$ is C^1 in zero, even if it is not C^2: this requires an argument that we develop now. By Riemann sum approximation we see that

$$1-\sum_n K(n)\exp(-xn) = \sum_n K(n)(1-\exp(-xn))$$
$$\stackrel{x\searrow 0}{\sim} \mathtt{c}_K x^{1/2}\int_0^\infty \frac{1-\exp(-t)}{t^{3/2}}\mathrm{d}t = 2\sqrt{\pi}\mathtt{c}_K x^{1/2}, \quad (2.19)$$

and since we already know that $\lim_{h\searrow 0}\mathrm{F}(h)=0$ we can apply this formula to (2.9) obtaining $2\sqrt{\pi}\mathtt{c}_K\mathrm{F}(h)^{1/2}\sim h$, that is

$$\mathrm{F}(h) \stackrel{h\searrow 0}{\sim} \frac{1}{4\pi\mathtt{c}_K^2}h^2. \quad (2.20)$$

Such an estimate is directly telling us that $\mathrm{F}(\cdot)$ is not C^2 at the origin and, together with convexity, is telling us also that $\mathrm{F}(\cdot)$ is C^1. In a standard statistical mechanics terminology this means that the system undergoes a *second order phase transition*, in the sense that the non-analiticity of the free energy comes from a singularity (in this case a jump discontinuity) in the second derivative of the free energy.

This description of the system in terms of contact density is only partially satisfactory, for example because we already know that the unperturbed random walk S ($h=0$) has zero contact density, but we know much more, namely that the number of contacts in a stretch N is of order $\sqrt{N}$ (a much sharper information). Can we get such a precise estimate also for the $h\neq 0$ case? The answer is yes and it is summed up in the next statement.

Proposition 2.5. *If $h<0$ then for every $n\in\mathbb{N}$*

$$\lim_{N\to\infty}\mathbf{P}_{N,h}(L_N(S)=n) = (1-\exp(h))^2\, n\exp(h(n-1)), \quad (2.21)$$

(note that the right-hand side is the discrete density of $X+Y+1$, with X and Y independent geometric variables of parameter $\exp(h)$*) while if $h>0$ we have that for every $\varepsilon>0$*

$$\lim_{N\to\infty}\mathbf{P}_{N,h}\left(\left|\frac{L_N(S)}{N}-\mathrm{F}'(h)\right|\geq\varepsilon\right) = 0. \quad (2.22)$$

Moreover for every h, every n, every $t_\star\in\mathbb{N}$ and every $t\in\mathbb{N}^n$ such that $0<t_1<t_2<\ldots<t_n\leq t_\star$ we have

$$\lim_{N\to\infty}\mathbf{P}_{N,h}\left(\tau\cap(0,t_\star]=\{t_1,\ldots,t_n\}\right) = \mathbf{P}\left(\widetilde{\tau}^{(h)}\cap(0,t_\star]=\{t_1,\ldots,t_n\}\right), \quad (2.23)$$

that is the sequence $\{\mathbf{P}_{N,h}\tau^{-1}\}_N$ of measures on $\mathscr{P}_0$ (cf. Sect. A.1.4 of Appendix A) converges weakly to $\mathbf{P}(\widetilde{\tau}^{(h)})^{-1}$.

Note that this statement includes global estimates, that is (2.21) and (2.22), and a local one, that is (2.23). Note that (2.22) holds also for $h\leq 0$, but of course for $h\leq 0$ one has much sharper estimates (like (2.21) for $h<0$!). These estimates are just instances of what one can obtain once the sharp asymptotic behavior of the partition function is known (see Fig. 2.2).

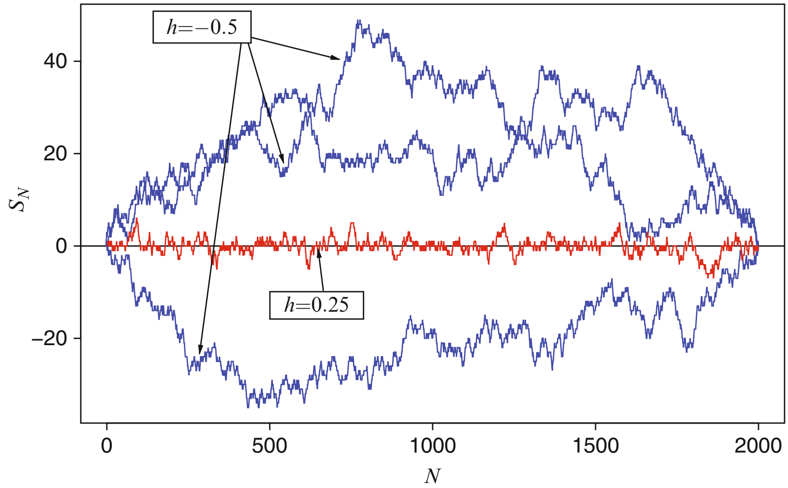

Fig. 2.2 Three trajectories for $h = -0.5$ and $N = 2{,}000$ (the underlying walk has $p = q = 1/2$) and one for $h = 0.25$. While the image clearly suggests that the first three trajectories are *delocalized*, i.e. they keep away from 0, and have a Brownian scaling, the maximum of the other trajectory is $+6$ and the minimum is -7, so the path is essentially localized at 0

Proof. The result follows from Proposition 2.4 by *routine* arguments. We go quickly through them, leaving some of the details to the reader. Let us point out that, in a sense, Proposition 2.4 is a detour and everything in the end boils down to renewal function estimates, but developing in arguments using the partition function has several advantages, like making connection with more general cases.

For what concerns (2.21) the result for $n = 1$ is immediate, since the probability we want to compute is $\exp(h)K(N)/Z_{N,h} \sim (1-p)^2$. For $n = 2$ instead one observes that for any choice of a sequence of natural numbers $\{a(N)\}_N$ such that $1 \ll a(N) \ll N$ we have

$$
\begin{aligned}
\mathbf{P}_{N,h}(L_N(S) = 2) &= \frac{\sum_{n=1}^{N} K(n)K(N-n)\exp(2h)}{Z_{N,h}} \\
&= \frac{\sum_{n=1}^{a(N)-1} \cdots + \sum_{n=a(N)}^{N-a(N)-1} \cdots + \sum_{n=N-a(N)}^{N} \cdots}{Z_{N,h}} \\
&= 2\exp(h)(1-\exp(h))^2(1+o(1)) \qquad (2.24) \\
&\quad + c_{h,K(\cdot)}^{-1} e^{2h}(1+o(1))N^{3/2} \sum_{n=a(N)}^{N-a(N)-1} n^{-3/2}(N-n)^{-3/2} \\
&= 2\exp(h)(1-\exp(h))^2(1+o(1)) + O(a(N)^{-1/2}),
\end{aligned}
$$

which is the result we were looking for. The general case is just a straightforward generalization which is best done by first observing that the configurations that contain points far from both 0 and N are negligible:

$$\mathbf{P}_{N,h}\left(\tau\cap[L,N-L]\neq\emptyset\right)\leq\frac{\sum_{n=L}^{N-L}Z_{n,h}Z_{N-n,h}}{Z_{N,h}}$$
$$\leq c_1\frac{\sum_{n=L}^{N-L}K(n)K(N-n)}{K(N)}\leq c_2\overline{K}(L)\,, \quad (2.25)$$

where c_1 and c_2 are suitable (h and $K(\cdot)$ dependent) positive constants. In words, the result is simply saying that in the limit the process comes back a finite number of times close to 0 (each time it attempts to come back it has a probability $1-\exp(h)$ of not making it) and the behavior near N is just mirror symmetric (in law).

A proof of (2.22) is an immediate consequence of the fact that for $h>0$ we have $\textsc{f}''(h)=\lim_N N^{-1}\mathrm{var}_{\mathbf{P}_{N,h}}(L_N(S))$, but this requires some work. So we take the cheaper path of observing that $\mathbf{P}_{N,h}$ actually coincides with the law of the $\widetilde{K}_h(\cdot)$-renewal conditioned to visit N: for every n and every $s\in\mathbb{N}^n$ such that $0<s_1<\ldots<s_n=N$ we have

$$\begin{aligned}\mathbf{P}_{N,h}\left(\tau\cap(0,N]=\{s_1,\ldots,s_n\}\right)&=\frac{e^{nh}K(s_1)K(s_2-s_1)\ldots K(N-s_{n-1})}{e^{N\textsc{f}(h)}\,\mathbf{P}\left(N\in\widetilde{\tau}^{(h)}\right)}\\&=\frac{\widetilde{K}_h(s_1)\widetilde{K}_h(s_2-s_1)\ldots\widetilde{K}_h(N-s_{n-1})}{\mathbf{P}\left(N\in\widetilde{\tau}^{(h)}\right)}\\&=\mathbf{P}\left(\widetilde{\tau}^{(h)}\cap(0,N]=\{s_1,\ldots,s_n\}\,\middle|\,N\in\widetilde{\tau}^{(h)}\right).\end{aligned}\quad (2.26)$$

But since, by the law of large numbers, $\widetilde{\tau}_j^{(h)}/j$ tends as $j\to\infty$ to $\mathbf{E}[\widetilde{\tau}_1^{(h)}]$ almost surely, one directly obtains that $N^{-1}|\widetilde{\tau}^{(h)}\cap(0,N]|\longrightarrow1/\mathbf{E}[\widetilde{\tau}_1^{(h)}]$ almost surely. Since the event $N\in\widetilde{\tau}^{(h)}$ has a probability bounded away from zero, for any sequence of events A_N such that $\mathbf{P}(A_N)\longrightarrow0$, we have also $\mathbf{P}(A_N|N\in\widetilde{\tau}^{(h)})\longrightarrow0$. Therefore, by using $A_N=\{|N^{-1}|\widetilde{\tau}^{(h)}\cap(0,N]|-1/\mathbf{E}[\widetilde{\tau}_1^{(h)}]|>\varepsilon\}$, we get (2.22).

For what concerns (2.23) consider first the case $t_n=t_\star$ and write much like for (2.26) (with $t_0:=0$ and assuming N larger than $t_\star$)

$$\mathbf{P}_{N,h}\left(\tau\cap(0,t_n]=\{t_1,t_2,\ldots,t_n\}\right)=$$
$$\exp(hn)\left(\prod_{j=1}^{n}K(t_j-t_{j-1})\right)\frac{Z_{N-t_n,h}}{Z_{N,h}}=\prod_{j=1}^{n}\widetilde{K}_h(t_j-t_{j-1})\frac{Z_{N-t_n,h}\exp(\textsc{f}(h)t_n)}{Z_{N,h}}$$
$$=\mathbf{P}\left(\widetilde{\tau}^{(h)}\cap(0,t_n]=\{t_1,t_2,\ldots,t_n\}\right)\left[\frac{\mathbf{P}\left(N-t_n\in\widetilde{\tau}^{(h)}\right)}{\mathbf{P}\left(N\in\widetilde{\tau}^{(h)}\right)}\right], \quad (2.27)$$

where we have applied the renewal property and (2.13). But the term between the square brackets tends to 1 as N tends to infinity (because of the Renewal Theorem if $h>0$, and because, if $h<0$, partition functions coincide with renewal functions to which (2.15) applies). The case $t_\star > t_n$ of (2.23) can be dealt with by decomposing the probability according to the values of the first contact site t larger than $t_\star$ and by applying (2.27), that is by writing

$$\mathbf{P}_{N,h}\left(\tau \cap (0,t_\star] = \{t_1,\ldots,t_n\}\right) = \sum_{t=t_\star+1}^{N} \mathbf{P}_{N,h}\left(\tau \cap (0,t] = \{t_1,\ldots,t_n,t\}\right). \quad (2.28)$$

Actually at this stage one can for example use the argument used in (2.25) to restrict the summation only to values of t that are either close to $t_\star$ or close to N and then apply (2.15) and (2.27). By performing the summation we recover (2.23). □

2.2 The General Homogeneous Pinning Model

The asymptotic arguments that we have developed up to here essentially rely only on the fact that the tail distribution of the first return to the origin of the random walk S has a power law decay with exponent $3/2$. The first generalization that comes to mind is, possibly, considering higher dimensional random walks. These cases can be treated precisely along the same line, in fact one can show (see e.g. [22, Appendix A.6]) that if the increment of the random walk is a ($\mathbb{Z}^d$-valued) centered random variable with finite variance σ^2 (and $\mathbf{P}(S_1=0)\in(0,1)$ to avoid periodicity and triviality) then

$$K(n) \overset{n\to\infty}{\sim} c_d(\sigma^2) \times \begin{cases} 1/\left(n(\log n)^2\right) & \text{if } d=2 \\ 1/n^{1+|(d/2)-1|} & \text{if } d=1,3,4\ldots \end{cases} \quad (2.29)$$

with $c_d(\sigma^2)>0$. Another important fact is that $\sum_n K(n)=1$ if $d=1$ and 2, but $\sum_n K(n)<1$ for $d=3,4,\ldots$ But since our model in the end depends only on the inter-arrival law it is very natural to look at the renewal process τ as the basic underlying process (the *free process*) and put conditions on it: much of the literature has been in fact developed for $K(\cdot)$ *regularly varying* and it is possibly also natural to look at the case is which $\overline{K}(\cdot)$ ($\overline{K}(n)=\sum_{j>n}K(j)$, $n=0,1,\ldots$) is regularly varying. In order to make our arguments lighter we will consider a subclass of regularly varying inter-arrival laws supported on $\mathbb{N}$, that is we will assume that there exists $\alpha>0$ such that

$$K(n) \overset{n\to\infty}{\sim} \frac{\mathsf{c}_K}{n^{1+\alpha}} \quad \text{and} \quad K(n)>0 \ \text{ for } n\in\mathbb{N}. \quad (2.30)$$

The positivity condition can be relaxed at the expense of a series of tedious remarks that we spare to the reader (of course $K(n)\sim \mathsf{c}_K/n^{1+\alpha}$ implies $K(n)>0$ for n

sufficiently large). The choice of restricting to *trivial* regularly varying behavior (pure power law) is instead more substantial, above all because it excludes from our analysis the $d = 2$ case (2.29) and, more generally, the $\alpha = 0$ case, in which an interesting phenomenon does happen. But the gain in simplicity of exposition is considerable.

Note that we have not assumed $\sum_n K(n) = 1$: in general we set (again) $K(\infty) := 1 - \sum_{n=1}^{\infty} K(n)$ and we stress that $\sum_{n=1}^{\infty} \ldots$ means $\sum_{n\in\mathbb{N}} \ldots$. Let us write down explicitly the model:

$$\frac{\mathrm{d}\mathbf{P}_{N,h}}{\mathrm{d}\mathbf{P}}(\tau) = \frac{1}{Z_{N,h}} \exp\left(h\,|\tau\cap(0,N]|\right) \mathbf{1}_{N\in\tau}. \tag{2.31}$$

As we have already stressed, τ can be terminating or persistent and the following remark, that is going to be repeated in the most general context later, turns out to be quite helpful.

Remark 2.6. If τ is terminating, then the model is equivalent on events that depend on $\tau\cap(0,N]$ to the model based on the persistent $\widetilde{\tau}$ renewal with inter-arrival law $n \mapsto \widetilde{K}(n) := K(n)/(1-K(\infty))$ and h replaced by $h + \log(1-K(\infty))$. This can be easily verified by writing explicitly the probability of the event $\tau\cap(0,N] = \{t_1, t_2, \ldots, t_n\}$, $n \in \{1,\ldots,N\}$ and $0 < t_1 < t_2 < \ldots < t_n = N$. In particular, the two partition functions coincide (we are talking of $Z_{N,h}$ not of $Z^{\mathtt{f}}_{N,h}$!). This allows us to restrict in most of the cases our attention to the case in which the underlying renewal is persistent.

The generalization of Proposition 2.4 is in a sense straightforward, but it does present some novelties both from the viewpoint of mathematical tools (in fact: renewal theory estimates) and for the novel behaviors arising (see Fig 2.3).

Theorem 2.7. *For the partition function $Z_{N,h} = \mathbf{E}[\exp(h\,|\tau\cap(0,N]|)\,; N\in\tau]$ and the companion free partition function $Z^{\mathtt{f}}_{N,h} = \mathbf{E}[\exp(h\,|\tau\cap(0,N]|)]$, both based on the $K(\cdot)$-renewal, with $K(\cdot)$ as in (2.30), we have that*

$$\lim_{N\to\infty} \frac{1}{N} \log Z_{N,h} = \lim_{N\to\infty} \frac{1}{N} \log Z^{\mathtt{f}}_{N,h} = \mathrm{F}(h), \tag{2.32}$$

where $\mathrm{F}(h)$ – the free energy – is the unique solution of (2.9) if such a solution exists (that is if $h \geq h_c := -\log(1-K(\infty))$) and $\mathrm{F}(h) := 0$ otherwise. Moreover

$$Z_{N,h} \overset{N\to\infty}{\sim} c_{h,K(\cdot)} \exp(\mathrm{F}(h)N) \times \begin{cases} N^0 & \text{if } h > h_c\,, \\ N^{\min(\alpha-1,0)} & \text{if } h = h_c \text{ and } \alpha \neq 1\,, \\ 1/\log N & \text{if } h = h_c \text{ and } \alpha = 1\,, \\ N^{-(1+\alpha)} & \text{if } h < h_c\,, \end{cases} \tag{2.33}$$

with the explicit value of the constant $c_{h,K(\cdot)} > 0$ given below.

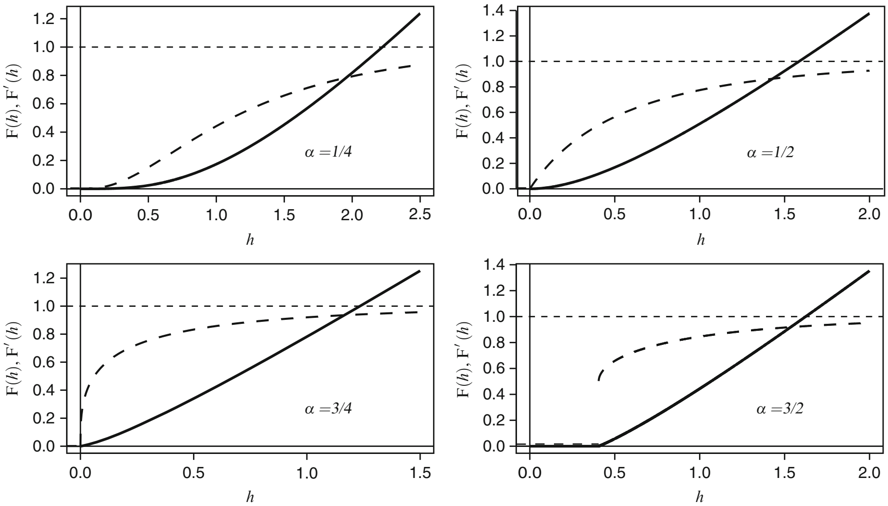

Fig. 2.3 Free energy ($\mathrm{F}(\cdot)$, *solid line*) and contact fraction ($\mathrm{F}'(\cdot)$, *dashed line*) for four values of α. The particular models we have chosen have $K(n) = \alpha\Gamma(n-\alpha)/(\Gamma(1-\alpha)n!) \overset{n\to\infty}{\sim} (\alpha/\Gamma(1-\alpha))n^{-1-\alpha}$ for $\alpha \in (0,1)$ and $K(n) = \Gamma(n-1/2)/(\sqrt{\pi}(n+1)!) \overset{n\to\infty}{\sim} (1/\sqrt{\pi})n^{-5/2}$ for the $\alpha = 3/2$ case. Such peculiar choices of $K(\cdot)$ are made because $\sum_n K(n)\exp(-\mathrm{F}n)$ can be made explicit by using By using the identity $\sum_{n=0}^{\infty}\Gamma(\beta+n)x^n/n! = \Gamma(\beta)(1-x)^{-\beta}$, that holds for $\beta \in \mathbb{R}\setminus\{0,-1,-2,\dots\}$: for example when $\alpha \in (0,1)$ we have $\sum_n K(n)\exp(-\mathrm{F}n) = 1-(1-\exp(-\mathrm{F}))^{\alpha}$. In the first three cases $\sum_n K(n) = 1$ so that $h_c = 0$, but for $\alpha = 3/2$ we have $\sum_n K(n) = 2/3$ (nothing sacred about $2/3$, it is just an arbitrary choice!), so that $h_c = \log(3/2) = 0.405\dots$. For the $\alpha = 1/2$ we have $K(n) = \mathbf{P}(\tau_1 = 2n)$, where τ the renewal set associated to the one dimensional symmetric simple random walk

Proof. All is of course in (2.13) [recall also (2.10)] and all we need are (sharp) renewal function estimates. These estimates are discussed at length in Appendix A, here we just recall the main results that can be summed up to: if $\widetilde{K}(\cdot)$ is an inter-arrival distribution (with $\widetilde{K}(\infty) := 1-\sum_n \widetilde{K}(n) \in (0,1)$) and $\widetilde{\tau}$ the corresponding renewal

1. If $\widetilde{K}(\infty) = 0$ and $\sum_n n\widetilde{K}(n) < \infty$ we have $\lim_{N\to\infty}\mathbf{P}(N \in \widetilde{\tau}) = 1/\sum_n n\widetilde{K}(n)$ (this is just the Renewal Theorem).
2. If $\widetilde{K}(\infty) = 0$ and $\widetilde{K}(n) \sim cn^{-1-\alpha}$ (with $c > 0$ and $\alpha \in (0,1)$) we have

$$\mathbf{P}(n \in \widetilde{\tau}) \overset{n\to\infty}{\sim} \frac{\alpha\sin(\pi\alpha)}{c\pi}n^{\alpha-1}. \tag{2.34}$$

3. If $\widetilde{K}(\infty) = 0$ and $\widetilde{K}(n) \sim c/n^2$ ($c > 0$) then

$$\mathbf{P}(n \in \widetilde{\tau}) \overset{n\to\infty}{\sim} \frac{1}{c\log n}. \tag{2.35}$$

4. If $\widetilde{K}(\infty) > 0$ and $\widetilde{K}(n) \sim cn^{-1-\alpha}$ (with $c > 0$ and $\alpha > 0$) then

$$\mathbf{P}(n \in \widetilde{\tau}) \stackrel{n\to\infty}{\sim} \frac{\widetilde{K}(n)}{\widetilde{K}(\infty)^2}. \tag{2.36}$$

Very little of the sharpness of these estimates is needed to establish (2.32) (see the proof of Proposition 2.4). Extracting (2.33) is instead a rather tedious book-keeping exercise with $\widetilde{K}(\cdot) = \widetilde{K}_h(\cdot)$ [cf. (2.10)]: let us go through it so that we determine $c_{h,K(\cdot)}$.

If $h > h_c$ we can apply point (1) and $\sum_n n\widetilde{K}_h(n) = 1/\mathrm{F}'(h)$ (use for example that the derivative of $\sum_n \widetilde{K}_h(n)$ with respect to h is zero), so that $c_{h,K(\cdot)} = \mathrm{F}'(h)$.

If $h < h_c$ we have $\widetilde{K}_h(\infty) > 0$ and we can apply point (4). The net result is that $c_{h,K(\cdot)} = \exp(h)/(1-\exp(h))^2$.

When $h = h_c$ instead notice first of all that $\widetilde{K}_h(\infty) = 0$, so $\widetilde{\tau}_h$ is persistent (regardless of the persistence properties of the reference renewal τ!). We distinguish the three cases $\alpha > 1$, $\alpha = 1$ and $\alpha < 1$. If $\alpha > 1$ we apply (1) and $c_{h,K(\cdot)} = \sum_n K(n)/\sum_n nK(n)$ (notice that we have used our convention that $\sum_n \ldots$ does not include $n = \infty$, so that $\sum_n nK(n) < \infty$). If $\alpha = 1$ we apply (3) and $c_{h,K(\cdot)} = \sum_n K(n)/\mathtt{c}_K$. If $\alpha \in (0,1)$ then (2) yields $c_{h,K(\cdot)} = (\alpha \sin(\pi\alpha)\sum_n K(n))/(\mathtt{c}_K \pi)$. $\square$

Remark 2.8. Extracting from (2.33) the sharp asymptotic behavior of $Z^{\mathrm{f}}_{N,h}$ is an even more tedious exercise. The result is however definitely instructive and not void of interest, both for the sequel and for the intuition. We do not want to make the exposition too heavy and we refer to [22, Chap. 2], but we point out that the fact that the constrained partition function $Z_{N,h}$ is invariant under the transformation $(\tau, h) \mapsto (\widetilde{\tau}, h + \log\sum_n K(n))$ of Remark 2.6, does not imply that also $Z^{\mathrm{f}}_{N,h}$ is invariant (in fact this is false and, in some cases, even the large N behavior is different).

Extracting path properties from Theorem 2.7 is an exercise: result and proof are absolutely parallel to Proposition 2.5.

Proposition 2.9. *If $h < h_c = -\log\sum_n K(n)$ then for every $n \in \mathbb{N}$*

$$\lim_{N\to\infty} \mathbf{P}_{N,h}(L_N(S) = n) = (1-\exp(h-h_c))^2 n \exp((h-h_c)(n-1)), \tag{2.37}$$

while if $h > h_c$ we have that for every $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbf{P}_{N,h}\left(\left|\frac{|\tau\cap(0,N]|}{N} - \mathrm{F}'(h)\right| \geq \varepsilon\right) = 0. \tag{2.38}$$

Moreover for every h, every n, every $t_\star$ and every $t \in \mathbb{N}^n$ such that $0 < t_1 < t_2 < \ldots < t_n \leq t_\star$ we have

$$\lim_{N\to\infty} \mathbf{P}_{N,h}(\tau\cap(0,t_\star] = \{t_1,\ldots,t_n\}) = \mathbf{P}\left(\tau^{(h)}\cap(0,t_\star] = \{t_1,\ldots,t_n\}\right), \tag{2.39}$$

that is the sequence $\{\mathbf{P}_{N,h}\tau^{-1}\}_N$ *of measures on* $\mathscr{P}_0$ *(cf. Sect. A.1.4 of Appendix A) converges weakly to* $\mathbf{P}(\widetilde{\tau}^{(h)})^{-1}$.

2.3 Phase Transition and Critical Behavior

This section focuses of the behavior of $\mathrm{F}(h)$ close to h_c. In view of (2.32) and of Remark 2.6, we can develop the arguments in the persistent set-up, that is when $h_c = 0$. The crucial estimate, like in Sect. 2.1.3, is understanding the asymptotic behavior of $\sum_n K(n)\exp(nx)$ for $x \searrow 0$. So let us set:

$$\Psi(x) \overset{x>0}{:=} 1 - \sum_{n=1}^{\infty} K(n)\exp(-nx), \tag{2.40}$$

and let us compute:

$$\begin{aligned} 1 - \sum_{n=1}^{\infty} K(n)\exp(-nx) &= 1 - \sum_{n=1}^{\infty} \left(\overline{K}(n-1) - \overline{K}(n)\right)\exp(-nx) \\ &= (1-\exp(-x)) \sum_{n=0}^{\infty} \exp(-nx)\overline{K}(n). \end{aligned} \tag{2.41}$$

Therefore, when $\alpha > 1$, one directly sees that $\Psi(x) \sim x\mathbf{E}[\tau_1]$ as $x \searrow 0$. If instead $\alpha \in (0,1)$, by Riemann sum approximation, one obtains

$$\Psi(x) \overset{x\searrow 0}{\sim} x\sum_n \frac{\mathtt{c}_K \exp(-xn)}{\alpha n^\alpha} \sim \frac{x^\alpha \mathtt{c}_K}{\alpha}\int_0^\infty t^{-\alpha}\exp(-t)\mathrm{d}t \sim \mathtt{c}_K \frac{\Gamma(1-\alpha)}{\alpha} x^\alpha. \tag{2.42}$$

For $\alpha = 1$ we set $\ell(n) := \sum_{j=1}^n 1/j$ for $n \in \mathbb{N}$ and $\ell(0) := 0$ so that

$$\frac{\Psi(x)}{\mathtt{c}_K} \overset{x\searrow 0}{\sim} x\sum_{n=1}^{\infty} \frac{\exp(-xn)}{n} = x(1-\exp(-x))\sum_{n=1}^{\infty} \ell(n)\exp(-xn), \tag{2.43}$$

Now note that $\sum_{n=1}^\infty \ell(n)\exp(-xn) \sim \sum_n \log(n)\exp(-xn)$ and since we have that $\sum_n \log(xn)\exp(-xn)$ is $O(1/x)$, we see that $\sum_{n=1}^\infty \ell(n)\exp(-xn)$ is asymptotically equivalent to $\log(1/x)\sum_{n=1}^\infty \exp(-xn) \sim x^{-1}\log(1/x)$. Therefore if $\alpha = 1$

$$\Psi(x) \overset{x\searrow 0}{\sim} \mathtt{c}_K x \log(1/x). \tag{2.44}$$

By recalling that $\Psi(\mathrm{F}(h)) = 1 - \exp(h)$, by inverting the asymptotic relations we obtain the behavior of $\mathrm{F}(h)$ for $h \searrow 0$: we sum up the result in the following statement.

Theorem 2.10. *For $K(\cdot)$ as in (2.30), $\mathtt{F}(\cdot)$ as in Theorem 2.7 and h_c equal to $-\log\sum_n K(n)$, we have that*

$$\mathtt{F}(h) \stackrel{h\searrow h_c}{\sim} C(K(\cdot)) \begin{cases} h-h_c & \text{if } \alpha>1\,,\\ (h-h_c)/\log(1/(h-h_c)) & \text{if } \alpha=1\,,\\ (h-h_c)^{1/\alpha} & \text{if } \alpha\in(0,1)\,, \end{cases} \tag{2.45}$$

where

$$C(K(\cdot)) = \begin{cases} \sum_n K(n)/\sum_n nK(n) & \text{if } \alpha>1\,,\\ 1/\mathtt{c}_K & \text{if } \alpha=1\,,\\ ((\alpha\sum_n K(n))/(\mathtt{c}_K\Gamma(1-\alpha)))^{1/\alpha} & \text{if } \alpha\in(0,1)\,. \end{cases} \tag{2.46}$$

If $\mathtt{F}(\cdot)$ is C^k (of course the issue is at h_c), then $\mathtt{F}(h)=o((h-h_c)^k)$ for $h\searrow h_c$. Therefore Theorem 2.10 directly implies that, for $k=2,3,\ldots$, $\mathtt{F}(\cdot)$ is not C^k for $\alpha\geq 1/k$. Moreover, it is not C^1 for $\alpha>1$ (but of course it is C^0). By using the convexity of $\mathtt{F}(\cdot)$ one directly extracts also that, for $\alpha\leq 1$, $\mathtt{F}(\cdot)$ is C^1: since $\mathtt{F}'(\cdot)$ is non-decreasing (and well-defined except possible at h_c), a discontinuity at h_c of $\mathtt{F}'(\cdot)$ implies $\mathtt{F}(h)\geq c(h-h_c)$ for $h>h_c$, with $c=\lim_{h\searrow h_c}\mathtt{F}'(h)>0$, which contradicts the estimate in Theorem 2.10. In general one has instead to resort to a direct estimate (that can be found Appendix A, Theorem A.8). The net result is summed up in the next statement in which we use the standard terminology: a phase transition is a point of non-analiticity of the free energy, this point is called *critical*, and the phase transition is said *of kth order* ($k\in\mathbb{N}$) if the free energy is, at the critical point, C^{k-1}, but not C^k.

Proposition 2.11. *The homogeneous pinning model (2.31) has a phase transition of kth order, $k=2,3,\ldots$, at $h=h_c$ if $\alpha\in[1/k,1/(k-1))$. The transition is of second order also if $\alpha=1$, while it is of first order for $\alpha>1$.*

2.4 A First Look at a Crucial Notion: The Correlation Length

The notion of correlation length plays a central role in the study of statistical mechanics systems. In general, even for a given system there are plenty of reasonable definitions of correlation length. Let us see for the homogeneous pinning system: we have seen (Proposition 2.5) that the $N\to\infty$ model is the renewal with inter-arrival law $\widetilde{K}_h(\cdot)$. In this case the first notion of correlation length that comes to mind is given by looking at the correlation

$$\mathbf{E}\left[\widetilde{\delta}_m\widetilde{\delta}_{m+n}\right]-\mathbf{E}\left[\widetilde{\delta}_m\right]\mathbf{E}\left[\widetilde{\delta}_{m+n}\right] = \mathbf{E}\left[\widetilde{\delta}_m\right]\left(\mathbf{E}\left[\widetilde{\delta}_n\right]-\mathbf{E}\left[\widetilde{\delta}_{m+n}\right]\right), \tag{2.47}$$

where $\widetilde{\delta}_n = \mathbf{1}_{n \in \widetilde{\tau}^{(h)}}$. If $h \le h_c$ (delocalized regime) then, thanks to Theorem A.2 and to Theorem A.4, one directly sees that for every m the correlation decays, as $n \to \infty$, with a power law: since the correlation length is naturally defined as the reciprocal of the exponential decay rate of the correlations, we see that in this case the correlation length is ∞. If instead $h > h_c$ one can, for the sake of simplicity, take the limit $m \to \infty$, so that one is effectively talking about the covariance of the stationary renewal: by the Renewal Theorem, applied to (2.47), the correlation length this time is read off

$$\mathbf{E}\left[\widetilde{\delta}_n\right] - \frac{1}{\widetilde{\mu}(h)}, \quad \text{with } \widetilde{\mu}(h) := \mathbf{E}\widetilde{\tau}_1^{(h)}. \tag{2.48}$$

The correlation length is therefore given by the reciprocal of the rate of convergence of the renewal function to its asymptotic value. The renewal equation comes to our help in order to compute it, but things are not as easy as one might think at first. It is a standard result [28] that if the inter-arrival law decays exponentially (more precisely: in the case of a recurrent $\widetilde{K}(\cdot)$-renewal such that $\sup_n \exp(cn)\widetilde{K}(n) < \infty$ for some $c > 0$), then the renewal function converges to its limit exponentially fast. As it is well known since a long time (see for example [26]) however, the relation between the rate of decay of $\widetilde{K}(\cdot)$ and the one of the renewal function are in general rather *unrelated* (see [23] for examples and several references). But what is going to be important for our discussion is that, in the context we consider, one can establish a general result. Namely that

Proposition 2.12. *[23] Choose an inter-arrival law $K(\cdot)$ that satisfies* (2.30)*. Then there exists $h_0 \in (h_c, \infty]$ such that for $h \in (h_c, h_0)$ we have*

$$\mathbf{P}\left(n \in \widetilde{\tau}^{(h)}\right) - \frac{1}{\widetilde{\mu}(h)} \stackrel{n\to\infty}{\sim} c(h)K(n)\exp(-\mathrm{F}(h)n), \tag{2.49}$$

with $c(h)$ a positive (explicit) constant. So, in particular, we have

$$\lim_{n\to\infty} -\frac{1}{n}\log\left(\mathbf{P}\left(n \in \widetilde{\tau}^{(h)}\right) - \frac{1}{\widetilde{\mu}(h)}\right) = \mathrm{F}(h). \tag{2.50}$$

This result can be read as saying that the correlation length $\kappa = \kappa(h)$ is equal to $1/\mathrm{F}(h)$, at least when the system is close to criticality. The fact that (in general) we can link the correlation length to the free energy only close to criticality is not a problem because the correlation length becomes important precisely close to criticality, that is when it diverges.

The role of $\kappa(h)$ emerges clearly also from (2.13): the exponential growth of the partition function sets in when N is about $\kappa(h)$ (look at Fig. 2.1). Figure 2.1 definitely suggests another correlation length: $\widetilde{\kappa}(h) := \inf\{N : Z_{N,h} > 1\}$, where the value 1 is a bit arbitrary (at this stage), but it is a natural reference point. Why this is not such a bad definition will be clear later on: for the moment we register the fact that $\log \kappa(h) \sim \log \widetilde{\kappa}(h)$ as $h \searrow h_c$.

2.5 Why Do People Look at Pinning Models? A Modeling Intermezzo

The main purpose of these notes is to investigate the effect of disorder on statistical mechanics models, notably on phase transitions and critical phenomena. Pinning models turn out to be a particularly favorable context to attack this daunting issue.

But pinning models have received widespread attention, notably in physics, chemistry and biology. Let us have a quick look at this direction by considering three different instances: this will serve also to motivate the introduction of *disorder*.

2.5.1 Polymer Pinning by a Defect

Polymers are chains of repetitive units (monomers) that may or may not be identical. Polymer modeling is tightly related to random walks in the sense that the most basic model of a polymer is the random walk. Less simplistic models include a self-avoiding constraint and/or increment correlation. In addition, polymers are often in interaction with an environment: the presence in the environment of an attractive (or repulsive) region may have a substantial effect on the polymer trajectory. When such a region is a point or a line (but it could also be a plane or a hyperplane) then a natural basic model, in which we either disregard self-avoidance or we implement it by looking at the so-called *directed polymers*, is precisely the pinning model. For more on this, see [22, Chap. 1] and references therein.

2.5.2 Interfaces in Two Dimensions

There is very deep link between interfaces in two dimensional (discrete spin) models and random walks (e.g. [1]). It is sketched in Fig. 2.4, both in the case in which the arising walk is free and when there is a pinning effect. The figure is based on the Ising model that we introduce also because it comes up later on in these notes. An Ising model in the rectangular box $\Lambda = \prod_{i=1}^{d}(-L_i, L_i) \cap \mathbb{Z}^d$ ($L_i \in \mathbb{N}$) is a measure on $\Omega_\Lambda := \{-1,+1\}^{\Lambda \cup \partial\Lambda}$, where $\partial\Lambda$ is the external boundary of Λ, that is $\partial\Lambda = \{x \in \mathbb{Z}^d \setminus \Lambda : |x-y| = 1 \text{ for a } y \in \Lambda\}$. The measure is determined once we fix a value $\beta \geq 0$ (the *inverse temperature*) and an element $\eta \in \{-1,+1\}^{\partial\Lambda}$, the boundary condition, and then we say that the probability $\mu_{\Lambda,\eta}(\sigma)$ of observing the configuration $\sigma \in \Omega_\Lambda$ is

$$\mu_{\Lambda,\eta}(\sigma) = \begin{cases} \exp(-\beta H_\Lambda(\sigma))/Z_{\Lambda,\beta,\eta} & \text{if } \sigma(x) = \eta(x) \text{ for } x \in \partial\Lambda\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{2.51}$$

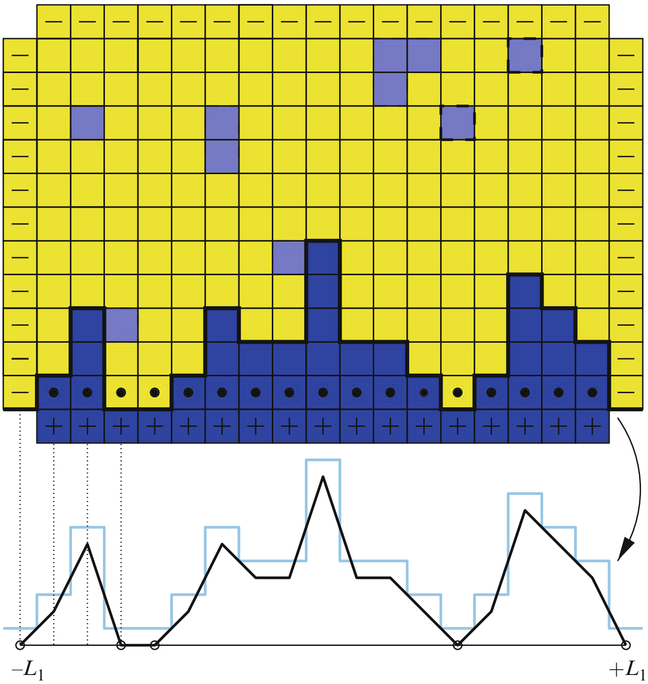

Fig. 2.4 We are drawing a configuration of the two-dimensional Ising model in the finite box with $L_1 = 9$, $L_2 = 6$ and boundary conditions that are $+1$ on $\{x : x_2 = -L_2, x_1 = -L_1 + 1, \ldots, L_1 - 1\}$ and -1 on the rest of $\partial^+\Lambda$. The up spins $(+)$ are identified by a *gray* or a *dark gray square*, while the down spins $(-)$ are *light gray*. The spins on which a boundary magnetic field acts are marked by *large black dots*. In the text, to which we refer for details, it is sketched the explanation of the reduction of the spin configuration to an *interface*: the interface is reproduced below the spin configuration with an equivalent but more natural representation if we look at it as a random walk path

where $Z_{\Lambda,\beta,\eta}$ is the normalization constant and $H_\Lambda(\sigma)$ is the energy of the model that we choose to be

$$H_\Lambda(\sigma) = -\frac{1}{2}\sum_{x,y} J(x,y)\sigma(x)\sigma(y) - \sum_x h(x)\sigma(x), \tag{2.52}$$

where the sums are over $\Lambda \cup \partial\Lambda$ and $J(x,y) = 0$ unless $|x-y| = 1$. Of course the most basic Ising model is the one in which $J(x,y) = 1$ for every $|x-y| = 1$ and $h(x)$ does not depend on x, but the general case here serves two purposes:

1. Later on we will discuss the generalization of what we develop for pinning models to more general cases and the disordered Ising model, that is the case in which $\{J(x,y)\}_{x,y}$ and/or $\{h(x)\}_x$ are realizations of families of IID random variables, is at the heart of the progress in the statistical mechanics of disordered

systems, much as the non-disordered Ising model is at the heart of the progress in statistical mechanics.

2. The *interface line*, or *phase separation line*, reduces to a random walk in the strongly anisotropic limit and a suitable choice of $h(\cdot)$ has the effect of a pinning potential: this is what we are going to explain next.

In (the upper part of) Fig. 2.4 we draw a configuration of the two dimensional Ising model in a finite box Λ: spins are drawn in small boxes that are either light gray, gray or dark gray. We are thinking of the case in which $J(x,y) = 0$ unless $|x-y| = 1$ and $J(x,y) = J_1 \geq 0$ (respectively $J(x,y) = J_2 \geq 0$) if $x - y = (\pm 1, 0)$ (respectively $x - y = (0, \pm 1)$). We also restrict our attention to $h(\cdot) \equiv 0$ for the moment, that is, we think of an Ising model without external (magnetic) field and with nearest neighbor interactions that can be different along the horizontal and vertical directions (the infinite volume limit of this model has been solved by Lars Onsager [4], a result that has deeply marked statistical mechanics).

As we are trying to convince the reader with the figure, such a spin configuration can be mapped to a set of contours (this is a very classical construction, see e.g. [18, Chap. 2]). All the contours are closed lines, except one that goes from the lower left corner to the lower right corner: we call such an open contour *interface* and we stress that the existence of an open contour is directly related to the boundary conditions (for example: all spins $+1$ on the boundary entails all contours are closed). Note that in the limit $J_1 \to \infty$ the configuration we have drawn has probability zero: a positive probability configuration is rather the one in which we switch to -1 all spins in the gray squares and in this case only one contour *survives*: the interface. More is true: in this limit the interface is a trajectory of a random walk with increments in $\mathbb{Z}$, starting in the lower left corner and ending on the lower right corner. As a matter of fact, it is an easy exercise to show that the law of such an Ising interface is just the law of the walk we have just mentioned (to be precise, the probability that the increment is equal to n is *const.*$\exp(-\beta J_2 |n|)$) conditioned not to exit the box Λ. If now we consider a very tall box ($L_2 \to \infty$) we are just dealing with a random walk bridge constrained not to go below the height of its starting (and arrival) point. The lower part of the figure draws the interface with a slightly different convention that has the advantage to be closer to the customary way of drawing random walk trajectories.

If now we allow what is usually called a *boundary magnetic field*, that is if we set for example $h((i, -L_2 + 2)) = -h < 0$ for $i = -L_1 + 1, \ldots, L_1 - 1$, spins of value -1 are favored in the sites on which we have put the field (the sites on which we put the boundary field are marked by large black dots). What is the effect of the boundary field on the random walk trajectory? The answer is simply that there is a reward of $h > 0$ for the walk to stick to the bottom line: we are therefore just dealing with a homogeneous pinning model.

Two remarks to close this issue are in order.

1. Of course all that we have discussed becomes more delicate and definitely not as straightforward if $J_1 < \infty$, nonetheless the simplified $J_1 = \infty$ case to a certain

extent turns out not to be an oversimplification (see e.g. [37] and references therein).
2. It is of course natural to choose $h(i, -L_2 + 2)$ with a non-trivial dependence on i (for example, we could choose them by coin tossing ($\pm h$)): the arising random walk model is a inhomogeneous pinning model that fits (and motivates!) the definition of inhomogeneous models of the next chapters.

2.5.3 DNA Denaturation: The Poland–Scheraga Model

Understanding the very complex geometrical structure in which two complementary DNA strands (two polymers) are found in cell nuclei is a long standing issue on which a lot of effort is invested. There are of course plenty of issues: we focus just on the fact that two complementary strands are not tightly bind all the times (as a matter of fact, unbinding is necessary in particular for copying the genetic code to another polymer, the RNA) and unbinding, above all local unbinding, happens *all the time* as a standard consequence of thermal fluctuations. Biologists and physicists have developed models for such a phenomenon and a basic, but apparently rather effective model, even on a quantitative level, is just based on pinning models [20]. Starting off in the most naive way we can model two-stranded DNA by two directed walks interacting via pinning potentials, see e.g. [30] and references therein. Since the difference of two independent random walks is still a random walk, we are dealing with a standard pinning model. However directed walk models lead to values of α that are in contrast with observations. In fact the three dimensional model, that corresponds to the walk in two dimensions plus the *fixed direction*, yields $\alpha = 0$, a case not treated in this notes that however leads to a C^∞ behavior of the free energy [22, Chap. 2], while there is a tendency to believe that the transition is first order, even if such a statement has to be taken with caution because real DNA experiments are not about infinitely long strands, see for example [6, 27]. To make a long story short, the bio-physical community seems to have settled that renewal pinning models with $\alpha \approx 1.15$ is a reasonably good model for DNA denaturation [6, 15]: however what is most important is that inhomogeneous interactions need to be taken into account, unless one is dealing with synthetic DNA made up by one strand containing only *Adenine* (respectively *Cytosine*) bases and the the other strand containing only *Thymine* (respectively *Guanine*) bases. A few more details can be found in Fig. 2.5 and its caption.

2.6 A Look at the Literature

Much of this chapter is devoted to the homogeneous pinning model. This model, at least in some random walk cases, has been the object of several works in the physical literature at the beginning of the 1980s proposing *different* exact solutions (e.g. [7, 29]), but the generalized model and a comprehensive view identifying the

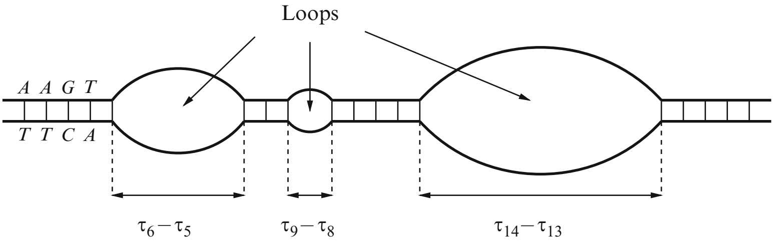

Fig. 2.5 A schematic, standard, view of DNA denaturation. The two *thick lines* are the DNA strands. They may be paired, gaining thus energetic contributions that depend on whether the base pair is A–T or G–C (the model is therefore inhomogeneous: A–T bonds are weaker than G–C bonds). The sections of unpaired bases are called *loops*. The DNA portion in the drawing corresponds to the renewal model trajectory with $\tau_j - \tau_{j-1} = 1$ except for three inter-arrivals (so loops correspond to inter-arrivals of length 2 or more)

general mechanism behind the *various exact solutions* are due to Fisher [19]. The approach given here, however, is not the one in [19], that goes by computing the series $\sum_N z^N Z_{N,h}$. We aim directly at $Z_{N,h}$ and at its interpretation in term of renewal function: this approach has been developed in [25, Appendix A] and [10]. It has been proven useful also beyond the renewal set-up, notably for Markov renewal processes that cover a very wide class of models: pinning and copolymers in periodic environments [11], pinning of directed semi-flexible polymers [9], pinning on layered interfaces [12] and pinning of random walks with continuous increments [33] (the Brownian motion case has been treated for example in [14, 31] by different techniques and we refer to [22] for further references on the vast literature on homogeneous pinning).

While of course the modeling aspects must not be neglected, the approach given here shows that the physical solution of the homogeneous pinning model (notably, free energy estimates) are just a subset of *classical* renewal theory developed in the 1950s and 1960s (e.g. [16, 17, 21], see [3] for further references).

Further considerations and references on path properties of the limit ($N \to \infty$) process can be found in [22, Chap. 2], notably the scaling limits at criticality that makes a link with the theory of *regenerative sets and subordinators* [5].

Dynamical issues have been left completely out and that will not change in the next chapters: these notes are about the equilibrium measure, but the dynamical issues are of great interest (see for example [8]).

In [2] the author provides a class of random walks with increments taking values ± 1 that have regularly varying return time distribution: therefore this work exhibits walks for which (2.30) holds, for arbitrary α.

Section 2.4 introduces the notion of correlation length: it is difficult to stress enough the role of such a concept in statistical mechanics. But it is also difficult to treat it in a satisfactory way without mentioning the *disordered* case: for here we content ourselves with adding the references [24, 34–36], that deal in part with the homogeneous case (and are very relevant for the disordered case), and [32] that develops a mathematical viewpoint on the *finite size scaling* properties of the homogeneous models, that is on the behavior of the system of correlation length size, close to criticality (when the correlation length diverges).

References

1. D.B. Abraham, *Surface structures and phase transitions, exact results*, in *Phase Transitions and Critical Phenomena*, vol. 10 (Academic, London, 1986), pp. 1–74
2. K.S. Alexander, Excursions and local limit theorems for Bessel-like random walks. Electron. J. Probab. **16**, 1–44 (2011)
3. S. Asmussen, *Applied Probability and Queues*, 2nd edn. (Springer, New York, 2003)
4. R.J. Baxter, *Exactly solved models in statistical mechanics* (Academic, London, 1982)
5. J. Bertoin, *Subordinators: examples and applications*. Lectures on probability theory and statistics (Saint-Flour, 1997), Lecture Notes in Mathematics, vol. 1717 (1999), pp. 1–91
6. R. Blossey, E. Carlon, Reparametrizing the loop entropy weights: effect on DNA melting curves. Phys. Rev. E **68**, 061911 (2003) (8 pages)
7. T.W. Burkhardt, Localization-delocalization transition in a solid-on-solid model with a pinning potential. J. Phys. A Math. Gen. **14**, L63–L68 (1981)
8. P. Caputo, F. Martinelli, F.L. Toninelli, On the approach to equilibrium for a polymer with adsorption and repulsion. Electron. J. Probab. **13**, 213–258 (2008)
9. F. Caravenna, J.-D. Deuschel, Pinning and wetting transition for (1+1)-dimensional fields with Laplacian interaction. Ann. Probab. **36**, 2388–2433 (2008)
10. F. Caravenna, G. Giacomin, L. Zambotti, Sharp asymptotic behavior for wetting models in (1+1)-dimension. Electron. J. Probab. **11**, 345–362 (2006)
11. F. Caravenna, G. Giacomin, L. Zambotti, A renewal theory approach to periodic copolymers with adsorption. Ann. Appl. Probab. **17**, 1362–1398 (2007)
12. F. Caravenna, N. Pétrélis, A polymer in a multi-interface medium. Ann. Appl. Probab. **19**, 1803–1839 (2009)
13. J.B. Conway, *Functions of One Complex Variable*, 2nd edn. Graduate Texts in Mathematics, vol. 11 (Springer, New York, 1978)
14. M. Cranston, L. Koralov, S. Molchanov, B. Vainberg, Continuous model for homopolymers. J. Funct. Anal. **256**, 2656–2696 (2009)
15. D. Cule, T. Hwa, Denaturation of heterogeneous DNA. Phys. Rev. Lett. **79**, 2375–2378 (1997)
16. W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. I, 3rd edn. (Wiley, New York, 1968)
17. W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. II, 2nd edn. (Wiley, New York, 1971)
18. R. Fernández, J. Frölich, A.D. Sokal, *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory*. Texts and Monographs in Physics (Springer, New York, 1992)
19. M.E. Fisher, Walks, walls, wetting, and melting. J. Stat. Phys. **34**, 667–729 (1984)
20. M. Fixman, J.J. Freire, Theory of DNA melting curves. Biopolymers **16**, 2693–2704 (1977)
21. A. Garsia, J. Lamperti, A discrete renewal theorem with infinite mean. Comment. Math. Helv. **37**, 221–234 (1963)
22. G. Giacomin, *Random Polymer Models* (Imperial College Press, London, 2007)
23. G. Giacomin, Renewal convergence rates and correlation decay for homogeneous pinning models. Electron. J. Probab. **13**, 513–529 (2008)
24. G. Giacomin, *Renewal sequences, disordered potentials, and pinning phenomena*, in Spin Glasses: Statics and Dynamics, Progress in Probability, vol. 62 (2009), pp. 235–270
25. G. Giacomin, F.L. Toninelli, Smoothing effect of quenched disorder on polymer depinning transitions. Commun. Math. Phys. **266**, 1–16 (2006)

26. C.R. Heathcote, Complete exponential convergence and some related topics. J. Appl. Probab., **4**, 217–256 (1967)
27. Y. Kafri, D. Mukamel, L. Peliti, Why is the DNA denaturation transition first order? Phys. Rev. Lett. **85**, 4988–4991 (2000)
28. D.G. Kendall, *Unitary dilations of Markov transition operators and the corresponding integral representation of transition probability matrices*, in Probability and Statistics, ed. by U. Grenander (Almqvist and Wiksell, Stockholm, 1959), pp. 138–161
29. J.M.J. van Leeuwen, H.J. Hilhorst, Pinning of rough interface by an external potential. Phys. A **107**, 319–329 (1981)
30. D. Marenduzzo, A. Trovato, A. Maritan, Phase diagram of force-induced DNA unzipping in exactly solvable models. Phys. Rev. E **64**, 031901 (2001) (12 pages)
31. B. Roynette, M. Yor, *Penalising Brownian Paths*. Lecture Notes in Mathematics, vol. 1969 (Springer, New York, 2009)
32. J. Sohier, Finite size scaling for homogeneous pinning models. ALEA Lat. Am. J. Probab. Math. Stat. **6**, 163–177 (2009)
33. J. Sohier, Phénomènes d'accrochage et théorie des fluctuations, PhD thesis, Univ. Paris Diderot, November 2010
34. F.L. Toninelli, Critical properties and finite-size estimates for the depinning transition of directed random polymers. J. Stat. Phys. **126**, 1025–1044 (2007)
35. F.L. Toninelli, Correlation lengths for random polymer models and for some renewal sequences. Electron. J. Probab. **12**, 613–636 (2007)
36. F.L. Toninelli, *Localization transition in disordered pinning models. Effect of randomness on the critical properties*, in Methods of Contemporary Mathematical Statistical Physics, Lecture Notes in Mathematics, vol. 1970, 129–176 (2009)
37. Y. Velenik, Localization and delocalization of random interfaces. Probab. Surv. **3**, 112–169 (2006)

http://www.springer.com/978-3-642-21155-3

Disorder and Critical Phenomena Through Basic
Probability Models
École d'Été de Probabilités de Saint-Flour XL – 2010
Giacomin, G.
2011, XI, 130 p. 12 illus., Softcover
ISBN: 978-3-642-21155-3